

Dihedral families of quarks, leptons, and Higgs bosonsK. S. Babu¹ and Jisuke Kubo²¹*Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078, USA*²*Institute for Theoretical Physics, Kanazawa University, Kanazawa 920-1192, Japan*

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We consider finite groups of small order for family symmetry. It is found that the binary dihedral group Q_6 , along with the assumption that the Higgs sector is of type II, predicts mass matrix of a nearest neighbor interaction type for quarks and leptons. We present a supersymmetric model based on Q_6 with spontaneously induced CP phases. The quark sector contains 8 real parameters with one independent phase to describe the quark masses and their mixing. Predictions in the $|V_{ub}| - \bar{\eta}$, $|V_{ub}| - \sin 2\beta(\phi_1)$, and $|V_{ub}| - |V_{td}/V_{ts}|$ planes are given. The lepton sector contains also 9 parameters. A normal as well as an inverted spectrum of neutrino masses is possible, and we compute V_{e3} . We find that $|V_{e3}|^2 > 10^{-4}$ in the case of a normal spectrum, and $|V_{e3}|^2 > 8 \times 10^{-4}$ in the case of an inverted spectrum. It is also found that Q_6 symmetry forbids all Baryon number violating terms of $d = 4$, and the contributions to EDMs from the A terms vanish in this model.

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I. INTRODUCTION

The gauge interactions of the standard model (SM) respect $U_L(3) \times U_R(3)$ family symmetry in both the leptonic and quark sectors. It is the Yukawa sector of the SM that breaks this family symmetry, and is responsible for the generation of the lepton and quark masses and their mixing. If no condition is imposed, there are $3 \times 3 \times 2 \times 8 = 28$ real free parameters in the quark sector alone, where 8 is the number of phases that can be absorbed into the phases of the quark fields. Of 28 only 10 parameters are physical parameters in the SM. That is, there are 18 redundant parameters in that sector. The presence of redundant parameters is not related to a symmetry in the SM. Even if they are set equal to zero at some energy scale, they will appear at different scales. These redundant parameters may become physical parameters when going beyond the SM, and, moreover, they can induce flavor changing neutral currents (FCNCs) and CP violating phenomena that are absent or strongly suppressed in the SM. Since the SM cannot control the redundant parameters, the size of the new FCNCs and CP violating phases may be unacceptably large unless there is some symmetry, or one fine tunes their values. This is a flavor problem that can occur when going beyond the SM, and the most familiar case is the minimal supersymmetric standard model (MSSM) [1].

Reducing phenomenologically the number of the free parameters in the mass matrices of quarks and leptons had started already decades ago. One of the successful Ansätze for the quark mass matrices, first proposed by Weinberg [2,3] and then extended by Fritzsch [4], is of a nearest neighbor interaction (NNI) type¹ [6–16]:

$$M = \begin{pmatrix} 0 & C & 0 \\ \pm C & 0 & B \\ 0 & B' & A \end{pmatrix}. \quad (1)$$

The complex parameters B, B', C , and A for each of the up- and down-quark sectors can be made real by an appropriate phase rotation on the quark fields, and as a consequence, there are only 8 real free parameters with two independent phases. The Ansatz (1) can successfully reproduce the quark masses and the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix V_{CKM} [5,11]. It has been also realized that the Ansatz (1) can be used in the leptonic sector, too [8–10,15,16]. Therefore, the Ansatz (1) is appropriate for unification, especially for the Pati-Salam type unification [17], in which the left-handed and right-handed fermion families can be separately unified.

It is known [6] that within the SM, any mass matrix for both up and down quarks can be simultaneously brought, without changing physics, to the form (1) with $|M_{12}| \neq |M_{21}|$. However, beyond the SM, this is no longer true, and to obtain (1) beyond the SM even with $|M_{12}| \neq |M_{21}|$, some principle should be required. In this paper we are motivated by a desire to derive the form (1) solely from a symmetry principle. To be definite, we assume that the responsible symmetry is (A) based on a non-Abelian discrete group, and (B) only spontaneously broken. As we will find, (i) the smallest group that satisfies our assumptions is Q_6 , a binary dihedral group with 12 elements, and (ii) the Higgs sector of the SM has to be so extended that the up- and down-type right-handed fermion families couple to their own $SU(2)_L$ Higgs doublets (type II Higgs). So, the Higgs sector of the MSSM fits the desired Higgs structure. Therefore, we are naturally led to consider a supersymmetric extension of the SM based on Q_6 , as we will do in this paper. (Frampton and Kephart [18] came to Q_6 , but

¹See [5] for review.

from different reasons.²) We will also discuss other important consequences of the supersymmetric Q_6 model such as a solution to the supersymmetry (SUSY) flavor problem [42]. We find that CP can be spontaneously broken, and thanks to Q_6 a phase alignment for each A term (trilinear coupling of bosonic superpartners) with the corresponding Yukawa term occurs. Note that the misalignment of the phases appearing in the Yukawa and A terms is the origin of a large contribution to the electric dipole moments (EDM) of neutron etc. [43]. We also find that Q_6 can forbid all the Baryon number violating $d = 3$ and 4 operators, and allows only one R -parity violating operator with $d \leq 4$.

After we discuss group theory on the dihedral groups D_N and the binary dihedral groups Q_N [18,44] in Sec. II, we consider a supersymmetric extension of the SM based on Q_6 in Sec. III. There we discuss the quark sector, lepton sector, and Higgs sector separately. We make predictions in the $|V_{ub}| - \bar{\eta}$, $|V_{ub}| - \sin 2\beta(\phi_1)$ and $|V_{ub}| - V_{td}/V_{ts}|$ planes as well as on the average neutrino mass $\langle m_{ee} \rangle$ in neutrinoless double β decay and the Dirac phase δ_{CP} in the neutrino mixing. In the last section, we briefly discuss the anomalies of discrete symmetries [45–47] introduced for the model, R parity violating operators and the SUSY flavor problem, respectively.

II. FINITE GROUPS D_N AND Q_N

A. Definitions

The group presentation for the dihedral groups D_N is given by

$$\{A_{D_N}, B_D; (A_{D_N})^N = B_D^2 = E, B_D^{-1}A_{D_N}B_D = A_{D_N}^{-1}\}, \quad (2)$$

and

$$\{A_{Q_N}, B_Q; (A_{Q_N})^N = E, B_Q^2 = (A_{Q_N})^{N/2}, B_Q^{-1}A_{Q_N}B_Q = A_{Q_N}^{-1}\} \quad (3)$$

for the binary dihedral group Q_N , where E is the identity element. $2N$ is the order of group (the number of the group elements). For the binary dihedral group Q_N , N should be even starting with 4, while N for D_N starts with 3. The $2N$ group elements are

$$\mathcal{G} = \{E, A, (A)^2, \dots, (A)^{N-1}, B, AB, (A)^2B, \dots, (A)^{N-1}B\} \quad (4)$$

²One of the first papers on discrete symmetries are [19–26]. Phenomenologically viable models based on non-Abelian discrete flavor symmetries A_4, S_3, D_4 , and Q_4 , which can partly explain the flavor structure of quarks and leptons such as large neutrino mixing, have been recently constructed, respectively, in [27–29], [30–33], [34,35], and [36]. In [37–41], non-Abelian discrete symmetries have been used to soften the SUSY flavor problem.

both for D_N and Q_N . Using the property that the product $(A^m B)(A^n B)$ with $m, n = 0, \dots, N-1$ can always be brought to one of the elements of \mathcal{G} , one can easily see that they form a group. A two-dimensional representation of A and B is given by

$$A_{D_N} = A_{Q_N} = \begin{pmatrix} \cos \phi_N & \sin \phi_N \\ -\sin \phi_N & \cos \phi_N \end{pmatrix} \text{ with } \phi_N = 2\pi/N, \quad (5)$$

$$B_D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } D_N, \quad (6)$$

$$B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ for } Q_N.$$

Note that $\det A_{Q_N} = \det B_{Q_N} = 1$, implying that Q_N is a subgroup of $SU(2)$. It follows that the dihedral group is a subgroup of $SO(3)$, which one sees if one embeds A_{D_N} and B_D into 3×3 matrices

$$A_{D_N} \rightarrow \begin{pmatrix} \cos \phi_N & \sin \phi_N & 0 \\ -\sin \phi_N & \cos \phi_N & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

$$B_D \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It also follows that D_N has only real representations, while Q_N can have real as well as pseudoreal representations [18,44]. However, the smallest binary dihedral group that contains both real and pseudoreal nonsinglet representations is Q_6 , because Q_4 has only pseudoreal nonsinglet representations. Note that the irreducible representations (irreps) of D_N and Q_N are either one- or two-dimensional.

Q_N is the “double-covering group” of D_N in the following sense. Consider the matrices of $D_{N/2}$, i.e., $A_{D_{N/2}}$ and B_D , and define

$$\tilde{A}_{Q_N} = A_{D_{N/2}}, \quad \tilde{B}_Q = B_D. \quad (8)$$

Note that \tilde{A}_{Q_N} have exactly the same properties as A_{Q_N} . Therefore, the set

$$\{E, \tilde{A}_{Q_N}, (\tilde{A}_{Q_N})^2, \dots, (\tilde{A}_{Q_N})^{N-1}, \tilde{B}_Q, \tilde{A}_{Q_N} \tilde{B}_Q, (\tilde{A}_{Q_N})^2 \tilde{B}_Q, \dots, (\tilde{A}_{Q_N})^{N-1} \tilde{B}_Q\} \quad (9)$$

is a set of Q_N elements. Since however $(\tilde{A}_{Q_N})^{N/2} = (A_{D_{N/2}})^{N/2} = E$ by definition, the $D_{N/2}$ elements appear twice in (9).

