PHYSICAL REVIEW D **71,** 054503 (2005)

(2 + 1)-dimensional lattice QCD

Peter Orland*

Kavli Institute for Theoretical Physics, The University of California, Santa Barbara, California 93106, USA
Physics Program, The Graduate School and University Center, The City University of New York,
365 Fifth Avenue, New York, New York 10016, USA
Department of Natural Sciences, Baruch College, The City University of New York,
17 Lexington Avenue, New York, New York 10010, USA
(Received 28 January 2005; published 9 March 2005)

We consider a (2 + 1)-dimensional SU(N) lattice gauge theory in an axial gauge with the link field U_1 set equal to one. The term in the Hamiltonian containing the square of the electric field in the 1-direction is nonlocal. Despite this nonlocality, we show that weak-coupling perturbation theory in this term gives a finite vacuum-energy density to second order, and suggest that this property holds to all orders. Heavy quarks are confined, the spectrum is gapped, and the spacelike Wilson loop has area decay.

DOI: 10.1103/PhysRevD.71.054503 PACS numbers: 11.15.Ha, 12.38.Mh

I. INTRODUCTION

The central problem of QCD is confinement. It is not enough to prove that lattice gauge theories have a confining phase—which is evident from strong-coupling expansions. It is necessary to see that the color is confined at arbitrarily weak lattice coupling.

We find here that (2 + 1)-dimensional lattice gauge theories confine for any dimensionless bare coupling. The technique used is a *weak-coupling* expansion in an anisotropic lattice gauge theory in an axial gauge. This is not the standard expansion utilizing Feynman diagrams. Though the coupling constant of a (2 + 1)-dimensional gauge theory is not infinitely renormalized in (2 + 1) dimensions, the dimensionless bare coupling on the lattice must vanish in the continuum limit. This is why a weak-coupling analysis is useful, even for this case. The dependence we find of the string tension and the mass gap on the coupling constant does not agree with conventional wisdom—our results for these physical quantities do not behave as anticipated, as the lattice spacing is taken to zero—but they are not zero.

The first analytic demonstration of confinement of heavy sources in (2 + 1)-dimensional gauge theories was given by Polyakov for lattice compact QED [1], and later for the Georgi-Glashow model [2]. The latter model is interesting in that color charges disappear completely from the spectrum. This is, however, different from the sort of confinement we expect for QCD, in that matter fields play an important role. Feynman argued that (2 + 1)-dimensional QCD is confining [3]. Unfortunately, the orbit-space distance estimates in Feynman's paper are incorrect. Nonetheless, his basic claim, that the diameter of gauge-orbit space of (2 + 1)-dimensional SU(N) Yang-Mills theory for small magnetic energy is finite, appears to be correct [4]. New nonperturbative methods which do not require a lattice have been derived by Karabali, Kim and Nair [5] (one of their formulations of the Hamiltonian has been obtained a different way in Ref. [6]).

There is one special set of assumptions we use to derive our results; the (1 + 1)-dimensional non-Abelian nonlinear sigma models (without topological terms) have a mass gap, exponentially decaying correlation functions, and their vacuum expectation values of local operators exhibit clustering [7–9]. Though no rigorous proof of these properties exists, we think that the evidence in their favor is overwhelming.

Our basic strategy is to write the lattice version of the Hamiltonian as the sum of two terms, namely

$$H_0 = \int d^2x \left(\frac{e^2}{2} \operatorname{Tr}\mathcal{E}_2^2 + \frac{1}{2e^2} \operatorname{Tr}B^2 \right),$$

and

$$\frac{e^2}{2} V = \frac{e^2}{2} \int d^2x \operatorname{Tr}\mathcal{E}_1^2,$$

where $\mathcal{E}_j$ are the components of the electric-field conjugate to the gauge field A_j, $[\mathcal{E}_j(x), A_k(y)] = i\delta_{jk}\delta^2(x-y)$ and $B = i[\partial_1 - iA_1, \partial_2 - iA_2]$ is the single space component of the magnetic field. We then pick the gauge $A_1 = 0$. When this is done on the lattice, H_0 is a set of decoupled chiral SU(N) × SU(N) nonlinear sigma models for which the S matrix and the spectrum are known. The quantity V is nonlocal, but we show that perturbation theory in this term is sensible to second order. The vacuum state in this perturbation series confines fundamental color charges. Our splitting of the Hamiltonian is not explicitly rotationally invariant, but if the method works to all orders of perturbation theory, rotational invariance should be restored.

