Post-1-Newtonian equations of motion for systems of arbitrarily structured bodies

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We give a surface-integral derivation of post-1-Newtonian translational equations of motion for a system of arbitrarily structured bodies, including the coupling to all the bodies' mass and current multipole moments. The derivation requires only that the post-1-Newtonian vacuum field equations are satisfied in weak field regions between the bodies; the bodies' internal gravity can be arbitrarily strong. In particular, black holes are not excluded. The derivation extends previous results due to Damour, Soffel, and Xu (DSX) for weakly self-gravitating bodies in which the post-1-Newtonian field equations are satisfied everywhere. The derivation consists of a number of steps: (i) The definition of each body's current and mass multipole moments and center-of-mass world line in terms of the behavior of the metric in a weak field region surrounding the body. (ii) The definition for each body of a set of gravitoelectric and gravitomagnetic tidal moments that act on that body, again in terms of the behavior of the metric in a weak field region surrounding the body. For the special case of weakly self-gravitating bodies, our definitions of these multipole and tidal moments agree with definitions given previously by DSX. (iii) The derivation of a formula, for any given body, of the second time derivative of its mass dipole moment in terms of its other multipole and tidal moments and their time derivatives. This formula was obtained previously by DSX for weakly self-gravitating bodies. (iv) A derivation of the relation between the tidal moments acting on each body and the multipole moments and center-of-mass world lines of all the other bodies. A formalism to compute this relation was developed by DSX; we simplify their formalism and compute the relation explicitly. (v) The deduction from the previous steps of the explicit translational equations of motion, whose form has not been previously derived.

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I. INTRODUCTION AND SUMMARY

A. Background and motivation

For slow motion sources in weak gravitational fields, general relativity can be accurately described in terms of a post-Newtonian approximation scheme. This approximation scheme is extremely useful in applications and is very well developed. Reviews of post-Newtonian theory can be found in Refs. [1–3] and in the textbook by Will [4].

There are several different types of equations that arise in post-Newtonian theory. First, one has continuum field equations, which are usually specialized to gravity coupled to perfect or imperfect fluids. These have been derived up to post-2.5-Newtonian order [5]. At post-1-Newtonian order, they have been extended beyond general relativity to encompass the class of theories of gravity described by the parametrized post-Newtonian framework [4].

A second type of equation of motion applies to systems consisting of N interacting, extended bodies moving under their mutual gravitational interactions, in the limit where the bodies' sizes are small compared to their mutual separations. For such systems one has ''point-particle'' equations of motion. Such equations were first derived [1] at post-1-Newtonian order by Lorentz and Droste [6], and later independently by Einstein, Infeld, and Hoffmann (EIH) [7]. They were also independently derived by Petrova [8] using a method devised by Fock [9]. These equations are usually called the EIH equations. In recent years the advent of gravitational-wave astronomy [10,11] has spurred renewed interest in such equations of motion. For coalescing binary systems, the waveforms of the emitted gravitational waves are expected to carry a great deal of information, and full exploitation of the expected observations will require accurate theoretical models of the waveforms [11]. This requirement has prompted the computation of point-particle equations of motion (as well as radiation reaction effects) to higher and higher post-Newtonian orders. Most recently the coalescence waveform's phase has been computed up to post-3.5-Newtonian order [12]; see also Refs. [13,14]. At post-5-Newtonian order and higher, the concept of point-particle equations of motion will break down due to effects related to the finite sizes of the bodies [15]. However an argument due to Damour [15] indicates that the point-particle equations should be well defined at lower orders, up to and including post-4.5-Newtonian.

A third type of equation of motion applies to systems of N interacting bodies whose sizes cannot be neglected. These equations consist of the point-particle equations of motion supplemented by tidal interaction terms. In principle, if one included tidal interactions to all multipole orders, and in addition coupled the equations of motion to a dynamical description of the internal degrees of freedom in each body, one would obtain a complete description of the system, equivalent to that provided by the continuum equations of motion (up to radiative effects).

For a system of bodies of typical size $\sim R$, of typical mass $\sim M$, and with typical separations $\sim D$, the force F that acts on one of the bodies can be written schematically as[1]

$$F \sim \frac{M^2}{D^2}\left\{1 + O\left(\frac{M}{D}\right) + O\left(\frac{M^2}{D^2}\right) + \ldots + O\left[\left(\frac{R}{D}\right)^l\right] + O\left[\frac{M}{D}\left(\frac{R}{D}\right)^l\right] + \ldots\right\}. \quad (1.1)$$

Here we use geometric units with $G = c = 1$. The terms inside the curly brackets are as follows. On the first line, the one is the usual Newtonian force between two point particles and the second and third terms are the post-1-Newtonian and post-2-Newtonian point-particle corrections. On the second line, the first term is the correction due to Newtonian tidal couplings; the minimum value of l allowed is $l = 2$ corresponding to quadrupolar coupling. The second term describes the post-1-Newtonian tidal couplings. Here the minimum allowed value l is lower than in the Newtonian case due to gravitomagnetic interactions which have no Newtonian analogs. This minimum value is $l = 1/2$, corresponding to spin-orbit couplings (assuming that the bodies' internal velocities are maximal, $v \sim \sqrt{M/R}$).

The purpose of this paper is to compute in detail the post-1-Newtonian tidal interaction terms in Eq. (1.1), for all values of l, for a system of N bodies. The explicit form of these terms has not been derived before, although there is substantial literature on this topic [2,16–26]. There are a number of motivations for this computation. First, as described in Refs. [2,20], in the area of celestial mechanics future experiments and observations in the solar system will provide very high precision data. For example, there are current plans to increase the accuracy of lunar laser ranging from the current centimeter level to the millimeter level [27]. The future astrometric missions Space Interferometry Mission (SIM) and Global Astrometric Interferometer for Astrophysics (GAIA) are expected to measure angles to an accuracy of a few microarcseconds, as compared to the current accuracy of milliarcseconds. In the radio, very long base line interferometry (VLBI) observations currently can yield precisions of order 10 microarcseconds [28]. Also, the proposed future laser astrometric test of relativity (LATOR) mission [29] would be sensitive to post-2-Newtonian effects, and therefore would likely require detailed modeling of post-1-Newtonian tidal effects.

Second, gravitational-wave measurements of coalescing binary compact stars will likely have some ability to detect finite size effects for sufficiently strong signals [30]. Although post-Newtonian tidal effects will in many cases be small compared to Newtonian tidal effects, there are some situations where the post-1-Newtonian effects dominate. An example is the gravitomagnetic resonant excitation of Rossby modes in neutron stars that are spinning at $\sim$100 Hz, which could be detectable with LIGO for moderately strong detected inspirals [31].

B. Tidal coupling in post-Newtonian theory

The textbook treatment of post-1-Newtonian gravity [4] is inadequate for the treatment of tidal interactions for several reasons, as explained by Damour *et al.* [2]. First, the standard treatment uses a single global coordinate system. Although one can write down the continuum equations of motion for a given body in that coordinate system, it is very difficult to separate out the gravitational influences of the other bodies from the self-field of the body, since the fractional distortions of the coordinate system produced by the other bodies can be large even when the tidal distortion of the star is negligible. The development of approximation schemes such as linear perturbations about an equilibrium state is hindered by the fact that the equilibrium state is not described in the usual way in the global coordinates.

