PHYSICAL REVIEW D **71,** 044021 (2005)

# Third post-Newtonian constrained canonical dynamics for binary point masses in harmonic coordinates

Raoul-Martin Memmesheimer* and Gerhard Schäfer†

*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany*

(Received 11 November 2004; published 22 February 2005)

The conservative dynamics of two point masses given in harmonic coordinates up to the third post-Newtonian order is treated within the framework of constrained canonical dynamics. A representation of the approximate Poincaré algebra is constructed with the aid of Dirac brackets. Uniqueness of the generators of the Poincaré group or the integrals of motion is achieved by imposing their action on the point mass coordinates to be identical with that of the usual infinitesimal Poincaré transformations. The second post-Coulombian approximation to the dynamics of two point charges as predicted by Feynman-Wheeler electrodynamics in Lorentz gauge is treated similarly.

DOI: 10.1103/PhysRevD.71.044021 PACS numbers: 04.25.Nx, 03.50.De, 04.20.Fy

## I. INTRODUCTION

High post-Newtonian accurate description of general relativistic dynamics of compact binaries in harmonic gauge has many applications in relativistic astrophysics, notably in connection with binary pulsars and future gravitational wave astronomy, see e.g. [1–3]. Inspiralling compact binaries are even the most promising sources to be detected by ground-based interferometers such as LIGO, VIRGO, and GEO600. The corresponding higher order post-Coulombian approximation offers a simpler analogue to post-Newtonian dynamics relevant also on its own [4,5].

The approximate analytical dynamics of compact binary systems in general relativity is most often treated in harmonic coordinates (see [6] and references therein). Quite recently the dynamics of binary point masses has been completed to the third post-Newtonian (3pN) order [7–13]. Hereby results derived by means of the canonical formalism of Arnowitt, Deser, and Misner have been confirmed [14–20]. In approaches based on the use of harmonic coordinates, the dynamics was first obtained under the form of ordinary second order 3pN equations of motion satisfied by the particle trajectories. The Lagrangian corresponding to the conservative part of the motion turns out to be of higher order in the time derivatives of the point mass coordinates [21]. This feature is shared by the Lagrangian of Feynman-Wheeler electrodynamics in Lorentz gauge derived by Kerner [22,23]. In both cases, Euler-Lagrange equations, of third order or higher, admit a wide class of solutions including physically irrelevant ones that do not reduce to the Newtonian solution in the limit where the speed of light $c$ tends to infinity. This can be seen directly from the number of independent initial data including higher than first order time derivatives of the position variable. When demanding the Newtonian limit, it becomes possible to rederive the ordinary second order equations of motion by an iterative order reduction procedure.

The higher order property of these Lagrangians actually arises from the fact that the dynamics in harmonic coordinates as well as in Lorentz gauge for electromagnetism are approximately Poincaré invariant. Indeed, the so-called no-interaction theorem by Martin and Sanz [24] states that Lagrangians of point particles derived by means of a slow-motion approximation from some classical field theory must contain higher order derivatives from second order level in the $1/c^2$ expansion, if approximate manifest Poincaré invariance is maintained. For arbitrary approximately Poincaré invariant point-particle dynamics, higher order derivatives must be contained in the Lagrangian only from the third order in powers of $1/c^2$ [25]. It can also be shown that, if exact Poincaré invariance of a system with finitely many degrees of freedom is required, (i) Lagrangians including interactions must depend on time derivatives of infinite order [26], (ii) for point-particle systems, the positions may not be chosen as canonical coordinates in Hamiltonian formalism (which reflects the time nonlocality due to retardation) [27–29].

For the approximate dynamics, appropriate contact transformations lead to an ordinary Lagrangian but in a nonharmonic grid [30] or in a non-Lorentzian gauge. The transformed representation of the dynamics can be described by means of an ordinary canonical formalism. Another approach consists in constructing the canonical formalism corresponding to the dynamics *directly* in the original frame.

The original higher order Lagrangian is of singular type because of the higher order derivatives occurring in "small" corrections, i.e., in terms of higher order in powers of $1/c^2$. The Hessian is thus multiplied with some positive power of $1/c^2$. Therefore, it is noninvertible on $\mathbb{R}[1/c^2]/(1/c^{2n+2})$, the ring of real polynomials in $1/c^2$ modulo $\mathcal{O}(1/c^{2n+2})$ we are working on at the $n$th order level of approximation. Indeed, in expressing the highest order derivative as a function of the others, the Euler-

*Electronic address: rmm@chaos.gwdg.de
†Electronic address: gos@tpi.uni-jena.de

 

Lagrange equations are multiplied with the inverse matrix of the Hessian, so that, in particular, the Newtonian part is multiplied with some power of $c^2$ and the Newtonian limit does not exist anymore. Independently from the $1/c^2$-power prefactor, what we shall call the ''matrix part'' of the Hessian may be singular by itself, leading to an additional singular structure.

The canonical formalism for singular Lagrangians goes back to Dirac [31,32], as well as Anderson and Bergmann [33]. The first canonical treatment of dynamics derived from a slow-motion approximation of a classical relativistic theory described by a singular Lagrangian of higher order (in the time derivatives) is due to Jaen, Llosa, and Molina [34]. Specializing on a class of approximate Lagrangians of a certain structure in the $1/c^2$-power expansion and demanding an invertible matrix part of the Hessian, they developed a method aimed at deriving an explicit expression for the Hamiltonian as well as the Dirac brackets. They applied their formalism to the 2pC dynamics in Lorentz gauge. However, the resulting Hamiltonian was not correct because of computational errors. The first correct 2pC Hamiltonian is due to Damour and Schäfer [35], their approach having been detailed in [30]. Later, Saito, Sugano, Ohta, and Kimura proposed a method how to treat general higher order singular Lagrangians in canonical formalism. They proved the equivalence of Lagrangian and Hamiltonian formulations for singular Lagrangians of higher order [36,37]. A similar analysis has later been performed by Gràcia, Pons, and Román-Roy in a geometrical framework [38]. The formalism given in [36] was applied to a class of 2pN Lagrangians to which the post-Newtonian Lagrangians in harmonic coordinates do not belong. It also was used by Ohta and Kimura for investigating aspects of 2pC Feynman-Wheeler dynamics in Lorentz gauge [39]. Note that the approaches of articles [34,36] are crucially different. The singularity arising from the fact that we are working on the ring $\mathbb{R}[1/c^2]/(1/c^{2n+2})$ is indeed not considered in Ref. [36].

The aim of this paper is to formulate the canonical formalism for the conservative part of 3pN dynamics in harmonic coordinates as well as for 2pC Feynman-Wheeler electrodynamics for two particles in Lorentz gauge and to analyze the dynamics in this framework. For the formulation, we use a similar method as the one developed in Ref. [34] generalized to 3pN conservative binary dynamics. For the first time we give the 3pN Hamiltonian in harmonic coordinates, the corresponding Dirac brackets, as well as a canonical representation of the Poincaré algebra of the 3pN and 2pC dynamics.

The canonical description is always helpful for a better understanding of the dynamics. It is an extremely elegant tool to derive features such as symmetries and integrals of motion. The latter quantities, computed in harmonic coordinates at the 3pN order and specialized to the center of mass frame [40], are useful for the description of inspiralling compact binaries relevant as sources of gravitational waves. They allow the derivation of an analytic parametric ''generalized quasi-Keplerian'' solution to the 3pN accurate conservative equations of motion for compact binaries moving in eccentric orbits [41]. This is relevant, in particular, to construct post-Newtonian search templates for the detection of gravitational waves or to compare the numerical and post-Newtonian descriptions of such systems. Our integrals of motion prove to be consistent with those computed by Andrade, Blanchet, and Faye [21] in Lagrangian formalism, providing a powerful cross check for the results.

Furthermore the inclusion of spin in post-Newtonian binary dynamics using covariant spin supplementary condition also results in a dynamics that is described by a higher order singular Lagrangian or Hamiltonian when staying in harmonic coordinates [42]. Especially, the investigation of its canonical description derived using the methods in the present article will likely be useful for the prediction of gravitational wave templates.

This article continues work initiated by Stachel and Havas who derived in 1976 the Hamiltonians describing a class of dynamics including the 1pN and 1pC ones and who computed the integrals of motion corresponding to the approximate Poincaré invariance [43]. They announced a further article, where special interactions allowing the choice of the spatial coordinates as canonical coordinates were to be treated up to second order, but this article was never published. We do not follow the program they had initially designed but rather concentrate on physically relevant interactions incompatible with the latter choice of canonical coordinates.

The plan of the paper is as follows: In Sec. II, we outline the general constrained Lagrangian formalism for Lagrangians containing higher order time derivatives. We also show how to derive a full-time stable set of Lagrangian constraints not only for Lagrangians having a similar structure as in [34], but also for cases where the matrix part of the Hessian is not invertible. In Sec. III, we outline the theory of the corresponding Hamiltonian formalism. Sec. IV is dedicated to a short description of the Poincaré algebra and its action on spatial coordinates. In Sec. V we apply the preceding results to the 3pN dynamics of two point masses, showing the explicit Poincaré invariance and deriving the corresponding integrals of motion. The 2pC Lagrangian of Feynman-Wheeler electrodynamics for two charged point masses is treated similarly in Sec. VI. Finally, in Sec. VII, we summarize and discuss our results.

## II. HIGHER ORDER SINGULAR LAGRANGIAN POINT MASS DYNAMICS

We start from the action integral of a higher order Lagrangian $L$ that does not depend explicitly on time. It simply reads [44,45]

$$S \equiv \int_{t_0}^{t_1} dt L(q, q^{(1)}, ..., q^{(n)}), \tag{1}$$

where $q$ is a short notation for the set of $f$ independent variables $\{q_\mu\}$, $\mu = 1, ..., f$, and where $q^{(i)}$ denotes the set $\{q_\mu^{(i)}\}$ of their $i$th derivatives with respect to time $t$. The highest order of time derivative appearing in $L$ is denoted by $n$. From the action principle $\delta S = 0$, we draw the generalized Euler-Lagrange equations,

$$\frac{\partial L}{\partial q_\mu} - \frac{d}{dt}\frac{\partial L}{\partial q_\mu^{(1)}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial q_\mu^{(2)}} - \ldots + (-1)^n \frac{d^n}{dt^n}\frac{\partial L}{\partial q_\mu^{(n)}} = 0, \tag{2}$$

with $\mu = 1, \ldots, f$, $d/dt$ denoting the total time derivative. By collecting the terms of the $\mu$th equation that do not depend on $q^{(2n)}$ into a single function $f_\mu$ and isolating the highest order derivatives $q^{(2n)}$, we may rewrite Eq. (2) as

$$\sum_{\nu=1}^{f} q_\nu^{(2n)} \frac{\partial^2 L}{\partial q_\mu^{(n)} \partial q_\nu^{(n)}} + f_\mu(q, ..., q^{(2n-1)}) = 0. \tag{3}$$

This relation shows that the highest order time derivatives always occur multiplied with the Hessian matrix

$$H_{\mu\nu} = \frac{\partial^2 L}{\partial q_\mu^{(n)} \partial q_\nu^{(n)}}, \tag{4}$$

so that the Euler-Lagrange equations can be solved for $q^{(2n)}$ as a function of the configuration space variables $q, ..., q^{(2n-1)}$ if and only if the Hessian is invertible.

We now want to specialize to Lagrangians derived within slow-motion approximation schemes of relativistic theories. These are for instance the Lagrangian describing the conservative part of post-Newtonian dynamics that has been determined up to third post-Newtonian order in Ref. [21], or the Lagrangian describing the $n$th post-Coulombian dynamics of two particles in the Feynman-Wheeler theory of electromagnetism [23]:

$$\begin{aligned} L(x, x^{(1)}, ..., x^{(n)}) = & -m_1 c^2 \left[1 - \frac{(x_1^{(1)})^2}{c^2}\right]^{1/2} \\ & - m_2 c^2 \left[1 - \frac{(x_2^{(1)})^2}{c^2}\right]^{1/2} \\ & - e_1 e_2 \sum_{k=0}^{n} \frac{1}{(2k)!} \frac{(-D_1 D_2)^k}{c^{2k}} \\ & \times \left[1 - \frac{(x_1^{(1)} x_2^{(1)})}{c^2}\right] r^{2k-1} + \mathcal{O}\left(\frac{1}{c^{2n+2}}\right), \end{aligned} \tag{5}$$

where the letter $x$ is used to refer to configuration space variables in order to emphasize the physical interpretation of $x_{ai}^{(j)}$ as the $j$th derivative of the $i$th component ($i = 1, 2, 3$) of the position vector of the particle $a = 1, 2$; where $x_a^{(j)}$ stands for the $j$th time derivative of the position vector of particle $a$, and $x^{(j)} = \{x_a^{(j)}\}$ for the set of all configuration space variables that are time derivatives of $j$th order ($x \equiv x^{(0)}$). Round brackets (. .) indicate the scalar product, e.g. $(x_1^{(1)} x_2^{(1)}) = \sum_{i=1}^{3} x_{1i}^{(1)} x_{2i}^{(1)}$; if both vectors are identical we denote, e.g. $(x_1^{(1)})^2$. We further introduced the operator $D_a$, which represents the time derivative acting exclusively on the configuration space variables belonging to the particle numbered $a$. $r$ is the absolute value of the relative separation vector, and $e_a$ denotes the charge of the particle $a$. Note that, since the square root may be expanded into a binomial series up to order $1/c^{2n+2}$, it makes sense to consider $N$-particle Lagrangians of the type [34]

$$L = \frac{1}{2} \sum_{a=1}^{N} m_a (x_a^{(1)})^2 + \sum_{s=0}^{n} \varepsilon^s V_s(x, ..., x^{(s)}) + \mathcal{O}(\varepsilon^{n+1}), \tag{6}$$

with $\varepsilon \equiv \frac{1}{c^2}$. Whereas for the 3pN dynamics there is actually no $x^{(3)}$ dependence, the formalism can be adapted to this case (see Sec. V). The Hessian of the above Lagrangian

$$H_{aibj} = \varepsilon^n \frac{\partial^2 V_n}{\partial x_{ai}^{(n)} \partial x_{bj}^{(n)}} \tag{7}$$

is not invertible on the ring $\mathbb{R}[\varepsilon]/(\varepsilon^{n+1})$ on which the approximation scheme is defined. Thus, the Lagrangian is singular and the system is subject to constraints which are now to be determined.

The primary Lagrangian constraints are given by all independent linear combinations of the Euler-Lagrange equations that do not contain $2n$th order time derivatives, imposing thereby conditions on the configuration space variables (cf., e.g., [46,47]). From Eq. (3) we see that they can be derived by contracting the Euler-Lagrange equations with some null vectors of the Hessian (7), we shall refer to as $\lambda_r$. Let us first suppose that the matrix part $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$ of $H_{aibj}$ is invertible [34], a restriction we will skip later. Then, in our approximation scheme, the null vectors of the Hessian are those that are multiples of $\varepsilon$. The contraction of the Euler-Lagrange equations with the canonical basis vectors of $\mathbb{R}^{3N}$ multiplied by $\varepsilon$ yields all the primary constraints. In the notation of Eq. (3) with the generalized coordinates being the spatial coordinates $x$ and their time derivatives, these are given by

$$\varepsilon f_{bj}(x, ..., x^{(2n-1)}) = \mathcal{O}(\varepsilon^{n+1}). \tag{8}$$

Since $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$ is invertible, there are no more independent ones. Requiring the special form (6) of the Lagrangian, the Euler-Lagrange equations read

$$-m_\alpha x_\alpha^{(2)} + \sum_{s=0}^{n} \varepsilon^s A_{\alpha s}(x, \ldots, x^{(2s)}) = \mathcal{O}(\varepsilon^{n+1}),$$
$$A_{\alpha s} = \sum_{r=0}^{s} \left(-\frac{d}{dt}\right)^r \frac{\partial V_s}{\partial x_\alpha^{(r)}}, \tag{9}$$

with $\alpha = ai$ and $m_a = m_\alpha$. Now, by means of Eq. (9), we rewrite the primary constraints (8) as

$$\varepsilon m_\alpha x_\alpha^{(2)} = \varepsilon \sum_{s=0}^{n-1} \varepsilon^s A_{\alpha s}(x, ..., x^{(2s)}) + \mathcal{O}(\varepsilon^{n+1}). \tag{10}$$

We observe that they coincide up to the factor $\varepsilon$ with the equations of motion of lower order in $\varepsilon$. Starting with the above equation, we can derive a minimal stable set of constraints as explained in Appendix A,

$$x_\alpha^{(2+r)} = \frac{1}{m_\alpha}\left[\sum_{s=0}^{n} \varepsilon^s B_{\alpha,2+r,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}), \tag{11}$$

for $r = 0, ..., 2n-3$. The precise definition of the functions $B_{\alpha,2+r,s}$ from the $A_{\alpha s}$'s is specified in the appendix. It is worth noticing that the constraints corresponding to the case $r = 0$ agree with the equations of motion after they have been iteratively reduced to order two in the time derivatives by removing higher order time derivatives with the help of the equations of motion of lower order in $\varepsilon$. Similarly, the additional constraints agree with the appropriately reduced time derivatives of the reduced equations of motion. This justifies the preceding statement saying that the constraints emerge by requiring the Newtonian limit.