B. Q_6 group theory

Before we consider a concrete model, we briefly outline the basic reasons why Q_6 . Let us find out what symmetry can explain the right-upper 2×2 block of the mass matrix (1). Clearly, no Abelian symmetry can explain it, whatever the Higgs structure is. In fact, that block with $M_{12} = -M_{21} = C$, is invariant under $SU(2)$. It is therefore invariant under its subgroups, too, in particular, under Q_N . Similarly, one can interpret that the 2×2 block is invariant under $O(2)$. Note that $O(2)$ is not Abelian (2×2 orthogonal matrices with $\det = \pm 1$ do not always commute with each other), and that $O(2)$ allows two different one-dimensional representations $\mathbf{1}$ and $\mathbf{1}'$, where only $\mathbf{1}$ is the

true singlet. The diagonal entries of the 2×2 block can be forbidden if $\mathbf{1}'$ is assigned to the responsible Higgs. In fact, $O(2)$ may be regarded as a subgroup of $SO(3)$, and in this case the non-Abelian finite subgroups are the dihedral groups, D_N . It turns out that to explain the whole structure of the mass matrix (1), we need to have real as well as pseudoreal representations.

The smallest group that contains both types of representations is Q_6 , which is the double-covering group of $S_3 \sim D_3$. The irreps of Q_6 are $\mathbf{2}, \mathbf{2}', \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$, where the $\mathbf{2}$ is pseudoreal, while $\mathbf{2}'$ is real. $\mathbf{1}, \mathbf{1}'$ are real representations, while $\mathbf{1}'', \mathbf{1}'''$ are complex conjugate to each other. The group multiplication rules are given as follows [18,44]:

$$\begin{aligned} \mathbf{1}' \times \mathbf{1}' &= \mathbf{1}, & \mathbf{1}'' \times \mathbf{1}'' &= \mathbf{1}', & \mathbf{1}''' \times \mathbf{1}''' &= \mathbf{1}', & \mathbf{1}'' \times \mathbf{1}''' &= \mathbf{1}, & \mathbf{1}' \times \mathbf{1}''' &= \mathbf{1}'', & \mathbf{1}' \times \mathbf{1}'' &= \mathbf{1}''', \\ \mathbf{2} \times \mathbf{1}' &= \mathbf{2}, & \mathbf{2} \times \mathbf{1}'' &= \mathbf{2}', & \mathbf{2} \times \mathbf{1}''' &= \mathbf{2}', & \mathbf{2}' \times \mathbf{1}' &= \mathbf{2}', & \mathbf{2}' \times \mathbf{1}'' &= \mathbf{2}, & \mathbf{2}' \times \mathbf{1}''' &= \mathbf{2}. \end{aligned} \quad (10)$$

Note that $\mathbf{1}'', \mathbf{1}'''$, and $\mathbf{2}$ are complex valued. The complex conjugate representation $\mathbf{2}^*$ transforms the same way as $\mathbf{2}$, when $\mathbf{2}^*$ is identified as $\mathbf{2}^* = i\tau_2\mathbf{2}$, with τ_2 being the second Pauli matrix. The Clebsch-Gordan coefficients for multiplying any of the irreps (which can be straightforwardly computed from the two-dimensional representation of A_{Q_N} and B_Q given in (5) and (6) are given by

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2} \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ x_1y_2 - x_2y_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}' \\ x_1y_1 + x_2y_2 \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ -x_1y_2 - x_2y_1 \\ x_1y_1 - x_2y_2 \end{pmatrix}, \quad (11)$$

$$\begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ a_1b_1 + a_2b_2 \end{pmatrix} + \begin{pmatrix} \mathbf{1}' \\ a_1b_2 - a_2b_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ -a_1b_1 + a_2b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix}, \quad (12)$$

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1}'' \\ x_1a_2 + x_2a_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}''' \\ x_1a_1 - x_2a_2 \end{pmatrix} + \begin{pmatrix} \mathbf{2} \\ x_1a_1 + x_2a_2 \\ x_1a_2 - x_2a_1 \end{pmatrix}. \quad (13)$$

In what follows we construct a concrete model with the group theory rules given above. Multiplication rules for $D_{3,4,6,8}, Q_{4,8}$ are given in the appendix.

III. SUPERSYMMETRIC EXTENSION OF THE STANDARD MODEL BASED ON Q_6

We would like to derive the mass matrix (1) solely from a symmetry principle. One finds that two conditions should be met, as discussed already: (i) There should be real as well as pseudoreal nonsinglet representations, and (ii) there should be the up- and down-type Higgs $SU(2)_L$ doublets (type II Higgs). The smallest finite group that allows both real and pseudoreal nonsinglet representations, is Q_6 , as already found out. One finds that all Higgs fields cannot be in real representations to obtain the mass matrix (1). So, certain Higgs fields are in pseudoreal representations. If a Higgs field H in a pseudoreal representation couples to a down-type fermion family, then its conjugate H^* couples to a up-type fermion family. Since H^* belongs to the conjugate representation, the mass matrix of the up-

type family differs from that of the down-type family. Therefore, H^* should be forbidden to couple to the up-type family in order to obtain the same form of the mass matrices for the up and down sectors. Accordingly, the Higgs sector of the MSSM fits the desired Higgs structure. Therefore, we concentrate on Q_6 and assume $N = 1$ supersymmetry in the following discussions.

A. Q_6 assignment

Let us denote the fermion and Higgs fields as

$$\begin{aligned} \psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, & \psi^c &= \begin{pmatrix} -\psi_1^c \\ \psi_2^c \end{pmatrix}, & \psi_3, & \psi_3^c, \\ H &= \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, & H_3, & & & \end{aligned} \quad (14)$$

and assume that the assignment under Q_6 is given by

$$\begin{aligned} \psi: \mathbf{2}, & \quad \psi^c: \mathbf{2}', & \psi_3: \mathbf{1}', & \psi_3^c: \mathbf{1}''', & H: \mathbf{2}', \\ & & & & H_3: \mathbf{1}''' \end{aligned} \quad (15)$$

for all sectors. Here H^d and H_3^d will couple to the down quarks and charged leptons, while H^u and H_3^u will couple to the up-quark and neutrino Dirac sectors. The assignment under Q_6 is the same in all sectors. So we focus on the down-quark sector. From the Clebsch-Gordan factors given in (10)–(13), the following Yukawa couplings are found to be invariant:

$$\begin{aligned} \mathcal{L}_Y = & a\psi_3\psi_3^c H_3 + b\psi^T\tau_1\psi_3^c H - b'\psi_3\psi^{cT}i\tau_2 H \\ & + c\psi^T\tau_1\psi^c H_3 + \text{H.c.} \end{aligned} \quad (16)$$

This leads to the mass matrix

$$M = \begin{pmatrix} 0 & c\langle H_3 \rangle & b\langle H_2 \rangle \\ -c\langle H_3 \rangle & 0 & b\langle H_1 \rangle \\ b'\langle H_2 \rangle & b'\langle H_1 \rangle & a\langle H_3 \rangle \end{pmatrix}. \quad (17)$$

If $\langle H_2 \rangle$ vanishes, the above mass matrix has exactly the desired form (1). If $\langle H_1 \rangle = \langle H_2 \rangle$, we can bring the mass matrix (17) to the desired form (1) by an overall 45° rotation on the fields of the Q_6 doublets fermions (the ψ and the ψ^c fields). So, the mass matrix (17) does not automatically lead to the desired form (1); we need to construct a Higgs sector that ensures the stability of the desired vacuum expectation value (VEV) structure $\langle H_1 \rangle = \langle H_2 \rangle$ (or $\langle H_2 \rangle = 0$). A parity invariance $H_1 \leftrightarrow H_2$ would ensure the VEV structure, but it is not a symmetry of (16). Therefore, the parity invariance can only be an accidental symmetry of the Higgs sector. In Sec. III E we will construct a Higgs sector that possesses accidentally the desired parity invariance, while the whole sector is invariant under $Q_6 \times Z_{12R}$. It would not work if the Q_6 doublets are replaced by doublets of $D_{3,4,5,6,7}$, Q_4 or $SU(2)$.

In Table I we write the Q_6 assignment (15) again to fix our notation, where Z_{12R} will be introduced later on when constructing the Higgs sector.

B. Spontaneous CP violation

The most stringent constraints on the soft supersymmetry breaking (SSB) parameters come from the electric dipole moments (EDM) of neutron, electron, and mercury atom. Recent experiments on EDMs require that $d_e < 64.3 \times 10^{-27} e \text{ cm}$ [48], $d_n < 6.3 \times 10^{-26} e \text{ cm}$ [49], and $d_{Hg} < 2.1 \times 10^{-28} e \text{ cm}$ [50]. A misalignment of the phases appearing in the Yukawa and A terms (trilinear couplings of bosonic superpartners) is the origin of EDMs and lead to a large contribution to EDMs, so that a very fine tuning to suppress the misalignment is required

[42,43,51,52]. Note, however, that a flavor symmetry cannot ensure the alignment of the phases. Therefore, we require that CP is spontaneously broken so that the origin of the CP phases in the Yukawa and A terms is the same. Consequently, all the coefficients appearing in the Lagrangian should be real. (Some of the couplings may be purely imaginary if the relevant fields are odd under CP .) Then thanks to Q_6 family symmetry the phases in the Yukawa and A terms after spontaneous symmetry breaking of CP are so aligned, that the contribution to EDMs coming from the A terms in this model vanishes. In the last section, we will discuss the phase alignment again when discussing the SUSY flavor problem. In the following discussions we simply assume that

$$\begin{aligned} \langle H_1^u \rangle = \langle H_2^u \rangle, \quad \langle H_1^d \rangle = \langle H_2^d \rangle, \quad \langle H_3^u \rangle \neq 0, \\ \langle H_3^d \rangle \neq 0, \end{aligned} \quad (18)$$

and denote the phases of VEVs as

$$\begin{aligned} \Delta\theta^u = \arg\langle H_3^u \rangle - \arg\langle H_1^u \rangle, \\ \Delta\theta^d = \arg\langle H_3^d \rangle - \arg\langle H_1^d \rangle. \end{aligned} \quad (19)$$

In Sec. III E, we will construct a Higgs sector that satisfies the assumptions (18) made on VEVs. The crucial point is that the superpotential (89) possess the symmetry (92), which can ensure the desired VEV structure (18). Note that the parity invariance (92) is not imposed by hand on the superpotential (89); it is an accidentally symmetry in the Higgs sector, while the whole theory possesses $Q_6 \times Z_{12R}$ symmetry only.