This difficulty has been comprehensively addressed in a series of papers by Brumberg and Kopeikin (BK) [22–24] and by Damour, Soffel, and Xu (DSX) [2,16–18]. These authors developed a detailed theory of post-1-Newtonian reference frames, in which each body has associated with it a coordinate system naturally adapted to that body. DSX also developed a formalism to compute translational equations of motion including the coupling to all the mass and current multipole moments of each body.[2] They applied their formalism to compute equations of motion including spin and quadrupole couplings. In this paper, we extend the DSX results in two ways. First, by simplifying their formalism we are able to compute the explicit form of the translational equations of motion, including all the multipole couplings. Second, we give a derivation that is valid for strongly self-gravitating objects as well as weakly self-gravitating objects.[3] We need only assume that the post-1-Newtonian field equations are satisfied in a weak field region surrounding each body. The bodies' internal gravitational fields can be arbitrarily strong; in particular, our assumptions do not exclude black holes. By contrast, DSX assumed the global validity of the post-1-Newtonian continuum field equations, and so their derivation applies only to weakly self-gravitating objects. Our result also generalizes existing derivations of the Newtonian [21] and post-1-Newtonian [33,34] equations of motion for strongly self-gravitating objects that incorporate only a few low-order

[1]These scalings apply to generic bodies; if the bodies are spherically symmetric the scalings are of course altered.

[2]The BK and DSX formalisms have recently been generalized to the parametrized post-Newtonian framework for scalar-tensor theories of gravity by Kopeikin and Vlasov [32].

[3]That is, we show that the dominant fractional errors scale as $O(M^2/D^2)$; global post-Newtonian methods [2,16,17] show only that these errors are $O(M^2/R^2)$ or smaller.

multipoles. Similar derivations to higher post-Newtonian orders including monopole terms only can be found in Refs. [35,36].

One of the key ideas underlying our derivation is that the equations of motion are determined entirely by the local field equations in weak field regions between the bodies. This was originally pointed out by Weyl and by Einstein and Grommer; see Thorne and Hartle [34] and references therein. Each body is surrounded by a vacuum, weak field region called a "buffer region" [34], and the quantities entering into the equations of motion are defined in terms of the behavior of the metric in those buffer regions (see Fig. 1). In particular, our multipole moments are defined in terms of the behavior of the metric in the buffer regions. Our definition of multipole moments is thus more general than the definition in terms of integrals over sources used by DSX. However, our multipole moments do coincide with those of DSX in the case of weakly self-gravitating bodies.

Our derivation consists of a number of steps: (i) The definition of each body's current and mass multipole moments and center-of-mass world line in terms of the behavior of the metric in that body's buffer region. (ii) The definition for each body of a set of gravitoelectric and gravitomagnetic tidal moments that act on that body, again in terms of the behavior of the metric in that body's buffer region. For the special case of weakly self-gravitating bodies, our definitions of these multipole and tidal moments agree with definitions given previously by DSX. (iii) The derivation of a formula, for any given body, of the second time derivative of its mass dipole moment in terms of its other multipole and tidal moments and their time derivatives. This formula was obtained previously by DSX for weakly self-gravitating bodies. (iv) A derivation of the relation between the tidal moments acting on each body and the multipole moments and center-of-mass world lines of all the other bodies. A formalism to compute this relation was developed by DSX; we simplify their formalism and compute the relation explicitly. (v) The deduction from the previous steps of the explicit translational equations of motion, whose form has not been previously derived.

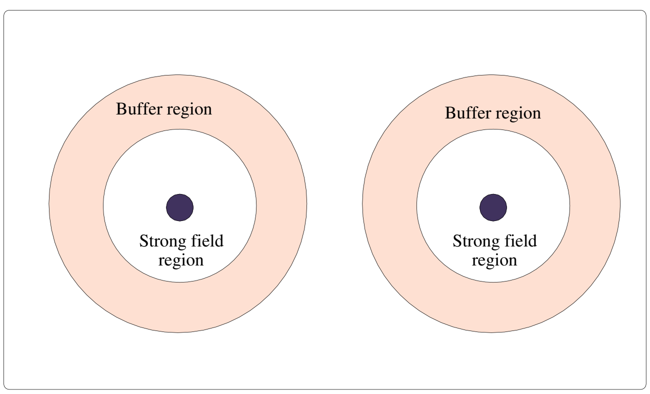

FIG. 1 (color online). An illustration of our assumptions for a system of N bodies. Each body is surrounded by a strong field region which is excluded from our analysis. Surrounding these strong field regions are weak field buffer regions. Each body's center-of-mass world line and mass and current multipole moments are defined in terms of the behavior of the metric in that body's buffer region. We assume that the vacuum post-1-Newtonian field equations are satisfied in all the buffer regions and in the regions of space between the buffer regions.

C. Results for equations of motion

We next describe our results for the equations of motion. We label each body by an integer A, with $1 \leq A \leq N$. We use a harmonic coordinate system (t, x^i) that covers all of spacetime except for the strong field regions near each body. The position of body A in this coordinate system is parametrized by a function $x^i = {}^{\rm cm}z_i^A(t)$ called the "center-of-mass world line." This function is defined precisely in Sec. V C below. It does not correspond to an actual world line in spacetime; rather it parametrizes the location of the local asymptotic rest frame (see below) attached to the Ath body. That is, it is encoded in the behavior of the metric in a weak field region surrounding body A in the same way that the actual center-of-mass world line of a weakly self-gravitating body would be encoded.

Associated with each body A is a coordinate system (s_A, y_A^i) which is defined only in that body's buffer region, and which is adapted to the body in the sense that it minimizes the coordinate effects of the external gravitational field due to the other bodies as much as possible. This coordinate system is discussed in detail in Secs. III D and V A below. We will call the corresponding reference frame the "body-frame" or, following Thorne and Hartle [34], the body's "local asymptotic rest frame." The details of the transformation between the body-adapted coordinates (s_A, y_A^i) and the global coordinates (t, x^i) are important for the purpose of deriving the translational equations of motion. However, for the purpose of using the equations of motion, one only needs to know the following. First, the time coordinate s_A is a "proper time" associated with body A. It corresponds to the proper time that would be measured by an observer in the local asymptotic rest frame of body A. In that local asymptotic rest frame it is related to the global-frame time coordinate t by

$$s_A = s_A(t), \tag{1.2}$$

where the function $s_A(t)$ is determined by a differential equation [Eqs. (1.7d) and (1.8) below]. Second, the leading order relation between the spatial coordinates y_A^i and x^i is just a translation together with a time-dependent rotation [cf. Eqs. (5.4), (5.10), and (5.38) below]:

$$x^i = {}^{\rm cm}z_i^A(t) + U_i^{Aj}(t) y_A^j. \tag{1.3}$$

The rotation matrix $U_i^{Aj}(t)$ describes dragging of inertial frames[4]; a differential equation for its evolution is given below. The body-adapted coordinates (s_A, y_A^i) rotate with respect to distant stars, while the global coordinates (t, x^i) do not.

Each body A has associated with it a unique set of mass multipole moments

$$M_L^A(s_A) = M_{a_1\ldots a_l}^A(s_A), \tag{1.4}$$

for $l = 0, 1, 2\ldots$ which are symmetric, trace-free, spatial tensors with l indices, of which mass dipole $M_i^A(s_A)$ vanishes identically. It also has a unique set of current multipole moments

$$S_L^A(s_A) = S_{a_1\ldots a_l}^A(s_A), \tag{1.5}$$

for $l = 1, 2, \ldots$. These quantities are functions of the body's proper time s_A. In the absence of interactions with other bodies the mass monopole M^A and the spin S_i^A are conserved.

We will obtain below coupled equations of motion for the center-of-mass world lines of all the bodies. Appearing in these equations as unknowns will be the mass multipole moments $M_L^A(s_A)$ for $l \geq 2$, and the current multipole moments $S_L^A(s_A)$ for $l \geq 2$. In order to obtain a closed system of equations, one would need to supplement the equations of this paper with equations determining the evolution of these multipole moments. We discuss further below various circumstances and approximations in which the evolution of the multipoles can be computed.