The matrix part $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$ of the Hessian of post-Newtonian Lagrangians linear in the accelerations, is not invertible. This can be cured by adding so-called double zeros. While this may change the rank of the matrix part of the Hessian, it does not influence the order-reduced equations of motion [48,49]. Because of the agreement between the latter equations and the constraints (11), established for an invertible matrix $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$, we do not expect double zeros to change the constraints either. A closer investigation shows that this is indeed the case. We can state even more generally that the expression for the constraints and the construction of the Hamiltonian remains unchanged if the matrix part of the Hessian is noninvertible.

We thus suppose that $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$ is not invertible and has a constant rank $3N - R$ all over the configuration space. Then, in the considered approximation scheme, all multiples of $\varepsilon$ are still null vectors of the Hessian, but there are also $R$ additional null vectors say $\lambda_{\rho\alpha}$, $\rho = 1, ..., R$, of order zero in $\varepsilon$. The primary constraints are obtained by contracting any of them with the equations of motion. We shall first consider the constraints emerging from the contraction of the Euler-Lagrange equations with the canonical basis vectors of $\mathbb{R}^{3N}$ times $\varepsilon$. Since the regularity of $(\partial^2 V_n)/(\partial x_{ai}^{(n)} \partial x_{bj}^{(n)})$ is actually not used in the derivation of the minimal time stable set they belong to, this set is again given by Eq. (11). The primary constraints generated by the additional null vectors $\lambda_{\rho\alpha}$ read

$$\sum_\alpha \lambda_{\rho\alpha}(x, ..., x^{(n)})\left[-m_\alpha x_\alpha^{(2)} + \sum_{s=0}^{n} \varepsilon^s A_{\alpha s}(x, ..., x^{(2s)})\right] = \mathcal{O}(\varepsilon^{n+1}), \tag{12}$$

where $\rho = 1, ..., R$ and where $\sum_\alpha$ denotes the sum over all pairs $\alpha = ai$ with $a = 1, 2$ and $i = 1, 2, 3$; $\lambda_{\rho\alpha}$ may depend on $x, ..., x^{(n)}$ for the Hessian itself possibly depends on these variables. We must now examine the additional restrictions imposed by Eqs. (12) to the already derived constraint surface given by the set of relations (11). On this surface, by definition all higher order derivatives $x^{(2)}, ..., x^{(2n-1)}$ entering Eqs. (12) may be expressed by means of the coordinates $x, x^{(1)}$ with the help of Eqs. (11). We find thus

$$\sum_\alpha \lambda_{\rho\alpha}(x, ..., x^{(n)})\left[-m_\alpha x_\alpha^{(2)} + \sum_{s=0}^{n} \varepsilon^s A_{\alpha s}(x, ..., x^{(2s)})\right] + \mathcal{O}(\varepsilon^{n+1}) \approx \sum_\alpha \lambda_{\rho\alpha}(x, x^{(1)})\left[-m_\alpha x_\alpha^{(2)} + \sum_{s=0}^{n} \varepsilon^s B_{\alpha,2,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}) \underset{(11)}{\approx} \mathcal{O}(\varepsilon^{n+1}). \tag{13}$$

We emphasize the fact that these relations only hold on the constraint surface by using the weak equality symbol "$\approx$". The system of Eqs. (13) tells us that the constraints resulting from the additional null vectors of the Hessian are already fulfilled on the surface defined by the constraints (11). It is satisfied for all times due to the time-stability property, so that the seemingly additional constraints are covered by the set (11). In short, the additional null vectors do not generate additional constraints. This fact enables us to perform the transition to Hamiltonian formalism regardless of the invertibility of the matrix part of the Hessian. Moreover, the Lagrangian constraints of the considered dynamics are still identical with the reduced equations of motion or their reduced derivatives. In particular, double zero terms, though they may change the rank of the Hessian, do not influence the formalism as long as the general structure (6) is maintained. We observe that according to above computation, unlike the usual theory, there are no arbitrary functions of time emerging in the dynamics, even if the contraction of some of the additional null vectors with the Euler-Lagrange equations vanishes identically [46,47]. This is ultimately a consequence of the linear independence of the equations of motion at lowest order.

## III. HIGHER ORDER SINGULAR CANONICAL FORMALISM

A system described by a Lagrangian of higher order allows for a canonical description with phase-space variables $q^{(j)}$ and canonically conjugate momenta $\Pi_j$ with $j = 0, ..., n-1$ [50,51]; $\Pi_j$ is the set of $j$th so-called Ostrogradski momenta $\Pi_{j1}, ..., \Pi_{jf}$ defined by

$$\Pi_{j\mu} := \sum_{k=0}^{n-j-1} \left(-\frac{d}{dt}\right)^k \frac{\partial L}{\partial q_\mu^{(k+j+1)}}, \tag{14}$$

with $j = 0, ..., n-1$, $\mu = 1, ..., f$, $n$ being the highest order derivative entering the Lagrangian and $f$ the number of degrees of freedom. For $j = 0, ..., n-2$ we may write alternatively

$$\Pi_{j\mu} = \sum_{\nu=1}^{f} (-1)^{n-j-1} q_\nu^{(2n-j-1)} H_{\nu\mu} + K_{j\mu}(q, ..., q^{(2n-j-2)}), \tag{15}$$

showing that the highest order time derivatives occur multiplied with the Hessian. For $j = n-1$ we have

$$\Pi_{n-1\mu} = \frac{\partial L}{\partial q_\mu^{(n)}}. \tag{16}$$

It is of the form (15) with $K_{j\mu} = 0$ if $L$ is quadratic in the $q_\mu^{(n)}$. (This special case has not been accounted for in Ref. [34].) Let us first assume that the Hessian is regular. In this case, the Ostrogradski transformation can be inverted by using an iterative algorithm. The implicit Eq. (16) is locally solvable for the $q^{(n)}$ and yields $q^{(n)}(q, ..., q^{(n-1)}, \Pi_{n-1})$. Having computed the variables $q^{(n+i)}$ with $i < j$, we can invert the equation for the $n-j-1$th Ostrogradski momentum for $q^{(n+j)}(q, ..., q^{(n-1)}, \Pi_{n-j-1}, ..., \Pi_{n-1})$.

The Hamiltonian of the system is

$$H = -L[q, ..., q^{(n-1)}, q^{(n)}(q, ..., q^{(n-1)}, \Pi_{n-1})] + \sum_{j=0}^{n-2} \sum_{\mu=1}^{f} \Pi_{j\mu} q_\mu^{(j+1)} + \sum_{\mu=1}^{f} \Pi_{(n-1)\mu} q_\mu^{(n)}(q, ..., q^{(n-1)}, \Pi_{n-1}), \tag{17}$$

while the Hamiltonian equations of motion take the familiar form

$$\frac{d}{dt} q_\mu^{(k)} = \frac{\partial H}{\partial \Pi_{k\mu}}, \tag{18a}$$

$$\frac{d}{dt} \Pi_{k\mu} = -\frac{\partial H}{\partial q_\mu^{(k)}}, \tag{18b}$$

with $k = 0, ..., n-1$. Introducing the (Ostrogradski-) Poisson bracket

$$\{F, G\} \equiv \sum_{j=0}^{n-1} \sum_{\mu=1}^{f} \frac{\partial F}{\partial q_\mu^{(j)}} \frac{\partial G}{\partial \Pi_{j\mu}} - \frac{\partial G}{\partial q_\mu^{(j)}} \frac{\partial F}{\partial \Pi_{j\mu}}, \tag{19}$$

the time evolution equations for a smooth function $F$ of the phase-space variables and time $t$ takes the form

$$\frac{d}{dt} F = \{F, H\} + \frac{\partial F}{\partial t}. \tag{20}$$

Let us now turn to the case where the Hessian is singular. Then, the Ostrogradski transformation $(q, ..., q^{(2n-1)}) \longrightarrow (q, ..., q^{(n-1)}, \Pi_0, ..., \Pi_{n-1})$ is not invertible anymore, or equivalently, the phase-space variables considered as functions of $q, ..., q^{(2n-1)}$ are not all independent. If the Hessian has rank $f - r$ (for a maximum rank $f$), they are linked by $r$ independent relations. These are the primary constraints

$$\bar{\Psi}_a(q, ..., q^{(n-1)}, \Pi_0, ..., \Pi_{n-1}) = 0, \tag{21}$$

following from the definition (14) of the momenta. To make sure that the resulting constraint surface $\Gamma$ be a submanifold of phase space, we impose the "regularity condition", demanding that zero be a regular value of $\bar{\Psi}$ regarded as a map on phase space to $\mathbb{R}^r$.

The Hamiltonian $H$ as a function of the configuration space variables reads

$$H(q, ..., q^{(2n-1)}) = -L(q, ..., q^{(n)}) + \sum_{j=0}^{n-1} \sum_{\mu=0}^{f} \Pi_{j\mu}(q, ..., q^{(2n-j-1)}) q_\mu^{(j+1)}. \tag{22}$$

The remarkable fact is that the coordinates $q^{(n)}, ..., q^{(2n-1)}$ appear only through the combinations $\Pi_{j\mu}(q, ..., q^{(2n-j-1)})$ due to the particular form of the dependence of $L$ and $\Pi_{j\mu}$ on those coordinates. Hence, $H$ actually depends only on $q, ..., q^{(n-1)}$ and $\Pi_0, ..., \Pi_{n-1}$. This can be verified in a similar way as in the absence of higher order derivatives [52,53]. Therefore, we may view the Hamiltonian as a function of the phase-space variables $H = H(q, ..., q^{(n-1)}, \Pi_0, ..., \Pi_{n-1})$ although it is not unique in the case where the Hessian is not invertible but defined modulo a linear combination of the primary constraints $\sum_{a=1}^{r} c^a \bar{\Psi}_a$, with $c^a$ being functions of the phase-space coordinates [52,53]. The time evolution of a smooth function $F$ of the phase-space coordinates and time is given by

$$\frac{d}{dt}F = \{F, H\} + \sum_{a=1}^{r} u^a\{F, \bar{\Psi}_a\} + \frac{\partial F}{\partial t}, \tag{23}$$

where $u^a$, $a = 1, ..., r$ are extra parameters, and $\{.,.\}$ refers to the Poisson bracket (19). Additionally, at this level, $\bar{\Psi}_a = 0$ has to be imposed on the motion. The relation $d\bar{\Psi}_a/dt = 0$, $a = 1, ..., r$ may entail restrictions to the $u^a$'s and/or lead to new, secondary, constraints. Using again the time-stability property, we may get further conditions, and so on. At the end we are left with a complete set of, say, $K$, time-stable constraints including the primary ones

$$\Psi_k(q, ..., q^{(n-1)}, \Pi_0, ..., \Pi_{n-1}) = 0, \qquad k = 1, ..., K, \tag{24}$$

subject to the same regularity assumptions.

In constrained dynamics, there is an important distinction between two types of constraints: (i) First-class constraints are characterized by the property that their Poisson brackets (19) with all the other constraints vanish on the constraint surface, (ii) second-class constraints have at least one nonvanishing Poisson bracket on $\Gamma$. If the Poisson-bracket matrix $D$, defined in components as $D_{kl} = \{\Psi_k, \Psi_l\}$ for $k, l = 1, ..., K$, has rank $A$ on $\Gamma$, then the set of constraints (24) can always be linearly transformed into an equivalent set consisting of $K - A$ first-class constraints and $A$ second-class constraints with invertible Poisson-bracket matrix on $\Gamma$ [52]. Conversely, if no first-class constraints can be obtained from Eqs. (24) by linear transformations, the Poisson-bracket matrix $D$ is invertible on $\Gamma$. In this case, the Dirac bracket of two phase-space functions may be defined by

$$\{F, G\}^* = \{F, G\} - \sum_{k,l=1}^{K} \{F, \Psi_k\}D^{-1}_{kl}\{\Psi_l, G\}, \tag{25}$$

where $D^{-1}$ is the inverse matrix of $D$, existing at least in a neighborhood of $\Gamma$.

The Dirac bracket keeps important features of the Poisson bracket; namely, it is bilinear, antisymmetric, acts as a derivation on each argument, and fulfills the Jacobi identity. It actually is the restriction of the Poisson bracket on the constraint surface [54]. Finally, it has by construction two additional important properties

$$\{\Psi_k, F\}^* = 0, \tag{26}$$

$$\{G, F\}^* \approx \{G, F\}, \tag{27}$$

valid for arbitrary phase-space functions $F$ and for functions $G$ of first-class, i.e., functions whose Poisson bracket with any constraint vanishes on $\Gamma$; "$\approx$" represents the weak equality holding only on $\Gamma$. We are thus allowed to simplify expressions entering the Dirac bracket by using the constraint equations, which amounts to setting the $\Psi_k$ to zero, before the final computation of the bracket. With the help of such a tool we are in position to reformulate the time evolution equations equivalent to Eq. (23) on the constraint surface as

$$\frac{d}{dt}F = \{F, H_{\mathrm{r}}\}^* + \frac{\partial F}{\partial t}, \tag{28}$$

where $H_{\mathrm{r}}$ is the reduced Hamiltonian, derived from $H = -L + \sum_{j=0}^{n-1}\sum_{\mu=0}^{f}\Pi_{j\mu}q_\mu^{(j+1)}$ by eliminating all the other coordinates in favor of the coordinates of the constraint surface with the help of the system's constraints [34,47,52,53]. The preceding canonical formalism has been shown to be equivalent to the corresponding Lagrangian formalism in Ref. [36,38].

We now want to specialize on systems described by a Lagrangian of the form (6). From Eq. (14) we compute the Ostrogradski momenta,

$$\begin{aligned}\Pi_{j\alpha} &= m_\alpha x_\alpha^{(1)}\delta_{j0} + \varepsilon^{j+1}\Phi_{j\alpha}(x, \ldots, x^{(2n-j-1)}) \\ &\quad + \mathcal{O}(\varepsilon^{n+1}), \\ \Phi_{j\alpha} &= \sum_{s=0}^{n-j-1}\varepsilon^s\sum_{l=0}^{s}\left(-\frac{d}{dt}\right)^l\frac{\partial V_{s+j+1}}{\partial x_\alpha^{(l+j+1)}}.\end{aligned} \tag{29}$$

The Lagrangian constraints (11) are already known. Owing to the equivalence theorems proved in [36,38], they can be translated into constraints of the canonical formalism by keeping the first $n-2$ relations and eliminating $x^{(2)}, ..., x^{(2n-1)}$ from the $n$ identities (29) by means of Eqs. (11) (cf. Ref. [34], in which this method is applied whereas the equivalence had not yet been formally stated). We find

$$\omega_{r\alpha} \equiv x_\alpha^{(r)} - \frac{1}{m_\alpha}\sum_{s=0}^{n}\varepsilon^s B_{\alpha,r,s}(x, x^{(1)}) = \mathcal{O}(\varepsilon^{n+1}), \tag{30a}$$

$$\omega_{1\alpha} \equiv x_\alpha^{(1)} - \frac{1}{m_\alpha}[\Pi_{0\alpha} - \varepsilon\Phi_{0\alpha}(x, x^{(1)})] = \mathcal{O}(\varepsilon^{n+1}), \tag{30b}$$

$$\chi_{j\alpha} \equiv \Pi_{j\alpha} - \varepsilon^{j+1}\Phi_{j\alpha}(x, x^{(1)}) = \mathcal{O}(\varepsilon^{n+1}), \tag{30c}$$

with $r = 2, ..., n-1$ and $j = 1, ..., n-1$. The $\Phi_{j\alpha}(x, x^{(1)})$'s are derived from the $\Phi_{j\alpha}(x, ..., x^{(2n-j-1)})$'s by eliminating higher order derivatives with the help of Lagrangian constraints. As the constraints (30a)–(30c) obviously fulfill the regularity conditions, they define a constraint surface $\Gamma$ that is a submanifold of phase space. As coordinates, we may choose e.g. $x$ and $\Pi_0$, another possibility would be $x$ and $x^{(1)}$.

