Noncommutative conformally coupled scalar field cosmology and its commutative counterpart

G. D. Barbosa
Centro Brasileiro de Pesquisas Físicas, CBPF, Rua Dr. Xavier Sigaud 150, 22290-180, Rio de Janeiro, Brazil
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We study the implications of a noncommutative geometry of the minisuperspace variables for the Friedmann-Robertson-Walker universe with a conformally coupled scalar field. The investigation is carried out by means of a comparative study of the universe evolution in four different scenarios: classical commutative, classical noncommutative, quantum commutative, and quantum noncommutative, the last two employing the Bohmian formalism of quantum trajectories. The role of noncommutativity is discussed by drawing a parallel between its realizations in two possible frameworks for physical interpretation: the NC frame, where it is manifest in the universe degrees of freedom, and in the C frame, where it is manifest through θ-dependent terms in the Hamiltonian. As a result of our comparative analysis, we find that noncommutative geometry can remove singularities in the classical context for sufficiently large values of θ. Moreover, under special conditions, the classical noncommutative model can admit bouncing solutions characteristic of the commutative quantum Friedmann-Robertson-Walker universe. In the quantum context, we find nonsingular universe solutions containing bounces or being periodic in the quantum commutative model. When noncommutativity effects are turned on in the quantum scenario, they can introduce significant modifications that change the singular behavior of the universe solutions or that render them dynamical whenever they are static in the commutative case. The effects of noncommutativity are completely specified only when one of the frames for its realization is adopted as the physical one. Nonsingular solutions in the NC frame can be mapped into singular ones in the C frame.

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I. INTRODUCTION

Over the last years a great deal of work and effort has been done in the direction of understanding canonical noncommutative field theories and quantum mechanics (see [1,2] and references therein). The recent interest in these theories is motivated by works that establish a connection between noncommutative geometry and string theory [3]. Intensive research is carried out to investigate their interesting properties, such as the IR-UV mixing and nonlocality [4], Lorentz violation [5], new physics at very short distances [1,2], and the equivalence between translations in the noncommutative directions and gauge transformations (see, e.g., [2,6]).

Several investigations have been pursued to verify the possible role of noncommutativity in a great deal of cosmological scenarios. Among them we quote Newtonian cosmology [7], cosmological perturbation theory and inflationary cosmology [8], noncommutative gravity [9], and quantum cosmology [10,11]. In a previous work [11], an investigation into the influence of noncommutativity of the minisuperspace variables in the early universe scenario was carried out for the Kantowski-Sachs universe. Although noncommutativity effects proved to be relevant to the universe history at intermediate times, they were shown not to be capable of removing the future and past cosmological singularities of that model in the classical context. In the quantum context, on the other hand, nonsingular universe solutions were shown to be present. However, since they exist for the commutative quantum Kantowski-Sachs universe, their presence in the ensemble of solutions of the noncommutative quantum model cannot be attributed to the noncommutativity effects.

Although the investigation carried out in [11] was restricted to a particular model, one expected that some of the results obtained there could be of general validity. Indeed, in this work we shall show that noncommutativity can appreciably modify the evolution of the Friedmann-Robertson-Walker (FRW) universe with a conformally coupled scalar field [12,13]. As in Ref. [11], our investigation is carried out by means of a comparative study of the universe evolution in four different scenarios: classical commutative, classical noncommutative, quantum commutative, and quantum noncommutative. The main motivation for the choice of the conformally coupled scalar field is that it admits exact solutions in the simpler cases discussed along this work and it is rich enough to be useful as a probe for the significant modifications noncommutative geometry introduces in classical and quantum cosmologies. The analytical treatment renders easy the study of the singular behavior of the model in its four versions. As we shall show later, even in the classical context noncommutative geometry can remove singularities. Moreover, depending on the value of the noncommutative parameter, noncommutative classical models can mimic quantum effects.

The application of the Copenhagen interpretation in quantum cosmology has many difficulties, as stressed along the years, e.g., by Everett [14], Gell-Mann, Hartle, Omnès and Griffiths [15], Bell [16], and 't Hooft [17].

*Email address: gbarbosa@cbpf.br

Among the variety of technical and conceptual problems that are present in quantum cosmology are the issue of time and the definition of probability [13,18,19], the latter related to the fact that the Wheeler-DeWitt equation is of Klein-Gordon type. One way to circumvent these problems is by adopting a nonepistemological interpretation for quantum theory, such as the one proposed by Bohm in 1952 [20,21]. This interpretation was further developed by several followers (see [22–26] and references therein), applied in quantum cosmology and quantum gravity [27], and is presently an object of interest for broad community (see, e.g., [28]). In addition to its capability to circumvent problems in quantum cosmology, a motivation for the adoption of the Bohmian interpretation in this work is the efficient framework it provides for comparison between the classical and quantum counterparts of a physical model in the common language of trajectories. We shall benefit from this facility in our study of the four versions of the FRW universe.

This work is organized as follows. Sections II and III are devoted to a comparative study of the classical FRW universe with a conformally coupled scalar field and its noncommutative counterpart. In Sec. IV we present a brief review of Bohmian formalism of quantum trajectories and apply it in the analysis of the commutative quantum version of the universe. A similar study is carried out in Secs. V and VI, which are concerned with the noncommutative quantum version of the model. In Sec. VII we end up with a general discussion and summary of the main results.

II. THE CONFORMALLY COUPLED SCALAR FIELD MODEL

As a reference for the identification of the noncommutative effects later, it is interesting to consider first the commutative classical FRW universe, which we describe as follows. We shall restrict our considerations to the case of constant positive curvature of the spatial sections. The action for the conformally coupled scalar field model in this case is [13]

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\phi_{;\mu}\phi^{;\mu} + \frac{1}{16\pi G}R - \frac{1}{12}R\phi^2\right], \tag{1}$$

where $g_{\mu\nu}$ is the four-metric, g its determinant, R is the scalar curvature, and ϕ is the scalar field. Units are chosen such that $\hbar = c = 1$ and $8\pi G = 3l_p^2$, where l_p is the Planck length. For the FRW model with a homogeneous scalar field the following ansatz of minisuperspace can be adopted,

$$\begin{cases} ds^2 = -N^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \\ \phi = \phi(t). \end{cases} \tag{2}$$

By substituting (2) in (1) and rescaling the scalar field as $\chi = \phi a l_p/\sqrt{2}$, we have the following minisuperspace action,[1]

$$S = \int dt\left(Na - \frac{a\dot{a}^2}{N} + \frac{a\dot{\chi}^2}{N} - \frac{N\chi^2}{a}\right). \tag{3}$$

The corresponding Hamiltonian is

$$H = N\left[-\frac{P_a^2}{4a} + \frac{P_\chi^2}{4a} - a + \frac{\chi^2}{a}\right] = N\mathcal{H}, \tag{4}$$

where the canonical momenta are

$$P_a = -\frac{2a\dot{a}}{N}, \qquad P_\chi = \frac{2a\dot{\chi}}{N}. \tag{5}$$

As the Poisson brackets for the classical phase space variables we have

$$\begin{aligned} \{a, \chi\} &= 0, & \{a, P_a\} &= 1, \\ \{\chi, P_\chi\} &= 1, & \{P_a, P_\chi\} &= 0. \end{aligned} \tag{6}$$

The equations of motion for the metric and matter field variables a, P_a, χ, and P_χ that follow from (4) and (6) are

$$\begin{cases} \dot{a} = \{a, H\} = -NP_a/2a, \\ \dot{P}_a = \{P_a, H\} = 2N, \\ \dot{\chi} = \{\chi, H\} = NP_\chi/2a, \\ \dot{P}_\chi = \{P_\chi, H\} = -2N\chi/a. \end{cases} \tag{7}$$

From now on we shall adopt conformal time gauge $N = a$. The general solution of (7) for a and χ in this gauge is

$$\begin{cases} a(t) = (A + C)\cos(t) + (B + D)\sin(t), \\ \chi(t) = (A - C)\cos(t) + (B - D)\sin(t), \end{cases} \tag{8}$$

where the super-Hamiltonian constraint $\mathcal{H} \approx 0$ imposes the relation

$$AC + BD = 0. \tag{9}$$

As it can be seen, the classical commutative solutions are necessarily singular in the past and in the future. Figures 1(a)–1(d) present plots of the solution for $a(t)$ in the dashed lines for given values of A, B, and C [D is fixed by (9)].