Next, we define the moments $\mathcal{M}_L^A(t)$ and $\mathcal{S}_L^A(t)$ to be the body's mass and current multipole moments, transformed to the nonrotating frame, and expressed as functions of the global time t. These moments are given by the equations [cf. Eqs. (5.35) and (5.36) below]

$$\mathcal{M}_{a_1\ldots a_l}^A(t) = U_{a_1}^{Aa_1'}(t)\ldots U_{a_l}^{Aa_l'}(t) M_{a_1'\ldots a_l'}^A[s_A(t)], \tag{1.6a}$$

$$\mathcal{S}_{a_1\ldots a_l}^A(t) = U_{a_1}^{Aa_1'}(t)\ldots U_{a_l}^{Aa_l'}(t) S_{a_1'\ldots a_l'}^A[s_A(t)]. \tag{1.6b}$$

We can now write down the schematic form of the equations of motion. They can be written as

$${}^{\rm cm}\ddot{z}_i^A(t) = \mathcal{F}_i^A[{}^{\rm cm}z_i^B, {}^{\rm cm}\dot{z}_i^B, \mathcal{M}_L^B, \dot{\mathcal{M}}_L^B, \ddot{\mathcal{M}}_L^B, \mathcal{S}_L^B, \dot{\mathcal{S}}_L^B], \tag{1.7a}$$

$$\dot{\mathcal{M}}^A(t) = \mathcal{F}^A[{}^{\rm cm}z_i^B, {}^{\rm cm}\dot{z}_i^B, \mathcal{M}_L^B, \dot{\mathcal{M}}_L^B], \tag{1.7b}$$

$$\dot{\mathcal{S}}_i^A(t) = \bar{\mathcal{F}}_i^A[{}^{\rm cm}z_i^B, \mathcal{M}_L^B], \tag{1.7c}$$

$$\frac{ds_A}{dt} = \bar{\mathcal{F}}^A[{}^{\rm cm}z_i^B, {}^{\rm cm}\dot{z}_i^B, \mathcal{M}_L^B], \tag{1.7d}$$

$$[\dot{\mathbf{U}}^A \cdot (\mathbf{U}^A)^{-1}]_{ij} = \mathcal{F}_{ij}^A[{}^{\rm cm}z_i^B, {}^{\rm cm}\dot{z}_i^B, \mathcal{M}_L^B, \dot{\mathcal{M}}_L^B, \mathcal{S}_L^B]. \tag{1.7e}$$

Here $\mathcal{F}_i^A$, $\mathcal{F}^A$, $\bar{\mathcal{F}}_i^A$, $\bar{\mathcal{F}}^A$, and $\mathcal{F}_{ij}^A$ are functions of their argument whose specific forms are discussed below. In these equations the dependencies on the time derivatives $\dot{\mathcal{M}}_L^B$, $\ddot{\mathcal{M}}_L^B$, and $\dot{\mathcal{S}}_L^B$ only occur for $l \geq 2$. Also the mass dipoles $\mathcal{M}_i^B$ vanish identically. Therefore, if we assume that the moments $M_L^A(s_A)$ and $S_L^A(s_A)$ are known for $l \geq 2$, Eqs. (1.6a), (1.6b), and (1.7a)–(1.7e) form a closed set of evolution equations which can be solved to obtain the center-of-mass world lines as well as the rotation matrices $\mathbf{U}^A$ and time functions $s_A(t)$.

We remark that the three Eqs. (1.7a)–(1.7c) by themselves form a closed set of evolution equations for the variables ${}^{\rm cm}z_i^A(t)$, $\mathcal{M}^A(t)$, and $\mathcal{S}_i^A(t)$, if we assume that the moments $\mathcal{M}_L^A(t)$ and $\mathcal{S}_L^A(t)$ are known for $l \geq 2$. However, approximation schemes for computing the multipole moments for $l \geq 2$ usually yield the variables $M_L^A(s_A)$, $S_L^A(s_A)$ rather than the variables $\mathcal{M}_L^A(t)$, $\mathcal{S}_L^A(t)$. This is because the moments $M_L^A(s_A)$ and $S_L^A(s_A)$ are the physical moments that would be measured by an observer in the local asymptotic rest frame of body A. In such cases we must enlarge the set of variables ${}^{\rm cm}z_i^A(t)$, $\mathcal{M}^A(t)$, $\mathcal{S}_i^A(t)$ to include the rotation matrices $\mathbf{U}^A(t)$ and time functions $s_A(t)$ in order to obtain a closed set of equations.

We also note that it is formally consistent to post-1-Newtonian accuracy to replace Eq. (1.6b) with the simpler relation $\mathcal{S}_L^A(t) = S_L^A(t)$. Nevertheless it can be useful in some circumstances to use the more accurate relation (1.6b), for example, for systems which evolve for sufficiently long times that the rotation matrices $U_a^{Aa'}$ become significantly different from unity.

We now discuss the functions $\mathcal{F}_i^A$, $\mathcal{F}^A$, $\bar{\mathcal{F}}_i^A$, $\bar{\mathcal{F}}^A$, and $\mathcal{F}_{ij}^A$ that appear in Eqs. (1.7a)–(1.7e). The functional form of $\mathcal{F}_i^A$ is one of our key results. It is given by Eq. (6.11) below, with the coefficients modified according to the substitutions given in Eq. (6.19) and in Appendix F. These modified coefficients are obtained by combining the results of this paper with those of the second paper in this series [37], which we will call paper II. The functions $\mathcal{F}^A$ and $\bar{\mathcal{F}}_i^A$ are standard functions that can be derived from Newtonian stress-energy conservation for weakly self-gravitating bodies, and their explicit functional forms are, respectively, given in section IV of paper II [37] and in Eq. (F1) below. The validity of Eqs. (1.7b) and (1.7c) for strongly self-gravitating bodies is derived in paper II [37]. The function $\bar{\mathcal{F}}^A$ is given by [cf. Eqs. (5.4), (5.32), (5.23a), and (5.20)

[4]The time derivative of this rotation matrix is actually of post-1-Newtonian order, so to Newtonian order this is a constant matrix. In the body of the paper we assumed that this constant, Newtonian order rotation matrix is the unit matrix. The description of our results given here allows this constant matrix to be arbitrary; this slight generalization would be useful to describe systems that evolve for a time long enough that the accumulated rotation due to frame dragging is of order unity.

below]

$$\bar{\mathcal{F}}^A = 1 - \frac{1}{2}\,{}^{\rm cm}\dot{z}_i^A\,{}^{\rm cm}\dot{z}_i^A - \sum_{B\neq A}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{k!}\frac{\mathcal{M}_K^B}{r_{BA}^{k+1}}n_K^{BA}. \tag{1.8}$$

Here K is the multi-index $b_1 \ldots b_k$, $r_{BA} = |{}^{\rm cm}\boldsymbol{z}^B - {}^{\rm cm}\boldsymbol{z}^A|$, $\boldsymbol{n}^{BA} = ({}^{\rm cm}\boldsymbol{z}^B - {}^{\rm cm}\boldsymbol{z}^A)/r_{BA}$, and $n_K^{BA} = n_{b_1}^{BA}\ldots n_{b_k}^{BA}$. Finally, the function $\mathcal{F}_{ij}^A$ is given by Eq. (6.20) below.

We next discuss various approximation schemes in which the equations of motion (1.7a)–(1.7e) can be supplemented by methods for obtaining the evolution of the mass and current multipole moments $\mathcal{M}_L^A(t)$ and $S_L^A(t)$ for $l \geq 2$ in order to obtain a complete, closed set of equations. Some examples of such approximations are as follows.