First-class Hamiltonian constraints occur only if there are configuration space variables that are arbitrary functions of time in Lagrangian formalism. We have already seen that these are not present in our case so that there are no first-class constraints in the dynamics under consideration. This result also can be proven by direct computation of the Poisson-bracket matrix (cf. Appendix C and [34]). It can be checked that

$$D = \begin{pmatrix} \{\chi_{k\alpha}, \chi_{r\beta}\} & \{\chi_{k\alpha}, \omega_{r\beta}\} \\ \{\omega_{k\alpha}, \chi_{r\beta}\} & \{\omega_{k\alpha}, \omega_{r\beta}\} \end{pmatrix} \tag{31}$$

is indeed invertible, which implies, in particular, that the number of constraints is even. Hence the constraint surface $\Gamma$ has an even dimension, say $g$. Now, by an iterative procedure, $D^{-1}$ may be expressed by means of submatrices of $D$ (cf. Appendix C and [34]). Adopting the splitting

$$D^{-1} = \begin{pmatrix} X & Y \\ -Y^T & Z \end{pmatrix}, \tag{32}$$

we may then write the Dirac bracket of two functions $f$ and $g$ on the phase space as

$$\begin{aligned} \{f, g\}^* = \{f, g\} &- \sum_{k,l=1}^{n-1} \sum_{\alpha,\beta} \{f, \chi_{k\alpha}\} X_{k\alpha,l\beta} \{\chi_{l\beta}, g\} \\ &- \sum_{k,l=1}^{n-1} \sum_{\alpha,\beta} \{f, \chi_{k\alpha}\} Y_{k\alpha,l\beta} \{\omega_{l\beta}, g\} \\ &+ \sum_{k,l=1}^{n-1} \sum_{\alpha,\beta} \{f, \omega_{k\alpha}\} Y_{l\beta,k\alpha} \{\chi_{l\beta}, g\} \\ &- \sum_{k,l=1}^{n-1} \sum_{\alpha,\beta} \{f, \omega_{k\alpha}\} Z_{k\alpha,l\beta} \{\omega_{l\beta}, g\}. \end{aligned} \tag{33}$$

This bracket defines a symplectic form on the constraint surface. According to the theorem of Darboux, we can locally find canonically conjugate coordinates $Q, P$ of $\Gamma$ such that it takes the familiar shape

$$\{F, G\}^* = \sum_{\alpha=1}^{g/2} \frac{\partial F}{\partial Q_\alpha} \frac{\partial G}{\partial P_\alpha} - \frac{\partial G}{\partial Q_\alpha} \frac{\partial F}{\partial P_\alpha}. \tag{34}$$

We shall see in Secs. V and VI that, for the dynamics investigated in this article, it will even be possible to exhibit global coordinates of this kind, which will greatly simplify computations involving Dirac brackets. The global existence of canonically conjugate coordinates entails that the group of generalized canonical transformations, i.e., the group of transformations that leave the Dirac bracket invariant, is necessarily a subgroup of the group of canonical transformations on the phase space [47].

## IV. POINCARÉ ALGEBRA

The symplectic structure on the phase space (or on the constraint surface) allows us to endow the vector space of scalar fields defined on $\Gamma$ with the structure of a Lie algebra. The Poisson-bracket (or Dirac-bracket) relations satisfied by the generators of the infinitesimal (generalized) canonical transformations corresponding to the action of a transformation group are known to be identical to the Lie-bracket relations of the generators of this group. In other words, there exists a Lie algebra homomorphism between the Lie algebra of the transformation group and that of the generators of the infinitesimal (generalized) canonical transformations (see, e.g., [55]).

Knowing the latter Lie algebra, we are enabled to reconstruct the symmetry group locally. This can be done globally only for simply connected groups (unlike the Poincaré group). As we aim at determining the symmetry group of some system in the phase space or on the constraint surface locally, it will be sufficient for us to consider the Lie algebra of the generators of infinitesimal (generalized) canonical transformations (see, e.g., [56]).

By construction, the conservative pN dynamics in harmonic coordinates as well as the pC dynamics of the Feynman-Wheeler theory in Lorentz gauge are approximately Poincaré invariant. Therefore, in a neighborhood of the identity there exists an approximate representation of the Poincaré group as a generalized canonical transformation group on $\Gamma$. This means that on the constraint surface, there are generators $H, P_i, J_i, G_i$ with $i = 1, 2, 3$ of infinitesimal generalized canonical transformations that approximately fulfill the Poincaré algebra with respect to Dirac bracket: We have thus

$$\{P_i, P_j\}^* = \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35a}$$

$$\{P_i, J_j\}^* = \sum_{k=1}^{3} \epsilon_{ijk} P_k + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35b}$$

$$\{J_i, J_j\}^* = \sum_{k=1}^{3} \epsilon_{ijk} J_k + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35c}$$

$$\{H, P_j\}^* = \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35d}$$

$$\{H, J_j\}^* = \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35e}$$

$$\{G_i, G_j\}^* = -\frac{1}{c^2} \sum_{k=1}^{3} \epsilon_{ijk} J_k + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35f}$$

$$\{G_i, H\}^* = P_i + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35g}$$

$$\{J_i, G_j\}^* = \sum_{k=1}^{3} \epsilon_{ijk} G_k + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35h}$$

$$\{G_i, P_j\}^* = \frac{1}{c^2} \delta_{ij} H + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{35i}$$

where $i, j = 1, 2, 3$ are spatial indices and where $n$ is the order of approximation ($n = 3$ for the 3pN, $n = 2$ for the 2pC dynamics). In the nonrelativistic limit $c \to \infty$, the algebra (35a)–(35i) reduces to the Galilean one. In particular Eq. (35i) becomes

$$\{G_i, P_j\}^* = \delta_{ij} M, \tag{36}$$

$M$ being the total nonrelativistic mass of the system. Since all quantities are defined on the ring $\mathbb{R}[1/c^2]/(1/c^{2(n+1)})$, strictly speaking, the term $Mc^2$ appearing in the generator $H$ is not allowed. However, because it is a mere constant, it does not change the dynamics of the system. We may keep it for it suggests the physical interpretation of $H$ as the total conserved energy. To harmonize the argument, we shall work with a slightly modified form of Eq. (35i), namely,

$$\{G_i, P_j\}^* = \delta_{ij}\left(M + \frac{1}{c^2} H\right) + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right). \tag{35j}$$

The statement that a dynamics is Poincaré invariant usually signifies more than the existence of generators satisfying the Poincaré algebra. Namely, that the generators of the infinitesimal canonical invariance transformations $H, P_i, J_i, G_i$ are the generators of the linear representation of infinitesimal time translations, spatial translations, spatial rotations, and Lorentz boosts acting on the particle components. In short, the members of the Poincaré group are to have the usual action on the physical system, when the phase-space coordinates $x(t)$ are interpreted as the positions of the particles in Cartesian coordinates parameterized by the time $t$. This is often emphasized by referring to this type of dynamics as ''manifestly'' Poincaré invariant.

In this perspective, we shall investigate one by one the actions of the preceding generators on the phase-space variables, acting on the coordinates $x(t)$ regarded as the spatial coordinates of the system of $N$ point particles at time $t$. The time evolution of $x^{(j)}(t), \Pi_j(t), j = 0, \ldots, n-1$ is given by the trajectories on the constraint surface; the argument $t$ will be dropped below for the sake of simplicity. Throughout this article we shall adopt the point of view of active transformations.

We start with the generator of infinitesimal time translations $H$. By definition it generates a transformation that can be interpreted, when regarded as active, as a translation of the particles along their trajectory. Now, the effect of an infinitesimal time translation about $\tau$ is

$$\tau \frac{d}{dt} x_{ai}^{(k)} = \delta_\tau x_{ai}^{(k)} = \tau\{x_{ai}^{(k)}, H\}^* + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{37a}$$

$$\tau \frac{d}{dt} \Pi_{kai} = \delta_\tau \Pi_{kai} = \tau\{\Pi_{kai}, H\}^* + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right). \tag{37b}$$

Note the use of Dirac brackets due to the fact that the trajectories are restricted to $\Gamma$. From the equations of motion (28), we see that $H$ may be chosen as the Hamiltonian or the reduced Hamiltonian of the system as has been anticipated by the notation. Since $H$ can be interpreted as the total energy of the system, the identity $\{H, H\}^* = 0$ traduces the conservation of energy in time.

The generators $P_i$ and $J_i$, interpreted as linear and angular momentum, are the generators of infinitesimal spatial translations and rotations. Even in absence of information on the dynamics, we know their Poisson bracket with arbitrary scalar fields defined on the phase space. We know furthermore that generalized velocities are contravariant while generalized momenta are covariant vectors. Therefore, under an infinitesimal translation about $\epsilon_i$ generated by $P_i$, and an infinitesimal rotation about $\varphi_i$, generated by $J_i$, $i = 1, 2, 3$, the phase-space coordinates representing the particle components transform as

$$\epsilon_i = \delta_\epsilon x_{ai} = \sum_{j=1}^{3} \epsilon_j \{x_{ai}, P_j\},$$

$$0 = \delta_\epsilon x_{ai}^{(l)} = \sum_{j=1}^{3} \epsilon_j \{x_{ai}^{(l)}, P_j\},$$

$$0 = \delta_\epsilon \Pi_{mai} = \sum_{j=1}^{3} \epsilon_j \{\Pi_{mai}, P_j\},$$

$$\sum_{j,k=1}^{3} \epsilon_{ijk} \varphi_j x_{ak}^{(m)} = \delta_\varphi x_{ai}^{(m)} = \sum_{j=1}^{3} \varphi_j \{x_{ai}^{(m)}, J_j\},$$

$$\sum_{j,k=1}^{3} \epsilon_{ijk} \varphi_j \Pi_{mak} = \delta_\varphi \Pi_{mai} = \sum_{j=1}^{3} \varphi_j \{\Pi_{mai}, J_j\},$$

with $l = 1, ..., n-1$ and $m = 0, ..., n-1$. They lead to the differential equations

$$\frac{\partial P_i}{\partial \Pi_{mak}} = \delta_{ik}\delta_{0m}, \qquad \frac{\partial P_i}{\partial x_{ak}^{(m)}} = 0, \tag{38a}$$

$$\frac{\partial J_i}{\partial \Pi_{mak}} = \sum_{j=1}^{3} \epsilon_{ijk} x_{aj}^{(m)}, \qquad \frac{\partial J_i}{\partial x_{aj}^{(m)}} = \sum_{k=1}^{3} \epsilon_{ijk} \Pi_{mak}, \tag{38b}$$

which fix the momenta $P_i$ and $J_i$ up to a constant. The Poisson-bracket relations of the these generators are well known; we have for instance

$$\{P_i, J_j\} = \sum_{k=1}^{3} \epsilon_{ijk} P_k, \tag{39a}$$

$$\{J_i, J_j\} = \sum_{k=1}^{3} \epsilon_{ijk} J_k. \tag{39b}$$

From Eqs. (38a), (38b), (39a), and (39b) we derive the unique result

$$P_i = \sum_{a=1}^{N} \Pi_{0ai}, \tag{40}$$

$$J_i = \sum_{m=0}^{n-1}\sum_{a=1}^{N}\sum_{j,k=1}^{3}\epsilon_{ijk}x_{aj}^{(m)}\Pi_{mak}. \tag{41}$$

If a given dynamics is invariant under spatial translations and rotations, so must be the constraint equations and the constraint surface. In this case, the generators of spatial translations and rotations are thus first-class functions. From Eq. (27), valid on the constraint surface, we see then that all Poisson-bracket relations including $P_i$ and $J_i$ also hold on $\Gamma$ as Dirac-bracket relations. This means that the momenta $P_i$ and $J_i$ displayed above are the generators of infinitesimal spatial translations and rotations represented as generalized canonical transformations on $\Gamma$.

The physical interpretation of $G_i$ as generator of infinitesimal Lorentz boosts allows us to determine its action on the particle coordinates $x_{ai}$ [43]. An infinitesimal boost about $v_j$, $j = 1, 2, 3$ acts on the space-time coordinates of a particle following

$$\delta t_a = -\frac{1}{c^2}\sum_{j=1}^{3} v_j x_{aj}, \tag{42a}$$

$$\delta x_{ai} = -v_i t. \tag{42b}$$

A particle, located at $x_{ai}$ at time $t$, is located after the active transformation at position $x'_{ai} = x_{ai} + \delta x_{ai}$ at time $t'_a = t + \delta t_a$. In the three-dimensional space, this results in

$$x'_{ai}(t'_a) = x_{ai}(t) + \delta x_{ai} = x_{ai}(t) - v_i t. \tag{43}$$

Let us stress that, because we interpret the transformation as active, we keep the same space-time coordinate system and simply boost the particles. We choose the time coordinate $t$ to be eliminated in favor of $t'_a$ up to first order in $v_i$. (However, since we are dealing with functional identities valid for all the times, we could also proceed by substituting for $t$.) Applying $t = t'_a + \frac{1}{c^2}\sum_{j=1}^{3} v_j x_{aj}(t)$ and expanding Eq. (43) up to the linear order in $v_i$, we arrive at the relation

$$x'_{ai}(t'_a) = x_{ai}(t'_a) + \frac{1}{c^2}\sum_{j=1}^{3} v_j x_{aj}(t'_a)\dot{x}_{ai}(t'_a) - v_i t'_a + \mathcal{O}(v^2). \tag{44}$$

We have now expressed both sides of Eq. (43) through the coordinate time $t'_a$. Since Eq. (44) is valid at any time, $t'_a$ is a mere ‘‘dummy’’ variable; it may be denoted by $t$ again or even be dropped as a function argument. This yields the following expression for the action of the generator $G_i$ on the spatial coordinates $x$ valid for any boost vector $v_i$

$$\sum_{j=1}^{3} v_j\left(\frac{1}{c^2}x_{aj}\dot{x}_{ai} - \delta_{ij}t\right) = \delta_v x_{ai} = \sum_{j=1}^{3} v_j\{x_{ai}, G_j\}^* + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right), \tag{45}$$

with $n$ being again the order of approximation. Inserting the equations of motion (28) into Eq. (45), we deduce the *world line condition*

$$\{x_{ai}, G_j\}^* = \frac{1}{c^2}x_{aj}\{x_{ai}, H\}^* - \delta_{ij}t + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right). \tag{46}$$

This condition joined to the Poincaré algebra, to the expression of the action of $P_i$, $J_i$ on the phase-space variables, and to the physical requirement that $H$ is the Hamiltonian of the system will be sufficient to determine $G_i$ uniquely (up to arbitrary generalized canonical transformations).

In the following, it will be useful to perform computations with the quantity $K_i = G_i + P_i t$ instead of $G_i$. Because of Eqs. (35a), (35b), (35d), and (35j) $K_i$ fulfills the same Dirac-bracket relations as $G_i$ but is not explicitly time dependent; it is not an integral of motion either. Indeed, we conclude from Eq. (35g) combined with the conservation law $\frac{dG_i}{dt} = 0$ that $\frac{\partial G_i}{\partial t} = -P_i$. Applying the definition of $K_i$ we find $\frac{\partial K_i}{\partial t} = 0$ and $\{K_i, H\}^* = P_i$. The world line condition for $K_j$ reads,

$$\{x_{ai}, K_j\}^* = \frac{1}{c^2}x_{aj}\{x_{ai}, H\}^* + \mathcal{O}\left(\frac{1}{c^{2(n+1)}}\right). \tag{47}$$

It can be derived from Eq. (46) and from the first of the two Eqs. (38a), which, due to the first-class property of $P_i$ and relation (27), also holds as Dirac-bracket relation on the constraint surface as mentioned above.