III. NONCOMMUTATIVE DEFORMATION OF THE CLASSICAL MODEL

Let us introduce a noncommutative classical geometry in our universe model by keeping the Hamiltonian with the same functional form as (4), but now valued on noncommutative variables,

[1] We have discarded total time derivatives and integrated out the spatial degrees of freedom since they are not relevant for the equations of motion.

$$H = N\left[-\frac{P_{a_{nc}}^2}{4a_{nc}} + \frac{P_{\chi_{nc}}^2}{4a_{nc}} - a_{nc} + \frac{\chi_{nc}^2}{a_{nc}}\right], \tag{10}$$

where a_{nc}, χ_{nc}, $P_{a_{nc}}$, and $P_{\chi_{nc}}$ satisfy the deformed Poisson brackets

$$\begin{aligned} \{a_{nc}, \chi_{nc}\} &= \theta, & \{a_{nc}, P_{a_{nc}}\} &= 1, \\ \{\chi_{nc}, P_{\chi_{nc}}\} &= 1, & \{P_{a_{nc}}, P_{\chi_{nc}}\} &= 0. \end{aligned} \tag{11}$$

By making the substitution

$$\begin{aligned} a_{nc} &= a_c - \frac{\theta}{2} P_{\chi_c}, & \chi_{nc} &= \chi_c + \frac{\theta}{2} P_{a_c}, \\ P_{a_{nc}} &= P_{a_c}, & P_{\chi_{nc}} &= P_{\chi_c}, \end{aligned} \tag{12}$$

the theory defined by (10) and (11) can be mapped into a theory where the metric and matter variables satisfy the Poisson brackets

$$\begin{aligned} \{a_c, \chi_c\} &= 0, & \{a_c, P_{a_c}\} &= 1, \\ \{\chi_c, P_{\chi_c}\} &= 1, & \{P_{a_c}, P_{\chi_c}\} &= 0. \end{aligned} \tag{13}$$

Written in terms of a_c, χ_c, P_{a_c}, and P_{χ_c}, the Hamiltonian (10) exhibits the noncommutative content of the theory through θ-dependent terms. Two distinct physical theories, one considering a_c and χ_c, and the other considering a_{nc} and χ_{nc}, as the physical scale factor and matter field can be assumed to emerge from (10)–(13). In the case where a_c and χ_c are assumed as the preferred variables for physical interpretation, the theory can be interpreted as a "commutative" one with a modified interaction. We shall refer to this theory as being realized in the "C frame." The other possible theory, which assumes a_{nc} and χ_{nc} as the constituents of the physical metric and matter field, we shall refer to as realized in the "NC frame." There are works that adopt the C-frame approach (e.g., [10]), and others the NC frame (e.g., [11,29,30]). Some works rely on the assumption that the difference between C and NC variables is negligible (e.g., [31]). However, as shown in [30], even in simple models the difference in behavior between these two types of variables can be appreciable. In what follows we will show that in the cosmological scenario the assumption of the NC- or C-frame point of view as preferential for physical interpretation leads to dramatic differences in the analysis of the universe history. A parallel between the theories in both frame realizations is drawn in the classical context here, and in the quantum context in Sec. VI.

Methodologically, the route we shall follow in the computation of the configuration variables in this classical context is the same as that adopted in the noncommutative quantum case discussed later on. We shall depart from the C frame and calculate $a_c(t)$, $\chi_c(t)$, $P_{a_c}(t)$, and $P_{\chi_c}(t)$. After that, we shall use (12) to obtain the corresponding $a_{nc}(t)$ and $\chi_{nc}(t)$ in the NC frame. The computation of the physical quantities is rendered simpler by the gauge choice $N = a_{nc}$, which from now on will be assumed as the gauge employed in all calculations.

The equations of motion for the variables $a_c(t)$, $\chi_c(t)$, $P_{a_c}(t)$, and $P_{\chi_c}(t)$ are

$$\begin{cases} \dot{a}_c = \{a_c, H\} = -(1/2)(1-\theta^2)P_{a_c} + \theta\chi_c, \\ \dot{P}_{a_c} = \{P_{a_c}, H\} = 2a_c - \theta P_{\chi_c}, \\ \dot{\chi}_c = \{\chi_c, H\} = (1/2)(1-\theta^2)P_{\chi_c} + \theta a_c, \\ \dot{P}_{\chi_c} = \{P_{\chi_c}, H\} = -2\chi_c - \theta P_{a_c}. \end{cases} \tag{14}$$

According to the values of θ, the system (14) allows three types of solutions, which we describe below.

A. Case $|\theta| < 1$

The general solutions for $a_c(t)$, $\chi_c(t)$, $P_{a_c}(t)$, and $P_{\chi_c}(t)$ in the $|\theta| < 1$ case are

$$\begin{cases} a_c(t) = (Ae^{\theta t} + Ce^{-\theta t})\kappa\cos(\kappa t) + (Be^{\theta t} + De^{-\theta t})\kappa\sin(\kappa t), \\ \chi_c(t) = (Ae^{\theta t} - Ce^{-\theta t})\kappa\cos(\kappa t) + (Be^{\theta t} - De^{-\theta t})\kappa\sin(\kappa t), \\ P_{a_c} = -2(Be^{\theta t} + De^{-\theta t})\cos(\kappa t) + 2(Ae^{\theta t} + Ce^{-\theta t})\sin(\kappa t), \\ P_{\chi_c} = 2(Be^{\theta t} - De^{-\theta t})\cos(\kappa t) + 2(-Ae^{\theta t} + Ce^{-\theta t})\sin(\kappa t), \end{cases} \tag{15}$$

where $\kappa = \sqrt{|1-\theta^2|}$. From (12) and (15) we can calculate $a_{nc}(t)$ and $\chi_{nc}(t)$ as

$$\begin{cases} a_{nc}(t) = [e^{\theta t}(A\kappa - B\theta) + e^{-\theta t}(C\kappa + D\theta)]\cos(\kappa t) + [e^{\theta t}(B\kappa + A\theta) + e^{-\theta t}(D\kappa - C\theta)]\sin(\kappa t), \\ \chi_{nc}(t) = [e^{\theta t}(A\kappa - B\theta) - e^{-\theta t}(C\kappa + D\theta)]\cos(\kappa t) + [e^{\theta t}(B\kappa + A\theta) - e^{-\theta t}(D\kappa - C\theta)]\sin(\kappa t). \end{cases} \tag{16}$$

The constraint $\mathcal{H} \approx 0$ in the present case can be written as

$$(AC + BD)\sqrt{1-\theta^2} + (AD - BC)\theta = 0. \tag{17}$$

The exponential factors that appear in (15) and (16) can model the shape of these solutions at intermediate times giving rise to bounces, as is depicted in the thin and thick solid lines in Fig. 1(a) for representative values of A, B, and C [D is fixed by (17)]. However, as in the commutative case, $a_{nc}(t)$ and $a_c(t)$ are unavoidably singular in the past and in the future. In the limit where $\theta \to 0$, Eq. (17) is reduced to Eq. (9). In the same limit, the NC- and C-frame

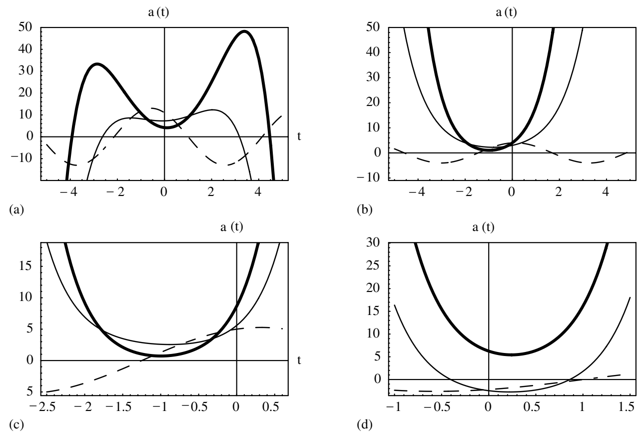

FIG. 1. The typical behavior of the scale factor of the noncommutative FRW universe in the NC-frame realization (thick lines) in contrast with C-frame realization (thin lines). The scale factor of the commutative counterpart appears plotted in the dashed lines. (a): $\theta = 3/4$, $A = 5$, $B = 3$, and $C = 6$. (b): $\theta = 1$, $A = 3$, $B = 2$, and $C = 1$. (c): $\theta = 3/2$, $A = 4$, $B = 3$, and $C = 1$. (d): $\theta = 3/2$, $A = -1.2$, $B = 2$, and $C = -1$.

solutions for the scale factor and matter field given by (15) and (16) coincide and match with the commutative solutions (8).

B. Case $\theta = \pm 1$

When $\theta = \pm 1$, the solutions for $a_c(t)$, $\chi_c(t)$, $P_{a_c}(t)$, and $P_{\chi_c}(t)$ are

$$\begin{cases} a_c(t) = A\cosh t + B\sinh t, \\ \chi_c(t) = \pm B\cosh t \pm A\sinh t, \\ P_{a_c} = 2(D + At)\cosh t + 2(C + Bt)\sinh t, \\ P_{\chi_c} = \mp 2(C + Bt)\cosh t \mp 2(D + At)\sinh t. \end{cases} \tag{18}$$

As the corresponding $a_{nc}(t)$ and $\chi_{nc}(t)$, we have

$$\begin{cases} a_{nc}(t) = (A + C + Bt)\cosh t + (B + D + At)\sinh t, \\ \chi_{nc}(t) = \pm(B + D + At)\cosh t \pm (A + C + Bt)\sinh t. \end{cases} \tag{19}$$

The constraint $\mathcal{H} \approx 0$ in the present case can be written as

$$A^2 - B^2 + 2(AC - BD) = 0. \tag{20}$$

As it can be seen, when $\theta = \pm 1$ the universe solutions are characterized by a qualitative behavior that differs from the one corresponding to the case where $|\theta| < 1$. There exist nonsingular bouncing solutions for both $a_{nc}(t)$ and $a_c(t)$, as depicted in Fig. 1(b). However, the correspondence between the NC and C frames can be broken for some values of the integration constants. Nonsingular solutions in the NC frame can correspond to singular solutions in the C frame. An interesting example is the case where $A = B = 0$ and $C > |D|$. This corresponds to a bouncing universe in the NC frame that has no counterpart in the C frame, where the universe is singular at all times.