C. Quark sector

Q_6 assignments of the left-handed quark supermultiplets Q and Q_3 , and the right-handed quark supermultiplets $U_R^c, U_{3R}^c, D_R^c, D_{3R}^c$ are given in Table I. The most general Q_6 invariant superpotential for the Yukawa interactions in the quarks sector is $W_Q = W_D + W_U$, where

$$\begin{aligned} W_D = & Y_a^d Q_3 D_3^c H_3^d + Y_b^d Q^T \tau_1 D_3^c H^d - Y_{b'}^d Q_3 D^{cT} i \tau_2 H^d \\ & + Y_c^d Q^T \tau_1 D^c H_3^d, \end{aligned} \quad (20)$$

and similarly for W_U (all the couplings are real). Note that neither Z_{12R} nor R parity is assumed to obtain W_Q . In fact, including the lepton sector, Q_6 alone forbids all the R parity violating renormalizable terms, except one term. It also forbids all the Baryon number violating Yukawa couplings. We will come to discuss this in the last section.

TABLE I. $Q_6 \times Z_{12R}$ assignment of the matter supermultiplets. Z_{12R} will be introduced to construct a desired Higgs sector later on.

	Q, L	U^c, D^c, E^c, N^c	H^u, H^d	Q_3, L_3	$U_3^c, D_3^c, E_3^c, N_3^c$	H_3^u, H_3^d
Q_6	2	2'	2'	1'	1'''	1'''
Z_{12R}	1	-1	-2	1	-1	-2

We assume that VEVs take the form (18), from which we obtain the mass matrix for the up and down quarks. We make an overall 45° rotation on the Q_6 doublets, Q , D^c and U^c , and phase rotations on the fields defined as

$$U \rightarrow P_u U, \quad U^c \rightarrow P_{u^c} U^c, \quad (21)$$

and similarly for D and D^c , where

$$P_{u,d} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i2\Delta\theta^{u,d}) & 0 \\ 0 & 0 & \exp(i\Delta\theta^{u,d}) \end{pmatrix}, \quad (22)$$

$$P_{u^c,d^c} = \begin{pmatrix} \exp(-i2\Delta\theta^{u,d}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-i\Delta\theta^{u,d}) \end{pmatrix} \\ \times \exp(-i \arg\langle H_3^{u,d} \rangle). \quad (23)$$

[$\Delta\theta^{u,d}$ are given in (19).] Then one obtains real mass matrices

$$M_{u,d} = m_{t,b} \begin{pmatrix} 0 & q_{u,d}/y_{u,d} & 0 \\ -q_{u,d}/y_{u,d} & 0 & b_{u,d} \\ 0 & b'_{u,d} & y_{u,d}^2 \end{pmatrix}, \quad (24)$$

where m_t and m_b are the top and bottom quark mass, respectively.³ We have changed our notation for the mass matrices to directly apply the result of [12] to our case. The CKM matrix V_{CKM} is given by

$$V_{\text{CKM}} = O_u^T P_q O_d, \quad (25)$$

where

$$O_u^T M_u M_u^T O_u = \begin{pmatrix} m_u^2 & 0 & 0 \\ 0 & m_c^2 & 0 \\ 0 & 0 & m_t^2 \end{pmatrix}, \quad (26)$$

$$O_d^T M_d M_d^T O_d = \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix},$$

$$P_q = P_u^\dagger P_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i2\theta_q) & 0 \\ 0 & 0 & \exp(i\theta_q) \end{pmatrix} \quad (27)$$

$$\text{with } \theta_q = \Delta\theta^d - \Delta\theta^u.$$

Note that there are only 8 real independent parameters, i.e.,

³Ma [30] considered a model based on a $S_3 \times Z_3$ family symmetry. In his model, the mass matrix for the down quarks is similar to M_d , but that for the up-quark sector is diagonal. So two models are distinguishable.

$q_{u,d}$, $b_{u,d}$, $b'_{u,d}$, $y_{u,d}$, and one independent phase θ_q to describe V_{CKM} and the quark masses. Using the result of [12] we find that V_{CKM} can be approximately written as

$$V_{ud} \simeq 1, \quad V_{cs} \simeq \exp i2\theta_q, \quad V_{tb} \simeq \exp i\theta_q, \quad (28)$$

$$V_{us} \simeq -y_d \sqrt{\frac{m_d}{m_s}} + y_u \sqrt{\frac{m_u}{m_c}} \exp i2\theta_q, \quad (29)$$

$$V_{cb} \simeq \frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp i2\theta_q - \frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp i\theta_q, \quad (30)$$

$$V_{ts} \simeq -\frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp i\theta_q + \frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp i2\theta_q, \quad (31)$$

$$V_{ub} \simeq \frac{\sqrt{1-y_d^4}}{y_d} \sqrt{\frac{m_d}{m_s}} \frac{m_s}{m_b} + y_u \sqrt{\frac{m_u}{m_c}} \left(\frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp i2\theta_q \right. \\ \left. - \frac{1}{y_u^2 \sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp i\theta_q \right), \quad (32)$$

$$V_{td} \simeq \frac{\sqrt{1-y_u^4}}{y_u} \sqrt{\frac{m_u}{m_c}} \frac{m_c}{m_t} + y_d \sqrt{\frac{m_d}{m_s}} \left(\frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp i2\theta_q \right. \\ \left. - \frac{1}{y_d^2 \sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp i\theta_q \right). \quad (33)$$

For the mass matrix parameters [defined in (24)]

$$\begin{aligned} \theta_q &= -1.15, & q_u &= 0.000\,2260, & b_u &= 0.045\,96, \\ b'_u &= 0.089\,59, & y_u &= 0.997\,46, & q_d &= 0.005\,110, \\ b_d &= 0.026\,09, & b'_d &= 0.7682, & y_d &= 0.8000, \end{aligned} \quad (34)$$

we obtain

$$\begin{aligned} m_u/m_t &= 1.24 \times 10^{-5}, & m_c/m_t &= 4.13 \times 10^{-3}, \\ m_d/m_b &= 1.22 \times 10^{-3}, & m_s/m_b &= 2.14 \times 10^{-2}, \end{aligned} \quad (35)$$

$$\begin{aligned}
|V_{ud}| &= 0.975(1.00), & |V_{us}| &= 0.222(0.231), & |V_{ub}| &= 3.64(3.59) \times 10^{-3}, & |V_{cb}| &= 4.20(3.73) \times 10^{-2}, \\
|V_{td}| &= 8.82(8.75) \times 10^{-3}, & |V_{ts}| &= 4.12(3.73) \times 10^{-2}, & |V_{tb}| &= 0.999(1.00), & \bar{\eta} &= 0.361(0.411), \\
\sin 2\beta(\phi_1) &= 0.704(0.740),
\end{aligned} \tag{36}$$

where $\bar{\eta}$ is one of the Wolfenstein parameters. The values in the parentheses are those obtained from the approximate formulas (28)–(33). The experimental values are [53]:

$$|V_{CKM}^{\text{exp}}| = \begin{pmatrix} 0.9739 \text{ to } 0.9751 & 0.221 \text{ to } 0.227 & 0.0029 \text{ to } 0.0045 \\ 0.221 \text{ to } 0.227 & 0.9730 \text{ to } 0.9744 & 0.039 \text{ to } 0.044 \\ 0.0048 \text{ to } 0.014 & 0.037 \text{ to } 0.043 & 0.9990 \text{ to } 0.9992 \end{pmatrix}, \tag{37}$$

and

$$\sin 2\beta(\phi_1) = 0.736 \pm 0.049, \quad \bar{\eta} = 0.33 \pm 0.05. \tag{38}$$

The quark masses at M_Z are given by [54]

$$\begin{aligned}
m_u/m_d &= 0.553 \pm 0.086(0.61), & m_s/m_d &= 18.9 \pm 1.6(17.5), & m_c &= 0.73 \pm 0.17 (0.72) \text{ GeV}, \\
m_s &= 0.058 \pm 0.015 (0.062) \text{ GeV}, & m_t &= 175 \pm 6 \text{ GeV}, & m_b &= 2.91 \pm 0.07 \text{ GeV},
\end{aligned} \tag{39}$$

where the values in the parentheses are the theoretical values obtained from (35) for $m_t = 174 \text{ GeV}$ and $m_b = 2.9 \text{ GeV}$. So, we see that the 9 independent parameters can well describe 10 physical observables. Our numerical analysis show that the experimental constraints can be satisfied if

$$-1.3 < \theta_q < -1.0 \quad \text{and} \quad m_s/m_d < 19. \tag{40}$$

In Fig. 1 we plot predicted points in the $|V_{ub}| - \bar{\eta}$ plane, where we have used the quark masses with the uncertainties given in (39) and $|V_{us}| = 0.2240 \pm 0.0036$, $|V_{cb}| =$

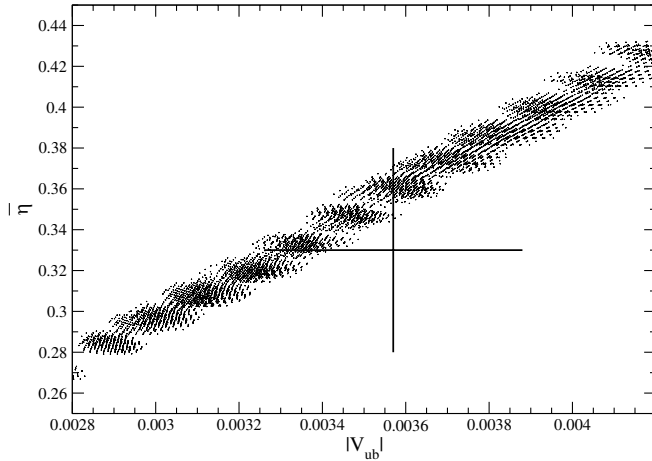


FIG. 1. Predictions in the $|V_{ub}| - \bar{\eta}$ plane. The uncertainties result from those in the quark masses (39) and in $|V_{us}|$ and $|V_{cb}|$, where we have used $|V_{us}| = 0.2240 \pm 0.0036$ and $|V_{cb}| = (41.5 \pm 0.8) \times 10^{-3}$ [54]. The vertical and horizontal lines correspond to the experimental values $\bar{\eta} = 0.33 \pm 0.05$ and $|V_{ub}| = (35.7 \pm 3.1) \times 10^{-4}$ [53,54].