## V. APPLICATION TO 3PN DYNAMICS

The conservative part of 3pN equations of motion for compact binaries in harmonic coordinates allows for a Lagrangian of the form [21]

$$L(x, x^{(1)}, x^{(2)}) = \frac{1}{2}\sum_{a=1}^{2} m_a(x_a^{(1)})^2 + V_0(x) + \frac{1}{c^2}V_1(x, x^{(1)}) + \frac{1}{c^4}V_2(x, x^{(1)}, x^{(2)}) + \frac{1}{c^6}V_3(x, x^{(1)}, x^{(2)}) + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{48}$$

with $V_0(x) = G\frac{m_1 m_2}{r}$. Beyond the fact it is restricted to a two-body system, it does not show any dependence on $x^{(3)}$ in the term $V_3$ by contrast to the general Lagrangian (6). The investigation of the dynamics can be performed in two different ways: First, since the original formalism as displayed in Secs. II and III does not require the invertibility of the Hessian, it may be applied on an artificially extended configuration space putting up however with lengthier computations. Second, we may consider $V_3(x, x^{(1)}, x^{(2)})$ as a higher order correction to $V_2(x, x^{(1)}, x^{(2)})$, gathering both functions into a new quantity,

$$\bar{V}_2(x, x^{(1)}, x^{(2)}) = V_2(x, x^{(1)}, x^{(2)}) + \frac{1}{c^2} V_3(x, x^{(1)}, x^{(2)}). \tag{49}$$

In this approach we write the 3pN Lagrangian (substituting $\varepsilon \equiv \frac{1}{c^2}$) as

$$L_{3\mathrm{pN}} = \frac{1}{2}\sum_{a=1}^{2} m_a (x_a^{(1)})^2 + V_0(x) + \varepsilon V_1(x, x^{(1)}) + \varepsilon^2 \bar{V}_2(x, x^{(1)}, x^{(2)}) + \mathcal{O}(\varepsilon^4), \tag{50}$$

without affecting the original accuracy. The formalism described in Secs. II and III has to be appropriately modified.

It has been checked explicitly that both methods lead to equivalent constraints, to identical expressions for the elementary Dirac brackets, and to identical conserved quantities on the constraint surface. However, since the second method is more adapted to the problem, it demands less computational effort. Furthermore it can be employed for the computations of 2pC dynamics when the terms of third order are neglected. We shall therefore use it for the subsequent calculations.

The equations of motion for the Lagrangian (50) read

$$\begin{aligned} &-m_\alpha x_\alpha^{(2)} + \sum_{s=0}^{2} \varepsilon^s A_{\alpha s}(x, \ldots, x^{(2s)}) = \mathcal{O}(\varepsilon^4), \\ &A_{\alpha 0} = \frac{\partial V_0}{\partial x_\alpha}, \qquad A_{\alpha 1} = \frac{\partial V_1}{\partial x_\alpha} - \frac{d}{dt}\frac{\partial V_1}{\partial x_\alpha^1}, \\ &A_{\alpha 2} = \frac{\partial \bar{V}_2}{\partial x_\alpha} - \frac{d}{dt}\frac{\partial \bar{V}_2}{\partial x_\alpha^{(1)}} + \frac{d^2}{dt^2}\frac{\partial \bar{V}_2}{\partial x_\alpha^{(2)}}. \end{aligned} \tag{51}$$

The computation of the constraints for a system described by the above Lagrangian is performed in Appendix B. The main difference to the formalism exposed in Sec. II is that the equations of motion have to be employed twice at the end to guarantee time stability. Finally we obtain a minimal set of time-stable Lagrangian constraints (B7)

$$m_\alpha x_\alpha^{(r)} - \sum_{s=0}^{2} \varepsilon^s B_{\alpha,r,s}(x, x^{(1)}) = \mathcal{O}(\varepsilon^4), \tag{52}$$

for $\alpha = ai$ and $r = 2, 3$. The Ostrogradski momenta derived from (29) are given by

$$\Pi_{0\alpha} = m_\alpha x_\alpha^{(1)} + \varepsilon \Phi_{0\alpha}(x, ..., x^{(3)}) + \mathcal{O}(\varepsilon^4), \tag{53a}$$

$$\Pi_{1\alpha} = \varepsilon^2 \Phi_{1\alpha}(x, x^{(1)}, x^{(2)}) + \mathcal{O}(\varepsilon^4), \tag{53b}$$

where the $\Phi_{0\alpha}$, $\Phi_{1\alpha}$ are now constructed with $V_1$ and $\bar{V}_2$:

$$\Phi_{0\alpha} = \frac{\partial V_1}{\partial x_\alpha^{(1)}} + \varepsilon \frac{\partial \bar{V}_2}{\partial x_\alpha^{(1)}} - \varepsilon \frac{d}{dt}\frac{\partial \bar{V}_2}{\partial x_\alpha^{(2)}} \tag{54a}$$

$$\Phi_{1\alpha} = \frac{\partial \bar{V}_2}{\partial x_\alpha^{(2)}}. \tag{54b}$$

The transformation of the Lagrangian constraints into constraints on the phase-space coordinates yields the Hamiltonian constraints

$$\chi_{1\alpha} \equiv \Pi_{1\alpha} - \varepsilon^2 \Phi_{1\alpha}(x, x^{(1)}) = \mathcal{O}(\varepsilon^4), \tag{55a}$$

$$\omega_{1\alpha} \equiv x_\alpha^{(1)} - \frac{1}{m_\alpha}[\Pi_{0\alpha} - \varepsilon \Phi_{0\alpha}(x, x^{(1)})] = \mathcal{O}(\varepsilon^4). \tag{55b}$$

The $\Phi_{j\alpha}(x, x^{(1)})$'s are derived from the $\Phi_{j\alpha}(x, ..., x^{(3-j)})$'s by eliminating higher order derivatives by means of Eqs. (52). The Poisson-bracket matrix $D$ of the constraints (55a) and (55b) is regular, as shown in Appendix C by an explicit computation of $D^{-1}$. It is thus possible to endow phase space with the Dirac brackets (25). Their explicit expression can be found at the end of Appendix C.

We are now in position to give the representation of the generators of the Poincaré group as generators of infinitesimal generalized canonical transformations with respect to the Dirac bracket for the conservative 3pN binary dynamics in harmonic coordinates. The phase-space variables $x_1$ and $x_2$ are the positions of the two point masses. The generators act as usual on them and on their conjugate momenta, in the way specified in Sec. IV.

The reduced Hamiltonian, generator of infinitesimal time translation, can be computed from

$$H_{3\mathrm{pN}} = -L_{3\mathrm{pN}} + \sum_{a=1}^{2}\sum_{i=1}^{3} \Pi_{0ai} x_{ai}^{(1)} + \sum_{a=1}^{2}\sum_{i=1}^{3} \Pi_{1ai} x_{ai}^{(2)}, \tag{56}$$

by using the constraints to eliminate $x^{(1)}$, $x^{(2)}$, and $\Pi_1$ in favor of the coordinates of the constraint surface. The latter are chosen to be $x$ and $\Pi_0$ because of their "Hamiltonian" character (a possible alternative is $x$ and $x^{(1)}$). In this grid, the reduced 3pN Hamiltonian explicitly reads

$$H_{\mathrm{r,3pN}} = {}^0H_{\mathrm{N}} + \frac{1}{c^2}{}^1H_{\mathrm{N}} + \frac{1}{c^4}{}^2H_{\mathrm{N}} + \frac{1}{c^6}{}^3H_{\mathrm{N}}, \tag{57}$$

$$
\begin{aligned}
{}^{0}H_{\mathrm{N}} &= \frac{\Pi_{01}{}^{2}}{2m_1} + \frac{\Pi_{02}{}^{2}}{2m_2} - G\frac{m_1 m_2}{r},\\
{}^{1}H_{\mathrm{N}} &= -\frac{\Pi_{01}{}^{4}}{8m_1{}^{3}} - \frac{\Pi_{02}{}^{4}}{8m_2{}^{3}} + \frac{G}{2r}\left[7(\Pi_{01}\Pi_{02}) - \frac{3m_1}{m_2}\Pi_{02}{}^{2} - \frac{3m_2}{m_1}\Pi_{01}{}^{2} + (\Pi_{01}n_{12})(\Pi_{02}n_{12})\right] + \frac{G^2 m_1 m_2(m_1+m_2)}{2r^2},\\
{}^{2}H_{\mathrm{N}} &= \frac{\Pi_{01}{}^{6}}{16m_1{}^{5}} + \frac{G}{16m_1{}^{2}m_2{}^{2}r}\left[\frac{10m_2{}^{3}}{m_1}\Pi_{01}{}^{4} - 15m_1m_2\Pi_{02}{}^{2}\Pi_{01}{}^{2} + 14m_1m_2(\Pi_{02}n_{12})^2\Pi_{01}{}^{2}\right.\\
&\quad - 4m_2{}^{2}(\Pi_{02}n_{12})(\Pi_{01}n_{12})\Pi_{01}{}^{2} + 4m_2{}^{2}(\Pi_{01}\Pi_{02})\Pi_{01}{}^{2} - 2m_1m_2(\Pi_{01}\Pi_{02})^2\\
&\quad \left. - 12m_1m_2(\Pi_{02}n_{12})(\Pi_{01}n_{12})(\Pi_{01}\Pi_{02}) - 3m_1m_2(\Pi_{01}n_{12})^2(\Pi_{02}n_{12})^2\right]\\
&\quad + \frac{G^2}{8m_1m_2r^2}\left[22m_2{}^{3}\Pi_{01}{}^{2} + 47m_1m_2{}^{2}\Pi_{01}{}^{2} - 4m_2{}^{3}(\Pi_{01}n_{12})^2 - 70m_1{}^{2}m_2(\Pi_{01}\Pi_{02}) + 16m_1{}^{2}m_2(\Pi_{01}n_{12})(\Pi_{02}n_{12})\right.\\
&\quad \left. - 13m_1m_2{}^{2}(\Pi_{01}n_{12})^2\right] - \frac{G^3m_1m_2}{8r^3}[19m_1m_2 + 4m_1{}^{2}] + 1\longleftrightarrow 2,\\
{}^{3}H_{N} &= -\frac{5\Pi_{01}{}^{8}}{128m_1{}^{7}} + \frac{G}{32m_1{}^{2}m_2{}^{2}r}\left[-\frac{14m_2{}^{3}\Pi_{01}{}^{6}}{m_1{}^{3}} + \frac{58m_2}{m_1}\Pi_{02}{}^{2}\Pi_{01}{}^{4} + \frac{28m_2{}^{2}}{m_1{}^{2}}(\Pi_{01}n_{12})(\Pi_{02}n_{12})\Pi_{01}{}^{4}\right.\\
&\quad - \frac{28m_2{}^{2}}{m_1{}^{2}}(\Pi_{01}\Pi_{02})\Pi_{01}{}^{4} - \frac{36m_2}{m_1}(\Pi_{02}n_{12})^2\Pi_{01}{}^{4} + \frac{12m_2}{m_1}(\Pi_{01}n_{12})^2(\Pi_{02}n_{12})^2\Pi_{01}{}^{2} + \frac{8m_2}{m_1}(\Pi_{01}\Pi_{02})^2\Pi_{01}{}^{2}\\
&\quad - 12(\Pi_{02}n_{12})^2(\Pi_{01}\Pi_{02})\Pi_{01}{}^{2} - \frac{20m_1}{m_2}(\Pi_{02}n_{12})^4\Pi_{01}{}^{2} - 4(\Pi_{02}n_{12})^3(\Pi_{01}n_{12})\Pi_{01}{}^{2} - 17\Pi_{02}{}^{2}(\Pi_{01}\Pi_{02})\Pi_{01}{}^{2}\\
&\quad - \frac{8m_2}{m_1}\Pi_{02}{}^{2}(\Pi_{01}n_{12})^2\Pi_{01}{}^{2} + \frac{16m_2}{m_1}(\Pi_{01}\Pi_{02})(\Pi_{02}n_{12})(\Pi_{01}n_{12})\Pi_{01}{}^{2} + 25\Pi_{02}{}^{2}(\Pi_{02}n_{12})(\Pi_{01}n_{12})\Pi_{01}{}^{2}\\
&\quad \left. - 10(\Pi_{01}n_{12})(\Pi_{01}\Pi_{02})^2(\Pi_{02}n_{12}) - 2(\Pi_{01}\Pi_{02})^3 + 5(\Pi_{01}n_{12})^3(\Pi_{02}n_{12})^3 + 15(\Pi_{01}n_{12})^2(\Pi_{02}n_{12})^2(\Pi_{01}\Pi_{02})\right]\\
&\quad + \frac{G^2}{144r^2}\left[-\frac{957m_2\Pi_{01}{}^{4}}{m_1{}^{2}} - \frac{261m_2{}^{2}\Pi_{01}{}^{4}}{m_1{}^{3}} - \frac{90m_2}{m_1{}^{2}}(\Pi_{01}\Pi_{02})\Pi_{01}{}^{2} + \frac{654}{m_2}(\Pi_{02}n_{12})^2\Pi_{01}{}^{2} + \frac{798}{m_1}\Pi_{02}{}^{2}\Pi_{01}{}^{2}\right.\\
&\quad + \frac{1848m_2}{m_1{}^{2}}(\Pi_{01}n_{12})^2\Pi_{01}{}^{2} - \frac{705}{m_1}(\Pi_{02}n_{12})^2\Pi_{01}{}^{2} + \frac{1938}{m_1}(\Pi_{01}\Pi_{02})\Pi_{01}{}^{2} - \frac{2310}{m_1}(\Pi_{01}n_{12})(\Pi_{02}n_{12})\Pi_{01}{}^{2}\\
&\quad + \frac{36m_2{}^{2}}{m_1{}^{3}}(\Pi_{01}n_{12})^2\Pi_{01}{}^{2} - \frac{1428}{m_1}(\Pi_{01}\Pi_{02})^2 - \frac{3192}{m_1}(\Pi_{01}n_{12})^2(\Pi_{01}\Pi_{02}) - \frac{1078}{m_1}(\Pi_{01}n_{12})^3(\Pi_{02}n_{12})\\
&\quad \left. + \frac{1146}{m_1}(\Pi_{01}n_{12})^2(\Pi_{02}n_{12})^2 + \frac{3660}{m_1}(\Pi_{01}n_{12})(\Pi_{01}\Pi_{02})(\Pi_{02}n_{12}) - \frac{104m_2}{m_1{}^{2}}(\Pi_{01}n_{12})^4\right]\\
&\quad + \frac{G^3}{20\,160r^3}\left[-501\,760m_2{}^{2}\Pi_{01}{}^{2} - \frac{85\,680m_2{}^{3}\Pi_{01}{}^{2}}{m_1} - 496\,736m_1m_2\Pi_{01}{}^{2} - 12\,915m_2{}^{2}\pi^2\Pi_{01}{}^{2}\right.\\
&\quad + 147\,840m_1m_2\ln\left(\frac{r}{r_1}\right)\Pi_{01}{}^{2} + 562\,256m_1{}^{2}(\Pi_{01}\Pi_{02}) + \frac{30\,240m_2{}^{3}}{m_1}(\Pi_{01}n_{12})^2 + 174\,720m_2{}^{2}(\Pi_{01}n_{12})^2\\
&\quad + 977\,808m_1m_2(\Pi_{01}n_{12})^2 - 982\,848m_1{}^{2}(\Pi_{01}n_{12})(\Pi_{02}n_{12}) - 38745m_1m_2\pi^2(\Pi_{01}n_{12})(\Pi_{02}n_{12})\\
&\quad + 38\,745m_2{}^{2}\pi^2(\Pi_{01}n_{12})^2 + 12915m_1m_2\pi^2(\Pi_{01}\Pi_{02}) + 547\,120m_1m_2(\Pi_{01}\Pi_{02})\\
&\quad - 149\,520m_1m_2(\Pi_{01}n_{12})(\Pi_{02}n_{12}) - 147\,840m_1{}^{2}\ln\left(\frac{r}{r_1}\right)(\Pi_{01}\Pi_{02})\\
&\quad \left. + 443\,520m_1{}^{2}\ln\left(\frac{r}{r_1}\right)(\Pi_{02}n_{12})(\Pi_{01}n_{12}) - 443\,520m_2m_1\ln\left(\frac{r}{r_1}\right)(\Pi_{01}n_{12})^2\right]\\
&\quad + \frac{1}{840}\frac{G^4}{r^4}\left[315m_1{}^{4}m_2 + 17\,427m_1{}^{3}m_2{}^{2} - 3080m_1{}^{3}m_2{}^{2}\lambda - 6160m_1{}^{3}m_2{}^{2}\ln\left(\frac{r}{r_1}\right)\right] + 1\longleftrightarrow 2.
\end{aligned}
$$

We have again adapted the notation of Ref. [21] introduced after Eq. (5). Furthermore, we have posed $n_{12} = \frac{x_1 - x_2}{r}$. The term "$+1 \longleftrightarrow 2$" represents the expression that precedes but with interchanged particle indices, including the contributions that are already symmetric in the particle indices, particularly $n_{12}$ must be changed into $n_{21} = -n_{12}$ there. Note that it is possible to remove the logarithm terms through a coordinate transformation preserving the harmonicity conditions outside the bodies [10].

