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Non-Abelian Meissner effect in Yang-Mills theories at weak coupling

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We present a weak-coupling Yang-Mills model supporting non-Abelian magnetic flux tubes and non-Abelian confined magnetic monopoles. In the dual description the magnetic flux tubes are prototypes of the QCD strings. Dualizing the confined magnetic monopoles we get gluelumps which convert a ''QCD string'' in the excited state to that in the ground state. Introducing a mass parameter m we discover a phase transition between the Abelian and non-Abelian confinement at a critical value $m = m_* \sim \Lambda$. Underlying dynamics are governed by a Z_N symmetry inherent to the model under consideration. At $m > m_*$ the Z_N symmetry is spontaneously broken, resulting in N degenerate Z_N (Abelian) strings. At $m < m_*$ the Z_N symmetry is restored, the degeneracy is lifted, and the strings become non-Abelian. We calculate tensions of the non-Abelian strings, as well as the decay rates of the metastable strings, at $N \gg 1$.

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I. INTRODUCTION

Ever since 't Hooft [1] and Mandelstam [2] put forward the hypothesis of the dual Meissner effect to explain color confinement in non-Abelian gauge theories people were trying to find a controllable approximation in which one could reliably demonstrate the occurrence of the dual Meissner effect in these theories. A breakthrough achievement was the Seiberg-Witten solution [3] of $\mathcal{N} = 2$ supersymmetric (SUSY) Yang-Mills theory. They found massless monopoles and, adding a small ($\mathcal{N} = 2$)-breaking deformation, proved that they condense creating strings carrying a chromoelectric flux. It was a great success in qualitative understanding of color confinement.

A more careful examination shows, however, that details of the Seiberg-Witten confinement are quite different from those we expect in QCD-like theories. Indeed, a crucial aspect of Ref. [3] is that the SU(N) gauge symmetry is first broken, at a high scale, down to $\mathrm{U}(1)^{N-1}$, which is then completely broken, at a much lower scale where monopoles condense. Correspondingly, the strings in the Seiberg-Witten solution are, in fact, Abelian strings [4] of the Abrikosov-Nielsen-Olesen (ANO) type which results, in turn, in confinement whose structure does not resemble at all that of QCD. In particular, the ''hadronic'' spectrum is much richer than that in QCD [5,6].

Thus, the problem of obtaining the Meissner effect in a more realistic regime in theories which are closer relatives of QCD remains open. A limited progress in this direction was achieved since the 1980's [7]; the advancement accelerated in recent years [8–14]. Our task is to combine and distill these advances to synthesize a relatively simple non-Abelian model exhibiting at least some features of bona fide non-Abelian confinement in a controllable setting.

What do we know of color confinement in QCD? At a qualitative level surprisingly much. We know that in the Yang-Mills theory chromoelectric flux tubes are formed between the probe heavy quarks (more exactly, between the quark and its antiquark), with the fundamental tension T_1 proportional to the square of the dynamical scale parameter, which does not scale with N at large N,

$$T_1 \sim \Lambda^2_{\mathrm{QCD}}.$$

If one pulls together N such flux tubes they can annihilate. This clearly distinguishes QCD flux tubes from the ANO strings. We know that for k-strings[1] (with $k > 1$) excitations lie very close to the ground state. For instance, if one considers two-index symmetric and antisymmetric sources, the corresponding string tensions $T_{[2]}$ and $T_{\{2\}}$ are split [15] by Λ^2/N^2. The decay rate of the symmetric string into antisymmetric (per unit length of the string per unit time) is

$$\Gamma_{\square\square \to \begin{smallmatrix}\square\\\square\end{smallmatrix}} \sim \Lambda^2 \exp(-\gamma N^2),$$

where γ is a positive constant of order 1. We would like to model all the above features at weak coupling, where all approximations made can be checked and verified. After extensive searches we found seemingly the simplest Yang-Mills model which does the job, at least to an extent. Our model seems to be minimal. It is nonsupersymmetric. It supports non-Abelian magnetic flux tubes and non-Abelian confined magnetic monopoles at weak coupling. In the dual description the magnetic flux tubes are prototypes of the QCD strings. Dualizing the confined magnetic monopoles we get gluelumps (string-attached gluons) which convert a ''QCD string'' in the excited state to that in the ground state. The decay rate of the excited string to its ground state is suppressed exponentially in N.

[1]Operationally, k-strings are defined as flux tubes attached to probe sources with k fundamental or k antifundamental indices.

It is worth noting that strings in non-Abelian theories at weak coupling were found long ago [16]—the so-called Z_N strings associated with the center of the SU(N) gauge group. However, in all these constructions the gauge flux was always directed along a fixed vector in the Cartan subalgebra of SU(N), and no moduli which would make the flux orientation a dynamical variable in the group space were ever found. Therefore, these strings are, in essence, Abelian.

Recently, non-Abelian strings were shown to emerge at weak coupling [10,11,13,14] in $\mathcal{N} = 2$ and deformed $\mathcal{N} = 4$ supersymmetric gauge theories (similar results in three dimensions were obtained in [9]). The main feature of the non-Abelian strings is the presence of orientational zero modes associated with the rotation of their color flux in the non-Abelian gauge group, which makes such strings genuinely non-Abelian. This is as good as it gets at weak coupling.

In this paper we extend (and simplify) the class of theories in which non-Abelian strings are supported. To this end we consider a ''minimal'' nonsupersymmetric gauge theory with the gauge group SU(N) $\times$ U(1). Our model is still rather far from real-world QCD. We believe, however, that our non-Abelian strings capture basic features of QCD strings to a much greater extent than the Abelian ANO strings.

Striking similarities between four-dimensional gauge theories and two-dimensional sigma models were noted long ago, in the 1970's and 80's. We continue revealing reasons lying behind these similarities: in fact, two-dimensional sigma models are effective low-energy theories describing orientational moduli on the world sheet of non-Abelian confining strings. A particular direct relation was found previously in $\mathcal{N} = 2$ supersymmetric theories [11,13,17,18] where the BPS kink spectrum in the two-dimensional $CP(N-1)$ model coincides with the dyon spectrum of a four-dimensional gauge theory given by the exact Seiberg-Witten solution. Pursuing this line of research we reveal a similar relationship between nonsupersymmetric two- and four-dimensional theories. The physics of nonsupersymmetric sigma models significantly differs from that of supersymmetric ones. We find interpretations of known results on nonsupersymmetric $CP(N-1)$ models in terms of non-Abelian strings and monopoles in four dimensions.

In particular, in parallel to the supersymmetric case [11–13], we interpret the confined monopole realizing a junction of two distinct non-Abelian strings, as a kink in the two-dimensional $CP(N-1)$ model. The argument is made explicit by virtue of an extrapolation procedure designed specifically for this purpose. Namely, we introduce mass parameters m_A ($A = 1, \ldots, N_f$, and $N_f = N$ is the number of bulk flavors) for scalar quarks in four dimensions. This lifts the orientational moduli of the string. Now the effective world-sheet description of the string internal dynamics is given by a massive $CP(N-1)$ model. In this quasiclassical limit the matching between the magnetic monopoles and kinks is rather obvious. Tending $m_A \to 0$ we extrapolate this matching to the quantum regime.

In addition to the four-dimensional confinement, that ensures that the magnetic monopoles are attached to the strings, they are also confined in the two-dimensional sense. Namely, the monopoles stick to antimonopoles on the string they are attached to, to form mesonlike configurations. The two-dimensional confinement disappears if the vacuum angle $\theta = \pi$. Some monopoles become deconfined along the string. Alternatively, one can say that strings become degenerate.

With nonvanishing mass terms of the type

$$\{m_A\} \to m\{e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}, 1\},$$

a discrete Z_N symmetry survives in the effective world-sheet theory. In the domain of large m [large compared to the scale of the $CP(N-1)$ model] we have Abelian strings and essentially the 't Hooft-Polyakov monopoles, while at small m the strings and monopoles we deal with become non-Abelian. We show that these two regions are separated by a phase transition (presumably, of the second order) which we interpret as a transition between the Abelian and non-Abelian confinement. We show that in the effective $CP(N-1)$ model on the string world sheet this phase transition is associated with the restoration of Z_N symmetry: Z_N symmetry is broken in the Abelian confinement phase and restored in the non-Abelian confinement phase. This is a key result of the present work which has an intriguing (albeit, rather remote) parallel with the breaking of the Z_N symmetry at the confinement/deconfinement phase transition found in lattice QCD at nonzero temperature.

Next, we consider some special features of the simplest SU(2) $\times$ U(1) case. In particular, we discuss the vacuum angle dependence. The $CP(1)$ model is known to become conformal at $\theta = \pi$, including massless monopoles/kinks at $\theta = \pi$.

Finally, we focus on the problem of the multiplicity of the hadron spectrum in the general SU(N) $\times$ U(1) case. As was already mentioned, the Abelian confinement generates too many hadron states as compared to QCD-based expectations [5,6,19]. In our model this regime occurs at large m_A.