(i) The simplest case is when the effect of all the $l \geq 2$ multipoles is negligible, and one can set $\mathcal{M}_L^A = S_L^A = 0$ for all $l \geq 2$. This yields the monopole-spin truncated equations of motion discussed in Refs. [16,34].

(ii) Another simple case is when the evolution of the multipoles of each body is dominated by dynamics internal to that body, and is negligibly influenced by the tidal fields of the other bodies. In this case, one can solve for the evolution of the multipoles $M_L^A(s_A)$, $S_L^A(s_A)$ of each body separately, and then insert those multipoles into the equations of motion (1.6a), (1.6b), and (1.7a)–(1.7e). This application will be valid only if the timescale over which the bodies' multipoles evolve is sufficiently long [38]; see Sec. I D below for further discussion of this point.

(iii) Another useful case to consider is that of rigid bodies. As noted by Thorne and Gürsel [39], in general relativity a body's rotation can be rigid only if its angular velocity (with respect to its local asymptotic rest frame) is constant. If the angular velocity is changing, for example, due to precession, then the body cannot be rigid due to Lorentz contraction effects. However, to linear order in the body's angular velocity the motion is rigid [39]. The analysis of Thorne and Gürsel can be adapted to the present context, if the bodies' rotations are slow enough that they can be idealized as rigid. In this case, the time dependence of the mass multipole moments $M_L^A(s_A)$ for $l \geq 2$ can be parametrized in terms of a time-dependent rotation matrix $\mathcal{U}_a^{A\bar{a}}(s_A)$:

$$M^A_{a_1\ldots a_l}(s_A) = \mathcal{U}_{a_1}^{A\bar{a}_1}(s_A)\ldots\mathcal{U}_{a_l}^{A\bar{a}_l}(s_A)M^A_{\bar{a}_1\ldots\bar{a}_l}.$$

Here the moments $M^A_{\bar{a}_1\ldots\bar{a}_l}$ are constant; these are the moments in the corotating frame which rotates with the body. We define the angular velocity $\Omega_a^A(s_A)$ in the usual way as $\dot{\mathcal{U}}_a^{A\bar{b}}\mathcal{U}_b^{A\bar{b}} = \epsilon_{acb}\Omega_c^A$. Then, the corotating frame spin $S_{\bar{a}}^A = \mathcal{U}_a^{A\bar{a}}S_a^A$ is related to the corotating frame angular velocity $\Omega_{\bar{a}}^A = \mathcal{U}_a^{A\bar{a}}\Omega_a^A$ via [39,40]

$$S_{\bar{a}}^A(s_A) = I_{\bar{a}\bar{b}}^A\Omega_{\bar{b}}^A(s_A),$$

where $I_{\bar{a}\bar{b}}^A$ is the (constant) moment of inertia tensor.[5] Similarly the higher-order current multipole moments are given by

$$S^A_{\bar{a}_1\ldots\bar{a}_l}(s_A) = I^A_{\bar{a}_1\ldots\bar{a}_l\bar{b}}\Omega^A_{\bar{b}}(s_A),$$

where $I^A_{\bar{a}_1\ldots\bar{a}_l\bar{b}}$ is a higher-order generalization of the moment of inertia tensor [39]. Combining these relations with the equations of motion(1.6a), (1.6b), and (1.7a)–(1.7e) yields a closed system of equations which can be solved for the center-of-mass world lines, the rotation $\mathcal{U}_a^{A\bar{a}}(s_A)$ of each body with respect to its local asymptotic rest frame (s_A, y_A^i), and the rotation $\mathbf{U}^A(t)$ of that local asymptotic rest frame with respect to distant stars. These equations describe torqued precession of relativistic objects, generalizing the free precession equations of Thorne and Gürsel [39].

(iv) For weakly self-gravitating bodies one can use the formalism developed by DSX [2,16,17] to obtain a post-1-Newtonian description of the internal dynamics of each body, for example, by using post-1-Newtonian stellar perturbation theory. Coupling such a description to the equations of motion yields a closed system of equations.

(v) Lastly, for fully relativistic, spherical stars, one can compute the leading order effects of tidal interactions by combining the results of this paper with linear relativistic stellar perturbation theory using matched asymptotic expansions; see, for example, Refs. [41–44]. For example, if one is interested only in the mass quadrupoles, and one restricts attention to the dominant, fundamental $l = 2$ modes with no radial nodes, then one has a relation of the form

$$M_{ij}^A(s_A) = \int ds'_A K(s_A - s'_A)G_{ij}^A(s'_A).$$

Here $K(s_A - s'_A)$ is a Green's function which can be computed from stellar perturbation theory, and G_{ij}^A is the body-frame gravitoelectric tidal moment that acts on body A, which is defined in Sec. V B below and which can be computed in terms of the world lines and multipole moments of the other bodies. Combining this relation with the equations of motion (1.7a)–(1.7e) again yields a closed system of equations, if one neglects the mass multipoles for $l \geq 3$ and the current multipoles.

D. Domain of validity of our results

As mentioned above, the key assumption which we make in deriving our results is that the post-1-Newtonian vacuum field equations are satisfied in a weak field region between the bodies; see Sec. VA below for more

[5]Thorne and Gürsel [39] have shown that for fully relativistic stars, as for Newtonian stars, the moment of inertia tensor is constant, independent of the angular velocity, up to linear order in the angular velocity.

details. We are unable to give a derivation of this assumption from first principles. However, in this subsection we discuss various physical effects which can cause our assumption to break down, and we make estimates of the sizes of these effects. We believe that the assumption should be generally valid aside from the effects discussed in this subsection.

The first type of correction are post-2-Newtonian corrections to the metric in the weak field, vacuum region between the bodies. These will give rise to fractional corrections of order M^2/D^2, where M is a typical mass and D a typical separation of the bodies, cf. the third term on the first line of Eq. (1.1). We can estimate as follows when these corrections will be larger than the tidal coupling terms which we retain. The estimate given in the last term of Eq. (1.1) can be refined by multiplying it by the dimensionless measure $\varepsilon_l = \mathcal{M}_L/(MR^l)$ of the lth mass multipole. Demanding that this quantity be larger than the post-2-Newtonian, point-particle term in Eq. (1.1) yields the criterion

$$D \lesssim \varepsilon_l^{1/(l-1)}\left(\frac{R}{M}\right)^{1/(l-1)} R. \tag{1.9}$$

Thus, the post-2-Newtonian terms will always dominate at sufficiently large separations D, but for any given $l \geq 2$ there will be a range of values of D for which the post-1-Newtonian tidal terms dominate, as long as $\varepsilon_l \gtrsim M/R$. In particular, this will be true for generic (nonsymmetric) bodies for which $\varepsilon_l \sim 1$. Similar estimates apply to current multipole couplings for $l \geq 2$. Thus, there is a nonempty regime in which the post-1-Newtonian tidal couplings computed here dominate over post-2-Newtonian, point-particle effects.

Note, however, that this range of values of D gets smaller as the strength of internal gravity $\sim M/R$ increases. In the limit of $M \sim R$ of a black hole, the post-2-Newtonian terms are always comparable to or larger than the post-1-Newtonian tidal terms. Therefore, our results cannot be applied consistently to black holes without including post-2-Newtonian and higher terms in the equations of motion.

