High overtones of Dirac perturbations of a Schwarzschild black hole

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Using the Frobenius method, we find high overtones of the Dirac quasinormal spectrum for the Schwarzschild black hole. At high overtones, the spacing for imaginary part of ω_n is equidistant and equals to $\mathrm{Im}(\omega_{n+1}) - \mathrm{Im}(\omega_n) = i/8M$, M being the black hole mass, which is twice less than that for fields of integer spin. At high overtones, the real part of ω_n goes to zero. This supports the suggestion that the expected correspondence between quasinormal modes and Barbero-Immirzi parameter in Loop Quantum Gravity is just a numerical coincidence.

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The Quasinormal Mode (QNM) spectrum is an important characteristic of a black hole. It dominates the late time response of a black hole to an external perturbation, and, at the same time, does not depend on the way of their excitation. Thus, being dependent on black hole parameters only, the QNMs provide us with the "fingerprints" of a black hole, feasible to be seen in the detection of gravitational waves (for reviews, see [1]).

The importance of the QN spectrum is not limited by the above observational aspects of gravitational waves phenomena, since QNMs have interpretation in conformal field theory through AdS/CFT and dS/CFT correspondence [2]. There is also suggestion that the asymptotic QNMs of black holes are connected with the so-called Barbero-Immirzi parameter in Loop Quantum Gravity, which must be fixed to reproduce the Bekenstein-Hawking entropy formula within this theory [3]. All this stimulated considerable interest to study the QNMs of black holes in flat, dS and AdS backgrounds [4]. In particular, the QNMs of the Dirac field for different black holes were considered in the papers [5–8]. Nevertheless, the study of the Dirac QNMs of four-dimensional black holes were limited by *low overtones*. The low overtones were found for a Schwarzschild black hole in [6], with the help of the third-order WKB method [9]. There it was observed that for low overtones the real part of the frequency decreases as the damping grows. Soon, the analysis was extended to the case of Schwarzschild-de Sitter black hole [7] using the sixth-order WKB method [10], and also to the case of the charged Dirac field in the background of a charged black hole [8].

In the present paper, we are trying to cover this gap in the study of the Dirac QNMs and shall investigate the *high* overtone behavior of the Dirac perturbations of a Schwarzschild black hole. We observed that at high overtones the imaginary part of the QN spectrum is equidistant with the spacing $\mathrm{Im}(\omega_{n+1}) - \mathrm{Im}(\omega_n) = i/8M$. It is also shown that when the overtone number n is increasing, the real part of ω_n approaches zero.

The Dirac equation in an arbitrary curved background has the form [11]:

$$[\gamma^a e_a^\mu(\partial_\mu + \Gamma_\mu) + m]\Psi = 0, \tag{1}$$

where m is the mass of the Dirac field, e_a^μ are tetrads and Γ_μ are spin connections [11]. The Schwarzschild metric has the form:

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

where $f(r) = 1 - (2M/r)$, M being the black hole mass. In this background, the equation for the massless Dirac field can be reduced, after some algebra, to the wavelike equation:

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r^*)\right]\Psi(r^*) = 0, \tag{3}$$

with the effective potential [12]

$$V(r) = f(r)\mu\left(\frac{\mu}{r^2} \pm \frac{d}{dr}\sqrt{\frac{f(r)}{r^2}}\right), \tag{4}$$

which vanishes at the both boundaries: $V(r^* = \pm\infty) = 0$. The parameter μ corresponds to a multipole number. Under the choice of the positive sign for the real part of ω ($\omega = \omega_{Re} - i\omega_{Im}$, $\omega_{Re} > 0$), QNMs satisfy the following boundary conditions

$$\Psi(r^*) \sim C_\pm \exp(\pm i\omega r^*), \qquad r^* \longrightarrow \pm\infty, \tag{5}$$

corresponding to purely in-going waves at the black hole event horizon and purely out-going waves at infinity. Then, following Leaver [13], we can choose

$$\Psi(r^*) = \exp(i\omega r^*)u(r^*), \tag{6}$$

where $u(r)$ has a regular singularity at the event horizon and is finite at $r^* \longrightarrow \infty$.

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The appropriate Frobenius series is

$$u(r) = f(r)^{2s} \sum_{n=0}^{\infty} a_n f(r)^{n/2}, \quad (7)$$

$$f(r)^s \propto \exp(-i\omega r^*), s = -2Mi\omega, \quad r^* \longrightarrow -\infty. \quad (8)$$

Substituting (6) and (7) in (3) we obtain the five-term recurrence relation, which, after a series of Gaussian eliminations, can be reduced to the three-term recurrence relation

$$a_{n+1}\alpha_n(\omega) + a_n\beta_n(\omega) + a_{n-1}\gamma_n(\omega) = 0.$$

When ω is the quasinormal frequency, the ratio of the series coefficients is finite and can be found from the standard continued fractions [13].