C. Case $|\theta| > 1$

The general solutions for $a_c(t)$, $\chi_c(t)$, $P_{a_c}(t)$, and $P_{\chi_c}(t)$ when $|\theta| > 1$ are

$$\begin{cases} a_c(t) = (A\kappa e^{\theta t} + C\kappa e^{-\theta t})\cosh(\kappa t) + (B\kappa e^{\theta t} + D\kappa e^{-\theta t})\sinh(\kappa t), \\ \chi_c(t) = (A\kappa e^{\theta t} - C\kappa e^{-\theta t})\cosh(\kappa t) + (B\kappa e^{\theta t} - D\kappa e^{-\theta t})\sinh(\kappa t), \\ P_{a_c} = 2(Be^{\theta t} + De^{-\theta t})\cosh(\kappa t) + 2(Ae^{\theta t} + Ce^{-\theta t})\sinh(\kappa t), \\ P_{\chi_c} = 2(-Be^{\theta t} + De^{-\theta t})\cos(\kappa t) + 2(-Ae^{\theta t} + Ce^{-\theta t})\sin(\kappa t), \end{cases} \tag{21}$$

where $\kappa = \sqrt{|1 - \theta^2|}$ as before.

From (12) and (21) we can calculate the corresponding $a_{nc}(t)$ and $\chi_{nc}(t)$ as

$$\begin{cases} a_{nc}(t) = [e^{\theta t}(A\kappa + B\theta) + e^{-\theta t}(C\kappa - D\theta)]\cosh(\kappa t) + [e^{\theta t}(B\kappa + A\theta) + e^{-\theta t}(D\kappa - C\theta)]\sinh(\kappa t), \\ \chi_{nc}(t) = [e^{\theta t}(A\kappa + B\theta) - e^{-\theta t}(C\kappa - D\theta)]\cosh(\kappa t) + [e^{\theta t}(B\kappa + A\theta) - e^{-\theta t}(D\kappa - C\theta)]\sinh(\kappa t). \end{cases} \tag{22}$$

When $|\theta| > 1$, the constraint $\mathcal{H} \approx 0$ is reduced to

$$(BD - AC)\sqrt{\theta^2 - 1} + (AD - BC)\theta = 0. \tag{23}$$

As in the case where $\theta = \pm 1$, there are present nonsingular bouncing solutions in both NC and C frames. Figure 1(c) depicts one example. Again we found that, depending on the values of the integration constants, nonsingular universes in the NC frame can correspond to singular universes in the C frame [Fig. 1(d)]. After this last case it is now clear that $\theta = \pm 1$ establishes a division between qualitatively different ensembles of noncommutative universe solutions: one admitting only singular solutions ($|\theta| < 1$) and the other admitting singular as well as nonsingular solutions ($|\theta| \geq 1$).

IV. MINISUPERSPACE QUANTIZATION

Here we present the quantum version of the commutative universe model discussed in Sec. II. The FRW universe with a conformally coupled scalar field has already been investigated on the basis of the Wheeler-DeWitt equation in [12,13], the latter using Bohmian trajectories. However, in Ref. [13] there was a restriction to the regime of small scale parameter, and the wave functions considered were different from the ones studied in this work.

The quantization of the minisuperspace model is carried out here by employing the Dirac formalism (for details see [13]). By making the canonical replacement $P_a = -i\partial/\partial a$ and $P_\chi = -i\partial/\partial\chi$ in (4) we obtain, applying the Dirac quantization procedure,

$$\left[-\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \chi^2} + 4(a^2 - \chi^2)\right]\Psi(a, \chi) = 0, \tag{24}$$

which is the Wheeler-DeWitt equation for the conformally coupled scalar field model.[2] This equation can be solved by separating the a and χ variables, as it has been done in the literature (see [13,18] and references therein). Here, however, we shall follow an alternative route that is interesting because of the ensemble of solutions it generates in this case and is also suitable for the noncommutative case later on.

By making the coordinate change

$$a = \xi\cosh\eta, \qquad \chi = \xi\sinh\eta, \tag{25}$$

we can rewrite (24) as

$$\left[\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{1}{\xi^2}\frac{\partial^2}{\partial\eta^2}\right) - 4\xi^2\right]\Psi(\xi, \eta) = 0. \tag{26}$$

By plugging in the ansatz

$$\Psi(\xi, \eta) = R(\xi)e^{i\alpha\eta}, \tag{27}$$

in (26) we obtain, after simplification,

$$\frac{\partial^2 R}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial R}{\partial\xi} + \left(\frac{\alpha^2}{\xi^2} - 4\xi^2\right)R = 0. \tag{28}$$

A solution to (28) is

$$R(\xi) = AK_{i\alpha/2}(\xi^2) + BI_{i\alpha/2}(\xi^2),$$

where $K_\nu(x)$ and $I_\nu(x)$ are Bessel functions of the second kind, A and B are constants, and α is a real number. The solution of the Wheeler-DeWitt equation (26) is therefore

$$\Psi(\xi, \eta) = AK_{i\alpha/2}(\xi^2)e^{i\alpha\eta} + BI_{i\alpha/2}(\xi^2)e^{i\alpha\eta}. \tag{29}$$

Such a kind of wave function also appears, e.g., in the study of quantum wormholes [32] and in quantum cosmology of the Kantowski-Sachs universe [10,33]. The contribution corresponding to $I_\nu(x)$ is usually discarded because it leads to a solution that is divergent in the classically forbidden region of the potential.[3] We shall therefore consider only the $K_\nu(x)$ contribution and write the solution of (26) as[4]

$$\Psi(\xi, \eta) = \sum_\alpha A_\alpha K_{i\alpha/2}(\xi^2)e^{i\alpha\eta}. \tag{30}$$

A. Bohmian trajectory formalism

In order to establish a framework where all versions of the universe model can be compared, it is interesting to appeal to a common language. This is provided by the Bohmian formalism of quantum trajectories, which we briefly describe in this section (a detailed account of the subject may be found in the references given in the introduction). In the formulation presented here, we shall benefit from ideas proposed in [24]. The wave function will be assumed not as a constituent of the physical system (as originally assumed by Bohm [20]), but as a generator of its evolution law. Quantum information theory tells us that the wave function has a nonphysical character [34]. Bohmian quantum physics should in some way be in accordance with this fact. As the object of ontology of the Bohmian interpretation in this work we have the primordial quantum universe characterized, in the minisuperspace formalism, by the configuration variables a and χ.[5] With the aid of the wave function, we can determine how they evolve in time. The procedure is best illustrated in the context of nonrelativistic quantum mechanics.

Bohmian nonrelativistic quantum mechanics is concerned with the behavior of point particles that move in space describing quantum trajectories. An evolution law is ascribed to them according to the rule

$$\dot{x}^i = \mathrm{Re}\left\{\frac{1}{m}\frac{[\Psi^*(-i\hbar\partial_i)\Psi]}{\Psi^*\Psi}\right\} = \frac{\nabla S}{m}, \tag{31}$$

where Ψ is the wave function and S is obtained from the polar decomposition $\Psi = A\exp(iS)$. As in the orthodox interpretation, the wave function satisfies the Schrödinger

[2]A particular factor ordering is being assumed.

[3]From the point of view of the quantum trajectories, as it will be clear in the next subsection, there is no fundamental reason for the solution $I_\nu(x)$ to be discarded. However, since in this work our main interest is in the influence of noncommutativity in cosmology, rather than in the foundations of quantum theory, we shall give preference for wave functions that are also admissible in interpretations other than the Bohmian one.

[4]Since α is a continuous index, in the most general case the summation can be replaced by an integral.

