New insight into the Witten-Dijkgraff-Verlinde-Verlinde equation

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We show that Witten-Dijkgraaf-Verlinde-Verlinde equation underlies the construction of $N = 4$ superconformal multi-particle mechanics in one dimension, including a $N = 4$ superconformal Calogero model.

I. INTRODUCTION

Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation was originally formulated in the context of $2d$ topological field theory [1,2]. Being a consequence of specific recursion relations for correlation functions which underlie a $2d$ topological model [1], the WDVV equation can be understood as providing a ring structure for local observables of a perturbed topological theory [2,3]. As within the topological framework the identity operator is physical, one coordinate in a parameter space of a perturbed topological model plays a distinguished role. In particular, some of the WDVV coefficients are identified with the two-point correlator on a sphere, and the latter is assumed to be a constant nondegenerate matrix (a metric on a physical Hilbert space) [2,3]. A very attractive link between the WDVV equation and differential geometry was established in Ref. [4] where it was demonstrated that locally any solution of the WDVV equation defines in the domain the structure of a Frobenius manifold and vice versa.

A generalization of the WDVV equation which treats all the coordinates on equal footing was proposed in Ref. [5] (see also Ref. [6]). The corresponding solution proved to be applicable to Seiberg-Witten theory where it was interpreted as the prepotential entering the low-energy effective action of $N = 2$ supersymmetric Yang-Mills theory [5].

The purpose of the present paper is to discuss one more physical application of the WDVV equation. As will be demonstrated below, in a slightly modified form it underlies the construction of $N = 4$ superconformal multi-particle models in one dimension, including a $N = 4$ superconformal Calogero model.

Notice also that there is an extensive literature on $N = 4$ supersymmetric mechanics in various dimensions. So far, the attention focused mainly on building appropriate Lagrangian and Hamiltonian models of supersymmetric mechanics and studying partial supersymmetry breaking. For approaches similar to ours see Refs. [7,8], which report on $N = 4$ supersymmetric mechanics in arbitrary D and also contain an extensive list of references on $N = 4$ super-

symmetric mechanics. In [9] $N = 4$, $D = 2$ supersymmetric mechanics described by chiral superfield actions was considered. Geometric models of $N = 4$ supersymmetric mechanics with $(2d|2d)_{\mathbb{C}}$ dimensional phase superspace have been proposed in [10]. These models reduce to the one considered in [9], in the simplest case of $d = 1$ and no central charges.

The interest in $N = 4$ superconformal multi-particle mechanics is due in part to a potential application in black hole physics. It was conjectured by Gibbons and Townsend [11] that the large n limit of a $N = 4$ superconformal Calogero model might be relevant for a microscopic description of the extreme Reissner-Nordström black hole, at least near the horizon. The conjecture originated from the fact that the near horizon geometry in this case has the isometry group $SU(1, 1|2)$ which includes the conformal group $SO(1, 2)$ as a subgroup. Taking into account that the extreme Reissner-Nordström black hole can be viewed as the configuration of four intersecting supergravity $D3$-branes wrapped on T^6 [12,13] and assuming that each of the supergravity $D3$-branes can be interpreted as a large number of coinciding microscopic $D3$-branes [11], one comes to the conjecture that there exists a $SU(1, 1|2)$-invariant mechanics governing the fluctuation of the branes in the region of intersection [11]. Being conformally invariant, the Calogero model seems to be a good candidate for the bosonic limit of such a $N = 4$ superconformal multiparticle mechanics.

Motivated by these issues, in the next section we construct a set of generators yielding a representation of $su(1, 1|2)$ superalgebra on a phase space which includes n copies of that corresponding to a conventional $N = 4$ superconformal particle [14]. These are designed to describe symmetries of a multiparticle mechanics in one dimension. As the Hamiltonian is part of the superalgebra, the dynamics is automatically taken into account. One reveals a central charge in the superalgebra which shows up in the Poisson brackets of the supersymmetry charges with the superconformal ones.

The explicit form of the generators involves two scalar functions V and F, which depend on position of particles only. The first of them specifies a potential of the problem, while the second function controls higher order fermionic

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contributions into the Hamiltonian. The structure relations of $N = 4$ superconformal algebra force V and F to obey a system of partial differential equations. One of them states that the vector $\partial^i V$ is covariantly constant with respect to the "connection" $\partial^i \partial^j \partial^k F \equiv W^{ijk}$. Then the WDVV equation, $W^{ijk} W^{kpl} = W^{ljk} W^{kpi}$, arises as the integrability condition. Notice that, similarly to the case of topological theories, the metric entering the equation is flat (the indices are contracted with the use of the ordinary Euclidean metric). However, there is no distinguished coordinate around and, as thus, no identification of some of the WDVV coefficients with the metric hold.