Let us review axial gauges in the continuum. If the SU(N)-Lie-algebra-valued gauge field A_1 is set to zero, then Gauss's law may be integrated to obtain

*Electronic address: orland@gursey.baruch.cuny.edu

$$\mathcal{E}_1(x) = -\int^{x^1} dy^1 \sum_{j=2}^{d-1} [\partial_j - iA_j(y^1, x^2, \ldots x^{d-1}), \mathcal{E}_j(y^1, x^2, \ldots, x^{d-1})] = -\int^{x^1} dy^1 D_\perp(y^1, x^\perp) \cdot \mathcal{E}_\perp(y^1, x^\perp), \quad (1.1)$$

where the dimension of space is $d-1$, $x^\perp = (x^2, \ldots, x^{d-1})$, and $D_\perp$ are the last $d-2$ components of the covariant derivative in the adjoint representation $(D_2, \ldots, D_{d-1})$. The term in the Hamiltonian

$$\frac{e^2}{2} V = \int d^{d-1}x \frac{e^2}{2} \mathrm{Tr}\mathcal{E}_1^2 \quad (1.2)$$

must have a vacuum expectation value proportional to the volume of $(d-1)$-dimensional space, if the theory is to be sensible. As discussed by Mandelstam [10] this means that the quantity

$$K(y^1, z^1, x^\perp) = \langle 0| \mathrm{Tr} D_\perp(y^1, x^\perp) \cdot \mathcal{E}_\perp(y^1, x^\perp) D_\perp(z^1, x^\perp) \cdot \mathcal{E}_\perp(z^1, x^\perp)|0\rangle$$

must have the property that $\int dy^1 dz^1 K(y^1, z^1, x^\perp)$ does not diverge with the spatial volume. One might think that if $K(y^1, z^1, x^\perp)$ falls off sufficiently fast with $|y^1 - z^1|$, the problem can be ameliorated. Rapid falloff of K, however, is not enough. Even if the falloff is exponential, the result may diverge as $(L^1)^2$ where L^1 is the range of x^1. Mandelstam recognized that the residual gauge invariance, remaining after solving for $\mathcal{E}_1$ in (1.1), namely

$$\int dx^1 D_\perp \cdot \mathcal{E}_\perp \Psi = 0, \quad (1.3)$$

must also be satisfied by the vacuum. Without both (1.3) and sufficiently rapid decay of $K(x^1, y^1, x^\perp)$, any conjecture for the vacuum may have an unacceptable infrared-divergent energy, coming from (1.2). Fortunately, we find that in our perturbation scheme, both the unperturbed vacuum energy and the first two corrections in our weak-coupling expansion obey the lattice versions of both the rapid-decay criterion and (1.3).

II. THE LATTICE GAUGE HAMILTONIAN

The purpose of this section is to establish our definitions and conventions. It is not an introduction to the Hamiltonian SU(N) gauge theory. Such introductions can be found in the review article by Kogut and in the book by Creutz [11].

Consider a lattice of sites x of size $L^1 \times L^2$, with sites x whose coordinates are x^1 and x^2. We require that x^1/a and x^2/a are integers, where a is the lattice spacing. There are two space directions, labeled $j = 1, 2$. Each link is a pair x, j, and joins the site x to $x + \hat{j}a$, where $\hat{j}$ is a unit vector in the jth direction.

We introduce basis vectors or generators t_α, $\alpha = 1, \ldots, N^2-1$, of the Lie algebra of SU(N). Sometimes we use Roman letters for the index, e.g., we may write t_b rather than t_α [the purpose of using different alphabets is to distinguish between coordinate indices on the SU(N) manifold and tangent-space vectors]. The generators are defined to be orthonormal, so that $\mathrm{Tr} t_\alpha t_\beta = \delta_{\alpha\beta}$. The structure coefficients of the Lie algebra, $f^\gamma_{\alpha\beta}$, $\alpha, \beta, \gamma = 1, \ldots, N^2-1$, are, as usual, the complex numbers defined by $[t_\alpha, t_\beta] = i f^\gamma_{\alpha\beta} t_\gamma$. The identity matrix will be denoted by $\mathbb{1}$.

The Hamiltonian lattice gauge theory is usually formulated in temporal gauge $A_0 = 0$. The basic degrees of freedom, before any further gauge fixing, are elements of the group SU(N) in the fundamental $(N \times N)$-dimensional matrix representation $U_j(x) \in \mathrm{SU}(N)$ at each link x, j. In addition, there are the N^2-1 electric-field operators at each link $l_j(x)_b$, $b = 1, \ldots, N^2-1$. The electric-field operators are self-adjoint by construction. The commutation relations on the lattice are

$$[l_j(x)_b, l_k(y)_c] = i\delta_{xy}\delta_{jk} f^d_{bc} l_j(x)_d, \qquad [l_j(x)_b, U_k(y)] = -\delta_{xy}\delta_{jk} t_b U_j(x), \quad (2.1)$$

all others zero. In the Schrödinger representation, with the components of $U_j(x)$ taken to be c numbers, the latter of (2.1) becomes

$$l_j(x)_b U_k(y) = -\delta_{xy}\delta_{jk} t_b U_j(x).$$

The lattice Hamiltonian is

$$H = \sum_x \sum_{j=1}^{2} \sum_{b=1}^{N^2-1} \frac{g_0^2}{2a} [l_j(x)_b]^2 - \sum_x \frac{1}{4g_0^2 a} [\mathrm{Tr} U_{12}(x) + \mathrm{Tr} U_{21}(x)], \quad (2.2)$$

where

$$U_{jk}(x) = U_j(x) U_k(x + \hat{j}a) U_j(x + \hat{k}a)^\dagger U_k(x)^\dagger,$$

and the bare coupling constant g_0 is dimensionless. Note that the coefficient of the kinetic term can be written in terms of the continuum coupling constant e, namely $g_0^2/(2a) = e^2/2$. It is for this reason that hadron masses and the string tension evaluated in lattice strong-coupling expansions all scale sensibly with e, in $(2+1)$ dimensions.