Another type of correction, which is also formally of post-2-Newtonian order, is that due to the time dependence of the mass and current multipole moments of the individual bodies [38]. The post-1-Newtonian solutions [Eqs. (3.5a)–(3.5c) below] do not exhibit the correct retarded dependence on these moments, that is, they are functions of $M_L(t)$ and $S_L(t)$ rather than $M_L(t-r)$ and $S_L(t-r)$. If these moments vary on a timescale τ, then the corresponding fractional corrections to the mass moments M_L^A scale as D^4/τ^4, and the fractional corrections to the current moments S_L^A scale as D^2/τ^2. Demanding that these corrections be smaller than the post-1-Newtonian accuracies of these quantities ($\sim M/D$ and ~ 1, respectively) yields the criterion $\tau \gg D$ for the current moments, and the more stringent criterion

$$\tau \gg D\left(\frac{D}{M}\right)^{1/4} \tag{1.10}$$

for the mass moments. Fractional corrections to the post-1-Newtonian tidal interactions will be of order unity for $\tau \sim D(D/M)^{1/4}$. The criterion $\tau \gg D$ essentially says that all of the bodies lie in the near zone of the gravitational radiation produced by any one body, and not in the wave zone [38], and the criterion (1.10) is a somewhat stronger requirement than this.

To illustrate the criterion (1.10) it is useful to consider some examples. First, if the time evolution of a body's moments is driven by tidal interactions with other bodies, then the timescale is of order $\tau \sim D(D/M)^{1/2}$, and the criterion is satisfied. Second, suppose that we have a three-body system consisting of two black holes in a tight binary, together with a third body in orbit around the binary. We can model such a system as a two-body system using the formalism of this paper, treating the black hole binary as a single body[6] whose mass and current multipole moments are evolving with time due to internal dynamics. Then, early in the gravitational-wave driven inspiral of the black hole binary, the criterion (1.10) will be satisfied and our results for the equation of motion will be valid. As the black hole binary gets tighter however, eventually the orbital period will become shorter than (1.10) and our results will no longer be applicable.

This second example illustrates that the post-1-Newtonian approximation can sometimes completely break down, even in the supposed weak field region between the bodies. During the final coalescence of the black holes the gravitational radiation metric perturbation will become temporarily as large as the Newtonian potential in the region between the binary and the loosely bound companion. Our results are not applicable to such systems, in which one of the bodies emits a strong burst of gravitational radiation. Further work is required to deduce the form of the translational equations of motion in this type of situation.

There are two other assumptions made in our derivation which slightly restrict the domain of validity of our results. First, we assume that a coordinate system which covers the weak field region between the bodies can be smoothly extended to cover the bodies' interiors (see Sec.III B below). This assumption essentially restricts the spatial to-

[6]Although our formalism cannot be consistently applied to individual black holes unless supplemented with post-2-Newtonian and higher-order point-particle terms, our formalism can be applied to black hole binaries treated as a single body. This is because binaries are less compact than black holes.

pology of the bodies' interiors, and excludes objects like eternal black holes, wormhole mouths, and naked singularities. It does not exclude realistic, astrophysical black holes for the reason explained in Sec. III B [45]. Second, in order for a body's multipole moments to be definable, it is necessary that there exist two concentric coordinate spheres surrounding the object, such that the region between the two spheres is vacuum, in a particular coordinate system centered on the body (see Sec. III B below). This assumption might break down when two bodies get within one or two radii of one another, slightly before they actually touch.

E. Organization of this paper

The structure of the paper is as follows. In Sec. II A we introduce our notations for the post-1-Newtonian continuum field equations, and following DSX we define a class of gauges (conformally Cartesian harmonic gauges) that we use throughout the paper. Section II B presents a simplified version of the theory of post-1-Newtonian reference systems of Refs. [2,16,17,22–25]. The key result of this section is the explicit parametrization (2.17) of the residual gauge freedom within conformally Cartesian harmonic gauge in terms of a number of freely specifiable functions of time and one harmonic function of time and space.

Section III is devoted to the definitions of the mass multipole moments $M_L(t)$, current multipole moments $S_L(t)$, gravitoelectric tidal moments $G_L(t)$, and gravitomagnetic tidal moments $H_L(t)$ associated with a given object and a given conformally Cartesian, harmonic coordinate system. These definitions are given in Sec. III B in terms of the general solution (3.5a)–(3.5c) of the post-1-Newtonian field equations in a vacuum region between two concentric coordinate spheres that surround the object (the object's "buffer region"). Section III C analyzes how all of these moments transform under the class of allowed gauge transformations discussed in Sec. II B. In Sec. III D we describe gauge specializations that fix the gauge freedom completely and accordingly determine the multipole and tidal moments uniquely. We call the resulting coordinate system a body-adapted coordinate system. Section III E gives a definition of multipole moments and tidal moments associated with a given object, a given world line, and a given coordinate system. These moments arise only in intermediate steps in the derivations of this paper and not in our final results. Finally, in Sec. III F we compare the moment definitions used here with other definitions in the literature.

Section IV derives the law of motion for a single body, that is, the relation between the second time derivative of its mass dipole moment and its other multipole and tidal moments and their time derivatives. The assumptions and result are described in Sec. IV A. A general description of the surface-integral method of derivation which we use is given in Sec. IV B. In Sec. IV C we give some of the post-2-Newtonian vacuum field equations which are needed for the derivation. Section IV D derives the single-body law of motion to Newtonian order, as a warm-up exercise. Finally, the post-1-Newtonian derivation is given in Sec. IV E. This derivation uses an idea due to Thorne and Hartle [34] to deduce the value of a complicated surface integral from previous weak field computations of DSX [2,16].

Section V lays the foundations for treating a system of N interacting, finite-sized bodies. Our assumptions are described and discussed in Sec. V A. In Sec. V B we define, for each body, a set of body-frame multipole and tidal moments associated with that body's adapted coordinate system. These are the moments that would be measured by an observer in that body's local asymptotic rest frame. Section V C defines the configuration variables that specify the location, orientation, etc., within the global coordinate system of the local asymptotic rest frame which is attached to that body. These variables include the center-of-mass world line and also the time functions and rotation matrices discussed in Sec. I C above. In Sec. V D we define for each body multipole and tidal moments associated with the global coordinate system. These quantities appear only in intermediate steps in our computations and not in our final results. The relation between the global-frame multipole and tidal moments and the body-frame multipole and tidal moments is computed in Sec. V E. Section V F defines the modified versions $\mathcal{M}_L$ and $\mathcal{S}_L$ of the body-frame multipole moments, discussed in Sec. I C above, which are defined with respect to a frame that is nonrotating with respect to distant stars, and which are expressed as functions of the global time coordinate. These are the moments that appear in the final equations of motion.

Finally, Sec. VI gives the derivation of the complete, explicit translational equations of motion for the N body system.

F. Notations and conventions

Throughout this paper we use geometric units in which $G = c = 1$. We use the sign conventions of Misner, Thorne, and Wheeler [46]; in particular, we use the metric signature $(-, +, +, +)$. Greek indices (μ, ν etc.) run from zero to three and denote spacetime indices, while Roman indices (a, b, i, j, etc.) run from one to three and denote spatial indices. The spacetime coordinates will generically be denoted by $(x^0, x^i) = (t, x^i)$. Spatial indices are raised and lowered using δ_{ij}, and repeated spatial indices are contracted regardless of whether they are covariant or contravariant indices. We denote by n^i the unit vector x^i/r, where $r = |\boldsymbol{x}| = \sqrt{\delta_{ij}x^i x^j}$.