As we discuss in more detail below, the arising system of partial differential equations can be treated in two ways. One can take a particular solution of the WDVV equation (the prepotential F) and then construct a $N = 4$ superconformal multiparticle model associated to it (the prepotential V). Alternatively, one can start with a bosonic conformally invariant multiparticle mechanics and then look for a solution of the WDVV equation that will provide a $N = 4$ superconformal extension.

Motivated by the search for a $N = 4$ superconformal Calogero model, in this paper we follow the second line. In Sec. III we take V which reproduces the Calogero potential and then move on to fix F. As will be shown below, in general, the solution is not unique. We treat in detail the two-body problem and show that at the classical level there are two families of $N = 4$ superconformal models which are parameterized by the value of the central charge in the $su(1, 1|2)$ superalgebra. We present also partial results on the three-body problem. In particular, the nonlinear WDVV equations are reduced to a system of four linear partial differential equations of the Euler type. In Sec. IV we summarize our results and discuss possible further developments.

II. ALGEBRAIC STRUCTURE

First of all, let us fix the notation. Working in phase space, different particles are labeled by the index $i = 1, \ldots, n$ attached to a pair (x, p). These variables obey the conventional Poisson bracket

$$\{x^i, p^j\} = \delta^{ij}. \tag{1}$$

Although we are primarily concerned with the construction of $N = 4$ superconformal many-body models, it seems natural to require that the conventional $N = 4$ superconformal mechanics (see Ref. [14] for details) be reproduced in the one-particle case. In particular, this suggests the correct way to introduce fermions. One assigns a $SU(2)$ spinor ψ_α, $\alpha = 1, 2$ and its complex conjugate to each particle. Our convention for the conjugation reads

$$(\psi_\alpha^i)^+ = \bar{\psi}^{i\alpha}. \tag{2}$$

In what follows we use the bracket

$$\{\psi_\alpha^i, \bar{\psi}^{j\beta}\} = -i\delta_\alpha^\beta \delta^{ij} \tag{3}$$

and sum up over repeated indices.

A. R-symmetry generators

In order to construct a representation of $su(1, 1|2)$ superalgebra on the phase space described above, it proves to be instructive to start with R-symmetry generators. For the case at hand, they form $su(2)$ subalgebra. Given the latter and half of the supercharges, one can consistently fix the other half. The same applies to superconformal generators. Besides, having chosen the $su(2)$ currents, one could also specify the dependence of the odd generators on the fermionic coordinates.

In this paper we adhere to the simplest representation of $su(2)$ which can be constructed out of the fermionic variables alone

$$J_+ = -i\psi_1^i \psi_2^i, \qquad J_- = -i\bar{\psi}^{i1}\bar{\psi}^{i2},$$
$$J_3 = \frac{1}{2}(\psi_1^i \bar{\psi}^{i1} + \psi_2^i \bar{\psi}^{i2}). \tag{4}$$

These generators obey the standard algebra

$$\{J_+, J_-\} = -2iJ_3, \quad \{J_+, J_3\} = iJ_+, \quad \{J_-, J_3\} = -iJ_-. \tag{5}$$

Notice that J_3 measures the $U(1)$ charge of the fermionic coordinates

$$\{\psi_\alpha^i, J_3\} = \frac{i}{2}\psi_\alpha^i, \qquad \{\bar{\psi}^{i\alpha}, J_3\} = -\frac{i}{2}\bar{\psi}^{i\alpha}. \tag{6}$$

It is worth mentioning also that J_+ and J_- are complex conjugates of each other, while J_3 is a self-conjugate generator.