$(41.5 \pm 0.8) \times 10^{-3}$ [54]. The experimental values of $|V_{ub}|$ and $\bar{\eta}$ are indicated by the solid lines. As we see from Figs. 1 and 2, accurate determinations of $\bar{\eta}$, $\sin 2\beta(\phi_1)$, and $|V_{ub}|$ are crucial to test the quark sector of the model.

In Fig. 3 we plot $|V_{td}/V_{ts}|$ against $|V_{ub}|$. From the figure we find that

$$\left| \frac{V_{td}}{V_{ts}} \right| = 0.205 - 0.230. \tag{41}$$

The present experimental upper bound is 0.25 [53], which is obtained from the ratio $\Delta M_{B_s}/\Delta M_{B_d}$, where ΔM_{B_d} is well measured, but only an lower bound on ΔM_{B_s} is experimentally known. Note that a direct comparison of the theoretical values given in (41) with the experimental

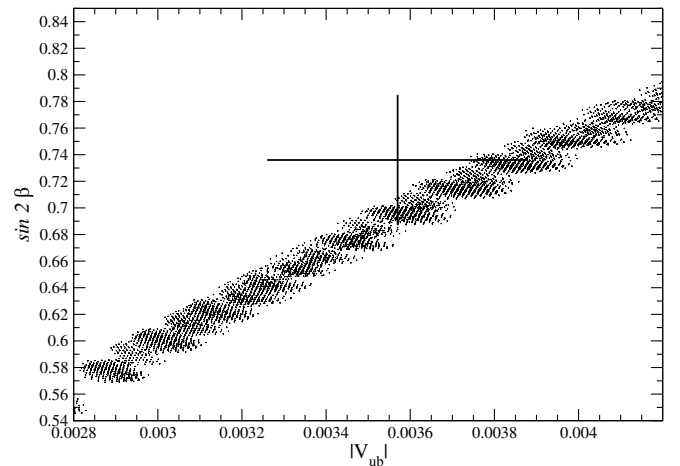


FIG. 2. Predictions in the $|V_{ub}| - \sin 2\beta(\phi_1)$ plane.

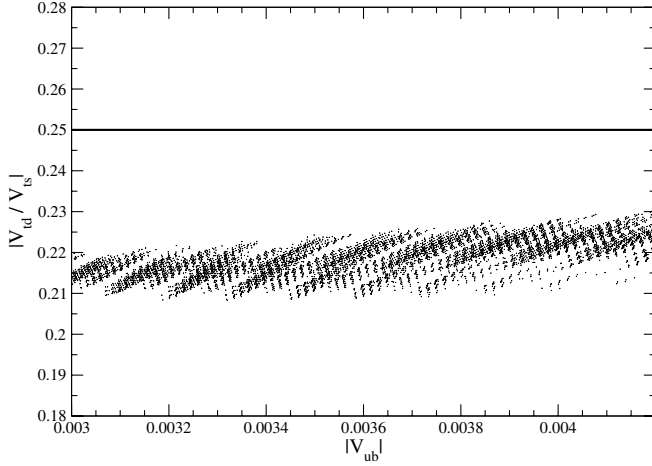


FIG. 3. Predictions in the $|V_{ub}| - |V_{id}/V_{ts}|$ plane. The horizontal line is the experiment upper bound.

value is possible, only if there is no contribution to ΔM_{B_s} and ΔM_{B_d} other than those of the SM. In supersymmetric extensions of the SM and in multi-Higgs models there are

$$O_e^T \simeq \begin{pmatrix} 1 & \sqrt{\frac{m_e \cos \phi}{m_\mu}} & -\sqrt{\frac{m_e}{m_\mu}} \frac{\sin \phi}{\sqrt{\cos \phi}} \\ -\sqrt{\frac{m_e}{m_\mu \cos \phi}} & \cos \phi \left(1 + \frac{m_\mu^2}{m_\tau^2}\right) & -\sin \phi \left(1 - \frac{m_\mu^2}{m_\tau^2 \tan^2 \phi}\right) \\ \sqrt{\frac{m_e}{m_\mu}} \frac{m_\mu^2}{m_\tau^2} \frac{\sqrt{\cos \phi}}{\sin \phi} & \sin \phi \left(1 - \frac{m_\mu^2}{m_\tau^2 \tan^2 \phi}\right) & \cos \phi \left(1 + \frac{m_\mu^2}{m_\tau^2}\right) \end{pmatrix}, \quad (44)$$

and

$$P_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i2\Delta\theta^d) & 0 \\ 0 & 0 & \exp(i\Delta\theta^d) \end{pmatrix}. \quad (45)$$

The angle ϕ is an independent free parameter, and it can be assumed without loss of generality that $\sin \phi$ and $\cos \phi$ are positive.

Now let us come to the neutrino sector. Since N^c belongs to $\mathbf{1}'''$ of Q_6 , N_3^c cannot form a Q_6 invariant mass term, while the Q_6 doublet N^c can do ($N_1^c N_1^c + N_2^c N_2^c$). The absence of a mass term for N_3^c would be phenomenologically inconsistent. In Sec. III E, when discussing the Higgs sector, we will give a way to obtain a mass term for N_3^c . Here we simply assume that there is a mass term for N_3^c . So, the mass matrix M_R of the right-handed neutrinos is diagonal: $M_R = \text{diag}(M_1, M_1, \kappa M_3)$, where $\kappa = \pm 1$ takes into account the difference of the Q_6 doublet N^c and Q_6 singlet N_3^c under CP . After an overall 45° rotation on the Q_6 doublets, and using the seesaw mechanism, we obtain a neutrino mass matrix. Next we make a phase rotation on the left-handed neutrinos with

$$P_\nu = \begin{pmatrix} \exp(-ir_1) & 0 & 0 \\ 0 & \exp(-ir_2) & 0 \\ 0 & 0 & \exp(-ir_3) \end{pmatrix}, \quad (46)$$

extra contributions in general. Nevertheless, $|V_{id}/V_{ts}|$ will be an important quantity for the Q_6 model.

D. Lepton sector

As in the quark sector, the Q_6 invariant superpotential for the Yukawa interactions in the lepton sector is given by $W_L = W_N + W_E$, where

$$W_N = Y_a^\nu L_3 N_3^c H_3^u + Y_b^\nu L^T \tau_1 N_3^c H^u - Y_{b'}^\nu L_3 N^{cT} i \tau_2 H^u + Y_c^\nu L^T \tau_1 N^c H_3^u, \quad (42)$$

and similarly for W_E .⁴ All the couplings are real, because we require that CP is only spontaneously broken. The charged lepton sector is very similar to the quark sector, and we find that the diagonalizing unitary matrix U_e rotating the left-handed fermions is given by

$$U_e^\dagger = O_e^T P_e^\dagger, \quad (43)$$

where the orthogonal matrix O_e can be approximately written as

$$r_1 = r_2 = \arg\langle H_3^u \rangle, \quad r_3 = \arg\langle H_1^u \rangle \quad (47)$$

to obtain the left-handed neutrino mass matrix

$$M_\nu = \begin{pmatrix} c_\nu^2 & 0 & c_\nu b'_\nu \\ 0 & c_\nu^2 + \kappa b_\nu^2 e^{-i\varphi} & \kappa a_\nu b_\nu \\ c_\nu b'_\nu & \kappa a_\nu b_\nu & \kappa a_\nu^2 e^{i\varphi} + b_\nu^2 \end{pmatrix}, \quad (48)$$

$$\varphi = 2(\arg\langle H_3^u \rangle - \arg\langle H_1^u \rangle) = 2\Delta\theta^u. \quad (49)$$

1. Inverted spectrum

First we discuss the case of an inverted spectrum of neutrino masses, i.e. $m_{\nu_2} > m_{\nu_1} > m_{\nu_3}$. To this end, we assume that b_ν is small, and we treat it as perturbation:

⁴An alternative assignment of the leptons is

$$L: \mathbf{2}, E^c: \mathbf{2}, L_3: \mathbf{1}''', E_3^c: \mathbf{1}''', N^c: \mathbf{2}, N_3^c: \mathbf{1},$$

where the assignment of the quarks and Higgs supermultiplets remains unchanged. The corresponding Yukawa sector will coincide with that of the S_3 model with Z_2 in the lepton sector [31,32]. It contains only 6 real parameters with two phases in the lepton sector and explains the maximal mixing of the atmospheric neutrinos. In that model, there exists not only Majorana phases but also a Dirac-type CP phase, which was overlooked in [31,32]. Because of the spontaneous CP violation in the present case, there is only one independent phase in the lepton sector. So, the Dirac and Majorana phases are related.

$$M_\nu = M_\nu^{(0)} + M_\nu^{(1)} + O(b_\nu^2) \quad (50)$$

where

$$M_\nu^{(0)} = \begin{pmatrix} c_\nu^2 & 0 & c_\nu b'_\nu \\ 0 & c_\nu^2 & 0 \\ c_\nu b'_\nu & 0 & \kappa a_\nu^2 e^{i\varphi} + b_\nu'^2 \end{pmatrix}, \quad (51)$$

$$M_\nu^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \kappa a_\nu b_\nu \\ 0 & \kappa a_\nu b_\nu & 0 \end{pmatrix}.$$