The generators of infinitesimal spatial translations and rotations are given on the whole phase space by Eqs. (40) and (41) as

$$P_i = \sum_{a=1}^{2} \Pi_{0ai} = \Pi_{01i} + \Pi_{02i}, \tag{58}$$

$$J_i = \sum_{m=0}^{1} \sum_{a=1}^{2} \sum_{j,k=1}^{3} \epsilon_{ijk} x_{aj}^{(m)} \Pi_{mak} = \sum_{a=1}^{2} \sum_{j,k=1}^{3} \epsilon_{ijk} x_{aj} \Pi_{0ak} + \sum_{a=1}^{2} \sum_{j,k=1}^{3} \epsilon_{ijk} x_{aj}^{(1)} \Pi_{1ak}. \tag{59}$$

For the dynamics, only their restrictions to the constraint surface are relevant as integrals of motion and generators of symmetry transformations. They can be computed by eliminating $x^{(1)}$ and $\Pi_1$ from Eqs. (58) and (59) with the help of the constraints (55a) and (55b). The result is

$$P_i|_\Gamma = \Pi_{01i} + \Pi_{02i}, \tag{60}$$

$$J_i|_\Gamma = \sum_{a=1}^{2} \sum_{j,k=1}^{3} \epsilon_{ijk} \bigg[ \bigg( x_{aj} - \varepsilon^2 \frac{1}{m_a} \Phi_{1aj}(x, x^{(1)}(x, \Pi_0)) \bigg) \Pi_{0ak} - \varepsilon^3 \frac{1}{m_a} \Phi_{0aj}(x, x^{(1)}(x, \Pi_0)) \Phi_{1ak}(x, x^{(1)}(x, \Pi_0)) \bigg] + \mathcal{O}(\varepsilon^4). \tag{61}$$

The angular momentum $J_i|_\Gamma$ of the post-Newtonian dynamics can be written explicitly as

$$J_{3\mathrm{pN}i} = {}^0J_{\mathrm{N}i} + \frac{1}{c^4} {}^2J_{\mathrm{N}i} + \frac{1}{c^6} {}^3J_{\mathrm{N}i}, \tag{62}$$

$${}^0J_{\mathrm{N}i} = \epsilon_{ijk} x_{1j} \Pi_{01k} + \epsilon_{ijk} x_{2j} \Pi_{02k},$$

$${}^2J_{\mathrm{N}i} = -\epsilon_{ijk} \frac{7G}{4m_1} (n_{12}\Pi_{01}) \Pi_{01j} \Pi_{02k} + \epsilon_{ijk} \frac{G}{8m_1 m_2 r} \Big[ (n_{12}\Pi_{02})^2 m_1 - 7m_1 {\Pi_{02}}^2 \Big] x_{1j} \Pi_{01k} - \epsilon_{ijk} \frac{G}{8m_1 m_2 r} \Big[ (n_{12}\Pi_{01})^2 m_2 - 7m_2 {\Pi_{01}}^2 \Big] x_{1j} \Pi_{02k} + 1 \longleftrightarrow 2,$$

$$\begin{aligned}
{}^3J_{\mathrm{N}i} = {} & \epsilon_{ijk} \frac{G}{24{m_1}^2{m_2}^2} \bigg[ 9m_2(n_{12}\Pi_{02})(n_{12}\Pi_{01})^2 - 3m_2{\Pi_{01}}^2(n_{12}\Pi_{02}) - \frac{3{m_2}^2}{m_1}{\Pi_{01}}^2(n_{12}\Pi_{01}) + 6m_2(n_{12}\Pi_{01})(\Pi_{01}\Pi_{02}) \\
& + \frac{10{m_2}^2}{m_1}(n_{12}\Pi_{01})^3 \bigg] \Pi_{01j}\Pi_{02k} + \epsilon_{ijk}\frac{G}{16{m_1}^2{m_2}^2 r} \bigg[ 7m_2{\Pi_{02}}^2{\Pi_{01}}^2 - m_2{\Pi_{01}}^2(n_{12}\Pi_{02})^2 \\
& + 8m_1{\Pi_{02}}^2(n_{12}\Pi_{01})(n_{12}\Pi_{02}) - 2m_1(n_{12}\Pi_{01})(n_{12}\Pi_{02})^3 - \frac{{m_1}^2}{m_2}(n_{12}\Pi_{02})^4 - 6m_1(n_{12}\Pi_{02})^2(\Pi_{01}\Pi_{02}) \\
& + \frac{14{m_1}^2}{m_2}{\Pi_{02}}^4 + 10m_2{\Pi_{02}}^2(n_{12}\Pi_{01})^2 + \frac{3{m_1}^2}{m_2}{\Pi_{02}}^2(n_{12}\Pi_{02})^2 \bigg] x_{1j}\Pi_{01k} \\
& + \epsilon_{ijk}\frac{G}{16{m_1}^2{m_2}^2 r} \bigg[ 2m_2(n_{12}\Pi_{02})(n_{12}\Pi_{01})^3 - \frac{3{m_2}^2}{m_1}{\Pi_{01}}^2(n_{12}\Pi_{01})^2 - 10m_1{\Pi_{01}}^2(n_{12}\Pi_{02})^2 - \frac{14{m_2}^2}{m_1}{\Pi_{01}}^4 \\
& + m_1{\Pi_{02}}^2(n_{12}\Pi_{01})^2 - 8m_2{\Pi_{01}}^2(n_{12}\Pi_{02})(n_{12}\Pi_{01}) + \frac{{m_2}^2}{m_1}(n_{12}\Pi_{01})^4 - 7m_1{\Pi_{02}}^2{\Pi_{01}}^2 \\
& + 6m_2(n_{12}\Pi_{01})^2(\Pi_{01}\Pi_{02}) \bigg] x_{1j}\Pi_{02k} + \epsilon_{ijk}\frac{G^2}{24r}\bigg[ 193(n_{12}\Pi_{01}) + \frac{17m_2}{m_1}(n_{12}\Pi_{01}) \bigg] \Pi_{01j}\Pi_{02k} \\
& + \epsilon_{ijk}\frac{G^2}{48r^2}\bigg[ \frac{21m_2}{m_1}(n_{12}\Pi_{01})^2 - \frac{408m_2{\Pi_{01}}^2}{m_1} - \frac{68m_1{\Pi_{02}}^2}{m_2} + 48(n_{12}\Pi_{01})(n_{12}\Pi_{02}) - 109{\Pi_{02}}^2 \\
& + \frac{100m_1}{m_2}(n_{12}\Pi_{02})^2 - 76(n_{12}\Pi_{02})^2 + 816(\Pi_{01}\Pi_{02}) \bigg] x_{1j}\Pi_{01k} + \epsilon_{ijk}\frac{G^2}{48r^2}\bigg[ -\frac{21m_1}{m_2}(n_{12}\Pi_{02})^2 - \frac{100m_2}{m_1}(n_{12}\Pi_{01})^2 \\
& + 76(n_{12}\Pi_{01})^2 - 816(\Pi_{01}\Pi_{02}) + \frac{68m_2{\Pi_{01}}^2}{m_1} + \frac{408m_1{\Pi_{02}}^2}{m_2} + 109{\Pi_{01}}^2 - 48(n_{12}\Pi_{01})(n_{12}\Pi_{02}) \bigg] x_{1j}\Pi_{02k} + 1 \longleftrightarrow 2,
\end{aligned}$$

where a sum over the indices $j$ and $k = 1, 2, 3$ must be understood. It has been checked that the above momenta satisfy the relations (35a)–(35e). In particular, $H$ is invariant under spatial translations and rotations and the components of the generalized total linear and angular momentum, $P_i$ and $J_i$, are integrals of motion.

The generator (62) takes a simple familiar form when written in coordinates that are canonically conjugate with respect to the Dirac bracket. We have obtained these coordinates, expressed in terms of $x, \Pi_0$ by guess work. They read

$$P_\alpha = \Pi_{0\alpha} - \varepsilon^3 \sum_\gamma \frac{1}{m_\gamma} \Phi_{1\gamma}(x, \Pi_0) \frac{\partial \Phi_{0\gamma}(x, \Pi_0)}{\partial x_\alpha} + \mathcal{O}(\varepsilon^4), \tag{63a}$$

$$Q_\alpha = x_\alpha - \varepsilon^2 \frac{1}{m_\alpha} \Phi_{1\alpha}(x, \Pi_0) + \varepsilon^3 \sum_\gamma \frac{1}{m_\gamma} \Phi_{1\gamma}(x, \Pi_0) \frac{\partial \Phi_{0\gamma}(x, \Pi_0)}{\partial \Pi_{0\alpha}} + \mathcal{O}(\varepsilon^4), \tag{63b}$$

where the sum $\sum_\gamma$ holds over all pairs of indices $\gamma = bj$, $b = 1, 2$, $j = 1, 2, 3$ and where $\Phi_{s\gamma}(x, \Pi_0)$, $s = 0, 1$, is a short notation for $\Phi_{s\gamma}[x, x^{(1)}(x, \Pi_0)]$. This result can be verified by using the explicit expressions (C12)–(C14) for the elementary Dirac brackets.

It is important to stress that there does not exist any harmonic-coordinate system in which $Q$ and more generally any other canonical coordinates represent the particle positions. If it were possible, we could go over to a frame where the particle positions are canonical coordinates while maintaining manifest Poincaré invariance. In terms of these new coordinates, the Hamiltonian as well as the Lagrangian would be ordinary although describing manifest Poincaré invariant dynamics, in contradiction to the no-interaction theorem [24]. A consequence is that we are not able to obtain canonically conjugate particle coordinates by means of Poincaré transformations. This can be seen from Eqs. (C15) and (C16), which imply that $\{x_\alpha, x_\beta\}^*$ cannot vanish everywhere for the 3pN and 2pC dynamics. The form of these two relations is maintained by Poincaré transformations that are also generalized canonical transformations. However, in the case of vanishing coupling constant, we have $Q \to x$, $P \to \Pi_0$. In other words, for the noninteracting systems, manifest Poincaré invariance becomes compatible with the choice of spatial coordinates as canonical coordinates.

The interest of using $Q$ and $P$ rather than $x$ and $\Pi_0$ for explicit calculations is that in these coordinates the Dirac bracket takes the simple standard form of a Poisson bracket. For simplicity, we shall note any function $F(x, \Pi_0)$ on the constraint surface expressed by means of the canonically conjugate variables $Q, P$ as $\tilde{F} \equiv F(Q, P) \equiv F(x(Q, P), \Pi_0(Q, P))$. The Dirac-bracket at a point $x(Q, P), \Pi_0(Q, P)$ of the constraint surface then reduces to

$$\{F, G\}^* \Big|_{x(Q,P),\Pi_0(Q,P)} = \{\tilde{F}, \tilde{G}\}_{Q,P} \equiv \sum_\alpha \frac{\partial \tilde{F}}{\partial Q_\alpha} \frac{\partial \tilde{G}}{\partial P_\alpha} - \frac{\partial \tilde{G}}{\partial Q_\alpha} \frac{\partial \tilde{F}}{\partial P_\alpha}. \tag{64}$$

In canonically conjugate coordinates, $P_i$ and $J_i$ have their usual expression on the constraint surface for both dynamics under consideration, namely,

$$\tilde{P}_i = \sum_{a=1}^{2} P_{ai} + \mathcal{O}(\varepsilon^{n+1}), \tag{65}$$

$$\tilde{J}_i = \sum_{a=1}^{2} \sum_{j,k=1}^{3} \epsilon_{ijk} Q_{aj} P_{ak} + \mathcal{O}(\varepsilon^{n+1}), \tag{66}$$

with $n = 3$ at 3pN ($n = 2$ at 2pC). Indeed, spatial translations and rotations leave the constraint surface invariant so that the restrictions of $P_i$ and $J_i$ to $\Gamma$ are the well-known generators of infinitesimal spatial translations and rotations on $\Gamma$ (cf. Sec. V) taking above shape in terms of canonically conjugate coordinates. This explains incidentally the absence of a first post-Newtonian contribution in the expressions (62) and (84) for $J_i|_\Gamma$ as a function of $x, \Pi_0$. If the term containing $\Pi_1$ does not contribute to the first order, this is because the $\Pi_1$'s are second order quantities; for the rest, the $x$'s and the $\Pi_0$'s are canonically conjugate modulo $1/c^4$ corrections.

To determine the generator of Lorentz boosts, it is useful to work with the canonical coordinates $Q, P$ on the constraint surface. In addition, it is more convenient to derive $\tilde{K}_i \equiv K_i(Q, P) = G_i(Q, P) + P_i(Q, P)t$ rather than $G_i(Q, P)$ itself. We compute $\tilde{K}_i$ with the help of the method of undetermined coefficients [17], thereafter fixing the uniqueness of the solution.

As already mentioned, $K_i$ fulfills the same Dirac-bracket relations (35f)–(35h) and (35j) as $G_i$, but it is not explicitly time dependent and it is not an integral of motion. The behavior of $\tilde{K}_i$ under spatial rotations is governed by Eq. (35h) with the angular momentum $\tilde{J}_i$ given by (66). We conclude that $\tilde{K}_i$ has the general structure

$$\tilde{K}_i = \sum_{a=1}^{2} M_a(Q, P) Q_{ai} + N_a(Q, P) P_{ai} + \mathcal{O}(\varepsilon^{n+1}), \tag{67}$$

where $M_a(Q, P)$, $N_a(Q, P)$ are two post-Newtonian scalar functions. The form of the differential equations for $\tilde{K}_i$ resulting from the Poincaré algebra and the world line condition suggests for $M_a(Q, P)$ and $N_a(Q, P)$ the ansatz

$$c_{n_0,\ldots,n_5} R^{n_0} P_1^{2n_1} P_2^{2n_2} (P_1 P_2)^{n_3} (N_{12} P_1)^{n_4} (N_{12} P_2)^{n_5} + d_{1,s_1} R^{s_1} \ln(R/r_1) + d_{2,s_2} R^{s_2} \ln(R/r_2), \tag{68}$$

with $R = |Q_1 - Q_2|$ and $N_{12} = \frac{Q_1 - Q_2}{R}$; the logarithm terms are only expected to appear in the function $M_a$ and the powers $n_0, s_1, s_2$ are presumed to be integers, while $n_1, \ldots, n_5$ are natural numbers. The admissible eight-tuples of $s_1, s_2, n_0, \ldots, n_5$ are restricted by demanding the correct physical dimension for $M_a$ and $N_a$.