[5]When dealing with a quantum theory one must have a clear picture of what it is essentially about, the *primitive ontology* of the theory [21]. The orthodox quantum theory based on the Copenhagen interpretation, e.g., is about observers that realize measurements. In the Bohmian interpretation, on the other hand, quantum theory is concerned with the physical systems, which can be particles, waves, strings, etc.

equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{32}$$

Equations (31) and (32) specify completely the theory. Without any other axiom, all phenomena governed by nonrelativistic quantum mechanics, from spectral lines and quantum interference phenomena to scattering theory, superconductivity and quantum computation follow from the analysis of the dynamical system defined by (31) and (32) [23]. The expectation value of a physical quantity associated with a Hermitian operator $\hat{A}(\hat{x}^i, \hat{p}^i)$ in the standard formalism can be computed in the Bohmian formulation by ensemble averaging the corresponding "beable"

$$\mathcal{B}(\hat{A}) = \mathrm{Re}\left\{\frac{[\Psi^*\hat{A}(\hat{x}^i, -i\hbar\partial_i)\Psi]}{\Psi^*\Psi}\right\} = A(x^i, t), \tag{33}$$

which represents the same quantity when seen from the Bohmian perspective.[6] In the context of nonrelativistic quantum mechanics, it can be shown from first principles that for an ensemble of particles obeying the evolution law (31) the associated probability density in the configuration space must be given by $\rho = |\Psi|^2$ [23]. This is why by computing the ensemble average of $A(x^i, t)$,

$$\int d^3x\rho A(x^i, t) = \int d^3x\Psi^*\hat{A}(\hat{x}^i, -i\hbar\partial_i)\Psi = \langle\hat{A}\rangle_t, \tag{34}$$

we arrive at the same results of the standard operatorial formalism. Notice that the law of motion (31) can itself be obtained from (33) by associating $\dot{x}^i$ with the beable corresponding to the velocity operator

$$\dot{x}^i = \mathcal{B}(i[\hat{H}, \hat{x}^i]) = \frac{\nabla S}{m}. \tag{35}$$

As a consequence of being an objective theory of point particles describing trajectories in space, Bohmian quantum mechanics does not give probability a prominent role. Instead, as discussed in [23], such a formulation probability is a derived concept, a decurrent of the law of motion of the point particles. The Bohmian formulation is therefore eminently suitable for the study of individual systems, such as the primordial quantum universe. In what follows we will be concerned with the application of the theory to the commutative quantum universe discussed in the beginning of this section. In the next sections a similar study will be carried out for the noncommutative quantum case.

[6] Holland [22] calls the procedure defined in Eq. (33) as "taking the local expectation value" of the observable $\hat{A}$. Such a nomenclature is not adopted here because it is unsuitable to be used in theories where the object of ontology is an individual system, as in quantum cosmology.

B. Application to quantum cosmology

This subsection is devoted to the application of Bohmian formalism in the determination of the evolution of the quantum FRW universe with a conformally coupled scalar field. In the description of quantum cosmology employing quantum trajectories we shall extend the evolution law (35) to the minisuperspace variables. In the commutative case the resulting Bohmian minisuperspace formalism matches with the minisuperspace version of the Bohmian quantum gravity proposed in [22], and employed to study the conformally coupled scalar model in [13]. From (35) we find, in the gauge $N = a$,

$$\begin{aligned}\dot{a} &= \mathrm{Re}\left\{\frac{[\Psi^*(-i\partial_a/2)\Psi]}{\Psi^*\Psi}\right\} = -\frac{1}{2}\frac{\partial S}{\partial a},\\ \dot{\chi} &= \mathrm{Re}\left\{\frac{[\Psi^*(-i\partial_\chi/2)\Psi]}{\Psi^*\Psi}\right\} = \frac{1}{2}\frac{\partial S}{\partial\chi}.\end{aligned} \tag{36}$$

By changing (36) into the (ξ, η) coordinates defined by (25) we obtain

$$\frac{d\xi}{dt} = -\frac{1}{2}\frac{\partial S(\xi, \eta)}{\partial\xi}, \qquad \frac{d\eta}{dt} = \frac{1}{2\xi^2}\frac{\partial S(\xi, \eta)}{\partial\eta}. \tag{37}$$

In what follows we shall solve the system (37) in two examples of a universe characterized by a wave function of the type (30).

1. Case 1

Let us consider first the example where there is a single Bessel function in (30). In this case the wave function is given by

$$\Psi(\xi, \eta) = AK_{i\alpha/2}(\xi^2)e^{i\alpha\eta}, \tag{38}$$

where A is a constant. Since the Bessel function $K_{i\nu}(x)$ is real for ν real and $x > 0$,[7] the phase can be read directly from the exponential in (38): $S = \alpha\eta$. The equations of motion (37) in this state are therefore reduced to

$$\frac{d\xi}{dt} = 0, \qquad \frac{d\eta}{dt} = \frac{\alpha}{2\xi^2}, \tag{39}$$

whose solutions are

$$\xi = \xi_0, \qquad \eta = \frac{\alpha}{2\xi_0^2}t + \eta_0. \tag{40}$$

The corresponding $a(t)$ and $\chi(t)$ obtained from (25) are

[7] This can be verified by looking at the integral representation (8.432) on page 958 of Ref. [35].

given by

$$a(t) = \xi_0 \cosh\left(\frac{\alpha}{2\xi_0^2} t + \eta_0\right), \qquad \chi(t) = \xi_0 \sinh\left(\frac{\alpha}{2\xi_0^2} t + \eta_0\right). \tag{41}$$

Quantum effects can therefore remove the cosmological singularity, giving rise to bouncing universes. Under suitable conditions, solution (41) can represent a quantum universe that is indistinguishable from a noncommutative classical universe. This is seen by noting that it can be mapped into the solution for $a_c(t)$ in the case where $\theta = \pm 1$ [Eq. (18)] by identifying $\alpha/2\xi_0^2 = 1$, $\xi_0 = A$, and $\eta_0 = B = 0$, or to the solution for $a_{nc}(t)$ [Eq. (19)] by identifying $\alpha/2\xi_0^2 = 1$, $\xi_0 = C$, and $\eta_0 = A = B = D = 0$. However, since (41) differ radically from circular trigonometric functions that characterize the classical commutative solutions, no classical commutative limit is admissible for the universe under consideration.

In the case where $\alpha = 0$, Eq. (41) describes a static universe with an arbitrary large scale factor, a highly nonclassical behavior.

2. Case 2

Let us now consider a wave function that is a superposition of two Bessel functions in (30), that is

$$\Psi(\xi, \eta) = A_1 K_{i\mu/2}(\xi^2) e^{i\mu\eta} + A_2 K_{i\nu/2}(\xi^2) e^{i\nu\eta}. \tag{42}$$

As the corresponding phase we find

$$S(\xi, \eta) = \arctan\left[\frac{A_1 K_{i\mu/2}(\xi^2)\sin(\mu\eta) + A_2 K_{i\nu/2}(\xi^2)\sin(\nu\eta)}{A_1 K_{i\mu/2}(\xi^2)\cos(\mu\eta) + A_2 K_{i\nu/2}(\xi^2)\cos(\nu\eta)}\right], \tag{43}$$

where the A_1 and A_2 were chosen as real coefficients. The equations of motion (37) for this state are

$$\frac{d\xi}{dt} = -\frac{A_1 A_2 [K'_{i\mu/2} K_{i\nu/2} - K_{i\mu/2} K'_{i\nu/2}]\xi \sin[(\mu - \nu)\eta]}{A_1^2 K_{i\mu/2}^2 + A_2^2 K_{i\nu/2}^2 + 2A_1 A_2 K_{i\mu/2} K_{i\nu/2}\cos[(\mu - \nu)\eta]}, \tag{44}$$

$$\frac{d\eta}{dt} = \frac{1}{2\xi^2}\frac{\mu A_1^2 K_{i\mu/2}^2 + \nu A_2^2 K_{i\nu/2}^2 + (\mu + \nu) A_1 A_2 K_{i\mu/2} K_{i\nu/2}\cos[(\mu - \nu)\eta]}{A_\mu^2 K_{i\mu/2}^2 + A_\nu^2 K_{i\nu/2}^2 + 2A_1 A_2 K_{i\mu/2} K_{i\nu/2}\cos[(\mu - \nu)\eta]}, \tag{45}$$

where prime means derivative with respect to the argument. The system (44) and (45) is a set of nonlinear coupled differential equations. Analytical solutions are difficult to find. Numerical solutions, on the other hand, can be easily computed for $\xi(t)$ and $\eta(t)$. Once the solutions for $\xi(t)$ and $\eta(t)$ are found, the corresponding $a(t)$ and $\chi(t)$ are determined from (25).

The qualitative properties of the solutions of an autonomous system such as (44) and (45) can be determined by analyzing the associated field of velocities. From the right hand side (RHS) of (44) and (45) we can see that the field of velocities has its direction inverted by the substitution $\mu \to -\mu$, $\nu \to -\nu$. Therefore, to have a qualitative picture of the associated flow one must be concerned only with the relative sign of μ and ν. For simplicity, let us fix $A_1 = A_2 = 1/\sqrt{2}$, and, without loss of generality, consider $\mu > 0$. The direction fields corresponding to the two possible combinations of relative sign between μ and ν are depicted in Figs. 2(a) and 2(b) for representative values of these constants, where the orbits of some solutions are also plotted. As it can be seen, the case with positive μ and negative ν favors the formation of closed orbits, which correspond to nonsingular cyclic universes. The case with positive μ and ν, on the other hand, tends to favor the open orbits.