We denote the adjoint representation of the SU(N) gauge field by $\mathcal{R}$:

$$\sum_{c=1}^{N^2-1} \mathcal{R}_b{}^c t_c = U t_b U^\dagger.$$

The matrix $\mathcal{R}$ lies in the group SU(N)/$\mathbb{Z}_N$. This is a special orthogonal matrix $\mathcal{R}^{\mathrm{T}}\mathcal{R} = 1$, $\det\mathcal{R} = 1$, and SU(N)/$\mathbb{Z}_N$ is a subgroup of SO(N^2-1).

Schrödinger wave functions are complex-valued functions of *all* the link degrees of freedom $U_j(x)$. Physical wave functions $\Psi(\{U\})$ satisfy Gauss's law

$$(\mathcal{D}\cdot l)(x)_b\Psi(\{U\}) = \sum_{j=1}^{2}[\mathcal{D}_j l_j(x)]_b\Psi(\{U\}) = 0, \quad (2.3)$$

where

$$[\mathcal{D}_j l_j(x)]_b = l_j(x)_b - \sum_{c=1}^{N^2-1}\mathcal{R}_j(x-\hat{k}a)_b{}^c l_j(x-\hat{k}a)_c. \quad (2.4)$$

Sometimes it is useful to introduce color charge operators at lattice sites, denoted by $q(x)_b$, which satisfy

$$[q(x)_b, q(y)_c] = i f^a_{bc}\delta_{xy}q(x)_a. \quad (2.5)$$

In the presence of charges, Gauss's law becomes

$$[(\mathcal{D}\cdot l)(x)_b - q(x)_b]\Psi(\{U\}) = 0. \quad (2.6)$$

Henceforth, we will drop the explicit summation symbol for repeated group indices and adopt the Einstein summation convention. Sometimes we omit the lattice site or link labels, provided no confusion should be caused by such omissions.

There is a natural geometric interpretation of the electric-field operator. The Maurer-Cartan vector $e_a{}^a$, on the manifold of SU(N) defined by

$$e_\alpha{}^a t_a = -iU^{-1}\partial_\alpha U,$$

is given explicitly by

$$e_\alpha{}^a = -i\left(\frac{\mathbb{1} - e^{i\mathcal{A}\cdot T}}{\mathcal{A}\cdot T}\right)_\alpha^{\ a},$$

in canonical coordinates $\mathcal{A}^\alpha$, $\alpha = 1, \ldots N^2 - 1$, defined by $U = e^{-i\mathcal{A}\cdot t}$, and $\partial_\alpha = \partial/\partial\mathcal{A}^\alpha$. The coordinates $\mathcal{A}$ are related to the continuum gauge field A by $\mathcal{A} = aA$. The matrix e is nonsingular (including at $\mathcal{A}^\alpha = 0$). One may view $e_a{}^a$ as the linear map from the group manifold to the tangent space; this is a particular choice of the vielbein, and in this case there is torsion. The electric-field operators are given by

$$l_a = -i(e^{-1})_a{}^\alpha\partial_\alpha.$$

III. THE AXIAL GAUGE ON A CYLINDER

By fixing an axial gauge, we will find the gauge-invariant degrees of freedom, up to coordinate singularities of measure zero. Such gauge fixings have been discussed many years ago, both in the continuum [12] and on a lattice [13], in the path-integral approach to gauge theories. The advantage of working with the Hamiltonian instead of the path integral is that unphysical components of the gauge fields may be more easily eliminated using Gauss's law [14] (this could also be done in a transfer matrix formalism).

We choose space to be a lattice cylinder of size $L^1 \times L^2$, with periodic boundary conditions in the 2-direction only. This means that for any function $f(x^1, x^2)$ of lattice sites $f(x^1, x^2 + L^2) = f(x^1, x^2)$. We take components of x to have the values $x^1 = 0, a, 2a, \ldots, L^1$, and $x^2 = 0, a, 2a, \ldots, L^2 - a$. Gauss's law is still given by (2.3), provided (2.4) is modified to

$$\begin{aligned}\mathcal{D}_1 l_1(x) &= (1-\delta_{x^1L^1})l_1(x) - (1-\delta_{x^10}) \\ &\quad\times\mathcal{R}_1(x^1-a, x^2)l_1(x^1-a, x^2), \quad (3.1)\\ \mathcal{D}_2 l_2(x) &= l_2(x) - \mathcal{R}_2(x^1, x^2-a)l_2(x^1, x^2-a),\end{aligned}$$

to take into account points on the boundary.