II. IN SEARCH OF NON-ABELIAN STRINGS AND MONOPOLES

A reference model which we suggest for consideration is quite simple. The gauge group of the model is SU(N) $\times$ U(1). Besides SU(N) and U(1) gauge bosons the model contains N scalar fields charged with respect to U(1) which form N fundamental representations of SU(N). It is convenient to write these fields in the form of $N \times N$ matrix $\Phi = \{\varphi^{kA}\}$ where k is the SU(N) gauge index while A is

the flavor index,

$$\Phi = \begin{pmatrix} \varphi^{11} & \varphi^{12} & \cdots & \varphi^{1N} \\ \varphi^{21} & \varphi^{22} & \cdots & \varphi^{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi^{N1} & \varphi^{N2} & \cdots & \varphi^{NN} \end{pmatrix}. \tag{2}$$

Sometimes we will refer to φ's as to scalar quarks, or just quarks. The action of the model has the form[2]

$$S = \int d^4x \Big\{ \frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \mathrm{Tr}(\nabla_\mu \Phi)^\dagger (\nabla^\mu \Phi) + \frac{g_2^2}{2}[\mathrm{Tr}(\Phi^\dagger T^a \Phi)]^2 + \frac{g_1^2}{8}[\mathrm{Tr}(\Phi^\dagger \Phi) - N\xi]^2 + \frac{i\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \Big\}, \tag{3}$$

where T^a stands for the generator of the gauge SU(N),

$$\nabla_\mu \Phi \equiv \left(\partial_\mu - \frac{i}{\sqrt{2N}} A_\mu - i A^a_\mu T^a \right) \Phi \tag{4}$$

[the global flavor SU(N) transformations then act on Φ from the right], and θ is the vacuum angle. The action (3) in fact represents a truncated bosonic sector of the $\mathcal{N} = 2$ model. The last term in the second line forces Φ to develop a vacuum expectation value (VEV) while the first term forces the VEV to be diagonal,

$$\Phi_{\mathrm{vac}} = \sqrt{\xi}\,\mathrm{diag}\{1, 1, \ldots, 1\}. \tag{5}$$

In this paper we assume the parameter ξ to be large,[3]

$$\sqrt{\xi} \gg \Lambda_4, \tag{6}$$

where Λ_4 is the scale of the four-dimensional theory (3). This ensures the weak-coupling regime as both couplings g_1^2 and g_2^2 are frozen at a large scale.

The vacuum field (5) results in the spontaneous breaking of both gauge and flavor SU(N)'s. A diagonal global SU(N) survives, however, namely

$$\mathrm{U}(N)_{\mathrm{gauge}} \times \mathrm{SU}(N)_{\mathrm{flavor}} \to \mathrm{SU}(N)_{\mathrm{diag}}. \tag{7}$$

Thus, color-flavor locking takes place in the vacuum. A version of this scheme of symmetry breaking was suggested long ago [20].

Now, let us briefly review string solutions in this model. Since it includes a spontaneously broken gauge U(1), the model supports conventional ANO strings [4] in which one can discard the $\mathrm{SU}(N)_{\mathrm{gauge}}$ part of the action. The topological stability of the ANO string is due to the fact that $\pi_1(\mathrm{U}(1)) = Z$. These are not the strings we are interested in. At first sight the triviality of the homotopy group, $\pi_1(\mathrm{SU}(N)) = 0$, implies that there are no other topologically stable strings. This impression is false. One can combine the Z_N center of SU(N) with the elements $\exp(2\pi i k/N) \in \mathrm{U}(1)$ to get a topologically stable string solution possessing both windings, in SU(N) and U(1). In other words,

$$\pi_1(\mathrm{SU}(N) \times \mathrm{U}(1)/Z_N) \neq 0. \tag{8}$$

It is easy to see that this nontrivial topology amounts to winding of just one element of Φ_{vac}, say, φ^{11}, or φ^{22}, etc., for instance,[4]

$$\Phi_{\mathrm{string}} = \sqrt{\xi}\,\mathrm{diag}(1, 1, \ldots, e^{i\alpha(x)}), \qquad x \to \infty. \tag{9}$$

Such strings can be called elementary; their tension is $1/N$th of that of the ANO string. The ANO string can be viewed as a bound state of N elementary strings.

More concretely, the Z_N string solution (a progenitor of the non-Abelian string) can be written as follows [10]:

$$\begin{aligned} \Phi &= \begin{pmatrix} \phi(r) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \phi(r) & 0 \\ 0 & 0 & \cdots & e^{i\alpha}\phi_N(r) \end{pmatrix}, \\ A_i^{\mathrm{SU}(N)} &= \frac{1}{N} \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -(N-1) \end{pmatrix} (\partial_i \alpha) \times [-1 + f_{NA}(r)], \\ A_i^{\mathrm{U}(1)} &= \frac{1}{N}(\partial_i \alpha)[1 - f(r)], \qquad A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0, \end{aligned} \tag{10}$$

where $i = 1, 2$ labels coordinates in the plane orthogonal to the string axis and r and α are the polar coordinates in this plane. The profile functions $\phi(r)$ and $\phi_N(r)$ determine the profiles of the scalar fields, while $f_{NA}(r)$ and $f(r)$ determine the SU(N) and U(1) fields of the string solutions, respectively. These functions satisfy the following rather obvious boundary conditions:

$$\phi_N(0) = 0, \qquad f_{NA}(0) = 1, \qquad f(0) = 1, \tag{11}$$

at $r = 0$, and

$$\begin{aligned} &\phi_N(\infty) = \sqrt{\xi}, \qquad \phi(\infty) = \sqrt{\xi}, \\ &f_{NA}(\infty) = 0, \qquad f(\infty) = 0 \end{aligned} \tag{12}$$

at $r = \infty$. Because our model is equivalent, in fact, to a bosonic reduction of the $\mathcal{N} = 2$ supersymmetric theory,

[2]Here and below we use a formally Euclidean notation, e.g. $F^2_{\mu\nu} = 2F^2_{0i} + F^2_{ij}$, $(\partial_\mu a)^2 = (\partial_0 a)^2 + (\partial_i a)^2$, etc. This is appropriate since we are going to study static (time-independent) field configurations, and $A_0 = 0$. Then the Euclidean action is nothing but the energy functional.

[3]The reader may recognize ξ as a descendant of the Fayet-Iliopoulos parameter.

[4]As explained below, α is the angle of the coordinate $\vec{x}_\perp$ in the perpendicular plane.

these profile functions satisfy the first-order differential equations obtained in [21], namely,

$$\begin{aligned} r\frac{d}{dr}\phi_N(r) - \frac{1}{N}(f(r) + (N-1)f_{NA}(r))\phi_N(r) = 0, \\ r\frac{d}{dr}\phi(r) - \frac{1}{N}(f(r) - f_{NA}(r))\phi(r) = 0, \\ -\frac{1}{r}\frac{d}{dr}f(r) + \frac{g_1^2 N}{4}[(N-1)\phi(r)^2 + \phi_N(r)^2 - N\xi] = 0, \\ -\frac{1}{r}\frac{d}{dr}f_{NA}(r) + \frac{g_2^2}{2}[\phi_N(r)^2 - \phi_2(r)^2] = 0. \end{aligned} \tag{13}$$

These equations can be solved numerically. Clearly, the solutions to the first-order equations automatically satisfy the second-order equations of motion. Quantum corrections destroy fine-tuning of the coupling constants in (3). If one is interested in the calculation of the quantum-corrected profile functions one has to solve the second-order equations of motion instead of (13).

The tension of this elementary string is

$$T_1 = 2\pi\xi. \tag{14}$$

As soon as our theory is not supersymmetric and the string is not BPS there are corrections to this result which are small and uninteresting provided the coupling constants g_1^2 and g_2^2 are small. Note that the tension of the ANO string is

$$T_{\rm ANO} = 2\pi N\xi \tag{15}$$

in our normalization.

The elementary strings are bona fide non-Abelian. This means that, besides trivial translational moduli, they give rise to moduli corresponding to spontaneous breaking of a non-Abelian symmetry. Indeed, while the ''flat'' vacuum is $\mathrm{SU}(N)_{\rm diag}$ symmetric, the solution (10) breaks this symmetry down[5] to $\mathrm{U}(1)\times\mathrm{SU}(N-1)$ (at $N>2$). This means that the world-sheet (two-dimensional) theory of the elementary string moduli is the $\mathrm{SU}(N)/(\mathrm{U}(1)\times\mathrm{SU}(N-1))$ sigma model. This is also known as the $CP(N-1)$ model.

To obtain the non-Abelian string solution from the Z_N string (10) we apply the diagonal color-flavor rotation preserving the vacuum (5). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge we have

$$\begin{aligned} \Phi &= U\begin{pmatrix} \phi(r) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \phi(r) & 0 \\ 0 & 0 & \cdots & \phi_N(r) \end{pmatrix} U^{-1}, \\ A_i^{\mathrm{SU}(N)} &= \frac{1}{N}U\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -(N-1) \end{pmatrix} U^{-1}(\partial_i\alpha) \\ &\quad\times f_{NA}(r), \\ A_i^{\mathrm{U}(1)} &= -\frac{1}{N}(\partial_i\alpha)f(r), \qquad A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0, \end{aligned} \tag{16}$$

where U is a matrix $\in \mathrm{SU}(N)$. This matrix parametrizes orientational zero modes of the string associated with flux rotation in $\mathrm{SU}(N)$. The presence of these modes makes the string genuinely non-Abelian. Since the diagonal color-flavor symmetry is not broken by the VEV's of the scalar fields in the bulk (color-flavor locking) it is physical and has nothing to do with the gauge rotations eaten by the Higgs mechanism. The orientational moduli encoded in the matrix U are *not* gauge artifacts. The orientational zero modes of a non-Abelian string were first observed in [9,10].

III. THE WORLD-SHEET THEORY FOR THE ELEMENTARY STRING MODULI

In this section we will present derivation of an effective low-energy theory for the orientational moduli of the elementary string and then discuss underlying physics. We will closely follow Refs. [10,11] where this derivation was carried out for $N=2$ which leads to the $CP(1)$ model. In the general case, as was already mentioned, the resulting macroscopic theory is a two-dimensional $CP(N-1)$ model [9–11,13].

A. Derivation of the $CP(N-1)$ model

First, extending the supersymmetric $CP(1)$ derivation of Refs. [10,11], we will derive the effective low-energy theory for the moduli residing in the matrix U in the problem at hand. As is clear from the string solution (16), not each element of the matrix U will give rise to a modulus. The $\mathrm{SU}(N-1)\times\mathrm{U}(1)$ subgroup remains unbroken by the string solution under consideration; therefore, as was already mentioned, the moduli space is

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1)\times\mathrm{U}(1)} \sim CP(N-1). \tag{17}$$

Keeping this in mind we parametrize the matrices entering Eq. (16) as follows:

[5] At $N=2$ the string solution breaks SU(2) down to U(1).

$$\frac{1}{N}\left\{U\begin{pmatrix}1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & -(N-1)\end{pmatrix}U^{-1}\right\}_p^l = -n^l n_p^* + \frac{1}{N}\delta_p^l, \tag{18}$$

where n^l is a complex vector in the fundamental representation of SU(N), and

$$n_l^* n^l = 1$$

($l, p = 1, \ldots, N$ are color indices). As we will show below, one U(1) phase will be gauged in the effective sigma model. This gives the correct number of degrees of freedom, namely, $2(N-1)$.