B. Supersymmetry charges

We now proceed to the construction of four supersymmetry charges G_1, G_2, $\bar{G}^1$, $\bar{G}^2$. Taking into account the brackets

$$\{G_1, J_+\} = 0, \qquad \{G_1, J_-\} = \bar{G}^2, \qquad \{G_1, J_3\} = \frac{i}{2}G_1,$$

$$\{G_2, J_+\} = 0, \qquad \{G_2, J_-\} = -\bar{G}^1, \qquad \{G_2, J_3\} = \frac{i}{2}G_2, \tag{7}$$

which make part of $N = 4$ superconformal algebra in one dimension, and the conjugation properties

$$(G_1)^+ = \bar{G}^1, \qquad (G_2)^+ = \bar{G}^2, \tag{8}$$

one concludes that it is instructive to start with an Ansatz for G_1. Then, the rest can be fixed with the use of Eq. (7), the $N = 4$ supersymmetry algebra

$$\{G_\alpha, \bar{G}^\beta\} = -2iH\delta_\alpha^\beta \tag{9}$$

and the conjugation relations. The function H appearing in the r.h.s. defines the Hamiltonian of a multiparticle system at which we aim.

As it was mentioned earlier, the one-particle case should reproduce the conventional $N = 4$ superconformal mechanics. Since the supersymmetry generators of the latter are cubic in the fermionic coordinates, for a generic multiparticle model it seems reasonable to terminate the expansion in fermions at that order. Then, a straightforward calculation yields ($\bar{G}^1$ and $\bar{G}^2$ follow by hermitian conjugation)

$$
\begin{aligned}
G_1 &= [p^i + i\partial^i V(x)]\psi_1^i + [p^i - i\partial^i V(x)]\psi_2^i \\
&\quad - iW^{ijk}(x)\psi_1^i\psi_2^j\bar{\psi}^{k1} + iW^{ijk}(x)\psi_1^i\psi_2^j\bar{\psi}^{k2}, \\
G_2 &= [p^i + i\partial^i V(x)]\psi_1^i - [p^i - i\partial^i V(x)]\psi_2^i \\
&\quad + iW^{ijk}(x)\psi_1^i\psi_2^j\bar{\psi}^{k1} + iW^{ijk}(x)\psi_1^i\psi_2^j\bar{\psi}^{k2},
\end{aligned} \tag{10}
$$

where the coefficient functions $V(x)$ and $W^{ijk}(x)$ obey the following restrictions:

$$
\partial^i\partial^j V - W^{ijk}\partial^k V = 0, \tag{11}
$$

$$
W^{ijk} = \partial^i\partial^j\partial^k F, \qquad W^{ijk}W^{kpl} = W^{ljk}W^{kpi}, \tag{12}
$$

where F is a scalar. Surprisingly enough, in the last line one recognizes the Witten-Dijkgraaf-Verlinde-Verlinde equation which underlies a $2d$ topological field theory [1,2].

In the context of the Calogero model considerations like the above ones were presented in Ref. [15], where a similar set of equations was derived. However, only partial solutions for V and F were examined which led the author to conclude that there was no a $SU(1,1|2)$-invariant extension of the Calogero model for generic values of coupling constant. Below we give a more rigorous procedure of solving these equations which relies upon general solutions for V and F.

According to Eq. (9) the Hamiltonian, which governs the dynamics of the model, has the form

$$
\begin{aligned}
H &= p^i p^i + \partial^i V(x)\partial^i V(x) - 2\partial^i\partial^j V(x)\psi_1^i\bar{\psi}^{j1} \\
&\quad + 2\partial^i\partial^j V(x)\psi_2^i\bar{\psi}^{j2} + 2\partial^i W^{jkl}(x)\psi_1^i\psi_2^j\bar{\psi}^{k1}\bar{\psi}^{l2}.
\end{aligned} \tag{13}
$$

Thus, given a solution of the WDVV Eq. (12), one can construct a $N = 4$ supersymmetric multiparticle mechanics associated to it. The bosonic part of the corresponding potential is of the form $\partial^i V\partial^i V$. It should be mentioned that Eqs. (11) and (12) hold also if one examines a smaller $N = 2$ subalgebra.

The inverse problem is also worthy to consider. One can start with a model governed by a potential $U(x^1, \ldots, x^n)$ and set

$$
U(x) = \partial^i V(x)\partial^i V(x), \tag{14}
$$

which will fix V. In general, the corresponding solution is not unique. Then, solving the system of partial differential Eqs. (11) and (12) for W^{ijk} will provide a $N = 4$ super-

symmetric extension of the model we started with. Below, we examine this possibility for the Calogero model.