Because the (1, 2) and (2, 1) elements of M_ν are exactly zero, the correction to the eigenvalues is $O(b_\nu^2)$. But the correction to the unitary matrix U_ν is $O(b_\nu)$. This correction is found to be

$$U_\nu = U_\nu^{(0)} + U_\nu^{(1)} + O(b^2), \quad (52)$$

$$U_\nu^{(1)} = \begin{pmatrix} 0 & 0 & \delta U_{13} \\ \delta U_{21} & \delta U_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (53)$$

$$\delta U_{13} = -\kappa a_\nu b_\nu / c b'_\nu,$$

$$\delta U_{21} = -(\cos\theta_\nu)(e^{-i(\varphi_1 - \phi_\nu)/2})\delta U_{13},$$

$$\delta U_{22} = (\sin\theta_\nu)(e^{-i(\varphi_2 - \phi_\nu)/2})\delta U_{13},$$

where $U_\nu^{(0)}$, $\varphi_{1,2}$, ϕ_ν and θ_ν are given below. The unitary matrix $U_\nu^{(0)}$ can be obtained from

$$U_\nu^{(0)T} M_\nu^{(0)} U_\nu^{(0)} = \begin{pmatrix} m_{\nu_1} & 0 & 0 \\ 0 & m_{\nu_2} & 0 \\ 0 & 0 & m_{\nu_3} \end{pmatrix}, \quad (54)$$

where m_{ν_i} are real and positive, and

$$U_\nu^{(0)} = \begin{pmatrix} c_\nu e^{-i(\varphi_1 - \phi_\nu)/2} & -s_\nu e^{-i(\varphi_2 - \phi_\nu)/2} & 0 \\ 0 & 0 & 1 \\ s_\nu e^{-i(\varphi_1 + \phi_\nu)/2} & c_\nu e^{-i(\varphi_2 + \phi_\nu)/2} & 0 \end{pmatrix} \quad (55)$$

with $c_\nu = \cos\theta_\nu$, etc. Note that we obtain an inverted spectrum

$$m_{\nu_1}, m_{\nu_2} > m_{\nu_3} \quad (56)$$

as long as b_ν is small. The phases and the masses satisfy to $O(b_\nu)$ the following relations:

$$m_{\nu_3} \sin\phi_\nu = m_{\nu_2} \sin\varphi_2 = m_{\nu_1} \sin\varphi_1, \quad (57)$$

$$\varphi = 2\Delta\theta^u = \varphi_1 + \varphi_2 - \frac{1}{2}(1 - \kappa)\pi + (\text{mod}2\pi)$$

$$\simeq \frac{1}{2}(1 + \kappa)\pi + (\text{mod}2\pi), \quad (58)$$

$$m_{\nu_3} = c_\nu^2, \quad \frac{m_{\nu_1} m_{\nu_2}}{m_{\nu_3}} = a_\nu^2, \quad (59)$$

$$\tan\phi_\nu = -\frac{\kappa a_\nu^2 \sin\varphi}{c_\nu^2 + b_\nu'^2 + \kappa a_\nu^2 \cos\varphi}, \quad (60)$$

$$\tan^2\theta_\nu = \frac{(m_{\nu_1}^2 - m_{\nu_3}^2 \sin^2\phi_\nu)^{1/2} - m_{\nu_3} |\cos\phi_\nu|}{(m_{\nu_2}^2 - m_{\nu_3}^2 \sin^2\phi_\nu)^{1/2} + m_{\nu_3} |\cos\phi_\nu|}. \quad (61)$$

The last Eq. (61) can be rewritten as

$$\frac{m_{\nu_1}^2}{\Delta m_{23}^2 - \Delta m_{21}^2} = \frac{(1 + 2t_\nu^2 + t_\nu^4 + r t_\nu^4)^2}{4t_\nu^2(1 + t_\nu^2)(1 + t_\nu^2 + r t_\nu^2)\cos^2\phi_\nu}$$

$$- \tan^2\phi_\nu$$

$$\simeq \frac{1}{\sin^2 2\theta_{12}\cos^2\phi_\nu} - \tan^2\phi_\nu \text{ for } |r| \ll 1, \quad (62)$$

where $t_\nu = \tan\theta_\nu$ and $r = \Delta m_{21}^2 / (\Delta m_{23}^2 - \Delta m_{21}^2)$.

The mixing matrix is obtained from $O_e^T P_e^\dagger P_\nu U_\nu$, which can be brought into the form

$$V_{\text{MNS}} = \hat{V}_{\text{MNS}} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}, \quad (63)$$

where (1, 1), (1, 2), (2, 3), and (3, 3) elements of \hat{V}_{MNS} are real. O_e^T , P_e , P_ν , and U_ν are given in (44), (45), (47), and (52). To $O(b_\nu)$, the mixing matrix \hat{V}_{MNS} and the Majorana phases are approximately given by

$$\alpha = \frac{1}{2}(\varphi_1 - \varphi_2), \quad (64)$$

$$\beta = -\Delta\theta^u - \Delta\theta^d + \frac{1}{2}(\varphi_1 + \phi_\nu), \quad (65)$$

$$\hat{V}_{e2} \simeq -\sin\theta_{12} \simeq -\sin\theta_\nu, \quad (66)$$

$$\hat{V}_{\mu 3} \simeq \cos\theta_{23} \simeq \cos\phi(1 + m_\mu^2/m_\tau^2), \quad (67)$$

$$\hat{V}_{\tau 3} \simeq \sin\theta_{23} \simeq \sin\phi(1 - m_\mu^2/m_\tau^2 \tan^2\phi), \quad (68)$$

$$\hat{V}_{e3} = -\sin\theta_{13} e^{-i\delta_{CP}} \quad (69)$$

$$\simeq -\kappa(a_\nu b_\nu / c_\nu b'_\nu) e^{-i\delta_{1CP}} + (m_e \cos\phi / m_\mu)^{1/2} e^{-i\delta_{2CP}}, \quad (70)$$

$$\delta_{1CP} = \phi_\nu - \Delta\theta^u - \Delta\theta^d, \quad (71)$$

$$\delta_{2CP} = \phi_\nu - \Delta\theta^u + \Delta\theta^d, \quad (72)$$

where ϕ is defined in (44), and ϕ_ν and θ_ν are defined in (55). Since $\varphi = 2\Delta\theta^u$ [see (49)] and because of (57) and (60), only $\Delta\theta^u$ and $\Delta\theta^d$ are independent phases. Note that $\varphi_2 + \varphi_1 \simeq \pi$ and $\varphi_2 \sim \pi$. If $\varphi_1 \sim \pi$, we obtain $\tan\theta_\nu > 1$.

Although b_ν is a free parameter, it is possible to give a minimum for $|V_{e3}|$, because the difference of the phases δ_{1CP} and δ_{2CP} depends only on $\Delta\theta^d$:

$$|V_{e3}| = |\hat{V}_{e3}| \simeq |\sin\theta_{13}| > (m_e \cos\theta_{23}/m_\mu)^{1/2} |\sin 2\Delta\theta^d| \quad (73)$$

$$> 0.057 |\sin 2\Delta\theta^d| \sim 0.03, \quad (74)$$

where we have used (67), (71), and (72), $-1.3 < \theta_q = \Delta\theta^d - \Delta\theta^u < -1.0$ [see (40)] and $2\Delta\theta^u \simeq \pi$ or 0 [see (58)].

Using (57) one can express to $O(b_\nu)$ $\sin 2\alpha$ as

$$\begin{aligned} \sin 2\alpha &= \sin(\varphi_1 - \varphi_2) \\ &= -\frac{m_{\nu_3} \sin\phi_\nu}{m_{\nu_1} m_{\nu_2}} \left(\sqrt{m_{\nu_2}^2 - m_{\nu_3}^2 \sin^2\phi_\nu} \right. \\ &\quad \left. + \sqrt{m_{\nu_1}^2 - m_{\nu_3}^2 \sin^2\phi_\nu} \right), \end{aligned} \quad (75)$$

where we have used $\varphi_2 + \varphi_1 \simeq \pi$ and $\varphi_2 \sim \pi$. Equation (60) can be rewritten in terms of measurable quantities:

$$\begin{aligned} \kappa \sin\varphi &= \kappa \sin 2\Delta\theta^u = \frac{m_{\nu_3}}{m_{\nu_1}} \sin\phi_\nu \left[1 - \left(\frac{m_{\nu_3}}{m_{\nu_2}} \sin\phi_\nu \right)^2 \right] \\ &\quad - \frac{m_{\nu_3}}{m_{\nu_2}} \sin\phi_\nu \left[1 - \left(\frac{m_{\nu_3}}{m_{\nu_1}} \sin\phi_\nu \right)^2 \right] \\ &\simeq \frac{m_{\nu_3}}{2m_{\nu_2}^3} \frac{\Delta m_{21}^2 \sin\phi_\nu}{[1 - (m_{\nu_3}/m_{\nu_2})^2 \sin^2\phi_\nu]^{1/2}}, \end{aligned} \quad (76)$$

which relates ϕ_ν with $\Delta\theta^u$. Although $\sin\varphi \simeq 0$ because of (58), $\sin\phi_\nu$ is not small and can vary from -1 to $+1$. So, it may be better to regard $\sin\phi_\nu$ as independent instead of $\Delta\theta^u$.

The average neutrino mass $\langle m_{ee} \rangle$ can be then predicted from

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^3 m_{\nu_i} V_{ei}^2 \right| \simeq |m_{\nu_1} c_\nu^2 + m_{\nu_2} s_\nu^2 \exp i 2\alpha|. \quad (77)$$

To obtain numerical values of $\langle m_{ee} \rangle$, we use the experimental values [55]

$$\begin{aligned} 7.2 \times 10^{-5} \text{ eV}^2 &\leq \Delta m_{21}^2 \leq 9.1 \times 10^{-5} \text{ eV}^2, \\ 1.4 \times 10^{-3} \text{ eV}^2 &\leq \Delta m_{23}^2 \leq 3.3 \times 10^{-3} \text{ eV}^2, \end{aligned} \quad (78)$$

$$\begin{aligned} 0.23 \leq \sin^2\theta_{12} \leq 0.38, \quad 0.34 \leq \sin^2\theta_{23} \leq 0.68, \\ |\sin^2\theta_{13}| < 0.047. \end{aligned} \quad (79)$$

Note that α is given in (75) and the absolute scale of the neutrino mass can be obtained from (62). In Fig. 4 we plot $\langle m_{ee} \rangle$ as a function of $\sin\phi_\nu$ for $\sin^2\theta_{12} = 0.3$, $\Delta m_{21}^2 = 8.1 \times 10^{-5} \text{ eV}^2$, and $\Delta m_{23}^2 = 1.4, 2.2, 3.3 \times 10^{-3} \text{ eV}^2$. As we can see from Fig. 2, the effective Majorana mass stays