The QNMs are the roots of the inverted continued fraction:

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-\alpha_{n-3}\gamma_{n-2}/\ldots}} = \frac{\alpha_n\gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-\alpha_{n+2}\gamma_{n+3}/\ldots}}$$

that can be solved numerically as soon as $\alpha_n(\omega)$, $\beta_n(\omega)$, $\gamma_n(\omega)$ are found.

Note, that for the above effective potential, we are not able to use the Nollert expansion [14]. That is why in spite of the slow convergence of the continued fractions in the unmodified Leaver procedure we had to be limited by it. This certainly requires much greater computing time to perform the computations.

Low overtones:— The low overtones can be obtained either with the help of the Frobenius method or, when the overtone number n is less than the multipole number μ, applying the WKB formula [9,10]. Let us remind that the potentials with opposite chirality produce the same spectrum (see for instance [7] and references therein). That is why we shall consider only positive values of μ. The WKB approach has been used recently when studying QNMs of different black holes (see [15] and references therein) and shown good agreement with numerical computations for lower overtones. As can be seen from Table I above we observe very good agreement between the WKB and Frobenius data, and, the higher the WKB order, the closer the obtained WKB values to accurate numerical results.

High overtones:— The main difference from what we know on the high damping regime for perturbations of fields of integer spin (scalar, gravitational, and electromagnetic) is that now the spacing in imaginary part is not $i/4M$, as it takes place for integer spin perturbations, but $i/8M$. The real part of ω_n (See Fig. 1 and 2) falls down quickly to tiny values which already cannot be found with reasonable accuracy by the Frobenius method. This occurs already at about $n = 400 - 500$ for the $\mu = 1$ multipole and at about $n = 8000$ for $\mu = 2$. Note that the greater the ratio $Im\omega_n/Re\omega_n$, the more slowly the Frobenius series converge, so we had to be careful when finding high overtones and to check the convergence by increasing of the "length" of the continued fraction until the result for ω_n will not change. This required, for instance, the length of the continued fraction up to five million for $n = 7000$. That is because we were not able to use the Nollert procedure for

TABLE I. First ten quasinormal frequencies of Dirac perturbations for $\mu = 1, 2$ found by the Frobenius method. For modes with $n < \mu$ the 6th order (∗∗) and 3rd order (∗) WKB values are given.

n	$\mu = 1$ Re(ω)	$-$Im(ω)
0	0.182963	0.096982
∗∗	0.182639	0.094938
∗	0.176452	0.100109
1	0.147822	0.316928
2	0.146458	0.424157
3	0.138885	0.549184
4	0.134650	0.674318
5	0.131726	0.799895
6	0.128822	0.925363
7	0.126064	1.050710
8	0.123570	1.176020

n	$\mu = 2$ Re(ω)	Im(ω)
0	0.380037	0.096405
∗∗	0.380068	0.096366
∗	0.378627	0.096542
1	0.355833	0.297497
∗∗	0.355857	0.297271
∗	0.353604	0.298746
2	0.326095	0.421677
3	0.313180	0.548007
4	0.305740	0.673165
5	0.300096	0.798686
6	0.294725	0.924309
7	0.289734	1.049817
8	0.285201	1.175236

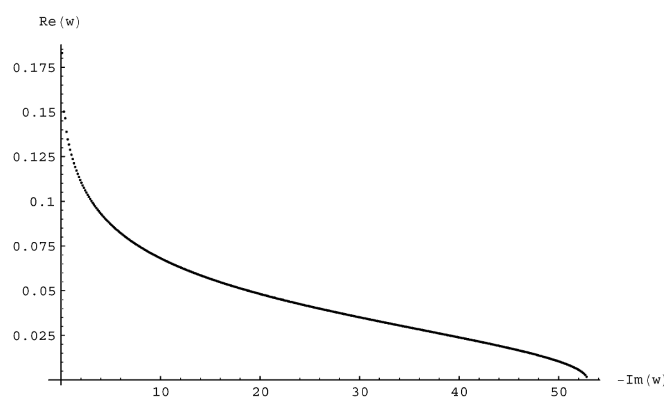

FIG. 1. Real part of ω as a function of imaginary part for $\mu = 1$ multipole.