With this parametrization the string solution (16) can be rewritten as

$$\begin{aligned}\Phi &= \frac{1}{N}[(N-1)\phi + \phi_N] - (\phi - \phi_N)\left(n\cdot n^* - \frac{1}{N}\right),\\ A_i^{\mathrm{SU}(N)} &= \left(n\cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\frac{x_i}{r^2}f_{NA}(r),\\ A_i^{\mathrm{U}(1)} &= \frac{1}{N}\varepsilon_{ij}\frac{x_i}{r^2}f(r),\end{aligned} \tag{19}$$

where for brevity we suppress all SU(N) indices. The notation is self-evident.

Assume that the orientational moduli are slowly varying functions of the string world-sheet coordinates x_α, $\alpha = 0, 3$. Then the moduli n^l become fields of a $(1+1)$-dimensional sigma model on the world sheet. Since n^l parametrize the string zero modes, there is no potential term in this sigma model.

To obtain the kinetic term we substitute our solution (19), which depends on the moduli n^l, in the action (3), assuming that the fields acquire a dependence on the coordinates x_α via $n^l(x_\alpha)$. In doing so we immediately observe that we have to modify the solution including in it the $\alpha = 0, 3$ components of the gauge potential which are no more vanishing. In the $CP(1)$ case, as was shown in [11], the potential A_α must be orthogonal [in the SU(N) space] to the matrix (18) as well as to its derivatives with respect to x_α. Generalization of these conditions to the $CP(N-1)$ case leads to the following ansatz:

$$A_\alpha^{\mathrm{SU}(N)} = -i[\partial_\alpha n\cdot n^* - n\cdot\partial_\alpha n^* - 2n\cdot n^*(n^*\partial_\alpha n)]\rho(r), \quad \alpha = 0, 3, \tag{20}$$

where we assume the contraction of the color indices inside the parentheses,

$$(n^*\partial_\alpha n) \equiv n_l^*\partial_\alpha n^l,$$

and introduce a new profile function $\rho(r)$.

The function $\rho(r)$ in Eq. (20) is determined through a minimization procedure [10,11] which generates ρ's own equation of motion. Now we derive it. But at first we note that $\rho(r)$ vanishes at infinity,

$$\rho(\infty) = 0. \tag{21}$$

The boundary condition at $r = 0$ will be determined shortly.

The kinetic term for n^l comes from the gauge and quark kinetic terms in Eq. (3). Using Eqs. (19) and (20) to calculate the SU(N) gauge field strength we find

$$\begin{aligned}F_{\alpha i}^{\mathrm{SU}(N)} = {}& (\partial_\alpha n\cdot n^* + n\cdot\partial_\alpha n^*)\varepsilon_{ij}\frac{x_j}{r^2}f_{NA}[1-\rho(r)]\\ &+ i[\partial_\alpha n\cdot n^* - n\cdot\partial_\alpha n^* - 2n\cdot n^*(n^*\partial_\alpha n)]\\ &\times\frac{x_i}{r}\frac{d\rho(r)}{dr}.\end{aligned} \tag{22}$$

In order to have a finite contribution from the term $\mathrm{Tr}F_{\alpha i}^2$ in the action we have to impose the constraint

$$\rho(0) = 1. \tag{23}$$

Substituting the field strength (22) in the action (3) and including, in addition, the kinetic term of the quarks, after a rather straightforward but tedious algebra we arrive at

$$S^{(1+1)} = 2\beta\int dtdz\{(\partial_\alpha n^*\partial_\alpha n) + (n^*\partial_\alpha n)^2\}, \tag{24}$$

where the coupling constant β is given by

$$\beta = \frac{2\pi}{g_2^2}I, \tag{25}$$

and I is a basic normalizing integral

$$I = \int_0^\infty rdr\left\{\left[\frac{d}{dr}\rho(r)\right]^2 + \frac{1}{r^2}f_{NA}^2(1-\rho)^2 + g_2^2\left[\frac{\rho^2}{2}(\phi^2+\phi_N^2) + (1-\rho)(\phi-\phi_N)^2\right]\right\}. \tag{26}$$

The theory in Eq. (24) is in fact the two-dimensional $CP(N-1)$ model. To see that this is indeed the case we can eliminate the second term in (24) by virtue of the introduction of a nonpropagating U(1) gauge field. We review this in Sec. IV, and then discuss the underlying physics of the model. Thus, we obtain the $CP(N-1)$ model as an effective low-energy theory on the world sheet of the non-Abelian string. Its coupling β is related to the four-dimensional coupling g_2^2 via the basic normalizing integral (26). This integral can be viewed as an "action" for the profile function ρ.

Varying (26) with respect to ρ one obtains the second-order equation which the function ρ must satisfy, namely,

$$-\frac{d^2}{dr^2}\rho - \frac{1}{r}\frac{d}{dr}\rho - \frac{1}{r^2}f_{NA}^2(1-\rho) + \frac{g_2^2}{2}(\phi_N^2+\phi^2)\rho - \frac{g_2^2}{2}(\phi_N-\phi)^2 = 0. \tag{27}$$

After some algebra and extensive use of the first-order

equations (13) one can show that the solution of (27) is given by

$$\rho = 1 - \frac{\phi_N}{\phi}. \tag{28}$$

This solution satisfies the boundary conditions (21) and (23).

Substituting this solution back into the expression for the normalizing integral (26) one can check that this integral reduces to a total derivative and is given by the flux of the string determined by $f_{NA}(0) = 1$. Therefore, we arrive at

$$I = 1. \tag{29}$$

This result can be traced back to the fact that our theory (3) is a bosonic reduction of the $\mathcal{N} = 2$ supersymmetric theory, and the string satisfies the first-order equations (13) [see [11] for the explanation why (29) should hold for the BPS non-Abelian strings in SUSY theories]. The fact that $I = 1$ was demonstrated previously for $N = 2$, where the $CP(1)$ model emerges. Generally speaking, for non-BPS strings, I could be a certain function of N (see Ref. [14] for a particular example). In the problem at hand it is N independent. However, we expect that quantum corrections slightly modify Eq. (29).

The relation between the four-dimensional and two-dimensional coupling constants (25) is obtained at the classical level. In quantum theory both couplings run. So we have to specify a scale at which the relation (25) takes place. The two-dimensional $CP(N-1)$ model (24) is an effective low-energy theory good for the description of internal string dynamics at small energies, much less than the inverse thickness of the string which is given by $\sqrt{\xi}$. Thus, $\sqrt{\xi}$ plays the role of a physical ultraviolet cutoff in (24). This is the scale at which Eq. (25) holds. Below this scale, the coupling β runs according to its two-dimensional renormalization-group flow, see the next section.

B. Penetration of θ from the bulk in the world-sheet theory

Now let us investigate the impact of the θ term that is present in our microscopic theory (3). At first sight, seemingly it cannot produce any effect because our string is magnetic. However, if one allows for slow variations of n^l with respect to z and t, one immediately observes that the electric field is generated via $A_{0,3}$ in Eq. (20). Substituting $F_{\alpha i}$ from (22) into the θ term in the action (3) and taking into account the contribution from $F_{\alpha\gamma}$ times F_{ij} ($\alpha, \gamma = 0, 3$ and $i, j = 1, 2$) we get the topological term in the effective $CP(N-1)$ model (24) in the form

$$S^{(1+1)} = \int dtdz \Big\{ 2\beta[(\partial_\alpha n^* \partial_\alpha n) + (n^* \partial_\alpha n)^2] - \frac{\theta}{2\pi} I_\theta \varepsilon_{\alpha\gamma} (\partial_\alpha n^* \partial_\gamma n) \Big\}, \tag{30}$$

where I_θ is another normalizing integral given by the formula

$$I_\theta = -\int dr \Big\{ 2f_{NA}(1-\rho)\frac{d\rho}{dr} + (2\rho - \rho^2)\frac{df}{dr} \Big\} = \int dr \frac{d}{dr}\{2f_{NA}\rho - \rho^2 f_{NA}\}. \tag{31}$$

As is clearly seen, the integrand here reduces to a total derivative, and the integral is determined by the boundary conditions for the profile functions ρ and f_{NA}. Substituting (11), (12), (21), and (23) we get

$$I_\theta = 1, \tag{32}$$

independently of the form of the profile functions. This latter circumstance is perfectly natural for the topological term.

The additional term (30) in the $CP(N-1)$ model that we have just derived is the θ term in the standard normalization. The result (32) could have been expected since physics is 2π periodic with respect to θ both in the four-dimensional microscopic gauge theory and in the effective two-dimensional $CP(N-1)$ model. The result (32) is not sensitive to the presence of supersymmetry. It will hold in supersymmetric models as well. Note that the complexified bulk coupling constant converts into the complexified world-sheet coupling constant,

$$\tau = \frac{4\pi}{g_2^2} + i\frac{\theta}{2\pi} \to 2\beta + i\frac{\theta}{2\pi}.$$

The above derivation provides the first direct calculation proving the coincidence of the θ angles in four and two dimensions.

Let us make a comment on this point from the brane perspective. Since the model under consideration is nonsupersymmetric, the usual brane picture corresponding to minimal surfaces in the external geometry is complicated and largely unavailable at present. However, a few statements insensitive to details of the brane picture can be made—the identification of the θ angles in the microscopic and macroscopic theories above is one of them. Indeed, in any relevant brane picture the θ angle corresponds to the distance between two M5 branes along the 11th dimension in M theory [22]. The four-dimensional theory is defined on the world volume of one of these M5 branes, while an M2 brane stretched between M5 branes corresponds to the non-Abelian string we deal with. It is clear that the θ angles are the same since it is just the same geometrical parameter viewed from two different objects: M5 and M2 branes (see also footnote 7 in Ref. [13]).