We gauge fix the links in the 1-direction by $U_1(x^1, x^2) = \mathbb{1}$ everywhere and use (2.3) and (3.1) to write

$$l_1(x^1, x^2) = -\sum_{y^1=0}^{x^1}(\mathcal{D}_2\cdot l_2)(y^1, x^2). \quad (3.2)$$

There is some non-Abelian gauge invariance remaining, namely, that

$$\Gamma(x^2)\Psi = \sum_{x^1=0}^{L^1}(\mathcal{D}_2\cdot l_2)(x^1, x^2)\Psi = 0. \quad (3.3)$$

We split the Hamiltonian into two terms $H = H_0 + \kappa V$, where eventually we set $\kappa = g_0^2/2a$

$$H_0 = \sum_{x^2=0}^{L^2-a}\left\{\sum_{x^1=0}^{L^1}\frac{g_0^2}{2a}[l_2(x^1,x^2)]^2 - \sum_{x^1=0}^{L^1-a}\frac{1}{2g_0^2 a}\times[\mathrm{Tr}U_2(x^1,x^2)^\dagger U_2(x^1+a,x^2) + \text{c.c.}]\right\}, \quad (3.4)$$

and

$$V = \sum_{x^2=0}^{L^2-a}\sum_{x^1=0}^{L^1}\left[\sum_{y^1=0}^{x^1}(\mathcal{D}_2\cdot l_2)(y^1, x^2)\right]^2. \quad (3.5)$$

It will be important for the discussion in the next section that the constraint (3.3) allows us to replace (3.5) by

$$V = -\sum_{x^2=0}^{L^2-a}\sum_{x^1=0}^{L^1-a}\left[\sum_{y^1=0}^{x^1}(\mathcal{D}_2\cdot l_2)(y^1, x^2)\right]^T \times\left[\sum_{z^1=x^1+a}^{L^1}(\mathcal{D}_2\cdot l_2)(z^1, x^2)\right]. \quad (3.6)$$

We have assumed until now that no charges are present. If a quark is placed at site u, then (2.6) may be solved to give

$$l_1(x^1, x^2) = q(u^1, u^2)\delta_{x^1\geq u^1}\delta_{x^2u^2} - \sum_{y^1=0}^{x^1}(\mathcal{D}_2\cdot l_2)(y^1, x^2). \quad (3.7)$$

The remaining gauge invariance is

$$\left[\sum_{x^1=0}^{L^1}(\mathcal{D}_2\cdot l_2)(x^1,x^2)-q(u^1,u^2)\delta_{x^2u^2}\right]\Psi=0. \quad (3.8)$$

IV. CONFINEMENT AT LEADING ORDER

The splitting (3.4) and (3.5) is not 90° rotation invariant. Nonetheless, if perturbation theory in V makes sense, this rotation invariance should be restored at sufficiently high orders. Notice that H_0 is a set of decoupled (1 + 1)-dimensional lattice chiral nonlinear sigma models, with global symmetry $SU(N)_L\times SU(N)_R$, plus an extra term at the boundary $x^1=0$ (this is a sum of unitary matrix-model Hamiltonians). The on-shell properties of these sigma models have been completely determined; the Bethe *Ansatz* [8] and analytic S-matrix theory [9] determine the spectrum in the renormalized continuum limit.

Let us briefly describe the particles of the chiral $SU(N)_L\times SU(N)_R$ model. There are fundamental particles with mass m_1 transforming as the fully antisymmetric tensor representation of $SU(N)_L\times SU(N)_R$. The particles are labeled by a quantum number $n=1,\ldots N-1$. The particle with $n>1$ is a bound state of n fundamental particles. We may regard the bound state of n particles as a bound state of $N-n$ antiparticles. There is no singlet in the one-particle spectrum (which would correspond to $n=N$). The particles have masses

$$m_n=m_1\frac{|\sin\frac{n\pi}{N}|}{\sin\frac{\pi}{N}},\qquad n=1,\ldots,N-1,$$

where the mass gap m_1 is of the form

$$m_1=\frac{C}{a}(g_0^{K_2}e^{-(K_1/g_0^2)}+\cdots)$$
$$=\frac{C}{a}\left[(e^2a)^{K_2/2}\exp-\frac{K_1}{e^2a}+\cdots\right], \quad (4.1)$$

where C is a nonuniversal constant, $K_1=-1$ and $K_2=4\pi$ are determined from the one- and two-loop coefficients of the chiral-model beta function [15], respectively, and the corrections are nonuniversal.

Suppose that there are no charges present. The remaining gauge invariance (3.3) means that we impose on the states Ψ of the chiral model at x^2 the constraints

$$\sum_{x^1=0}^{L^1}R_2(x^1,x^2-a)l_2(x^1,x^2-a)\Psi=\sum_{x^1=0}^{L^1}l_2(x^1,x^2)\Psi. \quad (4.2)$$

The meaning of (4.2) is that if the state of the chiral model at some particular x^2 transforms as a vector with some set of weights under $SU(N)_L$, then the state of the chiral model at x^2+a transforms the same way under $SU(N)_R$.

For the ground state $\Psi_0^{(0)}$, which is a product of chiral-model ground states, each side of (4.2) is automatically zero; for the Hohenberg-Mermin-Wagner theorem guarantees that it is a singlet under both the left global $SU(N)_L$ and under the right global $SU(N)_R$ invariances.