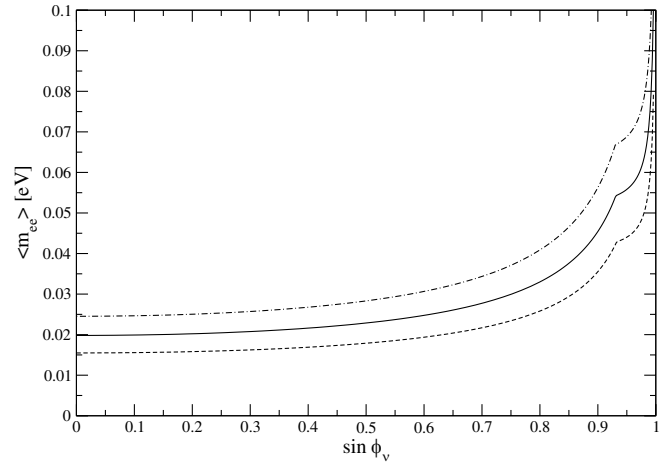


FIG. 4. The average neutrino mass $\langle m_{ee} \rangle$ in the limit $b_\nu \rightarrow 0$ as a function of $\sin\phi_\nu$ with $\sin^2\theta_{12} = 0.3$ and $\Delta m_{21}^2 = 8.1 \times 10^{-5} \text{ eV}^2$. The dashed, solid and dot-dashed lines correspond to $\Delta m_{23}^2 = 1.4, 2.2$ and $3.3 \times 10^{-3} \text{ eV}^2$, respectively. The Δm_{21}^2 dependence is very small. The neutrino spectrum is inverted.

at about its minimal value $\langle m_{ee} \rangle_{\min}$ for a wide range of $\sin\phi_\nu$. Since $\langle m_{ee} \rangle_{\min}$ is approximately equal to $\langle m_{ee} \rangle_{\min} \simeq \sqrt{\Delta m_{23}^2} / \tan 2\theta_{12} = (0.010\text{--}0.037) \text{ eV}$, it is consistent with recent experiments [56,57].

To make the case of an inverted spectrum of the model more transparent, we summarize the result in this sector. A set of the independent quantities are the following: three charged fermion masses, $\theta_{13}, \theta_{12} (\simeq \theta_\nu), \theta_{23} [\simeq \phi]$ defined in (44), $\Delta m_{21}^2, \Delta m_{23}^2$, and ϕ_ν , where $\theta_q = \Delta\theta^d - \Delta\theta^u$ can be determined in the quark sector, and $\Delta\theta^u$ is a function of ϕ_ν given in (76). Therefore, the absolute scale

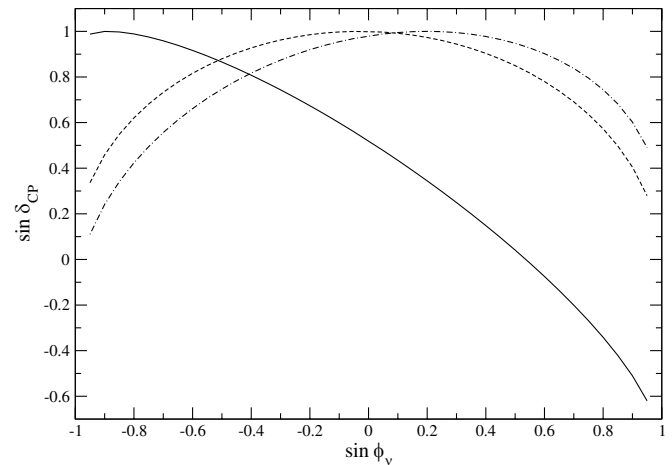


FIG. 5. $\sin\delta_{CP}$ in the limit $b_\nu \rightarrow 0$ versus $\sin\phi_\nu$ for different values of $\sin\theta_{13}$, where we use $\sin^2\theta_{12} = 0.3$, $\theta_q = -1.15$. The solid line corresponds to $\sin\theta_{13} = 0.0438$, which is the minimum for $\theta_q = -1.15$. The other lines are those for $\sin\theta_{13} = 0.10$ (dotted), 0.20 (dot-dashed), respectively. θ_q is defined in (48). The result does not depend on κ .

for neutrino masses can be computed from (62). One of the absolute scales is the effective neutrino mass $\langle m_{ee} \rangle$ in neutrinoless double β decay, which is shown in Fig. 4. Another prediction is the Dirac phase δ_{CP} defined in (69), which is plotted as a function of ϕ_ν for different values of $\sin\theta_{13}$ in Fig. 5, where we have assumed $\cos\theta_{23} = 1/\sqrt{2}$. As we can see from these figures, precise measurements of the physical parameters such as $\sin\theta_{13}$, δ_{CP} , and $\langle m_{ee} \rangle$ in the neutrino sector can in principle confirm the predictions of the model.

2. Normal spectrum

A normal hierarchal spectrum of neutrino masses with small $|V_{e3}|$ can be obtained if the mass parameters of M_ν (48) satisfy the following conditions:

$$c_\nu^2, b_\nu^2 \ll b_\nu^2, a_\nu^2 \quad \text{and} \quad \varphi = 2\Delta\theta^u \simeq 0. \quad (80)$$

So, in the lowest order approximation, we may neglect CP violating phases involved in the unitary matrix U_ν . Under the conditions (80), we find that the unitary matrix U_ν can be approximately written as

$$U_\nu^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_a & s_a \\ 0 & -s_a & c_a \end{pmatrix} \times \begin{pmatrix} c_\nu & -s_\nu & \epsilon \\ s_\nu & c_\nu & 0 \\ \epsilon c_\nu & \epsilon s_\nu & 1 \end{pmatrix}, \quad (81)$$

where $s_\nu = \sin\theta_\nu$, $s_a = \sin\theta_a$, etc., and

$$\epsilon \simeq \frac{R_\nu}{2} \cot\theta_a \sin 2\theta_\nu, \quad (82)$$

$$R_\nu = \frac{(m_{\nu_2} - m_{\nu_1})}{m_{\nu_3} - m_{\nu_2} \sin^2\theta_\nu - m_{\nu_1} \cos^2\theta_\nu}. \quad (83)$$

The mixing matrix V_{MNS} is obtained from $O_e^T P_e^\dagger P_\nu U_\nu$, where O_e^T , P_e , and P_ν are given in (44), (45), and (47). Note, however, that because of the condition (80), $\arg\langle H_1^u \rangle \simeq \arg\langle H_3^u \rangle$ so that $P_\nu \propto \mathbf{1}$. We find that the mixing matrix V_{MNS} is approximately given by

$$V_{e2} \simeq -\sin\theta_\nu, \quad (84)$$

$$V_{e3} \simeq \epsilon + \sqrt{\frac{m_e}{m_\mu}} \left(\sqrt{\cos\phi} \sin\theta_a e^{-i2\Delta\theta^d} - \frac{\sin\phi}{\sqrt{\cos\phi}} \times \cos\theta_a e^{-i\Delta\theta^d} \right), \quad (85)$$

$$V_{\mu 3} \simeq \sin\theta_a \cos\phi e^{-2i\Delta\theta^d} - \cos\theta_a \sin\phi e^{-i\Delta\theta^d}, \quad (86)$$

$$V_{\tau 3} \simeq \sin\theta_a \sin\phi e^{-2i\Delta\theta^d} + \cos\theta_a \cos\phi e^{-i\Delta\theta^d}, \quad (87)$$

where $\Delta\theta^d$ and ϕ are defined in (19) and (44), respectively.

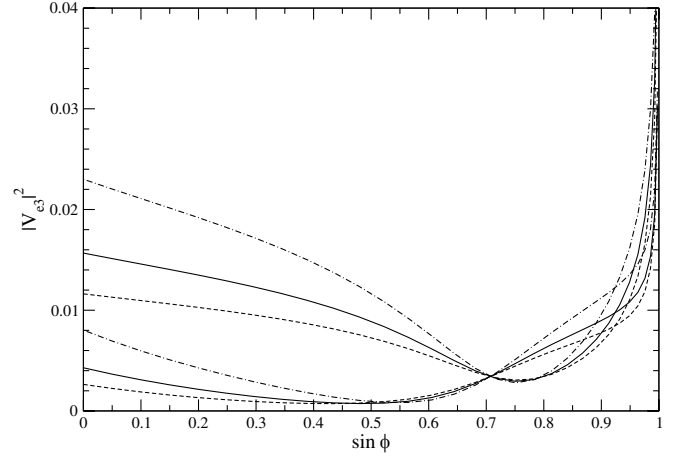


FIG. 6. $|V_{e3}|^2$ against $\sin\phi$ for $R_\nu = 0.15$ (dotted), 0.19 (solid), and 0.25 (dashed-dotted) with $\theta_q = -1.15$ and $\sin^2\theta_\nu = 0.3$, where R_ν and θ_q are defined in (83) and (27), respectively. The upper (lower) lines of each pair at $\sin\phi = 0$ correspond to $\sin\theta_\nu (\simeq \sin\theta_{12}) = -(+)\sqrt{0.3}$. The neutrino mass spectrum is normal.

The condition (80) in the normal hierarchal case requires that $\Delta\theta^u \simeq 0$, which implies that $\theta_q \simeq \Delta\theta^d$. We recall that θ_q is determined in the quark sector [realistic values of θ_q are between -1.3 and -1.0 as it is given in (40)]. Therefore, if we use the experimental information on $|V_{\mu 3}|$, we can relate two parameters ϕ and θ_a . Below we assume that the mixing of the atmospheric neutrino is close to maximal and that $\sin\theta_a$ is positive, and we regard $\sin\phi$ as independent. In Fig. 6 we plot $|V_{e3}|^2$ against $\sin\phi$ for different values of R_ν with $\theta_q = -1.15$. In Fig. 7, $\sin\delta_{CP}$ [defined in the standard form (69) under the assumption $\sin\theta_{13} > 0$] against $\sin\phi$ is plotted.