IV. DYNAMICS OF THE WORLD-SHEET THEORY

The $CP(N-1)$ model describing the string moduli interactions can be cast in several equivalent representations. The most convenient for our purposes is a linear

gauged representation (for a review see [23]). At large N the model was solved [24,25].

In this formulation the Lagrangian is built from an N-component complex field n^ℓ subject to the constraint

$$n^*_\ell n^\ell = 1. \tag{33}$$

The Lagrangian has the form

$$\mathcal{L} = \frac{2}{g^2}[(\partial_\alpha + iA_\alpha)n^*_\ell(\partial_\alpha - iA_\alpha)n^\ell - \lambda(n^*_\ell n^\ell - 1)], \tag{34}$$

where $1/g^2 \equiv \beta$ and λ is the Lagrange multiplier enforcing (33). Moreover, A_α is an auxiliary field which enters the Lagrangian with no kinetic term. Eliminating A_α by virtue of the equations of motion one arrives at Eq. (30).

At the quantum level the constraint (33) is gone; λ becomes dynamical. Moreover, a kinetic term is generated for the auxiliary field A_α at the quantum level, so that A_α becomes dynamical too.

As was shown above, the θ term which can be written as

$$\mathcal{L}_\theta = \frac{\theta}{2\pi}\varepsilon_{\alpha\gamma}\partial^\alpha A^\gamma = \frac{\theta}{2\pi}\varepsilon_{\alpha\gamma}\partial^\alpha(n^*_\ell\partial^\gamma n^\ell) \tag{35}$$

appears in the world-sheet theory of the string moduli provided the same θ angle is present in the bulk (microscopic) theory.

Now we have to discuss the vacuum structure of the theory (34). Basing on a modern understanding of the issue [26] (see also [27]) one can say that for each θ there are infinitely many ''vacua'' that are stable in the limit $N \to \infty$. The word vacua is in the quotation marks because only one of them presents a bona fide global minimum; others are local minima and are metastable at finite (but large) N. A schematic picture of these vacua is given in Fig. 1. All these minima are entangled in the θ evolution. If we vary θ continuously from 0 to 2π the depths of the minima

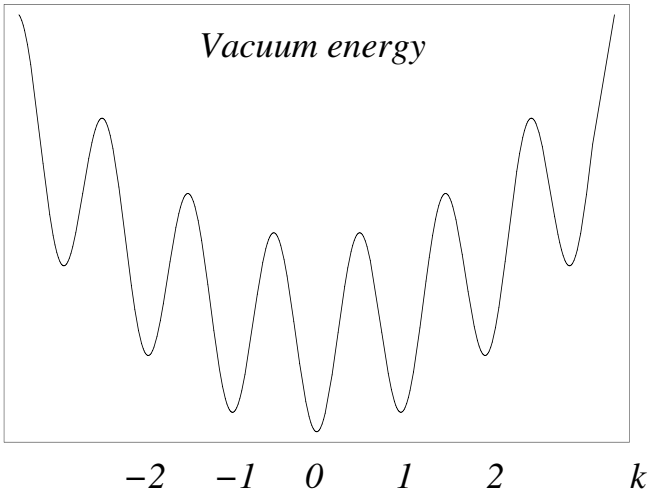

FIG. 1. The vacuum structure of the $CP(N-1)$ model at $\theta = 0$.

''breathe.'' At $\theta = \pi$ two vacua become degenerate (Fig. 2), while for larger values of θ the global minimum becomes local while the adjacent local minimum becomes global. The splitting between the values of the consecutive minima is of the order of $1/N$, while the probability of the false vacuum decay is proportional to $N^{-1}\exp(-N)$, see below.

As long as the $CP(N-1)$ model plays a role of the effective theory on the world sheet of non-Abelian string each of these vacua corresponds to a string in the four-dimensional bulk theory. For each given θ, the ground state of the string is described by the deepest vacuum of the world-sheet theory, $CP(N-1)$. Metastable vacua of $CP(N-1)$ correspond to excited strings.

As was shown by Witten [24], the field n^ℓ can be viewed as a field describing kinks interpolating between the true vacuum and its neighbor. The multiplicity of such kinks is N [28], they form an N plet. This is the origin of the superscript ℓ in n^ℓ.

Moreover, Witten showed, by exploiting $1/N$ expansion to the leading order, that a mass scale is dynamically generated in the model, through dimensional transmutation,

$$\Lambda^2 = M_0^2\exp\left(-\frac{8\pi}{Ng^2}\right). \tag{36}$$

Here M_0 is the ultraviolet cutoff (for the effective theory on the string world sheet $M_0 = \sqrt{\xi}$) and $g^2 = 1/\beta$ is the bare coupling constant given in Eq. (25). The combination Ng^2 is nothing but the 't Hooft constant that does not scale with N. As a result, Λ scales as N^0 at large N.

In the leading order, N^0, the kink mass M_n is θ independent,

$$M_n = \Lambda. \tag{37}$$

θ-dependent corrections to this formula appear only at the level $1/N^2$.

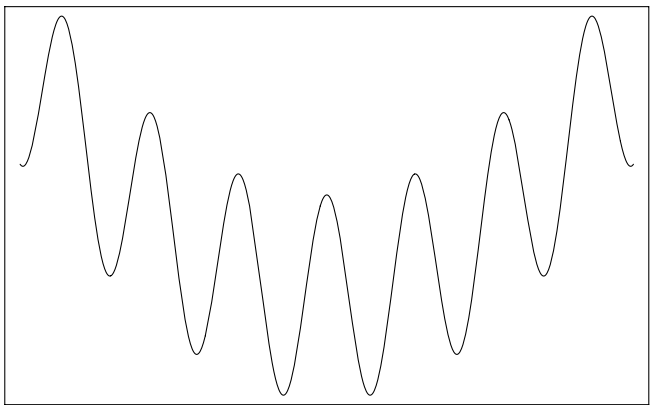

FIG. 2. The vacuum structure of the $CP(N-1)$ model at $\theta = \pi$.

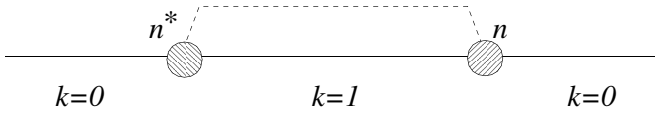

FIG. 3. Linear confinement of the n-n^* pair. The solid straight line represents the string. The dashed line shows the vacuum energy density (normalizing $\mathcal{E}_0$ to zero).

The kinks represented in the Lagrangian (34) by the field n^ℓ are not asymptotic states in the $CP(N-1)$ model. In fact, they are confined [24]; the confining potential grows linearly with distance,[6] with the tension suppressed by $1/N$. From the four-dimensional perspective the coefficient of the linear confinement is nothing but the difference in tensions of two strings: the lightest and the next one, see below. Therefore, we denote it as ΔT,

$$\Delta T = 12\pi\frac{\Lambda^2}{N}. \tag{38}$$

One sees that confinement becomes exceedingly weak at large N. In fact, Eq. (38) refers to $\theta = 0$. The standard argument that θ dependence does not appear at $N \to \infty$ is inapplicable to the string tension, since the string tension itself vanishes in the large-N limit. The θ dependence can be readily established from a picture of the kink confinement discussed in [14], see Fig. 3, which is complementary to that of [24]. This picture of the kink confinement is schematically depicted in this figure.

Since the kink represents an interpolation between the genuine vacuum and a false one, the kink-antikink configuration presented in Fig. 3 shows two distinct regimes: the genuine vacuum outside the kink-antikink pair and the false one inside. (The opposite case, see Fig. 4, is discussed below.) As was mentioned, the string tension ΔT is given by the difference of the vacuum energy densities, that of the false vacuum minus the genuine one. At large N, the k dependence of the energy density in the vacua (k is the excitation number), as well as the θ dependence, is well known [26],

$$\mathcal{E}_k(\theta) = -\frac{6}{\pi}N\Lambda^2\left\{1 - \frac{1}{2}\left(\frac{2\pi k + \theta}{N}\right)^2\right\}. \tag{39}$$

At $\theta = 0$ the genuine vacuum corresponds to $k = 0$, while the first excitation to $k = -1$. At $\theta = \pi$ these two vacua are degenerate, at $\theta = 2\pi$ their roles interchange. Therefore,

[6]Let us note in passing that corrections to the leading-order result (38) run in powers of $1/N^2$ rather than $1/N$. Indeed, as is well known, the θ dependence of the vacuum energy enters only through the combination of θ/N, namely $\mathcal{E}(\theta) = N\Lambda^2 f(\theta/N)$ where f is some function. As will be explained momentarily, $\Delta T = \mathcal{E}(\theta = 2\pi) - \mathcal{E}(\theta = 0)$. Moreover, $\mathcal{E}$, being CP even, can be expanded in even powers of θ. This concludes the proof that $\Delta T = (12\pi\Lambda^2/N)(1 + \sum_{k=1}^{\infty} c_k N^{-2k})$.

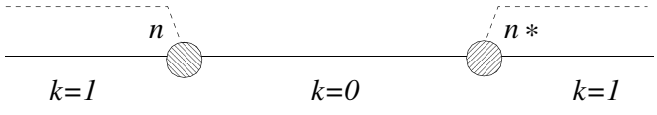

FIG. 4. Breaking of the excited string through the n-n^* pair creation. The dashed lines show the vacuum energy density.

$$\Delta T(\theta) = 12\pi\frac{\Lambda^2}{N}\left|1 - \frac{\theta}{\pi}\right|. \tag{40}$$

Note that at $\theta = \pi$ the string tension vanishes and confinement of kinks disappears.