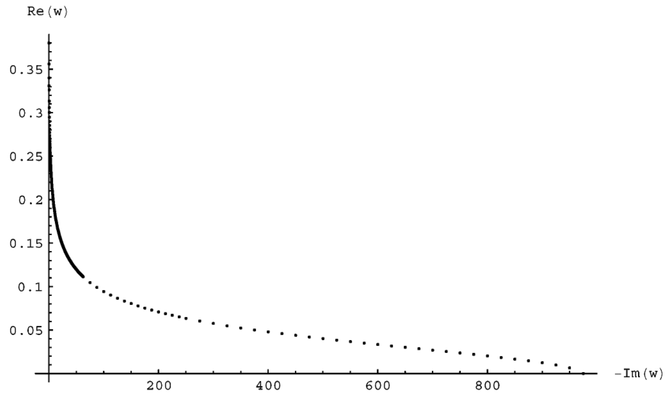

FIG. 2. Real part of ω as a function of imaginary part for $\mu = 2$ multipole.

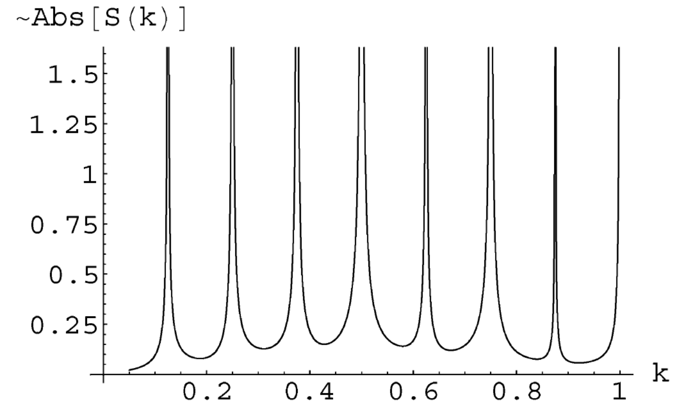

FIG. 3. The absolute value of the scattering amplitude $S(k)$ as a function of k up to a constant factor. $M = 1$, $\mu = 1$.

the considered effective potential, and therefore, a lot of computer time was required to get the convergence of the Leaver procedure. That is the reason why we could not extend our computation to higher n than that shown on Fig. 1 and 2. Note also, that for n larger then 500 it is very time consumingly to compute the modes one by one, so we had to ''skip''through the hundreds of overtones as it is shown on Fig. 2.

Thus, as can be seen from Fig. 1 and 2, QNMs demonstrate the following asymptotic behavior:

$$\mathrm{Re}(\omega)_n \approx 0 \quad \text{as} \quad n \rightarrow \infty, \tag{9}$$

$$\mathrm{Im}(\omega_{n+1}) - \mathrm{Im}(\omega_n) \approx -i/8M \quad \text{as} \quad n \rightarrow \infty. \tag{10}$$

The formula for the spacing of the imagianary part can also be reproduced following [16]. Thus, for highly damped modes we can use the Born approximation, where the scattering amplitude is given by the formula [16]:

$$S(k) = \int_{-\infty}^{+\infty} V[r(r^*)] e^{2ikr^*} dr^*. \tag{11}$$

The above integral gets significant contribution only near the event horizon. Using the Eq. (4) for $V[r(r^*)]$, we find

$$S(k) \sim \text{combinations of } \Gamma(4ikM) \quad \text{and} \quad \Gamma[(1/2) + 4ikM].$$

The poles of the amplitude occur when $(1/2) + 4iMk = -n$, or $4iMk = -n$ ($n \geq 0$ and is integer), i.e. $k_n = in/8M$ is the required high frequency asymptotic for imaginary part. The same result can be obtained either using the Taylor expansion of the effective potential near the event horizon, or without such expansion, but in the latter case the expression for the scattering amplitude will have a cumbersome form. The singularities of the scattering amplitude are shown on Fig. 3.

Conclusion:— In this paper we have studied the high overtones of the Dirac quasinormal spectrum of a Schwarzschild black hole. The spacing in the imaginary part is 2 times less than that for integer spin fields when $n \rightarrow \infty$. As was shown both numerically and analytically in [17] the real part of QNMs for scalar and gravitational perturbations asymptotically approaches a constant equal to $\log 3/8\pi M$. On the contrary, for the electromagnetic perturbations the real part goes to zero [18]. The real part for Dirac perturbations goes to zero, as it happens for electromagnetic perturbations. This supports the suggestion that the expected correspondence between quasinormal modes and the Barbero-Immirzi parameter in Loop Quantum Gravity is just a numerical coincidence [19].

The above analysis can easily be extended to the case of massive Dirac field [6,20]. At low overtones, massive Dirac pertubations [6], similar to massive scalar pertubations [21], lead to greater oscillation frequencies and slower damping. At high overtones, similarly to the treatment in [22] for massive scalar field, one can easily anticipate that the massive term will not affect the asymptotic quasinormal behavior.

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