Curiously enough, Eqs. (11) and (12) can be restated as the condition of the existence of a vector field $V^i(x)$ covariantly constant with respect to the connection $W^{ijk} = \partial^i\partial^j\partial^k F$

$$
\mathcal{D}^i V^j = \partial^i V^j - W^{ijk}V^k = 0. \tag{15}
$$

As W^{ijk} is completely symmetric, the previous line implies

$$
\partial^i V^j - \partial^j V^i = 0 \quad \longrightarrow \quad V^i = \partial^i V, \tag{16}
$$

for some scalar V, while the WDVV equation comes about as the integrability condition

$$
[\mathcal{D}^i, \mathcal{D}^j]V^k = 0. \tag{17}
$$

It would be interesting to find a geometrical interpretation behind these equations. Because the problem is formulated in a flat space, we anticipate that a geometric description, if any, is related to symplectic structures on the manifold (in this respect see Refs. [16,17]).

Concluding this section we notice that the $su(2)$ generators displayed in Eq. (4) commute with the Hamiltonian (13). Conservation of the supersymmetry charges follows from Eq. (9) and Jacobi identities.

C. Superconformal generators

Now let us see what restrictions on the form of the functions V and W^{ijk} follow if one wishes to make the model superconformal. Guided by the experience with the conformal mechanics [18], one takes the generators of dilatations and special conformal transformations in the form

$$
D = tH - \frac{1}{2}x^i p^i, \quad K = t^2 H - tx^i p^i + \frac{1}{2}x^i x^i. \tag{18}
$$

Being combined with the Hamiltonian they obey the conformal algebra in one dimension

$$
\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K, \tag{19}
$$

provided

$$
x^i\partial^i V = C_1, \qquad x^i W^{ijk} = C_2\delta^{jk}, \tag{20}
$$

where C_1 and C_2 are arbitrary real constants.

At this stage only superconformal generators, which in what follows we denote by S_α and $\bar{S}^\alpha$, are missing. Taking into account that the structure relations of $su(1,1|2)$ superalgebra involve the bracket

$$
\{K, G_\alpha\} = -S_\alpha, \tag{21}
$$

these can be consistently specified

$$S_1 = tG_1 - \frac{1}{2}x^i\psi_1^i - \frac{1}{2}x^i\psi_2^i,$$

$$S_2 = tG_2 - \frac{1}{2}x^i\psi_1^i + \frac{1}{2}x^i\psi_2^i. \tag{22}$$

The generators $\bar{S}^1$ and $\bar{S}^2$ follow by hermitian conjugation.

Finally, it remains to check that the entire algebra closes. This proves to be the case provided

$$C_2 = -1. \tag{23}$$

The last constraint follows immediately if one considers Poisson brackets of the superconformal generators with the supersymmetry ones. The explicit verification of the algebra turns out to be rather tedious. The use of Jacobi identities considerably simplifies the calculation. In the basis chosen one finds (vanishing brackets are omitted)

$$\{D, G_\alpha\} = -\frac{1}{2}G_\alpha, \quad \{D, S_\alpha\} = \frac{1}{2}S_\alpha, \quad \{H, S_\alpha\} = G_\alpha, \quad \{S_\alpha, \bar{S}^\beta\} = -2iK\delta_\alpha^\beta,$$

$$\{S_1, J_-\} = \bar{S}^2, \quad \{S_1, J_3\} = \frac{i}{2}S_1, \quad \{S_2, J_-\} = -\bar{S}^1, \quad \{S_2, J_3\} = \frac{i}{2}S_2,$$

$$\{G_1, S_2\} = -2iJ_+, \quad \{G_1, \bar{S}^1\} = -2iD + 2J_3, \quad \{G_1, \bar{S}^2\} = Z,$$

$$\{G_2, S_1\} = 2iJ_+, \quad \{G_2, \bar{S}^1\} = Z, \quad \{G_2, \bar{S}^2\} = -2iD + 2J_3, \tag{24}$$

plus their complex conjugates. Notice the appearance of the central charge

$$Z = -C_1 \tag{25}$$

with C_1 from Eq. (20), in the brackets involving the supersymmetry generators and the superconformal ones.