This formula requires a comment which we hasten to make. In fact, for each given θ, there are two types of kinks which are degenerate at $\theta = 0$ but acquire a splitting at $\theta \neq 0$. This is clearly seen in Fig. 5 which displays $\mathcal{E}_{0,\pm 1}$ for three minima: the global one ($k = 0$) and two adjacent local minima, $k = \pm 1$ (the above nomenclature refers to $|\theta| < \pi$). Let us consider, say, small and positive values of θ. Then the kink described by the field n can represent two distinct interpolations: from the ground state to the state $k = -1$ (i.e. the minimum to the left of the global minimum in Fig. 1); then

$$\Delta\mathcal{E} = \frac{12\pi\Lambda^2}{N}\left(1 - \frac{\theta}{\pi}\right).$$

Another possible interpolation is from the ground state to the state $k = 1$ (i.e. the minimum to the right of the global minimum in Fig. 1). In the latter case

$$\Delta\mathcal{E} = \frac{12\pi\Lambda^2}{N}\left(1 + \frac{\theta}{\pi}\right).$$

In the first scenario the string becomes tensionless,[7] i.e. the states $k = 0, -1$ degenerate, at $\theta = \pi$. The same consideration applies to negative values of θ. Now it is the vacua $k = 0, 1$ that become degenerate at $\theta = -\pi$, rendering the corresponding string tensionless. In general, it is sufficient to consider the interval $|\theta| \leq \pi$.

What will happen if we interchange the position of two kinks in Fig. 3, as shown in Fig. 4? The excited vacuum is now outside the kink-antikink pair, while the genuine one is inside. Formally, the string tension becomes negative. In fact, the process in Fig. 4 depicts a breaking of the excited string. As was mentioned above, the probability of such breaking is suppressed by $\exp(-N)$. Indeed, the master formula from Ref. [29] implies that the probability of the excited string decay (through the n-n^* pair creation) per unit time per unit length is

[7]Note that in Witten's work [24] there is a misprint in Eq. (18) and subsequent equations; the factor $\theta/2\pi$ should be replaced by θ/π. Two types of kinks correspond in this equation to $x > y$ and $x < y$, respectively.

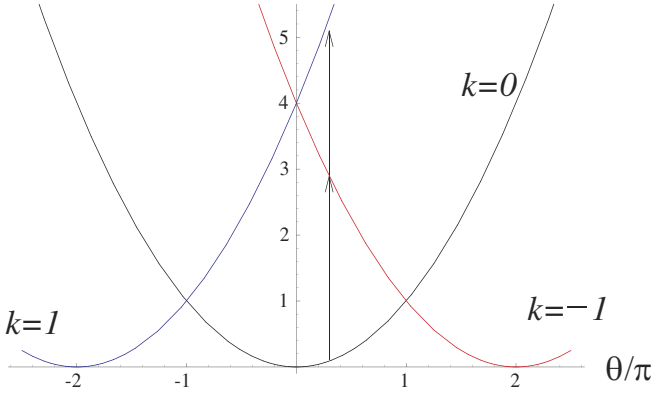

FIG. 5 (color online). The function $1 + \mathcal{E}_{0,\pm 1}/[(6N\Lambda^2)\pi^{-1}]$ in the units $\pi^2/(2N^2)$ versus θ/π.

$$\Gamma = \frac{\Delta T}{2\pi} \exp\left(-\frac{\pi M_n^2}{\Delta T}\right) = \frac{6\Lambda^2}{N} e^{-N/12} \tag{41}$$

at $\theta = 0$. At $\theta \neq 0$ the suppression is even stronger.

To summarize, the $CP(N-1)$ model has a fine structure of vacua which are split, with the splitting of the order of Λ^2/N. In four-dimensional bulk theory these vacua correspond to elementary non-Abelian strings. Classically all these strings have the same tension (14). Because of quantum effects in the world-sheet theory the degeneracy is lifted: the elementary strings become split, with the tensions

$$T = 2\pi\xi - \frac{6}{\pi} N\Lambda^2 \left\{1 - \frac{1}{2}\left(\frac{2\pi k + \theta}{N}\right)^2\right\}. \tag{42}$$

Note that (i) the splitting does not appear to any finite order in the coupling constants; (ii) since $\xi \gg \Lambda$, the splitting is suppressed in both parameters, $\Lambda/\sqrt{\xi}$ and $1/N$.

Let us also note that the identification of the θ terms and topological charges in two and four dimensions (see Sec. III B) allows us to address the issue of CP symmetry in four dimensions at $\theta = \pi$, and confront it with the situation in two dimensions, see Ref. [30]. In this work it was shown, on the basis of a strong coupling analysis, that there is a cusp in the partition function of the $CP(N-1)$ model at $\theta = \pi$, implying that the expectation value of the two-dimensional topological charge does not vanish at this point. This tells us that CP invariance is dynamically spontaneously broken at $\theta = \pi$.

The above result is in full agreement with Witten's picture of the vacuum family in the $CP(N-1)$ model, with N states—one global minimum, other local ones—entangled in the θ evolution. At $\theta = \pi$ two minima are degenerate, but they are characterized by opposite values of the topological charge VEV's,

$$\langle \varepsilon_{\alpha\gamma} \partial^\alpha n_\ell^* \partial^\gamma n^\ell \rangle = \pm\Lambda^2.$$

The kink (confined monopole) can be viewed as a barrier separating two domains (two degenerate strings) carrying opposite CP.

On the other hand, the bulk four-dimensional theory is weakly coupled, and for each given θ the bulk vacuum is unique. There is no spontaneous CP violation in the four-dimensional bulk theory at $\theta = \pi$. One can easily check this assertion by carrying out a direct instanton calculation.

V. FUSING STRINGS

As has been already mentioned, in QCD one can consider not only basic strings, but 2-strings, 3-strings, ..., k-strings, and their excitations. k-strings are composite flux tubes attached to color sources with N-ality k. Moreover, the N-string ensembles—i.e. N-strings—can decay into a no-string state. It is natural to ask how these phenomena manifest themselves in the model under consideration.

If the ansatz (9) defines a basic string, it is not difficult to generalize this definition to get an analog of 2-strings, 3-strings, etc., for instance,

$$\Phi_{2\text{-string}} = \sqrt{\xi}\,\mathrm{diag}(e^{i\alpha(x)}, e^{i\alpha(x)}, 1, \ldots, 1), \qquad x \to \infty. \tag{43}$$

The solution (43) breaks SU(N) symmetry down to U(1) $\times$ SU(2) $\times$ SU($N-2$) (at $N > 3$). This means that the world-sheet (two-dimensional) theory of the string moduli is the SU(N)/(U(1) $\times$ SU(2) $\times$ SU($N-2$)) sigma model. This is also known as the Grassmannian $G_{2,N}$ model. At large N it has more fields, by a factor of 2, than the $CP(N-1)$ model; other features are quite similar.

The statement that in our model the world-sheet theory for k-strings is the Grassmannian $G_{k,N}$ model has a clear-cut indirect confirmation. Indeed, the k-string ansatz of the type indicated in Eq. (43) tells us that the number of distinct classical strings is

$$\nu(k, N) = C_k^N = \frac{N!}{k!(N-k)!}, \tag{44}$$

since k phase factors $e^{i\alpha}$ can be distributed arbitrarily in N positions. From the two-dimensional perspective this number should match the number of distinct vacua of the world-sheet theory. The latter was calculated in a supersymmetric $G_{k,N}$ model in Ref. [31], where it was shown to be C_k^N, as in Eq. (44). In the supersymmetric $G_{k,N}$ model all these vacua are degenerate, i.e., we have degenerate strings. Introducing supersymmetry breaking we move away from the degeneracy. In the nonsupersymmetric $G_{k,N}$ model, the number $\nu(k, N) = C_k^N$ gives the number of states in the vacuum family: the genuine vacuum plus metastable ones entangled with the genuine vacuum in the θ evolution.

As soon as string tensions in our model are classically determined by their U(1) charges the tension of the k-string is given by

$$T_k = 2\pi k\xi + O(\Lambda^2), \tag{45}$$

where corrections of order of Λ^2 are induced by the quantum effects in the effective world-sheet theory.

If we add up N-strings, the resulting conglomerate is connected to the ANO string.

VI. KINKS ARE CONFINED MONOPOLES

The $CP(N-1)$ models are asymptotically free theories and flow to strong coupling in the infrared. Therefore, the non-Abelian strings discussed in the previous sections are in a highly quantum regime. To make contact with the classical Abelian strings we can introduce parameters which explicitly break the diagonal color-flavor $\mathrm{SU}(N)_{\mathrm{diag}}$ symmetry lifting the orientational string moduli. This allows us to obtain a quasiclassical interpretation of the confined monopoles as string junctions, and follow their evolution from (almost) 't Hooft-Polyakov monopoles to highly quantum sigma-model kinks. In the supersymmetric case this was done in Refs. [11–13].

A. Breaking $\mathrm{SU}(N)_{\mathrm{diag}}$

In order to trace the monopole evolution we modify our basic model (3) introducing, in addition to the already existing fields, a complex adjoint scalar field a^a,

$$\begin{aligned} S = \int d^4x \Big\{ & \frac{1}{4g_2^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_1^2}(F_{\mu\nu})^2 + \frac{1}{g_2^2}|D_\mu a^a|^2 \\ & + \mathrm{Tr}(\nabla_\mu \Phi)^\dagger (\nabla^\mu \Phi) + \frac{g_2^2}{2}[\mathrm{Tr}(\Phi^\dagger T^a \Phi)]^2 \\ & + \frac{g_1^2}{8}[\mathrm{Tr}(\Phi^\dagger \Phi) - N\xi]^2 + \frac{1}{2}\mathrm{Tr}|a^a T^a \Phi \\ & + \Phi\sqrt{2}M|^2 + \frac{i\theta}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} \Big\}, \end{aligned} \tag{46}$$

where D_μ is a covariant derivative acting in the adjoint representation of$\mathrm{SU}(N)$ and M is a mass matrix for scalar quarks Φ. We assume that it has a diagonal form

$$M = \begin{pmatrix} m_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & m_N \end{pmatrix}, \tag{47}$$

with the vanishing sum of the diagonal entries,

$$\sum_{A=1}^{N} m_A = 0. \tag{48}$$

Later on it will be convenient to make a specific choice of the parameters m_A, namely,

$$M = m \times \mathrm{diag}\{e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}, 1\}, \tag{49}$$

where m is a single common parameter, and the constraint (48) is automatically satisfied. We can (and will) assume m to be real and positive.