The leading-order Hamiltonian describes a theory which confines fundamental charges separated in the 2-direction. We will show that this is true by two different lines of reasoning. The first proof is more in line with the way people usually think about phenomena in gauge theories. The second proof is a direct utilization of the concepts we have used in the previous section and this one.

Here is the first proof: suppose that a quark is placed at u^1,u^2 and an antiquark at $u^1,y^2\gg u^2$. Gauge-invariant states are of the form

$$|C\rangle=A(u^1,y^2)^\dagger\prod_{\mathrm{link}\in C}U(\mathrm{link})B(u^1,u^2)^\dagger|0\rangle, \quad (4.3)$$

for some path C of links join the quark to the antiquark, whose creation operators are $A^\dagger$ and $B^\dagger$, respectively. The lowest-energy state in the presence of the sources is a superposition of such states. The Hohenberg-Mermin-Wagner theorem states that for a Hamiltonian with a global continuous symmetry, there is no spontaneous symmetry breaking. In the unperturbed vacuum, therefore, $\langle 0|U_2|0\rangle=0$. This means that the action of U_2 on the chiral-sigma-model ground state produces a superposition of excited states only. Thus the expectation value of H_0 in any state (4.3) must be bounded below by the gap times the separation of the fundamental charges, i.e.

$$\langle C|H_0|C\rangle\geq\frac{m_1}{a}|y^2-u^2|, \quad (4.4)$$

which means that there is confinement of fundamental charges, with string tension m_1/a. We call this phenomenon "vertical confinement," because confinement occurs in the 2-direction.

Now for the second proof: the constraint (3.8) has the form

$$\sum_{x^1=0}^{L^1}R_2(x^1,x^2-a)l_2(x^1,x^2-a)\Psi-q(u^1,u^2)\delta_{u^2x^2}\Psi$$
$$+q(u^1,u^2)\delta_{u^2y^2}\Psi$$
$$=\sum_{x^1=0}^{L^1}l_2(x^1,x^2)\Psi. \quad (4.5)$$

This tells us that if the chiral model at u^2-a is in an $SU(N)_L$ singlet state (such as the vacuum), then the chiral model at u^2 cannot be in an $SU(N)_R$ singlet. Thus the chiral model at u^2 is in an excited state. By continuing to use (4.5) we conclude that all the chiral models for x^2 satisfying $u^2\leq x^2\leq y^2$ are excited. In this way, we obtain the same result for the vertical string tension as that given above.

A rectangular Wilson loop of size $S_1 \times S_2$ is

$$A(S_1 \times S_2) = \mathrm{Tr} W(x^1, x^2; S_2)^\dagger W(x^1 + S_1, x^2; S_2), \quad (4.6)$$

in our gauge, where

$$W(x^1, x^2; S_2) = U_2(x^1, x^2 + S_2) \cdots U_2(x^1, x^1).$$

Correlation functions of U_2 decay exponentially. We expect that for large S_1

$$\langle 0|[U_2(x^1, x^2)^\dagger]_a{}^b U_2(x^1 + S_1, x^2)_c{}^d|0\rangle \simeq D^{bd}_{ac} \exp(-m_1 S_1).$$

The Wilson loop expectation value is a product of S_2/a such correlation functions, and therefore

$$\langle 0|A(S_1 \times S_2)|0\rangle \simeq \exp\left(-\frac{m_1}{a} S_1 S_2\right). \quad (4.7)$$

This is an area law, with the same string tension m_1/a found above.

There is no "horizontal confinement"—that is, there is no confinement in the 1-direction—yet. Horizontal confinement will only appear if the perturbation $\kappa V = g_0^2 V/(2a)$ is taken into account. This is because the constraint consistent with the presence of a quark at x^1, x^2 and an antiquark at y^1, x^2 with $y^1 \gg x^1$ is (3.3), which is satisfied by the unperturbed vacuum. Thus, if V is neglected, there is no force between a quark-antiquark in the 1-direction. The appearance of horizontal confinement in perturbation theory will be demonstrated in Sec. VI.

V. WEAK-COUPLING PERTURBATION THEORY AND INFRARED FINITENESS

If L^1 and L^2 are kept finite, the spectrum of the Hamiltonian $H_0 + \kappa V$ is purely discrete. Let us consider this spectrum to second order in Rayleigh-Schrödinger perturbation theory:

$$E_n = E_n^{(0)} + \kappa E_n^{(1)} + \kappa^2 E_n^{(2)} + \cdots,$$

where

$$E_n^{(1)} = \langle \Psi_n^{(0)}|V|\Psi_n^{(0)}\rangle, \qquad E_n^{(2)} = -\sum_{m \neq n} \frac{|\langle \Psi_n^{(0)}|V|\Psi_m^{(0)}\rangle|^2}{E_m^{(0)} - E_n^{(0)}}, \ldots, \quad (5.1)$$

and $|\Psi_n^{(0)}\rangle$ are the eigenvectors of H_0 with eigenvalues $E_n^{(0)}$. The purpose of this section is to show that the corrections to the vacuum energy $E_0^{(1)}$ and $E_0^{(2)}$ are proportional to $L^1 \times L^2$. This happens for two reasons: (1) the ground state of the chiral model is "disordered," i.e., two-point functions fall off exponentially, and (2) the unperturbed vacuum is a singlet, simplifies the form of V acting on this vacuum to (3.6). Our philosophy is close to that of Mandelstam [10] in this regard.