The presence of the trigonometric functions in the RHS of (44) and (45) is the reason for the repetitive pattern of the direction fields observed along the η direction with period $2\pi/|\mu - \nu| \simeq 5.32$ in Figs. 2(a) and 2(b). The systematic appearance of closed orbits along the η direc-

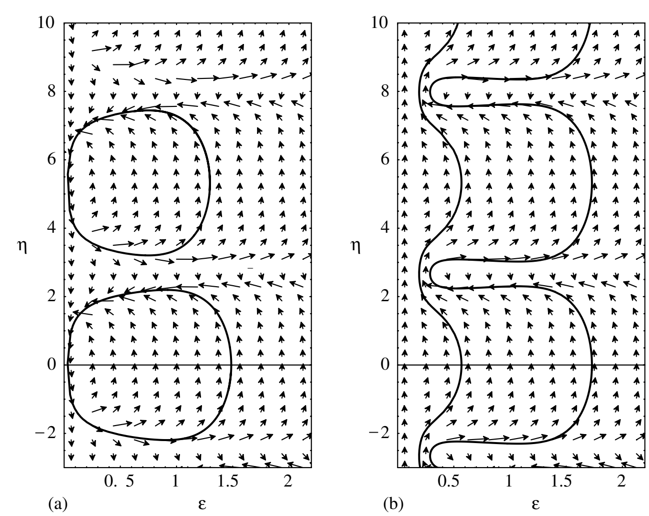

FIG. 2. The field of directions and selected orbits corresponding to the Bohmian differential equations for the commutative FRW universe in two cases: (a): $\mu = 0.6$, $\nu = -0.58$. Orbits: $\xi_0 = 1.2$, $\eta_0 = 2$, and $\xi_0 = 1.2$, $\eta_0 = 4$. (b): $\mu = 0.6$, $\nu = 1.78$. Orbits: $\xi_0 = 0.2$, $\eta_0 = 3$, and$\xi_0 = 1.7$, $\eta_0 = 4.5$.

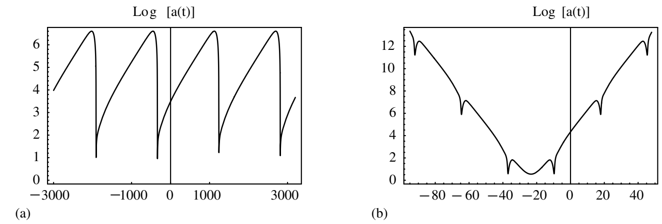

FIG. 3. The typical behavior of the scale factor of the commutative FRW universe. (a): $\mu = 0.6$, $\nu = -0.58$, $\xi_0 = 1.2$, and $\eta_0 = 4$. (b): $\mu = 0.6$, $\nu = 1.78$, $\xi_0 = 1.7$, and $\eta_0 = 4.5$.

tion in Fig. 2(a), and the possibility of varying their amplitudes with an appropriate choice of the initial conditions and the constants μ and ν, show us that the cyclic universe solutions can present a variable $a_{\min}$. Another quantity that is variable is the number of e folds between its maximum and minimum size configurations. This information can be read directly from the logarithmic plot of $a(t)$ in Fig. 3(a), where the solution depicted corresponds to one of the closed orbits of Fig. 2(a). Another interesting nonsingular solution type is depicted in Fig. 3(b). This corresponds to one of the open orbits of Fig. 2(b), and is an example of a universe that undergoes a sequence of bounces starting in the infinite past and never ending.

A different nonsingular solution type is present in the case where $\mu = -\nu$. Since the phase of such a kind of state is $S = 0$, the corresponding universe is necessarily static.

V. THE NONCOMMUTATIVE QUANTUM MODEL

After having studied the individual manifestation of noncommutative and quantum effects in the conformally coupled scalar field universe, we now study the combination of them in a unique model. This is achieved by considering the quantum version of Eq. (11):

$$\begin{aligned} &[\hat{a}_{nc}, \hat{\chi}_{nc}] = i\theta, \qquad [\hat{a}_{nc}, \hat{P}_{a_{nc}}] = i, \\ &[\hat{\chi}_{nc}, \hat{P}_{\chi_{nc}}] = i, \qquad [\hat{P}_{a_{nc}}, \hat{P}_{\chi_{nc}}] = 0. \end{aligned} \tag{46}$$

According to the Weyl quantization procedure [2,3], the commutation relation above between the observables $\hat{a}$ and $\hat{\chi}$ can be realized in terms of commutative functions by making use of the Moyal star product defined as

$$\begin{aligned} &f(a_c, \chi_c) \star g(a_c, \chi_c) \\ &\quad = f(a_c, \chi_c) e^{i(\theta/2)(\overleftarrow{\partial}_{a_c}\overrightarrow{\partial}_{\chi_c} - \overleftarrow{\partial}_{\chi_c}\overrightarrow{\partial}_{a_c})} g(a_c, \chi_c). \end{aligned} \tag{47}$$

The commutative coordinates a_c and χ_c are called Weyl symbols of the operators $\hat{a}$ and $\hat{\chi}$, respectively. A Wheeler-DeWitt equation that can be adopted for the noncommutative scalar field universe is

$$[\hat{P}^2_{a_c} - \hat{P}^2_{\chi_c}]\Psi(a_c, \chi_c) + 4(a_c^2 - \chi_c^2) \star \Psi(a_c, \chi_c) = 0, \tag{48}$$

which is obtained by Moyal deforming[8] (24). By using the properties of the Moyal product, it is possible to write (48) as

$$[\hat{P}^2_a - \hat{P}^2_\chi]\Psi(a_c, \chi_c) + 4(\hat{a}^2 - \hat{\chi}^2)\Psi(a_c, \chi_c) = 0, \tag{49}$$

where

$$\begin{aligned} &\hat{a}_{nc} = \hat{a}_c - \frac{\theta}{2}\hat{P}_{\chi_c}, \qquad \hat{\chi}_{nc} = \hat{\chi}_c + \frac{\theta}{2}\hat{P}_{a_c}, \\ &\hat{P}_{a_{nc}} = \hat{P}_{a_c}, \qquad \hat{P}_{\chi_{nc}} = \hat{P}_{\chi_c}. \end{aligned} \tag{50}$$

Equation (50) is nothing but the operatorial version of Eq. (12). The notations a_c and χ_c are now justified. These symbols match exactly with the canonical variables defined by (12). Here we are faced with the same situation as in the classical case. Two consistent cosmologies are possible. One considering $\hat{a}_c$ and $\hat{\chi}_c$ as the operators associated with the physical metric and matter field, and the other considering $\hat{a}$ and $\hat{\chi}$. In the worked examples we shall consider the two possibilities.

A. Noncommutative Bohmian formalism

In order to draw a parallel between the noncommutative quantum universe and the other three universe types, it is necessary to have a prescription of how to compute Bohmian trajectories in noncommutative quantum cosmology. The simplest way to do this is by extending the Bohmian formulation discussed in Sec. IV along the same lines proposed in [11]. The procedure consists of departing from the C frame and using the beable mapping (33) to ascribe an evolution law to the canonical variables. In our time gauge for the noncommutative cosmology, $N = a_{nc}$ (see Sec. III), the Hamiltonian (10) reduces simply to

$$h = \left[-\frac{P^2_{a_{nc}}}{4} + \frac{P^2_{\chi_{nc}}}{4} - a^2_{nc} + \chi^2_{nc}\right]. \tag{51}$$

We can therefore use h to generate time displacements and obtain the Bohmian equations of motion for $a_c(t)$ and $\chi_c(t)$ as

$$\begin{cases} \dot{a}_c = \mathcal{B}(i[\hat{h}, \hat{a}_c]) = -(1/2)(1-\theta^2)\partial_{a_c}S + \theta\chi_c, \\ \dot{\chi}_c = \mathcal{B}(i[\hat{h}, \hat{\chi}_c]) = (1/2)(1-\theta^2)\partial_{\chi_c}S + \theta a_c. \end{cases} \tag{52}$$

The connection between the C- and NC-frame variables is established by applying the beable mapping to the operatorial equations (50), that is, by defining $a \equiv \mathcal{B}(\hat{a})$ and $\chi \equiv \mathcal{B}(\hat{\chi})$. Once the trajectories are determined in the C frame, one can find their counterparts in the NC frame by evaluating the variables a and χ along the C-frame trajectories,

[8] For details about this procedure see [10,11].

$$\begin{cases} a_{nc}(t) = \mathcal{B}(\hat{a})|_{\substack{a_c=a_c(t)\\ \chi_c=\chi_c(t)}} = a_c(t) - (\theta/2)\partial_{\chi_c} S[a_c(t), \chi_c(t)], \\ \chi_{nc}(t) = \mathcal{B}(\hat{\chi})|_{\substack{a_c=a_c(t)\\ \chi_c=\chi_c(t)}} = \chi_c(t) + (\theta/2)\partial_{a_c} S[a_c(t), \chi_c(t)]. \end{cases} \tag{53}$$

In what follows we shall illustrate the application of the formalism in noncommutative quantum cosmology.