In fact, the model (46) presents a less reduced bosonic part of the $\mathcal{N}=2$ supersymmetric theory than the model (3) on which we dwelled above. In the $\mathcal{N}=2$ supersymmetric theory the adjoint field is a part of the $\mathcal{N}=2$ vector multiplet. For the purpose of the string solution the field a^a is sterile as long as $m_A = 0$. Therefore, it could be and was ignored in the previous sections. However, if one's intention is to connect oneself to the quasiclassical regime, $m_A \neq 0$, and the adjoint field must be reintroduced.

For the reason which will become clear shortly, let us assume that, although $m_A \neq 0$, they are all small compared to $\sqrt{\xi}$,

$$m \ll \sqrt{\xi},$$

but $m \gg \Lambda$. For generic nondegenerate values of m_A the adjoint field develops VEV's,

$$\langle a \rangle = -\sqrt{2}\begin{pmatrix} m_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & m_N \end{pmatrix}. \tag{50}$$

The vacuum expectation values of the scalar quarks Φ remain intact; they are given by Eq. (5). For the particular choice specified in Eq. (49)

$$\langle a \rangle = -\sqrt{2}m\,\mathrm{diag}\{e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}, 1\}. \tag{51}$$

Clearly the diagonal color-flavor group $\mathrm{SU}(N)_{\mathrm{diag}}$ is now broken by adjoint VEV's down to $\mathrm{U}(1)N-1 \times Z_N$. Still, the solutions for the Abelian (or Z_N) strings are the same as was discussed in Sec. II since the adjoint field does not enter these solutions. In particular, we have N distinct Z_N string solutions depending on what particular squark winds at infinity, see Sec. II. Say, the string solution with the winding last flavor is still given by Eq. (10).

What is changed with the color-flavor $\mathrm{SU}(N)_{\mathrm{diag}}$ explicitly broken by $m_A \neq 0$, the rotations (16) no more generate zero modes. In other words, the fields n^ℓ become quasimoduli: a shallow potential for the quasimoduli n^l on the string world sheet is generated. This potential is shallow as long as $m_A \ll \sqrt{\xi}$.

This potential was calculated in the $CP(1)$ case in Ref. [11]; the $CP(N-1)$ case was treated in [13]. It has the following form:

$$V_{CP(N-1)} = 2\beta\left\{\sum_l |m_l|^2|n^l|^2 - \left|\sum_l m_l |n^l|^2\right|^2\right\}. \tag{52}$$

The potential simplifies if the mass terms are chosen according to (49),

$$V_{CP(N-1)} = 2\beta m^2\left\{1 - \left|\sum_{\ell=1}^{N} e^{2\pi i\ell/N}|n^\ell|^2\right|^2\right\}. \tag{53}$$

This potential is obviously invariant under the cyclic substitution

$$\ell \to \ell + k, \qquad n^{\ell} \to n^{\ell+k}, \qquad \forall\ \ell, \tag{54}$$

with k fixed. This property will be exploited below.

Now our effective two-dimensional theory on the string world sheet becomes a massive $CP(N-1)$ model. The potential (52) or (53) has N vacua at

$$n^{\ell} = \delta^{\ell\ell_0}, \qquad \ell_0 = 1, 2, \ldots, N. \tag{55}$$

These vacua correspond to N distinct Abelian Z_N strings with $\varphi^{\ell_0\ell_0}$ winding at infinity, see Eq. (19).

B. Evolution of monopoles

Our task in this section is to trace the evolution of the confined monopoles starting from the quasiclassical regime, deep into the quantum regime. For illustrative purposes it will be even more instructive if we start from the limit of weakly confined monopoles, when in fact they present just slightly distorted 't Hooft-Polyakov monopoles (Fig. 6). For simplicity, in this section we will set $\theta = 0$. To further simplify the subsequent discussion we will not treat N as a large parameter in this section, i.e., we will make no parametric distinction between m and mN.

Let us start from the limit $|m_A| \gg \sqrt{\xi}$ and take all masses of the same order, as in Eq. (49). In this limit the scalar quark expectation values can be neglected, and the vacuum structure is determined by VEV's of the adjoint a^a field. In the nondegenerate case the gauge symmetry SU(N) of our microscopic model is broken down to U(1)$N-1$ modulo possible discrete subgroups. This is the textbook situation for occurrence of the SU(N) 't Hooft-Polyakov monopoles. The monopole core size is of the order of $|m|^{-1}$. The 't Hooft-Polyakov solution remains valid up to much larger distances of the order of $\xi^{-1/2}$. At distances larger than $\sim\xi^{-1/2}$ the quark VEV's become important. As usual, the U(1) charge condensation leads to the formation of the U(1) magnetic flux tubes, with the transverse size of the order of $\xi^{-1/2}$ (see the upper picture in Fig. 6). The flux is quantized; the flux tube tension is tiny in the scale of the square of the monopole mass. Therefore, what we deal with in this limit is basically a very weakly confined 't Hooft-Polyakov monopole.

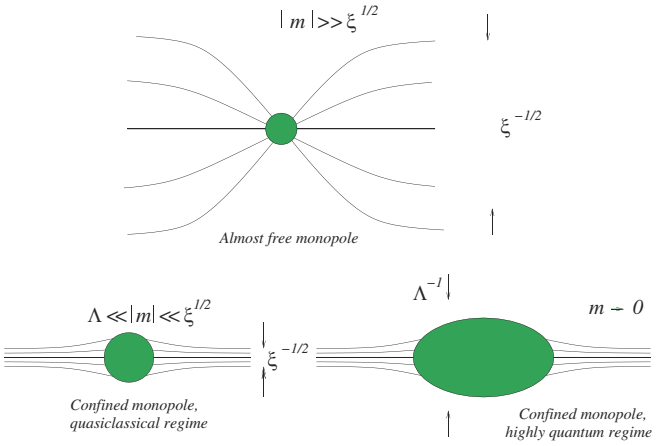

FIG. 6 (color online). Evolution of the confined monopoles.

Let us verify that the confined monopole is a junction of two strings. Consider the junction of two Z_N strings corresponding to two "neighboring" vacua of the $CP(N-1)$ model. For the ℓ_0th vacuum n^{ℓ} is given by (55) while for the ℓ_0+1th vacuum it is given by the same equations with $\ell_0 \to \ell_0 + 1$. The flux of this junction is given by the difference of the fluxes of these two strings. Using (19) we get that the flux of the junction is

$$4\pi \times \mathrm{diag}\frac{1}{2}\{\ldots 0, 1, -1, 0, \ldots\} \tag{56}$$

with the nonvanishing entries located at positions ℓ_0 and $\ell_0 + 1$. These are exactly the fluxes of $N-1$ distinct 't Hooft-Polyakov monopoles occurring in the SU(N) gauge theory provided that SU(N) is spontaneously broken down to U(1)$^{N-1}$. We see that in the quasiclassical limit of large $|m_A|$ the Abelian monopoles play the role of junctions of the Abelian Z_N strings. Note that in various models the fluxes of monopoles and strings were shown [21,32–35] to match each other so that the monopoles can be confined by strings in the Higgs phase. The explicit solution for the confined monopole as a 1/4 BPS junction of two strings was obtained in [11] for the $N=2$ case in the $\mathcal{N}=2$ supersymmetric theory. The general solution for 1/4 BPS junctions of semilocal strings was obtained in [36].

Now, if we reduce $|m|$,

$$\Lambda \ll |m| \ll \sqrt{\xi},$$

the size of the monopole ($\sim |m|^{-1}$) becomes larger than the transverse size of the attached strings. The monopole gets squeezed in earnest by the strings—it becomes a bona fide confined monopole (the lower left corner of Fig. 6). A macroscopic description of such monopoles is provided by the massive CP^{N-1} model, see Eqs. (52) or (53). The confined monopole is nothing but the massive sigma-model kink.

As we further diminish $|m|$ approaching Λ and then getting below Λ, the size of the monopole grows, and, classically, it would explode. This is where quantum effects in the world-sheet theory take over. This domain presents the regime of highly quantum world-sheet dynamics. While the thickness of the string (in the transverse direction) is $\sim\xi^{-1/2}$, the z-direction size of the kink representing the confined monopole in the highly quantum regime is much larger, $\sim\Lambda^{-1}$, see the lower right corner in Fig. 6. In passing from $m \gg \Lambda$ to $m \ll \Lambda$ we, in fact, cross a line of the phase transition from Abelian to non-Abelian strings. This is discussed in Sec. VII.

VII. ABELIAN TO NON-ABELIAN STRING PHASE TRANSITION

In this section we will restrict ourselves to the choice of the mass parameters presented in Eq. (49). Correspondingly, the potential of the massive CP^{N-1} model describing the quasimoduli has the form (53).

At large m, $m \gg \Lambda$, the model is at weak coupling, so the quasiclassical analysis is applicable. N quasiclassical vacua are presented in Eq. (55). The invariance of $V_{CP(N-1)}$ under the cyclic permutations (54) implies a Z_N symmetry of the world-sheet theory of the quasimoduli. In each given vacuum the Z_N symmetry is spontaneously broken. N vacua have strictly degenerate vacuum energies, which, as we already know, leads to the kinks *deconfinement*. From the four-dimensional point of view this means that we have N strictly degenerate Abelian strings (the Z_N strings).