When dealing with sequences of spatial indices, we use the multi-index notation introduced by Thorne [38] as modified slightly by Damour, Soffel, and Xu [2]. We use

L to denote the sequence of l indices $a_1 a_2 \dots a_l$, so that for any l-index tensor[7] T we have

$$T_L \equiv T_{a_1 a_2 \dots a_l}. \tag{1.11}$$

If $l = 0$, it is understood that T_L is a scalar. If $l < 0$ then $T_L \equiv 0$. We define $L - 1$ to be the sequence of $l - 1$ indices $a_1 a_2 \dots a_{l-1}$, so that

$$T_{L-1} \equiv T_{a_1 a_2 \dots a_{l-1}}. \tag{1.12}$$

If $l = 0$, then by convention $T_{L-1} \equiv 0$. We also define N to be the sequence of n indices $a_1 a_2 \dots a_n$, and $L - N$ to be the sequence of $l - n$ indices $a_{n+1} a_{n+2} \dots a_l$, so that we can write a relation like

$$G_{a_1 \dots a_l} = S_{a_1 \dots a_n} T_{a_{n+1} \dots a_l} \tag{1.13}$$

as $G_L = S_N T_{L-N}$, for any tensors G, S, and T. We define K, P, and Q to be the sequences of spatial indices $b_1 b_2 \dots b_k$, $c_1 c_2 \dots c_p$, and $d_1 d_2 \dots d_q$, respectively. Repeated multi-indices are subject to the Einstein summation convention, as in $S_L T_L$. We also use the notations

$$x^L \equiv x^{a_1 a_2 \dots a_l} \equiv x^{a_1} x^{a_2} \dots x^{a_l} \tag{1.14}$$

and

$$\partial_L \equiv \partial_{a_1 a_2 \dots a_l} \equiv \partial_{a_1} \partial_{a_2} \dots \partial_{a_l}. \tag{1.15}$$

We use angular brackets to denote the operation of taking the symmetric trace-free (STF) part of a tensor. Thus for any tensor T_L, we define

$$T_{\langle L \rangle} \equiv \mathrm{STF}_L(T_L). \tag{1.16}$$

where STF_L means taking the symmetric trace-free projection on the indices L. For example, if $l = 2$, we have

$$T_{\langle L \rangle} = T_{\langle a_1 a_2 \rangle} = \frac{1}{2}(T_{a_1 a_2} + T_{a_2 a_1}) - \frac{1}{3}\delta_{a_1 a_2} T_{jj}.$$

See Appendix A for the general definition of STF projection, and for a collection of useful relations involving STF tensors.

Throughout this paper, symbols will generally denote functions (as is common in mathematics) rather than physical quantities (as is common in physics). For example, in Sec. V we define a mass multipole moment $M_L^A(s_A)$ which is a function of a time coordinate s_A. In that section we also use a different time coordinate t. Then, the notation $M_L^A(t)$ will always mean $M_L(s_A)$ evaluated at $s_a = t$, rather than $M_L^A[s_a(t)]$.

Finally, for the aid of the reader an index of symbols is provided in Table II.

[7]Here by "tensor" we mean an object which transforms as a tensor under the symmetry group SO(3) of the zeroth order spatial metric δ_{ij}, not a spacetime tensor.

II. POST-1-NEWTONIAN CONTINUUM FIELD EQUATIONS AND GAUGE FREEDOM

In this section we summarize the form of the post-1-Newtonian field equations that we use, and analyze the residual gauge freedom left after imposing our gauge conditions. Our notation follows closely that of Weinberg [47], though we relate our conventions to those of DSX [2,16–18].

A. Metric expansion and field equations

In the post-1-Newtonian approximation, one considers a one-parameter family of solutions to Einstein's equations of the form

$$\begin{aligned} ds^2 = &-[1 + 2\varepsilon^2\Phi + 2\varepsilon^4(\Phi^2 + \psi) + O(\varepsilon^6)](dt/\varepsilon)^2 \\ &+ [2\varepsilon^3\zeta_i + O(\varepsilon^5)]dx^i(dt/\varepsilon) \\ &+ [\delta_{ij} + \varepsilon^2 h_{ij} + O(\varepsilon^4)]dx^i dx^j. \end{aligned} \tag{2.1}$$

Here the Newtonian potential Φ, the post-Newtonian potential ψ, the gravitomagnetic potential ζ_i, and the spatial metric perturbation h_{ij} are functions of the coordinates $x^0 = t$ and x^i, but are independent of the parameter ε. The corresponding expansion of the stress-energy tensor is

$$T^{\mu\nu} = \varepsilon^4[{}^{\mathrm{n}}T^{\mu\nu} + \varepsilon^2\, {}^{\mathrm{pn}}T^{\mu\nu} + O(\varepsilon^4)], \tag{2.2}$$

where ${}^{\mathrm{n}}T^{\mu\nu}$ is the Newtonian-order piece and ${}^{\mathrm{pn}}T^{\mu\nu}$ is the post-1-Newtonian-order piece. The post-Newtonian expansion parameter ε used here is equivalent to the expansion parameter c^{-1} used in some other treatments (we use units in which $G = c = 1$).

We note that many presentations of the post-1-Newtonian equations use a time coordinate $\hat{t}$ that is related to our time coordinate t by

$$\hat{t} = \frac{t}{\varepsilon}. \tag{2.3}$$

In $(\hat{t}, x^i)$ coordinate systems the various powers of ε that appear in the expansions (2.1) of the metric and (2.2) of the stress-energy tensor differ from those given here. The $(\hat{t}, x^i)$ coordinate systems have the advantage that the pointwise limit as $\varepsilon \to 0$ of the metric exists and is a flat, Minkowski metric, but have the disadvantage that the potentials Φ, ψ, and ζ_i become ε dependent. That ε dependence is usually accounted for by inserting an extra factor of ε whenever a time derivative of a potential is taken. By contrast, in the coordinates used here, the potentials are independent of ε and no such extra factors of ε are needed.

Assuming the validity of Einstein's equations for the metric (2.1) and stress-energy tensor (2.2) for all ε in some open interval $0 < \varepsilon < \varepsilon_0$ then implies that the metric $\delta_{ij} + \varepsilon^2(h_{ij} + 2\Phi\delta_{ij})$ is flat to $O(\varepsilon^2)$ [2]. Therefore one can always choose coordinate systems[8] in which $h_{ij} = -2\Phi\delta_{ij}$, for which metric expansion simplifies to

$$\begin{aligned} ds^2 = &-[1 + 2\varepsilon^2\Phi + 2\varepsilon^4(\Phi^2 + \psi) + O(\varepsilon^6)](dt/\varepsilon)^2 \\ &+ [2\varepsilon^3\zeta_i + O(\varepsilon^5)]dx^i(dt/\varepsilon) \\ &+ [\delta_{ij} - 2\varepsilon^2\Phi\delta_{ij} + O(\varepsilon^4)]dx^i dx^j. \end{aligned} \tag{2.4}$$

Such gauges are called conformally Cartesian [2]; we will restrict attention to conformally Cartesian gauges throughout this paper.

We will also assume the harmonic gauge condition

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0, \tag{2.5}$$

which at post-1-Newtonian order and for conformally Cartesian gauges reduces to

$$4\frac{\partial\Phi}{\partial t} + \frac{\partial\zeta^i}{\partial x^i} = 0. \tag{2.6}$$

Harmonic coordinate systems are usually conformally Cartesian, but one can have local coordinate patches which are harmonic but not conformally Cartesian.