The first correction to the vacuum energy is

$$E_0^{(1)} = -\langle \Psi_0^{(0)}|L^2 \sum_{x^1=0}^{L^1} \sum_{y^1=0}^{x^1} \sum_{z^1=x^1+a}^{L^1} [\mathcal{D}_2 l_2(y^1, x^2)]^T \times \mathcal{D}_2 l_2(y^1, x^2)|\Psi_0^{(0)}\rangle.$$

Correlation functions of l_2 and $R_2 l_2$ must decay exponentially with the distance, and therefore this quantity will have the form

$$E_0^{(1)} \simeq -L^2 \sum_{x^1=0}^{L^1-a} \sum_{y^1=0}^{x^1} \sum_{z^1=x^1+a}^{L^1} e^{-m_1|y^1-z^1|}. \quad (5.2)$$

The dominant contribution to this expression comes from $y^1 \approx z^1$. Since $y^1 \leq x^1 < z^1$, $E_0^{(1)}$ is proportional to the volume $L^1 L^2$.

Next we sketch the proof that the second-order correction to the vacuum energy also scales linearly with the volume. Notice that the coefficient of each energy denominator in the second correction (5.1) is nonpositive. Thus

$$|E_0^{(2)}| < \frac{1}{m_1} \sum_{m \neq 0} |\langle \Psi_0^{(0)}|V|\Psi_m^{(0)}\rangle|^2 = \frac{1}{m_1}[\langle \Psi_0^{(0)}|V^2|\Psi_0^{(0)}\rangle - (\langle \Psi_0^{(0)}|V|\Psi_0^{(0)}\rangle)^2]. \quad (5.3)$$

The connected vacuum expectation value on the right-hand side of (5.3) has the following form:

$$|E_0^{(2)}| < \frac{L^2}{m_1} \sum_{x^1=0}^{L^1} \sum_{w^1=0}^{L^1} C(x^1, w^1; x^2),$$

where

$$C(x^1, w^1; x^2) = \sum_{\mathrm{r}} \sum_{y^1=0}^{x^1} \sum_{z^1=x^1+a}^{L^1} \sum_{u^1=0}^{z^1} \sum_{v^1=w^1+a}^{L^1} [\langle \Psi_0^{(0)}|\mathcal{D}_2 l_2(y^1, x^2)^T \mathcal{D}_2 l_2(z^1, x^2) \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a)^T \mathcal{D}_2 l_2(v^1, x^2 + \mathrm{r}a)|\Psi_0^{(0)}\rangle - \langle \Psi_0^{(0)}|\mathcal{D}_2 l_2(y^1, x^2)^T \mathcal{D}_2 l_2(z^1, x^2)|\Psi_0^{(0)}\rangle\langle \Psi_0^{(0)}|\mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a)^T \mathcal{D}_2 l_2(v^1, x^2 + \mathrm{r}a)|\Psi_0^{(0)}\rangle], \quad (5.4)$$

and where $\mathrm{r} = 0, \pm 1$. The chiral model is a massive local quantum field theory, so that vacuum correlation functions must cluster for the dominant part of the summations in (5.4). Therefore this expression is approximated well by

$$C(x^1, w^1; x^2) \approx \sum_{\mathrm{r}} \sum_{y^1=0}^{x^1} \sum_{z^1=x^1+a}^{L^1} \sum_{u^1=0}^{z^1} \sum_{v^1=w^1+a}^{L^1} [\langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(y^1, x^2)^T \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(z^1, x^2)^T \times \mathcal{D}_2 l_2(v^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle + \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(y^1, x^2)^T \mathcal{D}_2 l_2(v^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(z^1, x^2)^T \times \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle]. \tag{5.5}$$

By using (3.3) we can write each term of (5.5) as something which vanishes exponentially away from $x^1 = w^2$. For example, consider the first factor of the first term:

$$\text{First Factor} = \sum_{y^1=0}^{x^1} \sum_{u^1=0}^{z^1} \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(y^1, x^2)^T \times \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle. \tag{5.6}$$

If $x^1 \leq z^1$, we may write this as

$$\text{F.F.} = -\sum_{y^1=0}^{x^1} \sum_{u^1=z^1+a}^{L^1} \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(y^1, x^2)^T \times \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle, \tag{5.7}$$

and we see that this expression is finite as $L^1 \to \infty$. On the other hand, if $x^1 \geq z^1$, we rewrite (5.6) as

$$\text{F.F.} = -\sum_{y^1=x^1+a}^{L^1} \sum_{u^1=0}^{z^1} \langle \Psi_0^{(0)} | \mathcal{D}_2 l_2(y^1, x^2)^T \times \mathcal{D}_2 l_2(u^1, x^2 + \mathrm{r}a) | \Psi_0^{(0)} \rangle, \tag{5.8}$$

and reach the same conclusion. Since each factor of each term behaves this way, we can conclude that the second-order correction to the vacuum energy can increase at most linearly with L^1.