As we see from Figs. 6–8 in the case of a normal neutrino mass spectrum, too, precise measurements of

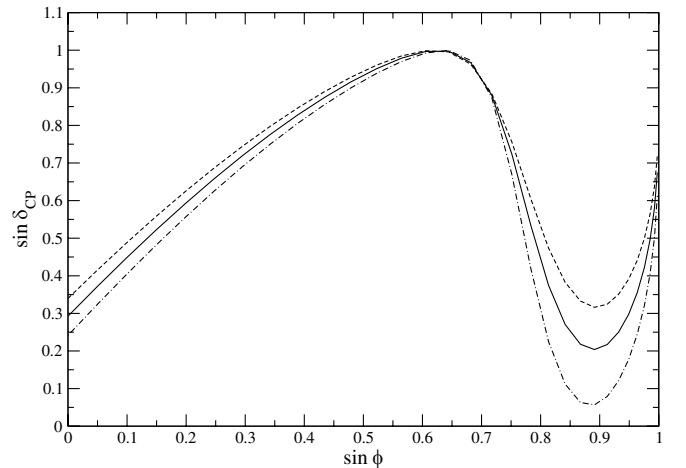
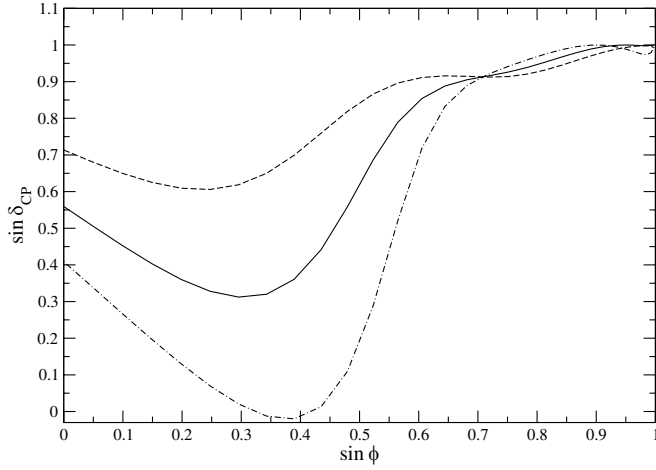


FIG. 7. $\sin\delta_{CP}$ against $\sin\phi$ for $R_\nu = 0.15$ (dotted), 0.19 (solid), and 0.25 (dashed-dotted) with $\sin\theta_\nu = -\sqrt{0.3}$.

FIG. 8. The same as Fig. 7 for $\sin\theta_\nu = +\sqrt{0.3}$.

the physical parameters in the neutrino sector can in principle confirm the predictions of the model.

E. Higgs sector

The Higgs supermultiplets $H^{u,d}$ and $H_3^{u,d}$ belong to $\mathbf{2}'$ and $\mathbf{1}'''$, respectively. Therefore, the singlets $H_3^{u,d}$ cannot form a mass term. So, the Q_6 invariant superpotential is

$$W_H = \mu_H(H_1^u H_1^d + H_2^u H_2^d), \quad (88)$$

which has an accidental $U(2)$ symmetry, implying that after spontaneous symmetry breaking of the electroweak gauge symmetry, there will appear quasi-Nambu-Goldstone supermultiplets. Moreover, the mass term for H_3 is absent, so that its fermionic component will remain massless even after SUSY breaking. So, they will conflict with experimental observations.

The origin of the so-called μ term is always a mysterious problem. It is usually assumed that it is related to SUSY breaking. In the present model with Q_6 family symmetry, we also worry about breaking of Q_6 , because our assumption on VEVs (18) means a complete breaking of Q_6 . It may be interesting to construct a Higgs sector, in which the origin of the μ term and Q_6 breaking are related to SUSY breaking. It turns out that to construct such a sector, we need to introduce a number of $SU(2)_L$ singlet Higgs multiplets, and also an additional Abelian discrete symmetry Z_{12R} , which acts as an R symmetry. $Q_6 \times Z_{12R}$ assignments are given in Tables I and II.

TABLE II. $Q_6 \times Z_{12R}$ assignment of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet Higgs supermultiplets.

	S	T	X	Y	Z
Q_6	$\mathbf{2}$	$\mathbf{2}'$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}'$
Z_{12R}	2	2	-2	6	0

Then we consider the following most general $Q_6 \times Z_{12R}$ invariant superpotential which contains the right-handed neutrino and Higgs supermultiplets:

$$W'_H = W_I + W_{II}, \quad (89)$$

where the Z_{12R} charge of W'_H is $-2(\text{mod}12)$, and

$$W_I = M_1(N_1^2 + N_2^2) + \lambda_N Z N_3^2 + \lambda_Z X(Z^2 - M^2), \quad (90)$$

$$\begin{aligned} W_{II} = & \lambda_1 X Y^2 + \lambda_2 (S_1 S_1 + S_2 S_2) Y \\ & + \kappa_1 (H_1^u S_2 + H_2^u S_1) H_3^d + \kappa_2 (H_1^d S_2 + H_2^d S_1) H_3^u \\ & + \kappa_3 [-(H_1^u H_1^d - H_2^u H_2^d) T_1 + (H_1^u H_2^d + H_2^u H_1^d) T_2]. \end{aligned} \quad (91)$$

We assume that $O(M_1) \sim O(M)$. In the SUSY limit, there is a local minimum, where $\langle Z \rangle = M$ and all other VEVs are zero. After SUSY breaking, X will acquire a VEV of order M_{SUSY} , which will induce effective μ terms for the other Higgs supermultiplets in (91). The charges of the fields are so chosen that the most general $Q_6 \times Z_{12R}$ invariant Higgs superpotential (89) has an accidental parity invariance:

$$H_1^{u,d} \leftrightarrow H_2^{u,d}, \quad S_1 \leftrightarrow S_2, \quad T_1 \rightarrow -T_1, \quad (92)$$

where $H_3^{u,d}$, X , Y , Z , and T_2 do not transform. This enables us to choose a VEV structure given by

$$\langle H_1^u \rangle = \langle H_2^u \rangle, \quad \langle H_1^d \rangle = \langle H_2^d \rangle, \quad \langle H_3^{u,d} \rangle \neq 0, \quad (93)$$

$$\langle S_1 \rangle = \langle S_2 \rangle, \quad \langle T_1 \rangle = 0, \quad \langle T_2 \rangle \neq 0. \quad (94)$$

Moreover, it is found that the superpotential W'_H (89) induces spontaneous CP violation, when all couplings are taken to be real.

The Higgs sector given in this subsection may not be minimal. It is however clear that the Higgs sector is related to SUSY breaking, so that there will be different modifications. The important point is that the effective superpotential for the $SU(2)_L$ doublet Higgs multiplets should possess the $H_1 \leftrightarrow H_2$ symmetry to ensure the VEV structure (18). Therefore, instead of introducing the S, T, \dots singlets fields, one may assume that Q_6 is explicitly, but softly broken by dimension 3 terms. If we demand that these soft terms do not break the $H_1 \leftrightarrow H_2$ symmetry, it must be of the form

$$\begin{aligned} W_{\text{eff}} = & \mu_1 (H_1^u H_1^d + H_2^u H_2^d) + \mu_3 H_3^u H_3^d \\ & + \mu_{13} (H_1^u + H_2^u) H_3^d + \mu_{31} H_3^u (H_1^d + H_2^d) \\ & + \mu_{12} (H_1^u H_2^d + H_2^u H_1^d). \end{aligned} \quad (95)$$

Since the μ parameter results from VEVs, they may be complex.

In models with more than one Higgs $SU(2)_L$ doublet, as in the case of the present model, tree-level FCNCs exist in the Higgs sector. They contribute, for instance, to the mass difference Δm_K of K^0 and \bar{K}^0 , which will give severe

constraints on the model, especially on the heavy Higgs masses. [See, for instance, [25,58].] This is one of the important questions, but detailed investigation of this question is beyond the scope of the present paper. So we would like to leave this problem to future work.

IV. DISCUSSIONS

Here we would like to discuss further features of the Q_6 model.

A. Anomalies of discrete symmetries

Quantum gravitational effects violate all global symmetries, while they respect all local symmetries [59]. When a local gauge group is spontaneously broken, a certain discrete subgroup may survive [60], which is respected by gravity. A necessary condition for a discrete symmetry group to be a subgroup of a gauge group is the absence of anomalies [45–47].

We start with Q_6 anomalies,⁵ $Q_6[SU(2)_L]^2$ and $Q_6[SU(3)_C]^2$. Since the anomaly in $Q_6[U(1)_Y]^2$ depends on the normalization of $U(1)_Y$, it does give a useful information. Similarly, heavy fields can contribute to the $[Q_6]^3$ anomaly so that it does not constrain the low-energy spectrum. To compute anomalies, we note that A_{Q_6} , $(A_{Q_6})^2$, ..., $(A_{Q_6})^5$ correspond to Z_6 rotations, and B_{Q_6} to Z_4 rotations. [A_{Q_6} and B_{Q_6} are defined in (5) and (6).] One can easily convince oneself that the doublets of Q_6 cannot contribute to the anomalies, because they always contain two components, one with a plus charge and the other one with a negative charge in the same amount. The charges of Q_6 singlets under (Z_6, Z_4) are given by

$$\mathbf{1}:(0, 0), \quad \mathbf{1}':(0, 2), \quad \mathbf{1}'':(3, -1), \quad \mathbf{1}''':(3, 1). \quad (96)$$

From Table I we can read off the corresponding charges of the matter supermultiplets, and we then compute the anomaly coefficients:

$$\begin{aligned} Z_6[SU(2)_L]^2:2A_2 &= 3(\text{mod}6), \\ Z_6[SU(3)_C]^2:2A_3 &= 3(\text{mod}6), \end{aligned} \quad (97)$$

$$\begin{aligned} Z_4[SU(2)_L]^2:2A_2 &= 2(\text{mod}4), \\ Z_4[SU(3)_C]^2:2A_3 &= 2(\text{mod}4). \end{aligned} \quad (98)$$

$Z_{12R}[SU(2)_L]^2$ and $Z_{12R}[SU(3)_C]^2$ anomalies can also be straightforwardly computed, and we find

$$\begin{aligned} Z_{12R}[SU(2)_L]^2:2A_2 &= 6(\text{mod}12), \\ Z_{12R}[SU(3)_C]^2:2A_3 &= 6(\text{mod}12). \end{aligned} \quad (99)$$

⁵ Q_6 anomalies are also considered in [18].