The metric (2.4), stress-energy tensor (2.2), and gauge condition (2.6) imply the standard[9] harmonic-gauge[10] field equations [47]

$$\nabla^2\Phi = 4\pi\,{}^{\rm n}T^{00}, \tag{2.8a}$$

$$\nabla^2\psi = \frac{\partial^2\Phi}{\partial t^2} + 4\pi({}^{\rm pn}T^{00} + {}^{\rm n}T^{jj}), \tag{2.8b}$$

$$\nabla^2\zeta^i = 16\pi\,{}^{\rm n}T^{0i}. \tag{2.8c}$$

For most of this paper we will be concerned with the vacuum versions of the field Eqs. (2.8a)–(2.8c). These are

$$\nabla^2\Phi = 0, \tag{2.9a}$$

$$\nabla^2\psi = \frac{\partial^2\Phi}{\partial t^2}, \tag{2.9b}$$

$$\nabla^2\zeta^i = 0. \tag{2.9c}$$

For later use we note the expansions of the electric and magnetic components $\mathcal{E}_{ij} \equiv R_{0i0j}$ and $\mathcal{B}_{ij} \equiv -\frac{1}{2}\epsilon_{ikl}R_{kl0j}$ of the curvature tensor. In vacuum regions these tensors are symmetric and traceless and can be expanded as

$$\mathcal{E}_{ij} = {}^{\rm n}\mathcal{E}_{ij} + \varepsilon^2\,{}^{\rm pn}\mathcal{E}_{ij} + O(\varepsilon^4) \tag{2.10}$$

and

$$\mathcal{B}_{ij} = \varepsilon^2\,{}^{\rm pn}\mathcal{B}_{ij} + O(\varepsilon^4). \tag{2.11}$$

Here ${}^{\rm n}\mathcal{E}_{ij} = \Phi_{,ij}$ is the Newtonian electric tidal tensor, and

$${}^{\rm pn}\mathcal{E}_{ij} = \dot\zeta_{\langle i,j\rangle} + 3\Phi_{\langle,i}\Phi_{,j\rangle} + 4\Phi\Phi_{,\langle ij\rangle} + \psi_{,\langle ij\rangle} \tag{2.12}$$

is the post-Newtonian electric tidal tensor. The angular brackets denote a STF projection, cf. Sec. I F above. The post-Newtonian magnetic tidal tensor is

$${}^{\rm pn}\mathcal{B}_{ij} = -\frac{1}{2}B_{(i,j)}, \tag{2.13}$$

where

$$\mathbf{B} \equiv \nabla\times\boldsymbol{\zeta} \tag{2.14}$$

is the so-called gravitomagnetic field.

For later use, we also note the definition of the gravitoelectric field $\mathbf{E}$ used by DSX [2]:

$$\mathbf{E} \equiv -\nabla(\Phi + \varepsilon^2\psi) - \varepsilon^2\dot{\boldsymbol{\zeta}}. \tag{2.15}$$

B. Parametrization of residual gauge freedom in conformally Cartesian harmonic gauge

In many applications of post-1-Newtonian theory, the elliptic harmonic-gauge field Eqs. (2.8a)–(2.8c) are valid in all of space. In such cases one normally solves the field equations by imposing the boundary condition that all the potentials go to zero at spatial infinity. This boundary condition determines a unique solution to the field equations and a unique choice of gauge.

However, in this paper we will be dealing with situations where the field Eqs. (2.8a)–(2.8c) do not have unique solutions, due to residual gauge freedom. There are two reasons for this residual gauge freedom [2]. First, even in cases where the field equations are valid in all of space, there is in fact no physical reason for imposing that the potentials go to zero at spatial infinity. Instead, the physical boundary condition to impose is that the components (2.10) and (2.11) of the Riemann curvature tensor go to zero at spatial infinity. One then finds that there is a large class of solutions of the harmonic-gauge field equations, when the

[8]This statement is always true locally, and is true globally if the spatial domain of the coordinates is simply connected. In this paper the spatial coordinate domains will always be simply connected.

[9]If we add ε^2 times Eq. (2.8b) to Eq. (2.8a) we can combine these two field equations into the single wave equation $-\partial^2 w/\partial\hat t^2 + \nabla^2 w = -4\pi\sigma + O(\varepsilon^4)$, where $w = -\Phi - \varepsilon^2\psi$ and

$$\sigma \equiv {}^{\rm n}T^{00} + \varepsilon^2\,{}^{\rm pn}T^{00} + \varepsilon^2\,{}^{\rm n}T^{jj}. \tag{2.7}$$

This is the notation used by DSX [2]. The potential w satisfies a flat-space wave equation, with hidden ε dependencies that come from the definition of w. For our purposes it will be more useful to keep the expansion in ε fully explicit by using the two elliptic Eqs. (2.8a) and (2.8b) for Φ and ψ instead of the above single hyperbolic equation for w.

[10]In the "standard post-Newtonian gauge" [4] the gauge condition (2.6) is replaced by $3\dot\Phi + \nabla\cdot\boldsymbol{\zeta} = 0$, and the field Eqs. (2.8b) and (2.8c) are replaced by $\nabla^2\psi = 4\pi\,{}^{\rm pn}T^{00} + 4\pi\,{}^{\rm n}T^{jj}$ and $\nabla^2\zeta^i = 16\pi\,{}^{\rm n}T^{0i} + \dot\Phi_{,i}$.

sources are fixed. This nonuniqueness is present even at Newtonian order; there are many solutions to the Newtonian Poisson Eq. (2.8a) with the boundary condition $\Phi_{,ij} \to 0$ as $r \to \infty$. In the Newtonian context, the new solutions are simply the standard solution transformed to accelerated frames. Similarly, in the post-Newtonian context, the additional solutions correspond to the original solution transformed to reference frames that are accelerated, rotating, or otherwise modified with respect to the standard reference frame.

In Sec. III D below, when considering a system of N interacting bodies, we will need to construct a coordinate system adapted to each body. Exploiting the additional freedom of allowing accelerated, rotating coordinate systems will be crucial for our construction of those adapted coordinate systems.

The second reason for residual gauge freedom in the situations considered in this paper is that we will be considering spacetimes containing strong field regions in which the post-Newtonian approximation is not valid. Therefore, we must analyze the field Eqs. (2.8a)–(2.8c) on some spatial region $\mathcal{D}$ which is not all of space. In this case, the boundary conditions imposed on the potentials Φ, ψ, and ζ_i on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ influence the solution. Part of the information inherent in those boundary conditions is gauge, and part of the information is physical.

DSX [2], Kopeikin [22], and Klioner and Voinov [25] have derived a complete parametrization of the residual gauge freedom present in some region $\mathcal{D}$ of space after the conformally Cartesian condition has been imposed; see Ref. [20] for a review. In this subsection we give a simplified, streamlined version of the DSX analysis in our somewhat different notation. We also specialize the DSX analysis by imposing in addition the harmonic gauge condition.

Our starting assumptions are as follows. We assume the existence of two different coordinate systems (t, x^i) and $(\bar{t}, \bar{x}^i)$ on $\mathcal{D}$, each of which is conformally Cartesian and harmonic. We also assume that both coordinate systems are such that the metric admits an expansion of the form (2.4). In particular, there exist potentials $\bar{\Phi}(\bar{t}, \bar{x}^j)$, $\bar{\psi}(\bar{t}, \bar{x}^j)$, and $\bar{\zeta}_i(\bar{t}, \bar{x}^j)$ such that

$$
\begin{aligned}
ds^2 = & -[1 + 2\varepsilon^2\bar{\Phi} + 2\varepsilon^4(\bar{\Phi}^2 + \bar{\psi}) + O(\varepsilon^6)](d\bar{t}/\varepsilon)^2 \\
& + [2\varepsilon^3\bar{\zeta}_i + O(\varepsilon^5)]d\bar{x}^i(d\bar{t}/\varepsilon) \\
& + [\delta_{ij} - 2\varepsilon^2\bar{\Phi}\delta_{ij} + O(\varepsilon^4)]d\bar{x}^i d\bar{x}^j. \qquad (2.16)
\end{aligned}
$$