VI. APPLICATION TO NONCOMMUTATIVE QUANTUM COSMOLOGY

By using the representations $P_{a_c} = -i\partial_{a_c}$ and $P_{\chi_c} = -i\partial_{\chi_c}$, we can write the noncommutative Wheeler-DeWitt equation (49) as

$$\left[(1-\theta^2)\left(-\frac{\partial^2}{\partial a_c^2} + \frac{\partial^2}{\partial \chi_c^2}\right) + 4(a_c^2 - \chi_c^2) + 4i\theta\left(\chi_c \frac{\partial}{\partial a_c} + a_c \frac{\partial}{\partial \chi_c}\right)\right]\Psi(a_c, \chi_c) = 0. \tag{54}$$

The separation of variables can be made by changing to the new set of coordinates

$$a_c = \xi \cosh\eta, \qquad \chi_c = \xi \sinh\eta, \tag{55}$$

which allow us to rewrite (54) as

$$\left[(1-\theta^2)\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial^2}{\partial \eta^2}\right) - 4i\theta\frac{\partial}{\partial \eta} - 4\xi^2\right]\Psi(\xi, \eta) = 0. \tag{56}$$

The computation of the Bohmian trajectories is rendered easy by expressing the equations of motion (52) in the same hyperbolic coordinates as the wave function. After the change of variables, the Bohmian equations of motion can be written as

$$\begin{aligned} \frac{d\xi}{dt} &= -\frac{1}{2}(1-\theta^2)\frac{\partial S(\xi, \eta)}{\partial \xi}, \\ \frac{d\eta}{dt} &= \frac{1}{2\xi^2}(1-\theta^2)\frac{\partial S(\xi, \eta)}{\partial \eta} + \theta. \end{aligned} \tag{57}$$

Equations (53), responsible for the NC-C-frame correspondence, can be written in the new set of coordinates as

$$\begin{cases} a_{nc}(t) = a_c(t) + (\theta/2)\sinh\eta\,\partial_\xi S[\xi(t), \eta(t)] - (\theta/2\xi)\cosh\eta\,\partial_\eta S[\xi(t), \eta(t)], \\ \chi_{nc}(t) = \chi_c(t) + (\theta/2)\cosh\eta\,\partial_\xi S[\xi(t), \eta(t)] - (\theta/2\xi)\sinh\eta\,\partial_\eta S[\xi(t), \eta(t)]. \end{cases} \tag{58}$$

Before starting our comparative study by computing Bohmian trajectories corresponding to specific solutions of (56), let us discuss the case of real wave functions.[9] While in the commutative Bohmian quantum cosmology real wave functions always represent static universes, in the noncommutative Bohmian quantum cosmology they can represent dynamical universes, a property pointed out in [11] for the Kantowski-Sachs model. In the FRW with a conformally coupled scalar field under consideration, Eqs. (57) tell us that real wave functions always correspond to a nontrivial and identical dynamics. Moreover, from (58) we can see that when $S = 0$ the NC- and C-frame realizations are indistinguishable, representing the same universe. This universe is determined by solving Eqs. (57) and substituting the solutions in (55). As a result, we find

$$\begin{cases} a_{nc}(t) = a_c(t) = \xi_0 \cosh(\theta t + \eta_0), \\ \chi_{nc}(t) = \chi_c(t) = \xi_0 \sinh(\theta t + \eta_0). \end{cases} \tag{59}$$

Real wave functions therefore always represent nonsingular bouncing universes. Complex wave functions, on the other hand, can give rise to a great variety of dynamics, where the distinction between the frames of physical realization can be crucial. In the same way as in the classical analog, we can distinguish three cases: $|\theta| < 1$, $\theta = \pm 1$, and $|\theta| > 1$.

A. Case $|\theta| < 1$

In this case, Eq. (56) can be solved by using the ansatz

$$\Psi(\xi, \eta) = R(\xi)e^{i\alpha\eta}. \tag{60}$$

As a result, we find

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial R}{\partial \xi} + \left(\frac{\alpha^2}{\xi^2} + \frac{4\theta\alpha}{1-\theta^2} - \frac{4\xi^2}{1-\theta^2}\right)R = 0, \tag{61}$$

whose solution is

$$R(\xi) = A_\alpha\left(\frac{\pi}{2}\right)^{1/2}\xi^{-1}W_{\alpha\theta/2\kappa, i\alpha/2}\left(\frac{2\xi^2}{\kappa}\right) + B_\alpha\left(\frac{\pi}{2}\right)^{1/2}\xi^{-1}M_{\alpha\theta/2\kappa, i\alpha/2}\left(\frac{2\xi^2}{\kappa}\right), \tag{62}$$

where $\kappa = \sqrt{|1-\theta^2|}$. $W_{u,\nu}(x)$ and $M_{\mu,\nu}(x)$ are Whittaker functions, A_α and B_α are constants, and α is a real number. The piece corresponding to the $M_{\mu,\nu}(x)$ contribution leads to a divergent wave function in the classically forbidden region. We shall therefore discard its contribution in a

[9] This particular case can be of special interest, since real wave functions are favored, e.g., by the nonboundary proposal for the initial conditions of the universe [18].

similar way as in the commutative model discussed before. We can thus write the solution of (56) as

$$\Psi(\xi, \eta) = \sum_{\alpha}\left[A_{\alpha}\left(\frac{\pi}{2}\right)^{1/2}\xi^{-1}W_{\alpha\theta/2\kappa, i\alpha/2}\left(\frac{2\xi^2}{\kappa}\right)e^{i\alpha\eta}\right]. \tag{63}$$

In the limit where $\theta = 0$, the Whittaker functions $W_{\mu,i\nu}(x)$ are reduced to the Bessel functions $K_{i\nu}$ through the relation

$$\xi^{-1}W_{0,i\alpha/2}(2\xi^2) = \left(\frac{2}{\pi}\right)^{1/2}K_{i\alpha/2}(\xi^2), \tag{64}$$

and therefore the wave function (63) matches with the commutative wave function (30).

In the sequel we present two examples of application of the Bohmian formalism in the investigation of the properties of the universe solutions.

1. Example 1

The wave function is of the type

$$\Psi(\xi, \eta) = A\xi^{-1}W_{\alpha\theta/2\kappa, i\alpha/2}\left(\frac{2\xi^2}{\kappa}\right)e^{i\alpha\eta}, \tag{65}$$

where A is a constant. Since the Whittaker function $W_{\mu,i\nu}(x)$ is real for μ and ν real and $x > 0$,[10] the phase can be read directly from the exponential: $S = \alpha\eta$. The equations of motion for ξ and η in this state are

$$\frac{d\xi}{dt} = 0, \qquad \frac{d\eta}{dt} = \frac{\alpha\kappa^2}{2\xi^2} + \theta. \tag{66}$$

As the solutions for $a_c(t)$ and $\chi_c(t)$, we have

$$a_c(t) = a_{c_0}\cosh(\sigma t), \qquad \chi_c(t) = a_{c_0}\sinh(\sigma t), \tag{67}$$

where $\sigma = \alpha\kappa^2/2a_{c_0}^2 + \theta$ and η_0 was absorbed by redefining the origin of time.

From (58) we can write

$$a_{nc}(t) = a_{nc_0}\cosh(\sigma t), \qquad \chi_{nc}(t) = a_{nc_0}\sinh(\sigma t), \tag{68}$$

where $a_{nc_0} = a_{c_0} - \alpha\theta/2a_{c_0}$. Although the solutions (67) and (68) in this case differ radically from their classical counterparts with $|\theta| < 1$ [see Eqs. (15) and (16)], they can be mapped into the noncommutative classical solutions with $\theta = \pm 1$ [Eqs. (18) and (19)] with a suitable identification of the integration constants. But this is not the only interesting property exhibited in this case. From (68) we can see that to each universe in the NC frame there corresponds at least one universe in the C frame, as it is shown in Fig. 4. For $\alpha\theta$ positive, the correspondence is one-to-one and exists only for $a_{c_0} > \sqrt{|\alpha\theta|/2}$. Smaller values of a_{c_0} would imply a singular universe in the NC frame.

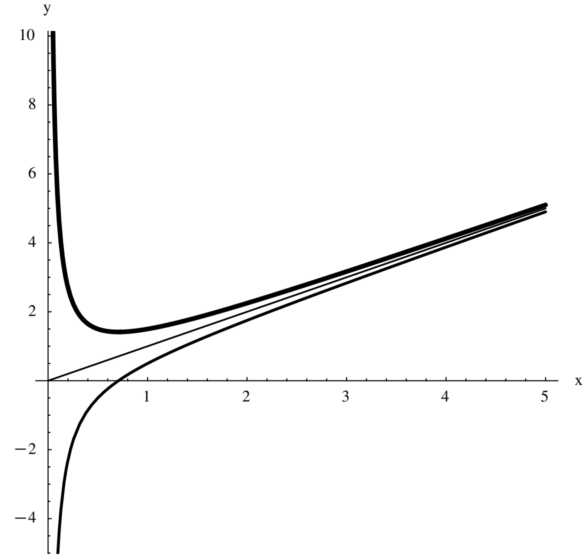

FIG. 4. The minimum value of the scale factor of the noncommutative quantum FRW universe with $|\theta| < 1$ in the NC frame $y = a_{nc_0}$ as a function of its value in the C frame $x = a_{c_0}$. The thick line refers to the case where $\alpha\theta < 0$, and the thin line to that where $\alpha\theta > 0$. The values adopted for $\alpha\theta$ are $\alpha\theta = -1$ and $\alpha\theta = 1$. The diagonal line is also plotted.