Infrared finiteness of the vacuum energy to first and second order in perturbation theory inspires confidence that it should hold to all orders. The main complication beyond the second order is the lack of nonpositivity or nonnegativity of products of matrix elements. We believe that careful application of the linked-cluster expansion, assuming clustering in the chiral-sigma model, can provide a proof to all orders.

VI. HORIZONTAL CONFINEMENT

In Sec. IV we showed that quarks are confined vertically, in the 2-direction, but not horizontally, in the 1-direction, at the zeroth order of the weak-coupling expansion. To see what happens beyond this order, it is necessary to examine the quark-antiquark potential in perturbation theory. This is very straightforward to do.

If a quark is located at u^1, u^2, and an antiquark is located at v^1, u^2 with $v^1 > u^1$, the electric-field operator in the 1-direction is given by

$$l_1(x^1, x^2) = q(u^1, u^2)\delta_{x^1 \geq u^1} \delta_{x^2 u^2} - q(v^1, u^2)\delta_{x^1 \geq v^1} \delta_{x^2 u^2} - \sum_{y^1=0}^{x^1} (\mathcal{D}_2 \cdot l_2)(y^1, x^2). \tag{6.1}$$

The constraint (3.3) is unmodified. Thus, the unperturbed states and energies are unaffected by these two charges, as we claimed in Sec. IV. However, to first order in perturbation theory, there is a new contribution to $E_0^{(1)}$ equal to

$$\Delta E_0^{(1)} = \kappa C_N |v^1 - u^1|, \tag{6.2}$$

where C_N is the smallest eigenvalue of the Casimir of SU(N), $q^2 = C_N \mathbb{1}$, by (2.5). Thus, to first order in perturbation theory, the horizontal string tension is $\frac{\kappa}{a} C_N$. What is especially remarkable about this result is that we can see clearly an electric string forming along the shortest path connecting the two quarks.

What is happening physically is that the vacuum remains undisturbed by the charges and prevents the penetration of electric flux. To this low order of perturbation theory, we have a cost of at least m_1 to excite the chiral model at x^2 and $x^2 - a$. Thus there is a string tension equal to the $(1 + 1)$-dimensional string tension through an electric Meissner effect. We do not have to appeal to the condensation of some kind of magnetic charge to make this interpretation. At higher orders of perturbation theory, the string of electric flux can presumably fluctuate; these corrections are needed to reliably set $\kappa = g_0^2/2a$.

VII. CONCLUSIONS

In this paper, we have shown that lattice gauge theories in two space and one time dimension confine charges, through an anisotropic weak-coupling expansion. Though we cannot exactly evaluate the terms in this expansion, by just using some general knowledge of the chiral nonlinear sigma models, we can make precise statements about these terms.

The astute reader may wonder if the methods developed here can work for the oldest known example of nontrivial confinement: lattice compact $(\mathrm{QED})_{2+1}$ [1]. The answer is that they do not. In this Abelian gauge theory, we would expand about the states of the U(1) nonlinear sigma model. This model has a massless phase at weak coupling, so we would not obtain vertical confinement and area-law behavior of the spacelike Wilson loop. In fact, our perturbation method makes no sense at all for lattice $(\mathrm{QED})_{2+1}$. The reason is that correlations of the operator $l(x^1, x^2) - l(x^1, x^2 - a)$ (the adjoint-representation covariant derivative is simply the ordinary lattice derivative) do not fall off sufficiently fast to make $\sum l_1^2$ directly proportional to the volume. The infrared divergence in the vacuum energy, which concerned us so much, really happens in the Abelian theory. This divergence is not real, but is an artifact of our methods. Our weak-coupling expansion seems peculiarly suited to non-Abelian theories in this regard.

We have assumed that a mass gap exists in the (1 + 1)-dimensional SU(N) chiral model. At strong coupling, this can be proved rigorously with a cluster expansion, in the Euclidean lattice formulation. Perhaps a fully rigorous proof can be made of confinement with g_0 large, but κ small.

Of the questions raised by our analysis we think that six stand out as important. We suspect, however, that only the first, second and possibly the third can be answered in the near future.

The first and probably easiest important question is whether the infrared finiteness of our perturbation series exists beyond the second order. We hope to be able to settle this issue soon. If settled affirmatively, the series probably does not converge, but may be Borel summable.

The second question is whether adjoint matter is confined for finite N. This is certainly happening at first order in the horizontal direction. We believe that this property will disappear at higher orders.

The third question is raised by the fact that our mass scales are set by (4.1), with one exception (the horizontal string tension). All these quantities are nonzero for any positive value of a. We believe, however, that we should still have a mass gap and gap confinement as $a \rightarrow 0$, provided the continuum coupling constant $g_0/(\sqrt{a})$ is kept fixed. Our vertical string tension, found in Sec. IV is too small, and our horizontal string tension found in Sec. VI is known only for small κ. These numbers should both be proportional to g_0^2/a, the square of the continuum coupling constant. Perhaps this difficulty can be removed by resummation of the perturbation series or by a renormalization-group argument.