Insertion of the preceding ansatz into the partial differential Eqs. (35f)–(35h) and (35j) yields the linear system of equations to be solved. The coefficients seem to be overdetermined, but the equations are not all independent so that there actually exists a solution. It is most easily derived by means of a computer-algebra program.[1] If we only impose that the Poincaré algebra should be satisfied, some coefficients remain undetermined even at the 1pN order. They are set by requiring the world line condition on $K_i$, reflecting the manifest Lorentz invariance of the system. As a result, the expression for $\tilde{K}_i$ obtained through this procedure is automatically consistent with both the Poincaré-algebra and the world line condition.

Now, as discussed before, the canonically conjugate coordinates simplifying our explicit calculation are not harmonic. Reexpressing $\tilde{K}_i$ by the coordinates $x$ and $\Pi_0$ of the constraint surface we finally arrive at $K_i$ in a harmonic-coordinate frame. The result is displayed up to third post-Newtonian order as

$$K_{3\mathrm{pN}i} = {}^0K_{\mathrm{N}i} + \frac{1}{c^2}{}^1K_{\mathrm{N}i} + \frac{1}{c^4}{}^2K_{\mathrm{N}i} + \frac{1}{c^6}{}^3K_{\mathrm{N}i}, \tag{69}$$

$${}^0K_{\mathrm{N}i} = m_1x_{1i} + m_2x_{2i}, \quad {}^1K_{\mathrm{N}i} = \frac{\Pi_{01}{}^2}{2m_1}x_{1i} + \frac{\Pi_{02}{}^2}{2m_2}x_{2i} - \frac{Gm_1m_2}{2r}(x_{1i}+x_{2i}),$$

$$\begin{aligned}
{}^2K_{\mathrm{N}i} = &-\frac{\Pi_{01}{}^4}{8m_1{}^3}x_{1i} + \frac{G}{8m_1m_2}[7m_2{}^2\Pi_{01}{}^2n_{12i} - m_2{}^2(n_{12}\Pi_{01})^2n_{12i} - 14m_1m_2(n_{12}\Pi_{02})\Pi_{01i} - 14m_2{}^2(n_{12}\Pi_{01})\Pi_{01i}]\\
&+\frac{G}{4r}\left[-\frac{6m_2}{m_1}\Pi_{01}{}^2x_{1i} + (n_{12}\Pi_{01})(n_{12}\Pi_{02})x_{1i} + 7(\Pi_{01}\Pi_{02})x_{1i}\right] + \frac{G^2m_1m_2}{4r^2}[-5m_1x_{1i} + 7m_2x_{1i}] + 1\longleftrightarrow 2,
\end{aligned}$$

$$\begin{aligned}
{}^3K_{\mathrm{N}i} = &\frac{\Pi_{01}{}^6}{16m_1^5}x_{1i} + \frac{G}{48m_1{}^2m_2{}^2}\left[-\frac{42m_2{}^3}{m_1}\Pi_{01}{}^4n_{12i} - \frac{6m_2{}^3}{m_1}(n_{12}\Pi_{01})\Pi_{01}{}^2\Pi_{01i} - 6m_2{}^2(n_{12}\Pi_{02})\Pi_{01}{}^2\Pi_{01i}\right.\\
&-30m_1m_2(n_{12}\Pi_{02})^2\Pi_{01}{}^2n_{12i} - 24m_2{}^2(n_{12}\Pi_{02})(n_{12}\Pi_{01})\Pi_{01}{}^2n_{12i} - \frac{9m_2{}^3}{m_1}(n_{12}\Pi_{01})^2\Pi_{01}{}^2n_{12i}\\
&+18m_1m_2(n_{12}\Pi_{02})^2(n_{12}\Pi_{01})\Pi_{01i} - 48m_1m_2\Pi_{02}{}^2(n_{12}\Pi_{01})\Pi_{01i} + 12m_1m_2(n_{12}\Pi_{02})(\Pi_{01}\Pi_{02})\Pi_{01i}\\
&+20m_1{}^2(n_{12}\Pi_{02})^3\Pi_{01i} + \frac{20m_2{}^3}{m_1}(n_{12}\Pi_{01})^3\Pi_{01i} + 18m_2{}^2(n_{12}\Pi_{02})(n_{12}\Pi_{01})^2\Pi_{01i} - 48m_1{}^2\Pi_{02}{}^2(n_{12}\Pi_{02})\Pi_{01i}\\
&+12m_2{}^2(n_{12}\Pi_{01})(\Pi_{01}\Pi_{02})\Pi_{01i} + \frac{3m_2{}^3}{m_1}(n_{12}\Pi_{01})^4n_{12i} + 6m_2{}^2(n_{12}\Pi_{02})(n_{12}\Pi_{01})^3n_{12i}\\
&\left.+18m_2{}^2n_{12i}(n_{12}\Pi_{01})^2(\Pi_{01}\Pi_{02})\right] + \frac{G}{16r}\left[\frac{21m_2}{m_1{}^3}\Pi_{01}{}^4x_{1i} + \frac{14}{m_1m_2}(n_{12}\Pi_{02})^2\Pi_{01}{}^2x_{1i} + \frac{2}{m_1{}^2}(\Pi_{01}\Pi_{02})\Pi_{01}{}^2x_{1i}\right.\\
&-\frac{15}{m_1m_2}\Pi_{02}{}^2\Pi_{01}{}^2x_{1i} - \frac{2}{m_1^2}(n_{12}\Pi_{01})(n_{12}\Pi_{02})\Pi_{01}{}^2x_{1i} - \frac{2}{m_1m_2}(\Pi_{01}\Pi_{02})^2x_{1i} - \frac{12}{m_1m_2}(n_{12}\Pi_{01})(n_{12}\Pi_{02})(\Pi_{01}\Pi_{02})x_{1i}\\
&\left.+\frac{2}{m_2^2}\Pi_{02}{}^2(\Pi_{01}\Pi_{02})x_{1i} - \frac{2}{m_2{}^2}\Pi_{02}^2(n_{12}\Pi_{02})(n_{12}\Pi_{01})x_{1i} - \frac{3}{m_1m_2}(n_{12}\Pi_{01})^2(n_{12}\Pi_{02})^2x_{1i} - \frac{11m_1}{m_2{}^3}\Pi_{02}{}^4x_{1i}\right]\\
&+\frac{G^2}{48r}\left[-299m_2\Pi_{01}^2n_{12i} + \frac{68m_2{}^2}{m_1}\Pi_{01}{}^2n_{12i} + \frac{34m_2{}^2}{m_1}(n_{12}\Pi_{01})\Pi_{01i} + 428m_2(n_{12}\Pi_{01})\Pi_{01i} + 386m_2(n_{12}\Pi_{02})\Pi_{01i}\right.\\
&+76m_1(n_{12}\Pi_{02})\Pi_{01i} + 48m_1(n_{12}\Pi_{02})(n_{12}\Pi_{01})n_{12i} - \frac{100m_2{}^2}{m_1}(n_{12}\Pi_{01})^2n_{12i} + 13m_2(n_{12}\Pi_{01})^2n_{12i}\\
&\left.+816m_1(\Pi_{01}\Pi_{02})n_{12i}\right] + \frac{1}{3360}\frac{G^2}{r^2}\left[21\,140m_2(\Pi_{01}\Pi_{02})x_{1i} - 50540m_1(\Pi_{01}\Pi_{02})x_{1i} + 21\,210m_2\Pi_{01}{}^2x_{1i}\right.\\
&-10640m_1(n_{12}\Pi_{01})(n_{12}\Pi_{02})x_{1i} - 1470m_1\Pi_{02}{}^2x_{1i} - 5670m_2(n_{12}\Pi_{01})^2x_{1i} - \frac{1680m_2{}^2}{m_1}(n_{12}\Pi_{01})^2x_{1i}\\
&\left.+\frac{9240m_2{}^2}{m_1}\Pi_{01}{}^2x_{1i} + 17360m_2(n_{12}\Pi_{01})(n_{12}\Pi_{02})x_{1i} + 210m_1(n_{12}\Pi_{02})^2x_{1i}\right] + \frac{1}{2520}\frac{G^3}{r^3}\left[-5985m_1{}^2m_2{}^2x_{1i}\right.\\
&\left.-28702m_1m_2{}^3x_{1i} - 18480m_1{}^3m_2\ln\left(\frac{r}{r_1}\right)x_{1i} + 27442m_1{}^3m_2x_{1i} + 18480m_1m_2{}^3\ln\left(\frac{r}{r_2}\right)x_{1i}\right] + 1\longleftrightarrow 2.
\end{aligned}$$

[1] We have used the software Maple 8 ©Waterloo Maple Inc.

The 3pN accurate generators $H_{3\mathrm{pN}}, P_{3\mathrm{pN}i}, J_{3\mathrm{pN}i}, G_{3\mathrm{pN}i} = K_{3\mathrm{pN}i} - P_{3\mathrm{pN}i}t$ we have derived in harmonic coordinates, constitute an approximate representation of the Poincaré Algebra given by the generators of infinitesimal canonical transformations with respect to the Dirac bracket, and thus, they generate the Poincaré transformation group on the phase-space coordinates. The quantities are integrals of motion that may be interpreted physically as the 3pN conserved total energy, the generalized total linear momentum, the generalized total angular momentum and the center of mass constant. Their expressions have been checked up to first order by comparing them with Ref. [43]. If we adapt the "Lagrangian-like" coordinates $x, x^{(1)}$ on the constraint surface, the above generators of infinitesimal canonical transformations representing conserved quantities reproduce the Noetherian constants of motion associated with the Poincaré symmetry of the dynamics that are derived in [21] by means of the Lagrangian formalism. Note that the definitions of $K_i$ and $G_i$ in this article are reverse.

It remains to show the uniqueness of the generators $P_i, J_i, G_i$ of the Poincaré group up to generalized canonical transformations provided that $H$ is a post-Newtonian Hamiltonian. Let us indicate that the following proof also is valid for the post-Coulombian dynamics.

We start with the case of the momenta $P_i$ and $J_i$. Their uniqueness as a function of the phase-space coordinates has been established on the whole phase space in Sec. IV. It entails the uniqueness of their restriction to the constraint surface.

We next consider the center of mass constant $G_i$. Its uniqueness is equivalent to that of $\tilde{K}_i \equiv K_i(Q,P) = G_i(Q,P) + P_i(Q,P)t$, since the coordinate transformation is a diffeomorphism and since $P_i$ is unique. Again, $Q, P$ denotes the set of canonically conjugate coordinates on the constraint surface while $P_i$ represents the total linear momentum of the system. Let us assume that there exist two solutions $\tilde{K}_i$ and $\tilde{K}'_i = \tilde{K}_i + f_i(Q,P)$ that both approximately fulfill the Poincaré algebra and the world line condition within the common coordinate frame. The relations (46), (35g), and (35h), written in terms of the coordinates $Q, P$ as given by Eqs. (63a) and (63b) read [cf. also (64)]:

$$\begin{aligned}\{\tilde{x}_{ai}, \tilde{K}_j\}_{Q,P} = \bigg\{&Q_{ai} + \varepsilon^2 \frac{1}{m_a}\tilde{\Phi}_{1ai} \\ &- \varepsilon^3 \sum_\gamma \frac{1}{m_\gamma}\tilde{\Phi}_{1\gamma}\frac{\partial\tilde{\Phi}_{0\gamma}}{\partial P_{ai}}, {}^0\tilde{K}_j + \varepsilon^1\tilde{K}_j \\ &+ \varepsilon^{22}\tilde{K}_j + \varepsilon^{33}\tilde{K}_j\bigg\}_{Q,P} + \mathcal{O}(\varepsilon^4) \\ = {}& \varepsilon\tilde{x}_{aj}\{\tilde{x}_{ai}, \tilde{H}\}_{Q,P} + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{70}$$

$$\{\tilde{K}_j, \tilde{H}\}_{Q,P} = \tilde{P}_j + \mathcal{O}(\varepsilon^4), \tag{71}$$

$$\{\tilde{J}_i, \tilde{K}_j\}_{Q,P} = \sum_{k=1}^{3} \epsilon_{ijk}\tilde{K}_k + \mathcal{O}(\varepsilon^4). \tag{72}$$

Note that, because $x_\alpha, \Pi_{0\beta}$ and $Q_\alpha, P_\beta$ agree at zeroth order, the Newtonian (or Coulombian) contribution to the Hamiltonian in canonically conjugate coordinates has the form

$${}^0\tilde{H} = \sum_{a=1}^{2}\frac{P_a^2}{2m_a} + U(Q). \tag{73}$$

The uniqueness of $\tilde{K}_i$ is then proved order by order using Eqs. (70)–(73). First, we may insert successively $\tilde{K}_j$ and $\tilde{K}'_j$ into (70) since they are both assumed to fulfill the latter relation. Taking the zeroth order of the difference, we find that

$$\frac{\partial^0 f_j(Q,P)}{\partial P_{ai}} = 0, \tag{74}$$

hence ${}^0f_j(Q,P) = {}^0f_j(Q)$. Next, we go over to Eq. (71), where we insert again $\tilde{K}_j$ and $\tilde{K}'_j$ successively before subtracting the ensuant equalities. By virtue of Eq. (73) we obtain at zeroth order:

$$\sum_{a=1}^{2}\sum_{k=1}^{3}\frac{P_{ak}}{m_a}\frac{\partial^0 f_j(Q)}{\partial Q_{ak}} = 0. \tag{75}$$

Because $Q_\alpha, P_\beta$ constitute a set of independent coordinates, and Eq. (75) holds for all $P_{ak}$, we are led to

$$\frac{\partial^0 f_j(Q)}{\partial Q_{ak}} = 0 \quad \Rightarrow \quad {}^0f_j(Q) = \text{const.} \tag{76}$$

From Eq. (72) taken at zeroth order, we conclude by applying the same procedure as explained above, that

$$\begin{aligned} 0 &= \{{}^0\tilde{J}_i, {}^0f_j\}_{Q,P} = \{{}^0\tilde{J}_i, {}^0\tilde{K}'_j - {}^0\tilde{K}_j\}_{Q,P} \\ &= \sum_{k=1}^{3}\epsilon_{ijk}({}^0\tilde{K}'_k - {}^0\tilde{K}_k) = \sum_{k=1}^{3}\epsilon_{ijk}{}^0f_k \\ &\Rightarrow \quad {}^0f_k = 0 \quad \Rightarrow \quad {}^0\tilde{K}'_i = {}^0\tilde{K}_i. \end{aligned} \tag{77}$$

The proof of uniqueness at the first order is similar. The world line condition Eq. (70) truncated at this level of approximation yields ${}^1f_j(Q,P) = {}^1f_j(Q)$. Equation (71) reduces to

$$\{{}^1\tilde{K}_j, {}^0\tilde{H}\}_{Q,P} + \{{}^0\tilde{K}_j, {}^1\tilde{H}\}_{Q,P} = 0, \tag{78}$$

and we have the same equation for $\tilde{K}'_j$. On the other hand, we know from Eq. (77) that $\tilde{K}'_i$ and $\tilde{K}_i$ differ at most from the first order on. Therefore taking the difference between the Eq. (78) for $\tilde{K}'_j$ and the same equation for $\tilde{K}_j$ at first order leads to

$$\{{}^1f_j(Q), {}^0\tilde{H}\}_{Q,P} = 0,$$
$$\sum_{a=1}^{2}\sum_{k=1}^{3}\frac{P_{ak}}{m_a}\frac{\partial {}^1f_j(Q)}{\partial Q_{ak}} = 0$$
$$\Rightarrow \quad \frac{\partial {}^1f_j(Q)}{\partial Q_{ak}} = 0, \tag{79}$$

so that ${}^1f_j(Q) = \text{const}$. Now, from the first order truncated version of Eq. (72), it follows

$$0 = \{{}^0\tilde{J}_i, {}^1f_j\}_{Q,P} = \{{}^0\tilde{J}_i, {}^1\tilde{K}'_j - {}^1\tilde{K}_j\}_{Q,P}$$
$$= \sum_{k=1}^{3}\epsilon_{ijk}({}^1\tilde{K}'_k - {}^1\tilde{K}_k) = \sum_{k=1}^{3}\epsilon_{ijk}{}^1f_k$$
$$\Rightarrow \quad {}^1f_k = 0 \quad \Rightarrow \quad {}^1\tilde{K}'_i = {}^1\tilde{K}_i. \tag{80}$$

The uniqueness of second, third or even higher orders follows analogously.