Let us recapitulate. On a phase space spanned by $2n$ real bosons (x^i, p^i) and $4n$ real fermions $(\psi_\alpha^i, \bar\psi^{i\alpha})$, $i = 1, \dots, n$, $\alpha = 1, 2$, one can build a representation of $N = 4$ superconformal algebra. The corresponding generators contain two scalar functions V and F which obey the system of partial differential equations

$$x^i\partial^iV = C_1, \qquad \partial^i\partial^jV - W^{ijk}\partial^kV = 0, \tag{26}$$

$$x^iW^{ijk} + \delta^{jk} = 0, \qquad W^{ijk}W^{kpl} = W^{ljk}W^{kpi}, \tag{27}$$

where $W^{ijk} = \partial^i\partial^j\partial^kF$ and C_1 is an arbitrary constant. The bosonic part of the Hamiltonian which governs the dynamics of the resulting N=4 superconformal many-body model reads

$$H_{\text{Bose}} = p^ip^i + \partial^iV\partial^iV. \tag{28}$$

It is worth mentioning that no contradiction results from contracting Eqs. (26) and (27) with x^i, ∂^iV, or W^{ijk}. This provides a naive compatibility check for the system.

III. $N = 4$ SUPERCONFORMAL CALOGERO MODEL

As it was mentioned earlier, there are two ways to treat the system (26) and (27). One can start with a particular solution of the WDVV equation and then construct a $N = 4$ superconformal multi-particle model associated to it. Alternatively, one can take a specific bosonic conformally invariant multi-particle mechanics and then search for a solution of the WDVV equation that will provide a $N = 4$ superconformal extension of the theory. For the rest of the paper we take the second path and consider the Calogero

model which is governed by the Hamiltonian [19]

$$H = p^ip^i + \sum_{i<j}\frac{g^2}{(x^i - x^j)^2}. \tag{29}$$

Here g is a dimensionless coupling constant. It was known for a long time that the Calogero model exhibits the conformal invariance [20,21]. Yet a consistent $N = 4$ superconformal extension has not been constructed. For earlier work on this subject, see Refs. [15,22–24].

A. One-particle case

For the one-particle case the system (26) and (27) reduces to ordinary differential equations which are easily integrated

$$V = \pm g\ln|x| + C_0, \qquad W = -\frac{1}{x},$$

$$F = -\frac{1}{2}x^2\ln|x|, \tag{30}$$

with C_0 a constant. Notice that by definition F is defined up to a polynomial quadratic in x. In what follows we disregard those terms as they do not contribute to W. The WDVV equation is trivial. The corresponding $N = 4$ superconformal mechanics coincides with that constructed in Ref. [14].

B. Two-body problem

Let us start with the linear inhomogeneous partial differential equation $x^i\partial^iV = C_1$, $i = 1, 2$. Making the substitution $V = C_1\ln|x^1| + \tilde{V}$ one reduces it to the homogeneous equation $x^i\partial^i\tilde{V} = 0$ which is equivalent to the system of ordinary differential equations $\frac{dx^1}{x^1} = \frac{dx^2}{x^2}$. The only integral of the latter, $\frac{x^1}{x^2} = \text{const}$, provides the general solution $\tilde{V} = \Lambda(\frac{x^1}{x^2})$ for the former, with Λ being an arbi-

trary function. As a result, for V one has

$$V = C_1 \ln|x^1| + \Lambda\left(\frac{x^1}{x^2}\right). \tag{31}$$

We next wonder what Λ corresponds to the Calogero potential. This information is encoded in the condition

$$(\partial^1 V)^2 + (\partial^2 V)^2 = \frac{g^2}{(x^1 - x^2)^2}, \tag{32}$$

which causes Λ to obey the ordinary differential equation

$$(1 + y^2)y^2\left(\frac{d\Lambda}{dy}\right)^2 + 2yC_1\frac{d\Lambda}{dy} + C_1^2 - \frac{g^2 y^2}{(y-1)^2} = 0, \tag{33}$$

where we denoted $\frac{x^1}{x^2} = y$. This is a quadratic algebraic equation with respect to the derivative which means that there are two distinct possibilities for Λ. The solutions explicitly depend on the constant C_1 and acquire the simplest form for the following choice:

$$\left(\frac{C_1}{g}\right)^2 = \frac{1}{2}. \tag{34}$$

For this particular value one finds

$$\Lambda_1 = C_1 \ln\left|1 - \frac{1}{y}\right| + \text{const}, \tag{35}$$

and

$$\Lambda_2 = C_1 \ln\left|\frac{y^2 + 1}{y(y-1)}\right| + \text{const}. \tag{36}$$

Thus, given the two-body Calogero model, the prepotential V generating a $N = 4$ superconformal extension is not unique. In fact, there are two families of models which are parameterized by the central charge C_1. For example, for the particular value of C_1 displayed in Eq. (34) above one gets either

$$V_1 = C_1 \ln|x^1 - x^2| + \text{const}, \tag{37}$$

which is of common use in the literature or

$$V_2 = C_1 \ln\left|\frac{(x^1)^2 + (x^2)^2}{x^1 - x^2}\right| + \text{const}, \tag{38}$$

which is an alternative. When constructing a $N = 4$ superconformal extension of the two-body problem both options are to be considered on equal footing. Worth mentioning also is that both solutions exhibit cyclic symmetry.