Anomalies of a discrete symmetry can be cancelled by the Green-Schwarz mechanism, if

$$\frac{A_2 + mN/2}{k_2} = \frac{A_3 + m'N/2}{k_3} \quad (100)$$

is satisfied, where k 's (the Kac-Moody levels) and m, m' are integers, and N stands for the order of Z_N [45–47]. Therefore, all the anomalies cancel if we choose

$$k_2 = k_3, \quad (101)$$

for instance.

B. R-parity and Baryon number violating operators

Just because of Q_6 symmetry, it turns out that R -parity violating couplings are almost absent. Out of the 96 R -parity breaking cubic couplings that are allowed in the MSSM superpotential, Q_6 allows only one coupling

$$\lambda'[(L_1 Q_2 + L_2 Q_1)D_1^c + (L_1 Q_1 - L_2 Q_2)D_2^c]. \quad (102)$$

Many couplings vanish because of color antisymmetry and $SU(2)_L$ antisymmetry. Furthermore, all baryon number violating cubic terms are forbidden by Q_6 alone. This means that there is no proton decay problem in the present model.⁶ The natural value of λ' is about 10^{-3} , consistent with Yukawa couplings of charged sector. This term will induce a neutrino mass proportional to the strange quark mass at one loop. The magnitude may be of order 10^{-2} eV.

C. The SUSY flavor problem and phase alignment

Since Q_6 is an intact symmetry at M_{SUSY} [except for the mass term of N_3^c , which results from the VEV of Z in (90)], it is natural to assume that the SSB sectors also respect Q_6 symmetry. From the results of [40,41], in which a detailed analysis on FCNCs and CP violations in a supersymmetric extension of the SM with an non-Abelian discrete flavor symmetry S_3 has been made, we may expect that Q_6 suppresses strongly FCNC and CP violating processes that are induced by the SSB terms. However, the constraints coming from the EDM of neutron, electron, and mercury atom [42,51–53] are very severe. Recent experiment on mercury atom [50] requires that the imaginary part of the dimensionless quantity $(\delta_{11}^d)_{LR}$ has to satisfy $|\text{Im}(\delta_{11}^d)_{LR}| < 6.7 \times 10^{-8} (\tilde{m}_q/100 \text{ GeV})^2$ [51], where \tilde{m}_q is the average of the squark masses. Similar constraints exist for $(\delta_{11}^u)_{LR}$ and $(\delta_{11}^e)_{LR}$, too. The quantity δ_{LR}^d is defined as [42]

$$\delta_{LR}^d = \frac{U_d^T \tilde{\mathbf{m}}_{dLR}^2 U_d^c}{\tilde{m}_q^2}, \quad (103)$$

⁶Anomaly-free Abelian discrete symmetries to suppress proton decay have been considered in [47].

where $\tilde{\mathbf{m}}^2_{dLR}$ resulting from the A terms is the so-called left-right mass matrix for the bosonic components of D and D^c . The unitary matrices are those that rotate the fermionic superpartners, i.e. $U_d = P_d O_d$, $U_{d^c} = P_{d^c} O_{d^c}$, where the phase matrices P_d and P_{d^c} are defined in (22) and (23), respectively. Since we assume that the CP phases are spontaneously induced, and thanks to Q_6 symmetry the mass matrix $\tilde{\mathbf{m}}^2_{dLR}$ has exactly the same structure as the mass matrix (17), we conclude that $(\delta_{11}^d)_{LR}$ is a real number. From the same reason, all δ_{LR} are real. Thus, we can satisfy the most stringent constraint on the A terms without any fine tuning. This is true not only at a particular energy scale, but also for the entire energy scale, which should be compared with the case of the MSSM [52].

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APPENDIX A: MULTIPLICATION RULES

Here we give the multiplication rules for D_3 , D_4 , D_6 , D_8 , Q_4 , and Q_8 . The multiplication rules (without the Clebsch-Gordan coefficients) for finite groups with larger orders are given in [44].

1. $D_3(S_3)$

The irreps of D_3 are $\mathbf{2}, \mathbf{1}, \mathbf{1}'$.

$$\begin{array}{l} \mathbf{2} \times \mathbf{2} = \mathbf{1}' + \mathbf{1} + \mathbf{2} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_1 y_1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} -x_1 y_1 + x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \end{array}$$

2. D_4

The irreps of D_4 are $\mathbf{2}, \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$.

$$\begin{array}{l} \mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1}'' \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 + x_2 y_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} \begin{pmatrix} x_1 y_1 - x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \\ + \mathbf{1}' + \mathbf{1}''' \\ \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \end{array}$$

3. D_6

The irreps of D_6 are $\mathbf{2}, \mathbf{2}', \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$.

$$\begin{array}{l} \mathbf{2} \times \mathbf{2} = \mathbf{1}' + \mathbf{1} + \mathbf{2}' \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_1 y_1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 y_1 - x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \\ \\ \mathbf{2}' \times \mathbf{2}' = \mathbf{1} + \mathbf{1}' + \mathbf{2}' \\ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a_2 b_2 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \begin{pmatrix} -a_1 b_1 + a_2 b_2 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} \\ \\ \mathbf{2} \times \mathbf{2}' = \mathbf{1}''' + \mathbf{1}'' + \mathbf{2} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 a_2 + x_2 a_1 \\ x_1 a_1 - x_2 a_2 \end{pmatrix} \begin{pmatrix} x_1 a_1 + x_2 a_2 \\ x_1 a_2 - x_2 a_1 \end{pmatrix} \end{array}$$

4. D_8

The irreps of D_8 are $\mathbf{2}, \mathbf{2}', \mathbf{2}'', \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$.

$$\begin{array}{l} \mathbf{2} \times \mathbf{2} = \mathbf{1}' + \mathbf{1} + \mathbf{2}' \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_1 y_1 + x_2 y_2 \end{pmatrix} \begin{pmatrix} x_1 y_1 - x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \end{array}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{1} \\ a_1 b_1 + a_2 b_2 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}'' \\ a_1 b_1 - a_2 b_2 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{1}' \\ a_1 b_2 - a_2 b_1 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}''' \\ a_1 b_2 + a_2 b_1 \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ x_1 a_1 + x_2 a_2 \\ x_1 a_2 - x_2 a_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}'' \\ x_1 a_1 - x_2 a_2 \\ x_1 a_2 + x_2 a_1 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ -t_1 a_1 - t_2 a_2 \\ t_1 a_2 - t_2 a_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}'' \\ t_1 a_1 - t_2 a_2 \\ -t_1 a_2 - t_2 a_1 \end{pmatrix}.$$

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1}'' \\ x_1 t_1 - x_2 t_2 \\ x_1 t_2 + x_2 t_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}''' \\ x_1 t_2 + x_2 t_1 \\ x_1 t_1 - x_2 t_2 \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ x_1 t_1 + x_2 t_2 \\ x_1 t_2 - x_2 t_1 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{2}'' \\ s_1 \\ s_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ s_1 t_1 + s_2 t_2 \\ s_1 t_2 - s_2 t_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}' \\ s_1 t_2 - s_2 t_1 \\ s_1 t_1 + s_2 t_2 \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ s_1 t_1 - s_2 t_2 \\ -s_1 t_2 - s_2 t_1 \end{pmatrix},$$

5. Q_4

The irreps of Q_4 are $\mathbf{2}, \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$, where $\mathbf{2}$ is pseudoreal, and the singlets are real.

$$\begin{aligned} \begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2} \\ y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{1}' \\ x_1 y_1 + x_2 y_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}''' \\ x_1 y_1 - x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{1} \\ x_1 y_2 - x_2 y_1 \\ x_1 y_1 + x_2 y_2 \end{pmatrix} + \begin{pmatrix} \mathbf{1}'' \\ x_1 y_2 + x_2 y_1 \\ x_1 y_1 - x_2 y_2 \end{pmatrix}, \end{aligned}$$

6. Q_6

The multiplication rules for Q_6 are given in Sec. II.

7. Q_8

The irreps of Q_8 are $\mathbf{2}, \mathbf{2}', \mathbf{2}'', \mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{1}'''$. All singlets and $\mathbf{2}'$ are real representations.

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2} \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ x_1 y_2 - x_2 y_1 \\ x_1 y_1 + x_2 y_2 \end{pmatrix} + \begin{pmatrix} \mathbf{1}' \\ x_1 y_1 + x_2 y_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ -x_1 y_2 - x_2 y_1 \\ x_1 y_1 - x_2 y_2 \end{pmatrix},$$

$$\begin{aligned} \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ b_1 \\ b_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{1} \\ a_1 b_1 + a_2 b_2 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}'' \\ a_1 b_1 - a_2 b_2 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} \\ &+ \begin{pmatrix} \mathbf{1}' \\ a_1 b_2 - a_2 b_1 \\ a_1 b_2 + a_2 b_1 \end{pmatrix} + \begin{pmatrix} \mathbf{1}''' \\ a_1 b_2 + a_2 b_1 \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ x_1 a_1 + x_2 a_2 \\ x_1 a_2 - x_2 a_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}'' \\ x_1 a_1 - x_2 a_2 \\ x_1 a_2 + x_2 a_1 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}' \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ -t_1 a_1 - t_2 a_2 \\ t_1 a_2 - t_2 a_1 \end{pmatrix} + \begin{pmatrix} \mathbf{2}'' \\ t_1 a_1 - t_2 a_2 \\ -t_1 a_2 - t_2 a_1 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{2} \\ x_1 \\ x_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1}''' \\ (x_1 t_1 - x_2 t_2) \end{pmatrix} + \begin{pmatrix} \mathbf{1}'' \\ (x_1 t_2 + x_2 t_1) \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ (-x_1 t_2 + x_2 t_1) \\ (x_1 t_1 + x_2 t_2) \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{2}'' \\ s_1 \\ s_2 \end{pmatrix} \times \begin{pmatrix} \mathbf{2}'' \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1}' \\ (s_1 t_1 + s_2 t_2) \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ (s_1 t_2 - s_2 t_1) \end{pmatrix} + \begin{pmatrix} \mathbf{2}' \\ (s_1 t_2 + s_2 t_1) \\ (s_1 t_1 - s_2 t_2) \end{pmatrix}.$$

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