For negative values of $\alpha\theta$, on the other hand, the correspondence is defined for all values of a_{c_0}. To each universe in the NC frame there correspond two universes in the C frame. An exception occurs for $a_{c_0} = \sqrt{\alpha\theta/2}$, where the curve $a_{nc_0} \times a_{c_0}$ achieves its minimum and the correspondence is one-to-one. This value of a_{c_0} marks the division

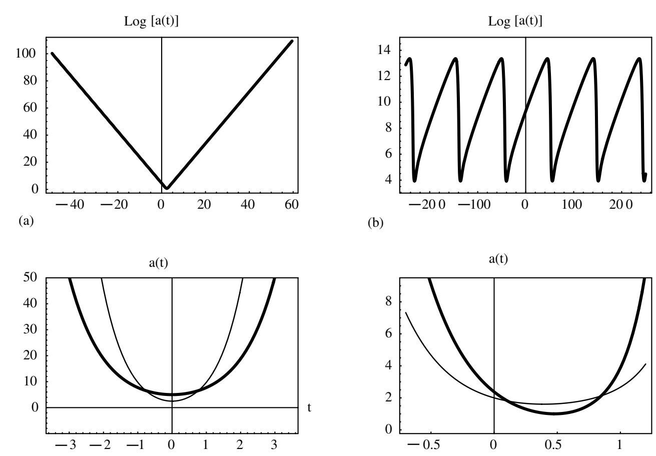

FIG. 5. Selected plots of the scale factor of the FRW universe in the NC-frame realization (thick lines) in contrast with C-frame realization (thin lines). (a): $\theta = -0.9$, $\mu = 0.6$, $\nu = 1.78$, $\xi_0 = 1.7$, and $\eta_0 = 4.5$. (b): $\theta = 0.1$, $\mu = 0.3$, $\nu = -0.3$, $\xi_0 = 1$, and $\eta_0 = 10$. (c): $\theta = 1$, $\xi_0 = 5$, and $\eta_0 = 0$. (d): $\theta = -1.5$, $\alpha = 1.5$, $a_0 = 2$, and $\eta_0 = 0$.

[10]This can be verified by looking at the integral representation (9.223) on page 1060 of Ref. [35].

between the two regimes that govern the NC-C-frame correspondence: large a_{nc_0} and small a_{c_0}, while the second deals with large $a_{nc_0} \simeq a_{c_0}$. A similar behavior was previously found in [30] in nonrelativistic Bohmian quantum mechanics when studying the harmonic oscillator. The capability of the noncommutativity effects to promote the interplay between large and small scale distances was interpreted in that reference as manifestation of a sort of "UR-UV mixing" in the oscillator orbits.

2. Example 2

The wave function (63) is a sum of two functions,

$$\Psi(\xi, \eta) = A_1 \xi^{-1} W_{\mu\theta/2\kappa, i\mu/2}\left(\frac{2\xi^2}{\kappa}\right) e^{i\mu\eta} + A_2 \xi^{-1} W_{\nu\theta/2\kappa, i\nu/2}\left(\frac{2\xi^2}{\kappa}\right) e^{i\nu\eta}. \tag{69}$$

The corresponding phase is

$$S(\xi, \eta) = \arctan\left[\frac{A_1 W_{\mu\theta/2\kappa,i\mu/2}\sin(\mu\eta) + A_2 W_{\nu\theta/2\kappa,i\nu/2}\sin(\nu\eta)}{A_1 W_{\mu\theta/2\kappa,i\mu/2}\cos(\mu\eta) + A_2 W_{\nu\theta/2\kappa,i\nu/2}\cos(\nu\eta)}\right], \tag{70}$$

where, as in the commutative case, A_1 and A_2 were chosen as real coefficients. The equations of motion (57) in this state are

$$\frac{d\xi}{dt} = -2\xi\kappa \frac{A_1 A_2 [W'_{\mu\theta/2\kappa,i\mu/2} W_{\nu\theta/2\kappa,i\nu/2} - W_{\mu\theta/2\kappa,i\mu/2} W'_{\nu\theta/2\kappa,i\nu/2}] \sin[(\mu-\nu)\eta]}{A_1^2 W^2_{\mu\theta/2\kappa,i\mu/2} + A_2^2 W^2_{\nu\theta/2\kappa,i\nu/2} + 2A_1 A_2 W_{\mu\theta/2\kappa,i\mu/2} W_{\nu\theta/2\kappa,i\nu/2}\cos[(\mu-\nu)\eta]}, \tag{71}$$

$$\frac{d\eta}{dt} = \frac{\kappa^2}{2\xi^2} \frac{\mu A_1^2 W^2_{\mu\theta/2\kappa,i\mu/2} + \nu A_2^2 W^2_{\nu\theta/2\kappa,i\nu/2} + A_1 A_2 (\mu+\nu) W_{\mu\theta/2\kappa,i\mu/2} W_{\nu\theta/2\kappa,i\nu/2}\cos[(\mu-\nu)\eta]}{A_1^2 W^2_{\mu\theta/2\kappa,i\mu/2} + A_2^2 W^2_{\nu\theta/2\kappa,i\nu/2} + 2A_1 A_2 W_{\mu\theta/2\kappa,i\mu/2} W_{\nu\theta/2\kappa,i\nu/2}\cos[(\mu-\nu)\eta]} + \theta, \tag{72}$$

where prime means derivative with respect to the argument. As in the commutative counterpart, we have an autonomous set of nonlinear coupled differential equations to solve. In order to render easy the comparison with that case, let us fix $A_1 = A_2 = 1/\sqrt{2}$. Again its is possible to find bouncing solutions, as it is shown in Fig. 5(a), where the effect of noncommutativity is manifest through the suppression of the sequence of bounces that appears in the commutative counterpart and by shifting the time where the universe achieves its minimum size [see Fig. 3(b)]. The case where $\mu = -\nu$, whose commutative counterpart is a static universe, here has a nontrivial dynamics. An example is depicted in Fig. 5(b), where a cyclic solution similar to that of Fig. 3(a) is induced by noncommutativity effects. The splitting between NC- and C-frame evolutions in both cases mentioned here is quantitatively irrelevant.

B. Case $\theta = \pm 1$

As in the classical counterpart, the case $\theta = \pm 1$ is characterized by a peculiar behavior. Equation (56) in this case is reduced to the first order partial differential equation

$$\left[i\frac{\partial}{\partial\eta} \pm \xi^2\right]\Psi(\xi, \eta) = 0, \tag{73}$$

whose general solution is

$$\Psi(\xi, \eta) = R(\xi) e^{\pm i\xi^2\eta}, \tag{74}$$

where $R(\xi)$ is any differentiable function of ξ. The equations of motion (57) in this case are reduced to

$$\frac{d\xi}{dt} = 0, \qquad \frac{d\eta}{dt} = \pm 1, \tag{75}$$

whose solutions are

$$\xi = \xi_0, \qquad \eta = \pm t + \eta_0. \tag{76}$$

As the solutions for $a_c(t)$ and $\chi_c(t)$, we have

$$a_c(t) = \xi_0 \cosh(\pm t + \eta_0), \qquad \chi_c(t) = \xi_0 \sinh(\pm t + \eta_0), \tag{77}$$

while the corresponding $a(t)$ and $\chi(t)$ are

$$\begin{cases} a_{nc}(t) = (\xi_0/2)\cosh(\pm t + \eta_0) + \xi_0(\pm t + \eta_0)\sinh(\pm t + \eta_0), \\ \chi_{nc}(t) = (\xi_0/2)\sinh(\pm t + \eta_0) + \xi_0(\pm t + \eta_0)\cosh(\pm t + \eta_0). \end{cases} \tag{78}$$

In both NC- and C-frame realizations of noncommutativity we find bounce solutions whenever $\xi_0 > 0$, while for $\xi_0 \leq 0$ the universe is necessarily singular [Fig. 5(c)]. The novelty here is that these universe solutions are the most general ones available. No matter what the wave function is, the fate of the universe is determined uniquely by ξ_0. At first sight it could seem strange that the wave function cannot have any influence on the fate of the universe, independent of its functional form. However, if we realize that the information provided by the wave function is about the universe evolution law, we find that the wave function is playing its role providing us Eqs. (75) in the same way as in all the other cases previously discussed. Since the kinetic term is quenched by noncommutativity effects, what we found in this case is exactly what could be expected: a poor and highly constrained dynamics, similar to the one that appears when a magnetic field projects a system onto its lowest Landau level (see [29] and references therein).