Having fixed V, let us discuss the equations which determine F. Taking into account that in the context of our problem F is defined up to a second order polynomial in the variable x, one can bring the first equation in (27) to the form

$$x^i\partial^i F - 2F + \frac{x^i x^i}{2} = 0, \tag{39}$$

which is an analog of the scaling law for the free energy of a perturbed topological theory (see Ref. [3] for more details). This is an inhomogeneous partial differential equation which is readily integrated by conventional means (see, e.g., Ref. [25]). The general solution reads

$$F = -\frac{1}{2}x^i x^i \ln|x^1| + \frac{1}{2}(x^1)^2\Delta\left(\frac{x^1}{x^2}\right), \tag{40}$$

where Δ is an arbitrary function of the ratio $\frac{x^1}{x^2}$.

With F at hand one can calculate the WDVV coefficients. These take the most readable form if one introduces into consideration a subsidiary function

$$\Sigma(y) = \frac{1}{2}y^3\frac{d^3\Delta}{dy^3} + 3y^2\frac{d^2\Delta}{dy^2} + 3y\frac{d\Delta}{dy}, \tag{41}$$

where as above we denoted $\frac{x^1}{x^2} = y$. Making use of the latter one derives the following expressions for W^{ijk}:

$$W^{111} = -\frac{1}{x^1}\left[1 + \frac{1}{y^2} - \Sigma(y)\right],$$

$$W^{112} = \frac{1}{x^1}\left[\frac{1}{y} - y\Sigma(y)\right],$$

$$W^{122} = -\frac{1}{x^1}[1 - y^2\Sigma(y)], \tag{42}$$

$$W^{222} = -\frac{1}{x^1}y^3\Sigma(y).$$

The convenience of such a representation becomes evident if one turns to analyze the WDVV equation. A simple calculation shows that the only nontrivial condition for the two-particle case

$$W^{11k}W^{k22} = W^{12k}W^{k12}, \tag{43}$$

is identically satisfied without imposing any restriction on the form of the function $\Delta(y)$.

Thus, it remains to solve the second equation in (26) which involves both V and F. No special effort is needed to demonstrate that three components forming the system are functionally dependent as there are two identities between them. This statement is easily verified with the use of the condition $x^i\partial^i V = C_1$. The only independent equation—the "22"-component-allows one to express Σ in terms of V algebraically

$$\Sigma = \frac{\partial^1 V + x^1\partial^2\partial^2 V}{y^2(\partial^1 V - y\partial^2 V)}. \tag{44}$$

Taking into account the explicit form of Σ from Eq. (41) one finally arrives at the inhomogeneous Euler equation which completely determines Δ and, as thus, F.

As we have seen above, given the value of C_1, there are two prepotentials V_1 and V_2 which reproduce the two–particle Calogero model in the bosonic limit. As an example, consider V_1 and V_2 displayed in Eqs. (37) and (38) above. For this particular choice of C_1 one readily gets

$$\Sigma_1(y) = \frac{1}{y^2(1-y^2)}, \qquad (45)$$

and

$$\Sigma_2(y) = \frac{1}{y^2(1-y^2)} + \frac{4y(1-y)}{(1+y)(1+y^2)^2}, \qquad (46)$$

respectively. In view of Eq. (42) the WDVV coefficients can be calculated by purely algebraic means. Thus, for the two-body problem actually there is no need to know the explicit form of the prepotential F.

C. Three-body problem

In the preceding section we have constructed a $N = 4$ superconformal extension for the two-particle Calogero model following the recipe proposed in Sec. II. However, that the extension exists might have been expected on general grounds. Indeed, employing the canonical transformation

$$\begin{aligned}
(x^1, p^1) &\longrightarrow \left[\frac{1}{\sqrt{2}}(x^1 - x^2), \frac{1}{\sqrt{2}}(p^1 - p^2)\right], \\
(x^2, p^2) &\longrightarrow \left[\frac{1}{\sqrt{2}}(x^1 + x^2), \frac{1}{\sqrt{2}}(p^1 + p^2)\right],
\end{aligned} \qquad (47)$$

one brings the Hamiltonian of the two–body Calogero model to the sum of a free particle (the center of mass) and the conventional conformal mechanics [18], both admitting a $N = 4$ superconformal extension. Thus, the most interesting situation occurs when the number of particles is greater than two. In this section we discuss the three-body problem.