## VI. APPLICATION TO 2PC DYNAMICS

The Lagrangian describing the binary dynamics of the Feynman-Wheeler theory in Lorentz gauge up to second post-Coulombian order takes the form

$$L(x, x^{(1)}, x^{(2)}) = \frac{1}{2}\sum_{a=1}^{2} m_a(x_a^{(1)})^2 + V_0(x) + \frac{1}{c^2}V_1(x, x^{(1)}) + \frac{1}{c^4}V_2(x, x^{(1)}, x^{(2)}), \tag{81}$$

with $V_0(x) = -\frac{e_1e_2}{r}$. We may thus adopt directly all general results derived for the 3pN dynamics by neglecting the third order contribution. The elementary Dirac brackets of the 2pC dynamics, for instance, may be inferred from Appendix C. The computation of the generators of the Poincaré group corresponding to the conserved quantities of the Feynman-Wheeler 2pC binary dynamics can be performed in a way similar to the post-Newtonian case. The results read as follows.

The reduced Hamiltonian is given by

$$H_{2\text{pC}} = {}^0H_\text{C} + \frac{1}{c^2}{}^1H_\text{C} + \frac{1}{c^4}{}^2H_\text{C}, \tag{82}$$

$${}^0H_\text{C} = \frac{{\Pi_{01}}^2}{2m_1} + \frac{{\Pi_{02}}^2}{2m_2} + \frac{e_1e_2}{r},$$
$${}^1H_\text{C} = -\frac{{\Pi_{01}}^4}{8{m_1}^3} - \frac{{\Pi_{02}}^4}{8{m_2}^3} - \frac{e_1e_2}{2m_1m_2r}[(\Pi_{01}\Pi_{02}) + (n_{12}\Pi_{02})(n_{12}\Pi_{01})],$$
$${}^2H_\text{C} = \frac{{\Pi_{01}}^6}{16{m_1}^5} + \frac{e_1e_2}{16{m_1}^2{m_2}^2r}\bigg[{\Pi_{02}}^2{\Pi_{01}}^2 - 2(\Pi_{02}n_{12})^2{\Pi_{01}}^2 + \frac{4m_2}{m_1}(\Pi_{02}n_{12})(\Pi_{01}n_{12}){\Pi_{01}}^2 + \frac{4m_2}{m_1}(\Pi_{01}\Pi_{02}){\Pi_{01}}^2 - 2(\Pi_{01}\Pi_{02})^2 + 3(\Pi_{01}n_{12})^2(\Pi_{02}n_{12})^2\bigg] + \frac{{e_1}^2{e_2}^2}{8{m_1}^2m_2r^2}[{\Pi_{01}}^2 + 3(\Pi_{01}n_{12})^2] + \frac{{e_1}^3{e_2}^3}{8m_1m_2r^3} + 1\longleftrightarrow 2.$$

This result confirms the one derived in Ref. [34] as corrected in Ref. [35].

The generators of spatial translations and rotations $P_i$ and $J_i$ can be derived from Eqs. (60) and (61). The explicit expressions for the restrictions on the constraint surface in terms of $x$ and $\Pi_0$ read

$$P_{2\text{pC}i} = \Pi_{01i} + \Pi_{02i}, \tag{83}$$

$$J_{2\text{pC}i} = {}^0J_{\text{C}i} + \frac{1}{c^4}{}^2J_{\text{C}i}, \tag{84}$$

$${}^0J_{\text{C}i} = \epsilon_{ijk}x_{1j}\Pi_{01k} + \epsilon_{ijk}x_{2j}\Pi_{02k},$$
$${}^2J_{\text{C}i} = \epsilon_{ijk}\frac{e_1e_2}{4m_2{m_1}^2}(\Pi_{01}n_{12})\Pi_{01j}\Pi_{02k} + \epsilon_{ijk}\frac{e_1e_2}{8{m_1}^2{m_2}^2r}[m_1{\Pi_{02}}^2 - m_1(\Pi_{02}n_{12})^2]x_{1j}\Pi_{01k} + \epsilon_{ijk}\frac{e_1e_2}{8{m_1}^2{m_2}^2r}\Big[-m_2{\Pi_{01}}^2 + m_2(\Pi_{01}n_{12})^2\Big]x_{1j}\Pi_{02k} - \epsilon_{ijk}\frac{{e_1}^2{e_2}^2}{4m_1m_2r^2}[x_{1j}\Pi_{01k} - x_{1j}\Pi_{02k}] + 1\longleftrightarrow 2,$$

where a sum over the indices $j$ and $k = 1, 2, 3$ must be understood.

The generator $G_i$ is computed as in the post-Newtonian case. For $K_i = G_i + P_it$ in terms of $x$ and $\Pi_0$, we have

$$K_{2\mathrm{pC}i} = {}^0K_{\mathrm{C}i} + \frac{1}{c^2}{}^1K_{\mathrm{C}i} + \frac{1}{c^4}{}^2K_{\mathrm{C}i}, \tag{85}$$

$$\begin{aligned}
{}^0K_{\mathrm{C}i} &= m_1 x_{1i} + m_2 x_{2i},\\
{}^1K_{\mathrm{C}i} &= \frac{\Pi_1{}^2}{2m_1}x_{1i} + \frac{\Pi_2{}^2}{2m_2}x_{2i} + \frac{e_1e_2}{2r}[x_{1i} + x_{2i}],\\
{}^2K_{\mathrm{C}i} &= -\frac{x_{1i}\Pi_{01}{}^4}{8m_1{}^3} + \frac{e_1e_2}{8m_1m_2}\left[\frac{m_2}{m_1}(n_{12}\Pi_{01})^2 n_{12i} - \frac{m_2}{m_1}\Pi_{01}{}^2 n_{12i} + 2(n_{12}\Pi_{02})\Pi_{01i} + \frac{2m_2}{m_1}(n_{12}\Pi_{01})\Pi_{01i}\right]\\
&\quad - \frac{e_1e_2}{4m_1m_2r}[(\Pi_{01}\Pi_{02})x_{1i} + (n_{12}\Pi_{01})(n_{12}\Pi_{02})x_{1i}] - \frac{e_1{}^2e_2{}^2}{4m_2r}n_{12i} + 1\longleftrightarrow 2.
\end{aligned}$$

The above generators $H, P_i, J_i$ and $G_i$ of infinitesimal generalized canonical transformations provide a representation of the Poincaré algebra on the constraint surface of 2pC Feynman-Wheeler binary dynamics with respect to the Dirac bracket. The uniqueness of $P_i$, $J_i$, and $G_i$ up to a generalized canonical transformation has been established in Sec. IV.

## VII. SUMMARY AND DISCUSSION

In the present article, we applied an appropriate canonical formalism to describe the third post-Newtonian dynamics of point mass binaries in harmonic coordinates. We treated the second order post-Coulombian dynamics of Feynman-Wheeler theory in Lorentz gauge analogously. In contrast to earlier works, we did not leave the coordinate conditions by performing a higher order contact transformation [30] but instead we generalized a method developed in Ref. [34] and constructed the dynamics directly in harmonic coordinates and Lorentz gauge within the framework of canonical formalism both singular and of higher order in the time derivatives. The canonical formulation opens the way to advanced investigations about the geometrical and physical interpretation of the motion. It is highly desirable for deriving the generators of infinitesimal generalized canonical symmetry transformations that provide the integrals of motion. We computed for the first time the generators of the Poincaré transformation group or, equivalently, the conserved quantities corresponding to the manifest Poincaré invariance, for third post-Newtonian conservative binary dynamics in harmonic coordinates as well as for Feynman-Wheeler second post-Coulombian binary dynamics in Lorentz gauge. An appropriate choice of coordinates of the constraint surface reveals that the 3pN conserved quantities we have obtained agree with those derived via the generalized Noether theorem in [21]. After being reduced to the center of mass frame [40], they can be used for the derivation of an analytic parametric solution to the third post-Newtonian equations of motion in harmonic gauge for compact binaries in eccentric orbits [41], which is in turn of high practical relevance for the construction of gravitational wave search templates and comparison with numerical simulations. A useful generalization of the present investigations is suggested to be the application of our method to post-Newtonian binary dynamics in harmonic coordinates including spin.

## ACKNOWLEDGMENTS

The authors are thankful to Guillaume Faye for a careful reading of the manuscript and many useful comments. The financial support of the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 "Gravitationswellenastronomie" is gratefully acknowledged.

## APPENDIX A: LAGRANGIAN CONSTRAINTS [34]

In this appendix we display for completeness the iterative method developed in Ref. [34] to derive a minimal set of time-stable constraints for a dynamics described by a Lagrangian with the structure shown in Eq. (6).

(i) The first stage of the method consists in transforming the primary constraints into a more restrictive set implied by the primary constraints through time stability, without making use of the time evolution equations. In this new set, the higher order time derivatives are found separately in the various equations.

The primary constraints (10) imply

$$\varepsilon^n x_\alpha^{(2)} = \varepsilon^n \frac{1}{m_\alpha} A_{\alpha 0}(x) + \mathcal{O}(\varepsilon^{n+1}). \tag{A1}$$

Thus, Eqs. (10) multiplied by $\varepsilon^{n-1}$ are fulfilled by the motion, as well as the relations following from (A1) through repeated time differentiation

$$\varepsilon^n x_\alpha^{(2+r)} = \varepsilon^n \frac{1}{m_\alpha} B_{\alpha,2+r,0}(x, x^{(1)}) + \mathcal{O}(\varepsilon^{n+1}), \tag{A2}$$

where $r = 0, ..., 2n-3$; the quantities $B_{\alpha,2+r,0}(x, x^{(1)})$ are derived from the $A_{\alpha 0}$'s by differentiating $r$ times and eliminating accelerations by virtue of Eq. (A1) whenever they occur; the relation (A2) reduces to (A1) when $r = 0$. Beware that, because of the occurrence of the time derivative $x_\alpha^{(2n)}$, Eq. (A2) with $r = 2n-2$ is not a constraint, but rather the condition of time-stability for the actual constraints given by the system (A2) for $r = 0, ..., 2n - 3$. It becomes a consequence of the equations of motion as soon as the dynamics is restricted to the constraint surface.

The next set of constraints implied by the primary ones can be found by multiplying Eqs. (10) by $\varepsilon^{n-2}$:

$$\varepsilon^{n-1}x_\alpha^{(2)} = \varepsilon^{n-1}\frac{1}{m_\alpha}[A_{\alpha 0}(x) + \varepsilon A_{\alpha 1}(x, x^{(1)}, x^{(2)})] + \mathcal{O}(\varepsilon^{n+1}). \tag{A3}$$

The acceleration dependence of $A_{\alpha 1}(x, x^{(1)}, x^{(2)})$ may be eliminated by means of Eqs. (A2)

$$\varepsilon^{n-1}x_\alpha^{(2)} = \varepsilon^{n-1}\frac{1}{m_\alpha}[A_{\alpha 0}(x) + \varepsilon B_{\alpha,2,1}(x, x^{(1)})] + \mathcal{O}(\varepsilon^{n+1}). \tag{A4}$$

Differentiating $r = 0, ..., 2n - 3$ times with respect to time and replacing all occurring accelerations by means of Eqs. (A4) results in

$$\varepsilon^{n-1}x_\alpha^{(2+r)} = \varepsilon^{n-1}\frac{1}{m_\alpha}[B_{\alpha,2+r,0}(x, x^{(1)}) + \varepsilon B_{\alpha,2+r,1}(x, x^{(1)})] + \mathcal{O}(\varepsilon^{n+1}). \tag{A5}$$

Carrying on with this procedure we arrive at the set of constraints

$$\varepsilon x_\alpha^{(2+r)} = \varepsilon\frac{1}{m_\alpha}\left[\sum_{s=0}^{n-1}\varepsilon^s B_{\alpha,2+r,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}), \tag{A6}$$

where $r = 0, ..., 2n - 3$.

(ii) In the second part of the method proposed by Jaen, Llosa, and Molina, the above system is completed to one that is stable under the time evolution ruled by the Euler-Lagrange equations. The time-stability condition for Eqs. (A6) which also covers the time stability of all previous sets of constraints, equals (A6) with $r = 2n - 2$:

$$\varepsilon x_\alpha^{(2n)} = \varepsilon\frac{1}{m_\alpha}\left[\sum_{s=0}^{n-1}\varepsilon^s B_{\alpha,2n,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}). \tag{A7}$$

Demanding time stability of the constraints for the motion on the constraint surface is equivalent to removing higher order time derivatives in the Euler-Lagrange equations with the help of Eqs. (A6) and eliminating $x_\alpha^{(2n)}$ by means of Eq. (A7). We obtain the new set of constraints

$$x_\alpha^{(2+r)} = \frac{1}{m_\alpha}\left[\sum_{s=0}^{n}\varepsilon^s B_{\alpha,2+r,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}), \tag{A8}$$

with $r = 0, ..., 2n - 3$. From the way this set of constraints has been obtained, we see that it must hold even if some of the $x_\alpha^{(2n)}$ do not occur in certain linear combinations of the Euler-Lagrange equations, which may be the case for noninvertible $(\partial^2 V_n)/(\partial x_{ai}^{(n)}\partial x_{bj}^{(n)})$.

For the set of constraints (A8), the time-stability condition reads

$$x_\alpha^{(2n)} = \frac{1}{m_\alpha}\left[\sum_{s=0}^{n}\varepsilon^s B_{\alpha,2n,s}(x, x^{(1)})\right] + \mathcal{O}(\varepsilon^{n+1}). \tag{A9}$$

This condition is satisfied in the sense that removing the higher derivatives in the Euler-Lagrange equations with the help of Eqs. (A8) and substituting $x_\alpha^{(2n)}$ by Eq. (A9) yields a constraint [Eq. (A8) with $r = 0$] fulfilled on the constraint surface. It is obvious, that the set (A8) is more restrictive than the original primary constraints. Nonetheless, we see from the derivation that it follows from them through the time-stability condition. As a result, with (A8), we have found a minimal set of constraints stable in time for the dynamics under investigation.

## APPENDIX B: 3PN CONSTRAINTS

In this appendix we derive the constraints for the two-body system at the third post-Newtonian order by modifying appropriately the method displayed in Appendix A.

Within our approximation scheme, any vector being a multiple of $\varepsilon^2$ is a null vector of the Hessian of the 3pN Lagrangian (50). Therefore, using Eq. (51) as initial primary constraints, we have

$$\varepsilon^2\left[m_\alpha x_\alpha^{(2)} - \sum_{s=0}^{1}\varepsilon^s A_{\alpha s}(x, ..., x^{(2s)})\right] = \mathcal{O}(\varepsilon^4). \tag{B1}$$

Primary constraints due to the possible existence of further null vectors are not precluded.

(i) We now transform the system (B1) into a more restrictive set obtained by implying time stability, without employing the explicit equations of motion yet.