C. Case $|\theta| > 1$

In this last case the most general wave function that satisfies (54) can be written as

$$\Psi(\xi, \eta) = \sum_{\alpha}\bigg[A_\alpha \xi^{-1} W_{i\alpha\theta/2\kappa, i\alpha/2}\bigg(\frac{2i\xi^2}{\kappa}\bigg)e^{i\alpha\eta} + B_\alpha \xi^{-1} M_{i\alpha\theta/2\kappa, i\alpha/2}\bigg(\frac{2i\xi^2}{\kappa}\bigg)e^{i\alpha\eta}\bigg]. \quad (79)$$

Contrary to the previous cases, the contribution corresponding to $M_{i\alpha\theta/2\kappa,i\alpha/2}$ is not divergent in the classically forbidden region. Moreover, each of the Whittaker functions $W_{\mu,\nu}(x)$ and $M_{\mu,\nu}(x)$ in this case is complex, and therefore can give rise to a dynamics that differs from the ones of the examples previously discussed. For simplicity, we shall consider only the example where

$$\Psi(\xi, \eta) = A W_{i\alpha\theta/2\kappa, i\alpha/2}\bigg(\frac{2i\xi^2}{\kappa}\bigg)e^{i\alpha\eta}. \quad (80)$$

For this wave function, the equations of motion (57) for $\xi(t)$ and $\eta(t)$ can be shown to be

$$\begin{cases} \dot{\xi} = \kappa\xi - \kappa\alpha\theta/2\xi - \kappa/\xi\, \mathrm{Im}[W_{i\alpha\theta/2\kappa+1,i\alpha/2}(2i\xi^2/\kappa)/W_{i\alpha\theta/2\kappa,i\alpha/2}(2i\xi^2/\kappa)], \\ \dot{\eta} = -\alpha\kappa^2/2\xi^2 + \theta. \end{cases} \quad (81)$$

Equations (81) can be solved analytically in the limit of large ξ, where the contribution coming from the term containing the Whittaker functions in the right hand side of (81) can be approximated by $-2\kappa\xi$. In this regime, (81) can be simplified to

$$\dot{\xi} = -\kappa\xi, \qquad \dot{\eta} = -\alpha\kappa^2/2\xi^2 + \theta. \quad (82)$$

The solutions of (82) are

$$\begin{cases} \xi(t) = \xi_0 e^{-\kappa t}, \\ \eta(t) = \alpha\kappa(1 - e^{2\kappa t})/4\xi_0^2 + \theta t + \eta_0. \end{cases} \quad (83)$$

As the expressions for $a_c(t)$ and $\chi_c(t)$ we have

$$\begin{cases} a_c(t) = a_{c_0} e^{-\kappa t}\cosh[g(t)], \\ \chi_c(t) = a_{c_0} e^{-\kappa t}\sinh[g(t)], \end{cases} \quad (84)$$

where η_0 was absorbed by redefining the origin of time and $g(t) = \alpha\kappa(1 - e^{2\kappa t})/4a_{c_0}^2 + \theta t$. The corresponding $a_{nc}(t)$ and $\chi_{nc}(t)$ obtained from (58) are

$$\begin{cases} a_{nc}(t) = f(t)e^{-\kappa t}\cosh[g(t)] - (\theta a_{c_0} e^{-\kappa t}/\kappa)\sinh[g(t)], \\ \chi_{nc}(t) = f(t)e^{-\kappa t}\sinh[g(t)] - (\theta a_{c_0} e^{-\kappa t}/\kappa)\cosh[g(t)], \end{cases} \quad (85)$$

where $f(t) = a_{c_0} - \alpha\theta e^{2\kappa t}/2a_{c_0}$.

From (83) we can see that the physical meaning of the approximation assumed is that of early times. Figure 5(d) depicts the scale factors $a_{nc}(t)$ and $a_c(t)$ in an interval where the approximation proposed is accurate. Depending on the values of θ, α, and a_{c_0}, the deviation in their behavior can be very large. The f function plays here a role similar to that of a_{nc_0} in Case 1, Sec. VI. In each instant of time $t = T$, the graph of $f(T)$ as a function of a_{c_0} is identical in shape to that of Fig. 4. The singular behavior of the solutions, however, can differ from that of Case 1, Sec. VI due to the presence of the $\sinh[g(t)]$ in the expression for $a_{nc}(t)$.

VII. DISCUSSION

In this work we carried out an investigation into the role of noncommutative geometry in the cosmological scenario by introducing a noncommutative deformation in the algebra of the minisuperspace variables along the same lines proposed in [10] and followed in [11]. As a cosmological model to carry out such an investigation, we chose a Friedmann-Robertson-Walker universe with a conformally coupled scalar field. A parallel was drawn between the realizations of noncommutativity in two possible frameworks for physical interpretation: the C frame, where it manifests the Hamiltonian through θ-dependent terms, and in the NC frame, where it is manifest directly in the universe degrees of freedom.

The influence of noncommutativity in the universe evolution and its capability to remove cosmological singularities was investigated by means of a comparative study of the FRW model in four different versions: classical commutative, classical noncommutative, quantum commutative, and quantum noncommutative. The confrontation between the classical and quantum versions was rendered easy by the Bohmian interpretation of quantum theory, which provided a common language for comparison through the quantum trajectory formalism. An extension of the Bohmian formulation to comprise noncommutative effects was previously proposed for the Kantowski-Sachs model in [11]. In our comparative study we have dealt with the noncommutative quantum model along the same lines. The beable mapping commonly employed in Bohmian quantum mechanics was extended to noncommutative quantum cosmology. In the commutative context, our formulation is reduced to the one proposed by Holland [22] in the minisuperspace approximation.

In the classical context, the main result of our investigation is that, contrary to the noncommutative Kantowski-Sachs model, for sufficiently large θ the noncommutative FRW can be nonsingular. When $|\theta| \geq 1$, noncommutativity can give rise to bouncing universes in the NC- and C-frame realizations. The $\theta = \pm 1$ case is of particular interest since it reveals that the capability noncommutativity has to mimic quantum effects under special conditions. The bouncing solutions that appear in both NC- and C-frame realizations [Eq. (18)] can be mapped into the commutative quantum solution (41) with an appropriate identification of the integration constants. Therefore a noncommutative classical quantum universe can be indistinguishable from a commutative quantum universe. Concerning the investigation into the NC- and C-frame correspondence, it was shown in Sec. III that when $\theta = \pm 1$ there exist classical universe solutions that are nonsingular in the NC frame and have no correspondent in the C frame, where they are singular at all times. Therefore the description of the universe evolution provided by these two possible scenarios for the realization of noncommutativity can differ radically.

While in the classical context nonsingular universe solutions can exist only in the noncommutative universe model and for $|\theta| \geq 1$, in the commutative quantum context one may find nonsingular universe solutions even in the commutative case. The qualitative behavior of the universe solutions in noncommutative quantum cosmology was discussed in Sec. IV, where examples were presented that contain nonsingular periodic and bouncing solutions. As in the classical cosmology, examples involving the NC-C frame were worked out in the noncommutative quantum cosmology for the cases $|\theta| < 1$, $\theta = \pm 1$, and $|\theta| > 1$. When noncommutativity effects are turned on in the quantum scenario, they give rise to dynamical universes in situations where Bohmian commutative quantum cosmology admits only static universes. In the model under consideration we showed that real wave functions always represent nonsingular bouncing universes in both NC and C frames. Complex wave functions, on the other hand, can give rise to a great variety of dynamics, where the distinction between the frames in the noncommutative quantum context can be crucial. An example was presented illustrating how a universe with large $a_{nc\,\min}$ in the NC frame can correspond to two universes in the C frame, one with large $a_{c\,\min} \simeq a_{nc\,\min}$, and the other with very small $a_{c\,\min}$. Such an interplay between small and large scale distances was previously reported in [30], where it was interpreted as a sort of "IR-UV mixing," in analogy with noncommutative field theory. As in the classical analog, the case where $\theta = \pm 1$ was shown to have peculiar properties. When $\theta = \pm 1$ noncommutativity effects act to drop the kinetic term from the Wheeler-DeWitt equation. This justifies the poor and highly constrained dynamics found in this case. No matter what the wave function is, if the initial conditions are nonsingular the universe is nonsingular and undergoes a single bounce in both NC- and C-frame descriptions [Fig. 5(c)].

If the noncommutativity of the minisuperspace variables has in fact played a role in the evolution of the primordial quantum universe (as proposed in [10]), the study carried out in this work renders evident the need of an ontology for the theory. This seems to be imperative in order that the essential features of the noncommutative universe models can be understood. The correspondence between degrees of freedom in two different frames of realization is not sufficient to define the theory completely, which is only fixed by assuming one of them as the physical frame. This necessity does not seem to be an exclusive feature of the cosmological model considered here, where the dramatic difference in the universe evolution can be attributed, in part, to the fact that the noncommutativity in question is that of the system's degrees of freedom—the minisuperspace variables. In the models where noncommutativity does not directly involve the system's degrees of freedom, as the canonical noncommutative field theories that come from string theory [3], the study of the correspondence between NC- and C-frame descriptions is also a relevant subject. In the context of gauge theories, where the connection between the NC and C frames is via the Seiberg-Witten map, an investigation into the properties of the theory that have resemblance with gravity was carried out, e.g., in [6], where the equivalence between spacetime translations and gauge transformations is shown to occur in the NC frame. In the C frame, on the other hand, where such an equivalence seems to be lost, noncommutative fields can be interpreted as ordinary theories immersed in a gravitational background generated by the gauge field, as shown in the interesting work by Rivelles [36], and further in [37].

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