The flux of the Abelian 't Hooft-Polyakov monopole equals to the difference of the fluxes of two neighboring strings, see (56). Therefore, the confined monopole in this regime is obviously a junction of two distinct Z_N strings. It is seen as a quasiclassical kink interpolating between the neighboring ℓ_0th and $(\ell_0 + 1)$th vacua of the effective massive $CP(N-1)$ model on the string world sheet. A monopole can move freely along the string as both attached strings are tension degenerate.

Now if we further reduce m tending it to zero, the picture changes. At $m = 0$ the global symmetry $\mathrm{SU}(N)_{\mathrm{diag}}$ is unbroken, and so is the discrete Z_N of the massive $CP(N-1)$ model with the potential (53). N degenerate vacua of the quasiclassical regime give place to N nondegenerate vacua depicted in Fig. 1 (see Sec. IV). The fact that $\langle n^\ell \rangle = 0$ in the quantum regime signifies that in the limit $m \to 0$ the Z_N symmetry of the massive model gets restored. Now kinks are confined, as we know from Sec. IV.

From the standpoint of the four-dimensional microscopic theory the tensions of N non-Abelian strings get a split, and the non-Abelian monopoles, in addition to the four-dimensional confinement (which ensures that the monopoles are attached to the strings) acquire a two-dimensional confinement along the string: a monopole-antimonopole forms a mesonlike configuration, with necessity, see Fig. 3.

Clearly these two regimes at large and small m are separated by the phase transition at some critical value m_*. We interpret this as a phase transition between the Abelian and non-Abelian confinement. In the Abelian confinement phase at large m, the Z_N symmetry is spontaneously broken, all N-strings are strictly degenerate, and there is no two-dimensional confinement of the 4D-confined monopoles. Instead, in the non-Abelian confinement phase occurring at small m, the Z_N symmetry is fully restored, all N elementary strings are split, and the 4D-confined monopoles combine with antimonopoles to form a mesonlike configuration on the string, see Fig. 3. We show schematically the dependence of the string tensions on m in these two phases in Fig. 7.

It is well known [37] that the two-dimensional $CP(N-1)$ model can be obtained as a low-energy limit of a U(1) gauge theory with N flavors of complex scalars n^ℓ and the potential

$$e^2\beta^2(|n^\ell|^2 - 1)^2, \tag{57}$$

where e^2 is U(1) gauge coupling. Classically the $CP(N-1)$ model corresponds to the Higgs phase of this gauge theory. The potential (57) forces n^ℓ to develop VEV's breaking the U(1) gauge symmetry. Then the U(1) photon becomes heavy and can be integrated out. Namely, in the low-energy limit the gauge kinetic term can be ignored which leads us to the model (34).

To include the masses m_A in this theory we add, following [37], a neutral complex scalar field σ and consider the U(1) gauge theory with the potential

$$S^{(1+1)} = \int dtdz\Big\{2\beta|\nabla_\alpha n|^2 + \frac{1}{4e^2}F_{\alpha\gamma}^2 + \frac{1}{e^2}|\partial_\alpha\sigma|^2 + 4\beta\left|\left(\sigma - \frac{m_\ell}{\sqrt{2}}\right)n^\ell\right|^2 + 2e^2\beta^2(|n^\ell|^2 - 1)^2\Big\}, \tag{58}$$

where $\nabla_\alpha = \partial_\alpha - iA_\alpha$ [A_α is the two-dimensional U(1) gauge potential].

At large m_A this theory is in the Higgs phase. Moreover, quantum effects do not destroy the Higgs phase because the coupling constant is small. Namely, σ develops a VEV,

$$\langle\sigma\rangle = m_{\ell_0},$$

while VEV's of n^ℓ are given by (55). In this phase both the U(1) gauge field and the scalar field σ become heavy and can be integrated out leading to the massive $CP(N-1)$ model with the potential (52).

At small m_A this theory is in the Coulomb phase. The VEV's of n^ℓ vanish, and the photon becomes massless.

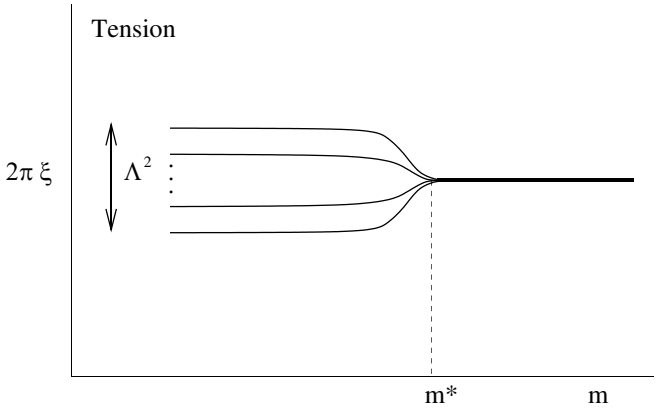

FIG. 7. Schematic dependence of string tensions on the mass parameter m. At small m in the non-Abelian confinement phase the tensions are split while in the Abelian confinement phase at large m they are degenerative.

Since the Coulomb potential in two dimensions is linear, the photon masslessness results in the confinement of kinks [24]. Thus, the phase transition which we identified above separates the Higgs and Coulomb phases of the two-dimensional U(1) gauge theory (58). The Higgs phase is characterized by a broken Z_N symmetry and degenerate vacua, while in the Coulomb phase the Z_N symmetry gets restored, and the vacua split. In four dimensions the former phase is an Abelian confinement phase with degenerate Abelian strings and 2D deconfinement of monopoles. The latter phase is a non-Abelian confinement phase with N split non-Abelian strings and non-Abelian 2D-confined monopoles forming mesonlike configurations on these strings. Note that the description of the $CP(N-1)$ theory on the string world sheet as a U(1) gauge theory (58) was used in [13] in a supersymmetric setting.

In particular, we expect that in the $N=2$ case the massive $CP(1)$ model is in the same universality class as the two-dimensional Ising model. Therefore, we conjecture that the phase transition from the Abelian confinement phase to the non-Abelian one is of the second order, and is described (at $N=2$) by conformal field theory with the central charge $c=1/2$, which corresponds to a free Majorana fermion.

To conclude this section we would like to stress that we encounter a crucial difference between the non-Abelian confinement in supersymmetric and nonsupersymmetric gauge theories. For BPS strings in supersymmetric theories we do not have a phase transition separating the phase of the non-Abelian strings from that of the Abelian strings [11,13]. Even for small values of the mass parameters supersymmetric theory strings are strictly degenerate, and the Z_N symmetry is spontaneously broken. In particular, at $m_A=0$ the order parameter for the broken Z_N, which differentiates N degenerate vacua of the supersymmetric $CP(N-1)$ model, is the bifermion condensate of two-dimensional fermions living on the string world sheet of the non-Abelian BPS string.

An example of the deconfinement phase transition at a critical mass is known [38] in four-dimensional softly broken $\mathcal{N}=2$ supersymmetric quantum chromodynamics (SQCD); in this model the order parameter is the Seiberg-Witten monopole condensate, and the collision of vacua happens in the parameter space, which is absent in our model. Note, that in some two-dimensional supersymmetric theories both Coulomb and Higgs branches are present and they have distinct R symmetries and different renormalization-group flows in the infrared domain [39]. Interpolation between two branches is a rather delicate issue since the transition region is described by a nontrivial geometry in the moduli space. A recent analysis of the supersymmetric case [40] shows that the two phases can even coexist on the world sheet and, moreover, integration over the form of the boundary is necessary to make the theory self-consistent.

We do *not* expect such a picture in the nonsupersymmetric case under consideration.

VIII. THE SU(2) × U(1) CASE

The $N=2$ case is of special importance, since the corresponding world-sheet theory, $CP(1)$, is exactly solvable. In this section we discuss special features of this theory in more detail. The Lagrangian on the string world sheet is

$$S^{(1+1)} = 2\beta \int dtdz\{(\partial_\alpha n^* \partial_\alpha n) + (n^* \partial_\alpha n)^2 + m^2[1 - (|n^1|^2 - |n^2|^2)^2]\} - \frac{\theta}{2\pi} \int dtdz \varepsilon_{\alpha\gamma}(\partial_\alpha n^* \partial_\gamma n), \tag{59}$$

where in the case at hand the mass parameter $m = m_1 = -m_2$, see (49). In this theory the mass term breaks $\mathrm{SU}(2)_{\mathrm{diag}}$ symmetry down to $\mathrm{U}(1) \times Z_2$ since the potential is invariant under the exchange $n^1 \leftrightarrow n^2$. It has two minima: the first one located at $n^1=1$, $n^2=0$, and the second minimum at $n^1=0$, $n^2=1$.

Now let us discuss the $m=0$ limit, i.e., non-Abelian strings, in more detail. Setting $N=2$ we arrive at a non-Abelian string with moduli forming a $CP(1)$ model on the world sheet. The very same string emerges in the supersymmetric model [14] which supports non-BPS string solutions. It is instructive to discuss how the pattern we have established for the $CP(N-1)$ string is implemented in this case.

Unlike $CP(N-1)$, the $CP(1)$ model has only one parameter, the dynamical scale Λ. The small expansion parameter $1/N$ is gone. Correspondingly, the kink-antikink interaction becomes strong, which invalidates quasiclassical-type analyses. On the other hand, this model was exactly solved [41]. The exact solution shows that the SU(2) doublets (i.e. kinks and antikinks) do not show up in the physical spectrum, and the only asymptotic states present in the spectrum are SU(2) triplets, i.e., bound states of kinks and antikinks. [Note that there are no bound states of the SU(2)-singlet type.] As was noted by Witten [24] passing from large N to $N=2$ does not change the picture qualitatively. In the quantitative sense it makes little sense now to speak of the kink linear confinement, since there is no suppression of the string breaking. The metastable vacuum entangled with the true vacuum in the θ evolution, is, in fact, grossly unstable. Attempting to create a long string, one just creates multiple kink-antikink pairs, as shown in Fig. 8. We end up with pieces of broken string of a typical length $\sim\Lambda^{-1}$.