The acceleration dependences in orders higher than $\varepsilon^2$ can be removed from Eq. (B1) by means of the constraints obtained through multiplying the primary constraints with $\varepsilon$. We are led to

$$\varepsilon^2\left[m_\alpha x_\alpha^{(2)} - \sum_{s=0}^{1}\varepsilon^s B_{\alpha,2,s}(x, x^{(1)})\right] = \mathcal{O}(\varepsilon^4). \tag{B2}$$

After differentiating and eliminating the occurring accelerations by making use of Eqs. (B2), we find

$$\varepsilon^2\left[m_\alpha x_\alpha^{(3)} - \sum_{s=0}^{1}\varepsilon^s B_{\alpha,3,s}(x, x^{(1)})\right] = \mathcal{O}(\varepsilon^4). \quad \text{(B3)}$$

Requiring that the above constraints are satisfied for all times yields the time-stability condition

$$\varepsilon^2\left[m_\alpha x_\alpha^{(4)} - \sum_{s=0}^{1}\varepsilon^s B_{\alpha,4,s}(x, x^{(1)})\right] = \mathcal{O}(\varepsilon^4). \quad \text{(B4)}$$

(ii) Additional constraints emerge from the latter time-stability condition as well as from the equations of motion. They are derived by multiplying the Euler-Lagrange equations by $\varepsilon$ and thereafter eliminating all second and higher order time derivatives and found beyond the leading term with the help of Eqs. (B2)–(B4). A last time differentiation and elimination of the newly appeared accelerations leads to the constraints

$$\varepsilon\left[m_\alpha x_\alpha^{(r)} - \sum_{s=0}^{2}\varepsilon^s B_{\alpha,r,s}(x, x^{(1)})\right] = \mathcal{O}(\varepsilon^4), \quad r = 2, 3, \quad \text{(B5)}$$

which are not yet time stable. The time-stability condition

$$\varepsilon\left[m_\alpha x_\alpha^{(4)} - \sum_{s=0}^{2}\varepsilon^s B_{\alpha,4,s}(x, x^{(1)})\right] = \mathcal{O}(\varepsilon^4), \quad \text{(B6)}$$

inserted together with (B5) into the equations of motion yields the more restrictive set of constraints

$$m_\alpha x_\alpha^{(r)} - \sum_{s=0}^{2}\varepsilon^s B_{\alpha,r,s}(x, x^{(1)}) = \mathcal{O}(\varepsilon^4), \quad r = 2, 3. \quad \text{(B7)}$$

These constraints are stable, for inserting all constraints plus the time-stability condition into the equations of motion yields a constraint [namely, (B7) with $r = 2$] that is actually satisfied on the constraint surface. In contrast to Appendix A, the time-stability condition, and thus the equations of motion had to be employed twice to determine the constraints of 3pN dynamics. Finally, we note that it is now possible to argue as in Sec. II that further null vectors of the Hessian do not lead to further constraints.

## APPENDIX C: 3PN ELEMENTARY DIRAC BRACKETS

The issue of this appendix is to present the explicit computation of the Dirac brackets for the dynamics of interest in this article. The Dirac bracket is bilinear and acts as a derivation on each argument. To know its action on any two functions of the phase space, it is therefore sufficient to know all elementary Dirac brackets, i.e., the Dirac brackets of the constraint surface coordinates. In order to compute them, we have first to determine the Poisson-bracket matrix $D$ of the constraints up to the needed third order in $\varepsilon$, appropriately adapting the general method given by [34] to the 3pN case. All expressions reduced to second order in $\varepsilon$ can be directly used for the post-Coulombian dynamics case.

We start by splitting $D$ into $6 \times 6$ submatrices as

$$D = \begin{pmatrix} S & T \\ -T^T & U \end{pmatrix}, \quad \text{(C1)}$$

$$S_{\alpha,\beta} = \{\chi_{1\alpha}, \chi_{1\beta}\} = \varepsilon^2 \frac{\partial \Phi_{1\beta}}{\partial x_\alpha^{(1)}} - \varepsilon^2 \frac{\partial \Phi_{1\alpha}}{\partial x_\beta^{(1)}} + \mathcal{O}(\varepsilon^4), \quad \text{(C2)}$$

$$T_{\alpha,\beta} = \{\chi_{1\alpha}, \omega_{1\beta}\} = -\delta_{\alpha\beta} - \varepsilon \frac{1}{m_\beta}\frac{\partial \Phi_{0\beta}}{\partial x_\alpha^{(1)}} + \varepsilon^2 \frac{1}{m_\beta}\frac{\partial \Phi_{1\alpha}}{\partial x_\beta} + \mathcal{O}(\varepsilon^4), \quad \text{(C3)}$$

$$U_{\alpha,\beta} = \{\omega_{1\alpha}, \omega_{1\beta}\} = \varepsilon \frac{1}{m_\alpha m_\beta}\left(\frac{\partial \Phi_{0\beta}}{\partial x_\alpha} - \frac{\partial \Phi_{0\alpha}}{\partial x_\beta}\right) + \mathcal{O}(\varepsilon^4). \quad \text{(C4)}$$

This matrix is clearly invertible. To obtain iteratively the inverse matrix of $D$, we write the expansion in powers of $\varepsilon$ as

$$D = \sum_{s=0}^{n} \varepsilon^s\, {}^sD + \mathcal{O}(\varepsilon^{n+1}). \quad \text{(C5)}$$

It can be easily verified, that $D^{-1}$ up to $n$th order is given by

$$D^{-1} = \left(\mathbf{1} + \sum_{s=1}^{n}\varepsilon^s\, {}^sN\right){}^0D^{-1} + \mathcal{O}(\varepsilon^{n+1}), \quad \text{(C6)}$$

where ${}^sN$ is computed iteratively from

$${}^sN = -{}^0D^{-1}\, {}^sD - \sum_{r=1}^{s-1} {}^0D^{-1}\, {}^{s-r}D\, {}^rN. \quad \text{(C7)}$$

In our problem, we need $D^{-1}$ up to the third order. Its explicit expression reads

$$D^{-1} = \left(\mathbf{1} + \sum_{s=1}^{3} \varepsilon^s\, {}^sN\right) {}^0D^{-1} + \mathcal{O}(\varepsilon^4)$$
$$= {}^0D^{-1} - \varepsilon\, {}^0D^{-1}\, {}^1D\, {}^0D^{-1} + \varepsilon^2(-{}^0D^{-1}\, {}^2D\, {}^0D^{-1} + {}^0D^{-1}\, {}^1D\, {}^0D^{-1}\, {}^1D\, {}^0D^{-1}) + \varepsilon^3(-{}^0D^{-1}\, {}^3D\, {}^0D^{-1}$$
$$+ {}^0D^{-1}\, {}^2D\, {}^0D^{-1}\, {}^1D\, {}^0D^{-1} + {}^0D^{-1}\, {}^1D\, {}^0D^{-1}\, {}^2D\, {}^0D^{-1} - {}^0D^{-1}\, {}^1D\, {}^0D^{-1}\, {}^1D\, {}^0D^{-1}\, {}^1D\, {}^0D^{-1}) + \mathcal{O}(\varepsilon^4). \quad \text{(C8)}$$

At last, we split $D^{-1}$ into submatrices,

$$D^{-1} = \begin{pmatrix} X & Y \\ -Y^T & Z \end{pmatrix}, \quad \text{(C9)}$$

and account for

$$\{x_\alpha, \omega_{1\beta}\} = \mathcal{O}(\varepsilon^0), \qquad \{x_\alpha, \chi_{1\beta}\} = \mathcal{O}(\varepsilon^4),$$
$$\{\Pi_{0\alpha}, \omega_{1\beta}\} = \mathcal{O}(\varepsilon), \qquad \{\Pi_{0\alpha}, \chi_{1\beta}\} = \mathcal{O}(\varepsilon^2).$$

From the expression (33) for the Dirac bracket we conclude directly that, to determine the elementary Dirac brackets up to the desired third order, we need $Y$ up to order 1, $Z$ up to order 2, and that we do not need $X$ at all. The components of the relevant submatrices are

$$Y_{\alpha,\beta} = \delta_{\alpha\beta} - \varepsilon \frac{1}{m_\alpha} \frac{\partial \Phi_{0\alpha}}{\partial x_\beta^{(1)}} + \mathcal{O}(\varepsilon^2), \quad \text{(C10)}$$

$$Z_{\alpha,\beta} = \varepsilon^2\left[\frac{\partial \Phi_{1\beta}}{\partial x_\alpha^{(1)}} - \frac{\partial \Phi_{1\alpha}}{\partial x_\beta^{(1)}}\right] - \varepsilon^3\left[\sum_\gamma \left(\frac{\partial \Phi_{1\gamma}}{\partial x_\alpha^{(1)}} - \frac{\partial \Phi_{1\alpha}}{\partial x_\gamma^{(1)}}\right) \times \left(\frac{1}{m_\gamma}\frac{\partial \Phi_{0\gamma}}{\partial x_\beta^{(1)}}\right) - \left(\frac{\partial \Phi_{1\gamma}}{\partial x_\beta^{(1)}} - \frac{\partial \Phi_{1\beta}}{\partial x_\gamma^{(1)}}\right)\left(\frac{1}{m_\gamma}\frac{\partial \Phi_{0\gamma}}{\partial x_\alpha^{(1)}}\right)\right] + \mathcal{O}(\varepsilon^4). \quad \text{(C11)}$$

Hence the elementary Dirac brackets read

$$\{x_\alpha, x_\beta\}^* = \frac{1}{m_\alpha m_\beta} Z_{\alpha,\beta} + \mathcal{O}(\varepsilon^4), \quad \text{(C12)}$$

$$\{x_\alpha, \Pi_\beta\}^* = \delta_{\alpha\beta} + \varepsilon^2 \frac{1}{m_\alpha}\frac{\partial \Phi_{1\alpha}}{\partial x_\beta} + \varepsilon^3 \sum_\gamma \frac{1}{m_\alpha m_\gamma}\left[\left(\frac{\partial \Phi_{1\gamma}}{\partial x_\alpha^{(1)}} - \frac{\partial \Phi_{1\alpha}}{\partial x_\gamma^{(1)}}\right)\frac{\partial \Phi_{0\gamma}}{\partial x_\beta} - \frac{\partial \Phi_{0\gamma}}{\partial x_\alpha^{(1)}}\frac{\partial \Phi_{1\gamma}}{\partial x_\beta}\right] + \mathcal{O}(\varepsilon^4), \quad \text{(C13)}$$

$$\{\Pi_\alpha, \Pi_\beta\}^* = \epsilon^3 \sum_\gamma \frac{1}{m_\gamma}\left(\frac{\partial \Phi_{0\gamma}}{\partial x_\alpha}\frac{\partial \Phi_{1\gamma}}{\partial x_\beta} - \frac{\partial \Phi_{0\gamma}}{\partial x_\beta}\frac{\partial \Phi_{1\gamma}}{\partial x_\alpha}\right) + \mathcal{O}(\varepsilon^4). \quad \text{(C14)}$$

Finally, we explicitly display the relation (C12) up to second order for both 3pN and 2pC dynamics

$$\text{2pN: } \{x_{ai}, x_{bj}\}^* = \varepsilon^2 \frac{G}{4}\left[7\left(\frac{\Pi_{0ai}}{m_a} + \frac{\Pi_{0bi}}{m_b}\right)n_{abj} - 7\left(\frac{\Pi_{0aj}}{m_a} + \frac{\Pi_{0bj}}{m_b}\right)n_{abi} + (7\delta_{ij} + n_{abi}n_{abj})\left(\sum_{k=1}^{3} n_{abk}\left(\frac{\Pi_{0ak}}{m_a} + \frac{\Pi_{0bk}}{m_b}\right)\right)\right], \quad \text{(C15)}$$

$$\text{2pC: } \{x_{ai}, x_{bj}\}^* = -\varepsilon^2 \frac{e_a e_b}{m_a m_b}\left[\left(\frac{\Pi_{0ai}}{m_a} + \frac{\Pi_{0bi}}{m_b}\right)n_{abj} - \left(\frac{\Pi_{0aj}}{m_a} + \frac{\Pi_{0bj}}{m_b}\right)n_{abi} + (\delta_{ij} + n_{abi}n_{abj})\left(\sum_{k=1}^{3} n_{abk}\left(\frac{\Pi_{0ak}}{m_a} + \frac{\Pi_{0bk}}{m_b}\right)\right)\right]. \quad \text{(C16)}$$


[1] A. G. Lyne *et al.*, astro-ph/0401086.
[2] L. Blanchet, Living Rev. Relativity **5** (2002) http://relativity.livingreviews.org/Articles/lrr-2002-3/index.html.
[3] J. Hough and S. Rowan, Living Rev. Relativity **3** (2000) http://relativity.livingreviews.org/Articles/lrr-2000-3/index.html.
[4] A. Buonanno, Phys. Rev. D **62**, 104022 (2000).
[5] M. Kunze and H. Spohn, J. Nonlinear Sci. **11**, 321 (2001).
[6] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland Publishing Company, Amsterdam, 1983).
[7] L. Blanchet and G. Faye, Phys. Lett. A **271**, 58 (2000).
[8] L. Blanchet and G. Faye, J. Math. Phys. (N.Y.) **41**, 7675 (2000).
[9] L. Blanchet and G. Faye, J. Math. Phys. (N.Y.) **42**, 4391 (2001).

[10] L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001).
[11] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D **69**, 124007 (2004).
[12] Y. Itoh and T. Futamase, Phys. Rev. D **68**, 121501 (2003).
[13] Y. Itoh, Phys. Rev. D **69**, 064018 (2004).
[14] P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998).
[15] P. Jaranowski and G. Schäfer, Phys. Rev. D **63**, 029902 (2001).
[16] P. Jaranowski and G. Schäfer, Phys. Rev. D **60**, 124003 (1999).
[17] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **62**, 021501(R) (2000).
[18] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 029903 (2001).
[19] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **63**, 044021 (2001).
[20] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Lett. B **513**, 147 (2001).
[21] V. C. Andrade, L. Blanchet, and G. Faye, Classical Quantum Gravity **18**, 753 (2001).
[22] A. Wheeler and R. Feynman, Rev. Mod. Phys. **21**, 425 (1949).
[23] E. H. Kerner, J. Math. Phys. (N.Y.) **3**, 35 (1962).
[24] J. Martin and J. L. Sanz, J. Math. Phys. (N.Y.) **20**, 25 (1979).
[25] J. Martin and J. L. Sanz, J. Math. Phys. (N.Y.) **19**, 780 (1978).
[26] R. P. Gaida, Y. B. Kluchkovsky, and V. I. Tretyak, in *Constraint's Theory and Relativistic Dynamics*, edited by G. Longhi and L. Lusanna (World Scientific, Singapore, 1987).
[27] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).
[28] J. T. Cannon and T. F. Jordan, J. Math. Phys. (N.Y.) **5**, 299 (1964).
[29] H. Leutwyler, J. Math. Phys. (N.Y.) **5**, 219 (1964).
[30] T. Damour and G. Schäfer, J. Math. Phys. (N.Y.) **32**, 127 (1991).
[31] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[32] P. A. M. Dirac, Proc. R. Soc. Edinburgh, Sect. A: Math **246**, 326 (1958).
[33] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).
[34] X. Jaen, J. Llosa, and A. Molina, Phys. Rev. D **34**, 2302 (1986).
[35] T. Damour and G. Schäfer, Phys. Rev. D **37**, 1099 (1988).
[36] Y. Saito, R. Sugano, T. Ohta, and T. Kimura, J. Math. Phys. (N.Y.) **30**, 1122 (1989).
[37] Y. Saito, R. Sugano, T. Ohta, and T. Kimura, J. Math. Phys. (N.Y.) **34**, 3775 (1989).
[38] X. Gràcia, J. M. Pons, and N. Román-Roy, J. Math. Phys. (N.Y.) **32**, 2745 (1991).
[39] T. Ohta and T. Kimura, J. Math. Phys. (N.Y.) **34**, 4655 (1993).
[40] B. R. Iyer and L. Blanchet, Classical Quantum Gravity **20**, 755 (2003).
[41] R.-M. Memmesheimer, A. Gopakumar, and G. Schäfer, Phys. Rev. D **70** 104011 (2004).
[42] T. Damour, C.R. Acad. Sci. Paris, Série II **294**, 1355 (1982).
[43] J. Stachel and P. Havas, Phys. Rev. D **13**, 1598 (1976).
[44] M. Borneas, Phys. Rev. **186**, 1299 (1969).
[45] T. Damour and G. Schäfer, Gen. Relativ. Gravit. **17**, 879 (1985).
[46] K. Sundermeyer, *Constrained Dynamics* (Springer, Heidelberg, 1982).
[47] E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics—A Modern Perspective* (Krieger, Malabar, 1983).
[48] B. M. Barker and R. F. O'Connell, Ann. Phys. (N.Y.) **129**, 358 (1980).
[49] B. M. Barker and R. F. O'Connell, Can. J. Phys. **58**, 1659 (1980).
[50] M. V. Ostrogradski, Mem. Acad. St. Petersburg **6**, 85 (1850).
[51] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, 1999).
[52] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
[53] R.-M. Memmesheimer Diploma thesis, Friedrich-Schiller University Jena, 2004.
[54] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1999).
[55] C.-M. Marle, Université Pierre et Marie Curie, (1998) www.institut.math.jussieu.fr/~marle/.
[56] R. U. Sexl and H. K. Urbantke, *Relativity, Groups, Particles* (Springer, Vienna, 2000).