The fourth question is whether we can do a better job of calculating energies and states. Perhaps we could accomplish this, if Bethe's *Ansatz* for the chiral model could be carried out in a formalism where both the left- and right-handed SU(N) symmetries are manifest in the Hamiltonian. In the work of Polyakov and Wiegmann [8] only one of these is manifest; the other appears in the S matrix, but its interpretation is obscure. If a version of Bethe's *Ansatz* with both symmetries manifest can be found, there is the possibility of a better understanding of the (2 + 1)-dimensional gauge theory. One could use whatever regularization is most expedient for diagonalizing the Hamiltonian, instead of the lattice. It may be a long time before this question can be seriously addressed (perhaps never). We believe a more likely path to success is to expand some version of the axial-gauge Hamiltonian about a system of (1 + 1)-dimensional field theories other than chiral-sigma models. It would be a stroke of good luck, to have an expansion about exactly solvable field theories where the symmetries are easy to understand.

The fifth question is whether our results can be understood in the context of condensation of magnetic charge. If a picture of condensing composite operators could work in the (1 + 1)-dimensional chiral models (no one has succeeded in showing this), then operators defined on sets of points of one dimension higher should be important for confinement in (2 + 1) dimensions.

The last and most important question is whether $(\mathrm{QCD})_{3+1}$ could be studied by our methods. This is, we hope to no one's surprise, a much harder problem. A lattice gauge theory in (2 + 1) dimensions is particularly amenable to the methods discussed here, because if the square of electric field in the 1-direction is dropped from the Hamiltonian, it easily breaks apart into (1 + 1)-dimensional Hamiltonians we know a lot about. This does not happen in (3 + 1) dimensions. The Hamiltonian breaks into (2 + 1)-dimensional Hamiltonians with both gauge fields and matter in the adjoint representation. These models are probably not even renormalizable, but seem worthy of investigation.

ACKNOWLEDGMENTS

I thank Mike Creutz, John Kogut, Herbert Neuberger, and Tilo Wettig for organizing the workshop "Modern Challenges for Lattice Field Theory," at the Kavli Institute for Theoretical Physics, where this work was completed amid the discussion of many interesting topics. I would also like to thank David Adams for discussions and a careful reading of the manuscript. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949. It was also supported in part by a grant from the PSC-CUNY.

[1] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975).
[2] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[3] R. P. Feynman, Nucl. Phys. **B188**, 479 (1981).
[4] P. Orland and G. W. Semenoff, Nucl. Phys. **B576**, 627 (2000).
[5] D. Karabali and V. P. Nair, Nucl. Phys. **B464**, 135 (1996); Phys. Lett. B **379**, 141 (1996); D. Karabali, C. Kim, and V. P. Nair, Phys. Lett. B **524**, 661 (1998); **434**, 103 (1998);

Nucl. Phys. **B566**, 331 (2000); Phys. Rev. D **64**, 025011 (2001).
[6] P. Orland, Phys. Rev. D **70**, 045014 (2004).
[7] A. M. Polyakov, Phys. Lett. **59B**, 79 (1975); E. Brezin, J. Zinn-Justin, and J. C. Le Guillou, Phys. Rev. D **14**, 2615 (1976); E. Brezin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976); W. A. Bardeen, B. W. Lee, and R. E. Schrock, Phys. Rev. D **14**, 985 (1976); Y. Y. Goldschmidt and E. Witten, Phys. Lett. **91B**, 392 (1980); A. B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979); P. Wiegmann, Phys. Lett. **152B**, 209 (1985).
[8] A. M. Polyakov and P. Wiegmann, Phys. Lett. **131B**, 121 (1983); P. Wiegmann, Phys. Lett. **141B**, 217 (1984); L. D. Fadeev and N. Yu. Reshetikhin, Ann. Phys. (N.Y.) **167**, 227 (1986).
[9] E. Abdalla, M. C. B. Abdalla, and A. Lima-Santos, Phys. Lett. **140B**, 71 (1984); P. Wiegmann, Phys. Lett. **142B**, 173 (1984).
[10] S. Mandelstam, Bull. Am. Phys. Soc. **22**, 541 (1977); Phys. Rev. D **19**, 2391 (1979).
[11] J. Kogut, Rev. Mod. Phys. **51**, 659 (1979); M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, England, 1983).
[12] M. B. Halpern, Phys. Rev. D **19**, 517 (1979).
[13] G. G. Batrouni, Nucl. Phys. **B208**, 12 (1982); M. B. Halpern and G. G. Batrouni, Phys. Rev. D **30**, 1782 (1984).
[14] S. G. Rajeev, Phys. Lett. B **212**, 203 (1988); E. Langmann and G. W. Semenoff, Phys. Lett. B **296**, 117 (1992); J. Hetrick, Int. J. Mod. Phys. A **9**, 3153 (1994); V. Müller and W. Rühl, Nucl. Phys. **B230**, 49 (1984); J. B. Bronzan, Phys. Rev. D **31**, 2020 (1985); J. B. Bronzan and T. E. Vaughan, Phys. Rev. D **47**, 3543 (1993).
[15] A. McKane and M. Stone, Nucl. Phys. **163B**, 169 (1980).