For the case of three particles the system of ordinary differential equations corresponding to the homogeneous

equation $x^i \partial^i \tilde{V} = 0$ has two independent integrals

$$\frac{x^1}{x^2} = \text{const}, \qquad \frac{x^1}{x^3} = \text{const}'. \qquad (48)$$

As a result, the general solution of the first equation in (26) reads

$$V = C_1 \ln|x^1| + \Lambda(y, z), \qquad (49)$$

where Λ is an arbitrary function of two variables $y = \frac{x^1}{x^2}$ and $z = \frac{x^1}{x^3}$. In order to pick out Λ which corresponds to the Calogero model one has to impose the constraint

$$(\partial^1 V)^2 + (\partial^2 V)^2 + (\partial^3 V)^2 = \frac{g^2}{(x^1 - x^2)^2} + \frac{g^2}{(x^1 - x^3)^2} + \frac{g^2}{(x^2 - x^3)^2}, \qquad (50)$$

which implies a partial differential equation for Λ. Like the two-body case there are many solutions to the latter equation. The simplest one reads

$$V = \frac{C_1}{3}(\ln|x^1 - x^2| + \ln|x^1 - x^3| + \ln|x^2 - x^3|), \quad (51)$$

where $(C_1)^2 = (3g)^2/2$.

In much the same way the general solution of Eq. (39) is found to be of the form

$$F = -\frac{1}{2}x^i x^i \ln|x^1| + \frac{1}{2}(x^1)^2 \Delta(y, z), \qquad (52)$$

where $\Delta(y, z)$ is an arbitrary function. Triple differentiation of F then gives ten WDVV coefficients

$$\begin{aligned}
W^{111} &= -\frac{1}{x^1}\left(1 + \frac{1}{y^2} + \frac{1}{z^2} - \Sigma_1 - \Sigma_2 - 3\Sigma_3 - 3\Sigma_4\right), \\
W^{112} &= \frac{1}{x^1}\left(\frac{1}{y} - y\Sigma_1 - y\Sigma_3 - 2y\Sigma_4\right), \quad W^{113} = \frac{1}{x^1}\left(\frac{1}{z} - z\Sigma_2 - 2z\Sigma_3 - z\Sigma_4\right), \\
W^{122} &= -\frac{1}{x^1}(1 - y^2\Sigma_1 - y^2\Sigma_4), \quad W^{123} = \frac{1}{x^1}yz(\Sigma_3 + \Sigma_4), \\
W^{133} &= -\frac{1}{x^1}(1 - z^2\Sigma_2 - z^2\Sigma_3), \quad W^{222} = -\frac{1}{x^1}y^3\Sigma_1, \quad W^{223} = -\frac{1}{x^1}zy^2\Sigma_4, \\
W^{233} &= -\frac{1}{x^1}yz^2\Sigma_3, \quad W^{333} = -\frac{1}{x^1}z^3\Sigma_2,
\end{aligned} \qquad (53)$$

which are conveniently expressed in terms of four subsidiary functions

$$\Sigma_1 = \frac{1}{2}y^3\frac{\partial^3\Delta}{\partial y^3} + 3y^2\frac{\partial^2\Delta}{\partial y^2} + 3y\frac{\partial\Delta}{\partial y},$$

$$\Sigma_2 = \frac{1}{2}z^3\frac{\partial^3\Delta}{\partial z^3} + 3z^2\frac{\partial^2\Delta}{\partial z^2} + 3z\frac{\partial\Delta}{\partial z},$$

$$\Sigma_3 = \frac{1}{2}yz^2\frac{\partial^3\Delta}{\partial y\partial z^2} + yz\frac{\partial^2\Delta}{\partial y\partial z},$$

$$\Sigma_4 = \frac{1}{2}zy^2\frac{\partial^3\Delta}{\partial z\partial y^2} + yz\frac{\partial^2\Delta}{\partial y\partial z}. \qquad (54)$$