There is a special interval of θ where long strings do exist, however, and, hence, we can apply the approach developed in the previous sections to obtain additional information. The $CP(1)$ model at $\theta=\pi$ turns out to be integrable [42,43], much in the same way as at $\theta=0$.

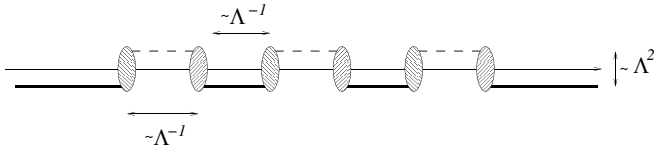

FIG. 8. Breaking of a would-be string through the kink-antikink pair creation in $CP(1)$. The thick solid lines show the energy density of the true vacuum, while the dashed ones indicate the energy density of the ''metastable'' vacuum.

From the exact solution [42,43] it is known that at $\theta = \pi$ there are no localized asymptotic states in the physical spectrum—the model becomes conformal. The exact solution confirms the presence of deconfined kinks (doublets) at $\theta = \pi$ and their masslessness. The S matrix for the scattering of these massless states has been found in [42,43].

We will focus on a small interval of θ in the vicinity of $\theta = \pi$. It is convenient to introduce a new small parameter

$$\varepsilon = |\pi - \theta|. \tag{60}$$

If $\varepsilon \ll 1$, our model again becomes two parametric. We will argue that in this regime the string tension in the $CP(1)$ model is

$$\Delta T_{CP(1)} \sim \Lambda^2 \varepsilon, \tag{61}$$

while the kink mass and the string size scale as

$$M_n \sim \Lambda \varepsilon^{1/2}, \qquad L \sim \Lambda^{-1} \varepsilon^{-1/2}. \tag{62}$$

The mass of the kink-antikink bound state also scales as

$$M \sim \Lambda \varepsilon^{1/2}, \tag{63}$$

so that at $\theta = \pi$ the string tension vanishes allowing the model to become conformal.

Let us elucidate the above statements starting from the string tension. In the $CP(1)$ model the vacuum family consists of two states: one true vacuum, and another—local—minimum, a companion of the true vacuum in the θ evolution. This fact can be confirmed by consideration of the supersymmetric $CP(1)$ model which has two degenerate vacua. Upon a soft SUSY breaking deformation, a small fermion mass term, the above vacua split: one minimum moves to a higher energy while another to a lower one. The roles of these nondegenerate minima interchange in the process of the θ evolution from zero to 2π; at $\theta = \pi$ they get degenerate.

Returning to the nonsupersymmetric $CP(1)$ model, it is not difficult to derive that in the vicinity of $\theta = \pi$ the vacuum energy densities $\mathcal{E}_{1,2}$ of the two vacua behave as

$$\mathcal{E}_{1,2} = \mathcal{E}_0 \pm \Lambda^2(\theta - \pi). \tag{64}$$

This formula proves that the difference of the vacuum energy densities (also known as the string tension) scales as indicated in Eq. (61).

Now the validity of Eq. (62) can be checked with ease. Indeed, the kink momentum (which is $\sim L^{-1}$) is of the order of its mass. Therefore, the kink and the antikink in the bound pair are right at the border of nonrelativistic and ultrarelativistic regimes. No matter which formula for their potential energy we use, we get $E_n \sim \Lambda \varepsilon^{1/2}$, so that the potential energy of the bound state is of the order of the kinetic energy of its constituents. The total mass of the bound state is then given by Eq. (63). This is in full agreement with the fact [43] that the conformal theory one arrives at in the limit $\theta = \pi$ has the Virasoro central charge $c = 1$. At $c = 1$ the spectrum of the scaling dimensions is given by $(1/4) \times (\text{integer})^2$.

It is easy to verify that any regime other than (62) is inconsistent. Here we note in passing that our result contradicts the analyses of Refs. [44,45]. In these papers a deformation of the exact $\theta = \pi$ solution of the $CP(1)$ model was considered, with the conclusion that $M \sim \Lambda \varepsilon^{2/3}$. This scaling regime is in contradiction with our analysis.

An alternative analysis of the $CP(1)$ model at generic θ can be carried out using the quasiclassical picture developed by Coleman a long time ago [46]. Namely, in the dual fermionic version of the model θ corresponds to the constant electric field created by two effective charges located at the ends of the strings. The value of the electric field experiences jumps at the kinks' positions, since the kinks are charged too. Generically the system is in a 2D Coulomb phase, with the vanishing photon mass. Coleman's analysis is qualitatively consistent with the description of the $CP(N-1)$ model as a Coulomb phase of the U(1) gauge theory (58) reviewed in Sec. VII and with the solution [24] of the $CP(N-1)$ models at large N.

IX. DUAL PICTURE

It is instructive to compare properties of the QCD strings summarized at the end of Sec. I with those emerging in the model under consideration. First of all, let us mention the most drastic distinction. In QCD, the string tension, excitation energies, and all other dimensionful parameters are proportional to the only scale of the theory, the dynamical scale parameter $\Lambda_{\rm QCD}$. In the model at hand we have two mass scales, $\sqrt{\xi}$ and Λ. To ensure full theoretical control we must assume that $\xi \gg \Lambda^2$.

The transverse size of the string under consideration is proportional to $1/\sqrt{\xi}$. Correspondingly, a large component in the string tension is proportional to ξ, see Eq. (42). It is only a fine structure of the string that is directly related to Λ, for instance, the splittings between the excited strings and the string ground state. The decay rates of the excited strings are exponentially suppressed, $\sim \exp(-\gamma N^2)$ in the QCD case and $\sim \exp(-\gamma N)$ in our model. The confined

monopoles in our model are in one-to-one correspondence with gluelumps of QCD (remember, in the model at hand we deal with the Meissner effect, while it is the *dual* Meissner effect that is operative in QCD).

N-strings in QCD can combine to produce a no-string state, while N non-Abelian strings in our model can combine to produce an ANO string, with no structure at the scale Λ. The only scale of the ANO string is ξ.

Moreover, confinement in our model should be thought of as dual to confinement in pure Yang-Mills theory with *no sources* because there are no monopoles attached to the string ends in our model. If we started from a $SU(N+1)$ gauge theory spontaneously broken to $SU(N) \times U(1)$ at a very high scale, then in that theory there would be extra very heavy monopoles that could be attached to the ends of our strings. However, in the $SU(N) \times U(1)$ model *per se* these very heavy monopoles become infinitely heavy. The $SU(N)$ monopoles we have considered in the previous sections are junctions of two elementary strings dual to gluelumps, rather than to the end-point sources.

Our model exhibits a phase transition in m between the Abelian and non-Abelian types of confinement. As is well known [5,6], the Abelian confinement leads to proliferation of hadronic states: the bound state multiplicities within the Abelian confinement are much higher than they ought to be in QCD-like theories.

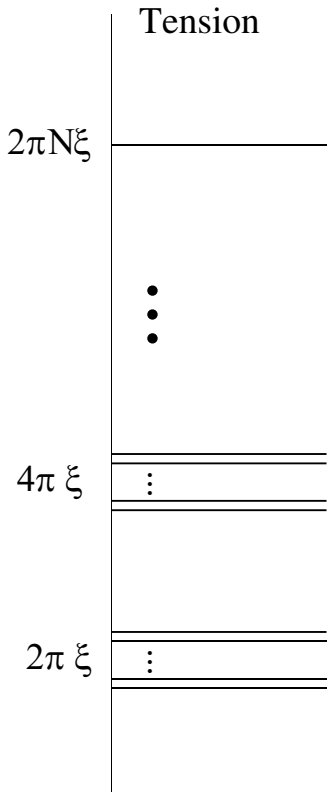

FIG. 9. The string spectrum in the non-Abelian confinement phase. The k-strings at each level are split.

In our model the Abelian confinement regime occurs at large m ($m \gg \Lambda$). In this region we have N degenerate Z_N strings with the tensions given in (14). If we extend our model to introduce superheavy monopoles (see above) as the end-point source objects, it is the N-fold degeneracy of the Z_N strings occurring in this phase that is responsible for an excessive multiplicity of the ''meson'' states.

Now, as we reduce m and eventually cross the phase transition point m_*, so that $m < m_*$, the strings under consideration become non-Abelian. The world-sheet $CP(N-1)$ model becomes strongly coupled, and the string tensions split according to Eq. (42). The splitting is determined by the $CP(N-1)$ model and is $\sim \Lambda^2/N$. Thus, (at $\theta = 0$) we have one lightest string—the ground state—as expected in QCD. Other $N-1$ excited strings become metastable. They are connected to the ground-state string through the monopole-antimonopole pairs. At large N their decay rates are $\sim \exp(-N)$.

Besides N elementary strings we also have k-strings which can be considered as a bound states of k elementary strings. Their tensions are given in Eq. (45). At each level k we have $N!/k!(N-k)!$ split strings. The number of strings at the level k and $N-k$ are the same. At the highest level $k = N$ we have only the ANO string. The string spectrum in our theory is shown in Fig. 9.

The dual of this phenomenon is the occurrence of the k-strings in QCD-like theories. If $k \gg 1$ we have a large number of metastable strings, with splittings suppressed by inverse powers of N, which are connected to the ground-state string through a gluelump.

At small N all metastable strings become unstable and practically unobservable.

X. CONCLUSIONS

Our main task in this work was developing a simple reference setup which supports non-Abelian strings and confined monopoles at weak coupling. We construct a simple nonsupersymmetric $SU(N) \times U(1)$ Yang-Mills theory which does the job. The advantages of the large-N limit (i.e. $1/N$ expansion) are heavily exploited. We discover, *en route*, a phase transition between Abelian and non-Abelian confinement regimes. We discuss in detail a dual picture where the confined monopoles turn into string-attached gluelumps; these gluelumps separate excited strings from the ground state. The non-Abelian strings we obtain in non-Abelian regime have many common features with QCD k-strings; however, they have significant distinctions as well. At the present level of understanding, this is as good as it gets on the road to quantitative theory of QCD strings.

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