As we see from Eq. (53) the first six functions can be expressed in terms of the latter four. This observation simplifies an analysis of the WDVV equations considerably. Actually, it is a matter of straightforward calculation to verify that out of six WDVV equations which are present for the case under consideration

$$W^{11k}W^{k22} = W^{12k}W^{k12}, \quad W^{11k}W^{k23} = W^{12k}W^{k13},$$

$$W^{11k}W^{k33} = W^{13k}W^{k13}, \quad W^{12k}W^{k23} = W^{13k}W^{k22},$$

$$W^{12k}W^{k33} = W^{13k}W^{k23}, \quad W^{22k}W^{k33} = W^{23k}W^{k23}, \qquad (55)$$

only the last one is independent while the first five equations contain just the same information.

Thus, the only constraint on the model coming on the WDVV side reads

$$W^{22k}W^{k33} = W^{23k}W^{k23}, \qquad (56)$$

which can be viewed as an algebraic equation involving four subsidiary functions $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$.

As the last step we analyze the second equation in (26). Similar to the two-body problem one can demonstrate that among six relations available for this case only half, namely, the "22"–, "23"–, and "33"–components, are independent. These are simple algebraic equations which allow one to express Σ_1, Σ_2 and Σ_3 in terms of V and Σ_4. Substituting the resulting expressions in Eq. (56) one finally gets a quadratic algebraic equation which completely determines Σ_4 in terms of V.

Thus, similar to the two-body case, all the subsidiary functions are fixed by purely algebraic means and Eqs. (54) become a system of four linear partial differential equations for the function $\Delta(y, z)$. In order to integrate this system, one can proceed with the first equation which is of the Euler type. The corresponding solution will fix the dependence of Δ on y and will involve three arbitrary functions of z. Their explicit form will be determined from three equations remaining in (54).

When constructing an explicit solution, it should be remembered that there are nontrivial integrability conditions associated with the system (54). For example, from Eq. (54) one readily gets the following restrictions:

$$z\frac{\partial\Sigma_1}{\partial z} - y\frac{\partial\Sigma_4}{\partial y} - 2\Sigma_4 = 0,$$

$$y\frac{\partial\Sigma_2}{\partial y} - z\frac{\partial\Sigma_3}{\partial z} - 2\Sigma_3 = 0, \qquad (57)$$

$$y\frac{\partial\Sigma_3}{\partial y} - z\frac{\partial\Sigma_4}{\partial z} + \Sigma_3 - \Sigma_4 = 0.$$

As was explained above, all the subsidiary functions are expressed in terms of V. So, the integrability conditions (57) will be either identically satisfied, or they will give rise to a nontrivial additional constraint on the form of the prepotential V. A closed expression for such a constraint is still to be found.

Thus, we have demonstrated that for the three-body Calogero model the problem of constructing a $N = 4$ superconformal extension reduces to the integration of four linear partial differential Eqs. (54).

IV. CONCLUSION

To summarize, in this paper we have constructed a set of generators which comprise symmetries of a multi-particle mechanics in one dimension and form a $N = 4$ superconformal algebra. The WDVV equation was shown to play the central role in this framework. We treated in detail the two-body $N = 4$ superconformal Calogero model and presented some partial results on the three-body problem.

Since the connection between the WDVV equation and $N = 4$ supeconformal multiparticle mechanics seems quite intriguing, a few related problems deserve further investigation. First of all, it would be interesting to find a topological field theory interpretation of a generalization of the WDVV equation considered in this paper. Then, as mentioned above, the WDVV equation may be viewed as the integrability condition for the existence of a vector field covariantly constant with respect to the connection W^{ijk}. The geometry behind this equation remains to be clarified.

When considering the Calogero model in Sec. III, we found that the corresponding $N = 4$ superconformal extension is not unique. A similar observation for a $N = 2$ Calogero model was recently made in Ref. [26]. It would be interesting to investigate whether the different extensions are equivalent or related by a kind of a duality transformation. As we have seen above, the arbitrariness is controlled by the value of the central charge in the $su(1, 1|2)$ superalgebra. Then, an interesting question is whether the central charge is fixed upon quantization. It would be also interesting to construct multiparticle generalizations of the $N = 8$ supersymmetric mechanics models constructed recently [27].

Finally, $N = 4$ superconformal algebra examined in this work can be consistently reduced to $N = 2$ superconformal algebra. Then, one may wonder how this reduction is related to $N = 2$ superconformal Calogero model constructed by Freedman and Mende [28].

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