In Appendix B we show that the most general relation between the two coordinate systems that is compatible with these assumptions is[11]

$$
\begin{aligned}
x^i &= \bar{x}^i + z^i(\bar{t}) + \varepsilon^2 h^i(\bar{t}, \bar{x}^j) + O(\varepsilon^4), \\
t &= \bar{t} + \varepsilon^2\alpha(\bar{t}, \bar{x}^j) + \varepsilon^4\beta(\bar{t}, \bar{x}^j) + O(\varepsilon^6),
\end{aligned}
\qquad (2.17)
$$

where

$$
\alpha(\bar{t}, \bar{x}^j) = \alpha_c(\bar{t}) + \bar{x}_i\dot{z}^i(\bar{t}), \qquad (2.18a)
$$

$$
\begin{aligned}
h^i(\bar{t}, \bar{x}^j) = & h_c^i(\bar{t}) + \epsilon^{ijk}\bar{x}_j R_k(\bar{t}) + \frac{1}{2}\ddot{z}^i(\bar{t})\bar{x}_j\bar{x}^j - \bar{x}^i\dot{\alpha}_c(\bar{t}) \\
& - \bar{x}^i\bar{x}_j\ddot{z}^j(\bar{t}) + \frac{1}{2}\bar{x}^i\dot{z}_j(\bar{t})\dot{z}^j(\bar{t}) + \frac{1}{2}\dot{z}^i(\bar{t})\dot{z}^j(\bar{t})\bar{x}_j,
\end{aligned}
\qquad (2.18b)
$$

$$
\beta(\bar{t}, \bar{x}^j) = \bar{x}_j\bar{x}^j\left[\frac{1}{10}\dddot{z}^k(\bar{t})\bar{x}_k + \frac{1}{6}\ddot{\alpha}_c(\bar{t})\right] + \beta_h(\bar{t}, \bar{x}^j), \qquad (2.18c)
$$

and where overdots mean derivatives with respect to the time argument.

We next discuss the meaning of the various freely specifiable functions $\alpha_c(\bar{t})$, $z^i(\bar{t})$, $h_c^i(\bar{t})$, $R_k(\bar{t})$, and $\beta_h(\bar{t}, \bar{x}^j)$ that appear in the coordinate transformation (2.17) and (2.18a)–(2.18c). At Newtonian order there appears the function of time $\alpha_c(\bar{t})$, which governs the normalization of the time coordinate at $O(\varepsilon^2)$. In standard treatments of Newtonian gravity, this freedom is fixed by the usual assumption $\Phi \to 0$ as $r \to \infty$. In the present context, however, this coordinate freedom is not fixed. There also appears the spatial 3-vector $z^i(\bar{t})$, which parametrizes the translational motion of the new frame with respect to the original frame, to Newtonian order. At post-1-Newtonian order, one has the 3-vector $h_c^i(\bar{t})$, which parametrizes the post-Newtonian translational motion of the new frame with respect to the original frame. There also appears the spatial 3-vector $R_k(\bar{t})$, whose time derivative is an angular velocity that parametrizes the slow, post-Newtonian rotation of the coordinate system. Finally, there is the function $\beta_h(\bar{t}, \bar{x}^j)$ which governs the normalization of the time coordinate at $O(\varepsilon^4)$. This function is not completely freely specifiable but must be harmonic, i.e.,

$$
\nabla^2\beta_h = 0. \qquad (2.19)
$$

It is straightforward to compute how the potentials transform by combining the metric expansions (2.4) and (2.16) with the coordinate transformation given by Eqs. (2.17) and (2.18a)–(2.18c) and using the tensor transformation law for the components of the metric. The results are

[11]Up to constant displacements in time and time-independent spatial rotations. We also assume that the coordinate transformation is orientation-preserving and time-orientation-preserving.

$$\bar{\Phi}(\bar{t}, \bar{x}^j) = \hat{\Phi}(\bar{t}, \bar{x}^j) + \dot{\alpha} - \frac{1}{2}\dot{z}_j\dot{z}^j, \tag{2.20a}$$

$$\bar{\zeta}_i(\bar{t}, \bar{x}^j) = \hat{\zeta}_i(\bar{t}, \bar{x}^j) - [4\hat{\Phi}(\bar{t}, \bar{x}^j) + \dot{\alpha}]\dot{z}_i + \frac{\partial h_i}{\partial \bar{t}} + \frac{\partial h^j}{\partial \bar{x}^i}\dot{z}_j - \frac{\partial \beta}{\partial \bar{x}^i}, \tag{2.20b}$$

$$\bar{\psi}(\bar{t}, \bar{x}^j) = \hat{\psi}(\bar{t}, \bar{x}^j) - \hat{\zeta}_i(\bar{t}, \bar{x}^j)\dot{z}^i + \alpha\frac{\partial \hat{\Phi}(\bar{t}, \bar{x}^j)}{\partial \bar{t}} + 2\hat{\Phi}(\bar{t}, \bar{x}^j)\dot{z}_i\dot{z}^i - (\alpha\ddot{z}^i - h^i)\frac{\partial \hat{\Phi}(\bar{t}, \bar{x}^j)}{\partial \bar{x}^i} - \frac{1}{4}(\dot{z}^i\dot{z}_i)^2 + \dot{\alpha}\dot{z}^i\dot{z}_i - \dot{z}^i\frac{\partial h_i}{\partial \bar{t}} + \frac{\partial \beta}{\partial \bar{t}} - \frac{1}{2}\dot{\alpha}^2. \tag{2.20c}$$

Here the function $\hat{\Phi}(\bar{t}, \bar{x}^j)$ is defined as

$$\hat{\Phi}(\bar{t}, \bar{x}^j) = \Phi[\bar{t}, \bar{x}^j + z^j(\bar{t})], \tag{2.21}$$

where the right-hand side is Φ evaluated at the point $x^i = \bar{x}^i + z^i(\bar{t})$, $t = \bar{t}$. We define the functions $\hat{\psi}(\bar{t}, \bar{x}^j)$ and $\hat{\zeta}^i(\bar{t}, \bar{x}^j)$ similarly.

The transformation laws (2.20a)–(2.20c) are expressed in terms of the functions $\alpha(\bar{t}, \bar{x}^j)$, $\beta(\bar{t}, \bar{x}^j)$, and $h^i(\bar{t}, \bar{x}^j)$ defined by Eqs. (2.17) and (2.18a)–(2.18c). More explicit versions of the transformation laws, in which they are expressed in terms of the freely specifiable functions $\alpha_c(\bar{t})$, $z^i(\bar{t})$, $h^i_c(\bar{t})$, $R_k(\bar{t})$, and $\beta_h(\bar{t}, \bar{x}^j)$, can be obtained by substituting the definitions (2.17) and (2.18a)–(2.18c) into Eqs. (2.20a)–(2.20c).

There are two special subgroups of the group (2.17) of transformations that will be of importance later. The first subgroup applies when the spatial domain $\mathcal{D}$ is all of space, and when in addition one imposes the boundary condition that all the potentials vanish at spatial infinity. In this case it is easy to show that only uniform relative motion of the two frames is allowed, $\ddot{z}^i = \ddot{h}^i_c = 0$, that $\beta_h = 0$, $\alpha_c = \dot{\mathbf{z}}^2/2$, and that R^k is constant. These well-known "post-Galilean" transformations are discussed in, for example, Sec. 39.9 of Ref. [46].

The second important subgroup is the subgroup parametrized by the harmonic function β_h, for which $\alpha_c = z^i = h^i_c = R^k = 0$. The corresponding coordinate transformations are

$$x^i = \bar{x}^i + O(\varepsilon^4), \qquad t = \bar{t} + \varepsilon^4\beta_h + O(\varepsilon^6), \tag{2.22}$$

and the potentials transform according to

$$\bar{\Phi} = \Phi, \qquad \bar{\psi} = \psi + \frac{\partial \beta_h}{\partial \bar{t}}, \qquad \bar{\zeta}^i = \zeta^i - \frac{\partial \beta_h}{\partial \bar{x}^i}. \tag{2.23}$$

One can show that the Newtonian and post-Newtonian pieces of the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ and of the Riemann tensor components $R_{\alpha\beta\gamma}{}^\delta$ are invariant under this subgroup. As noted by DSX [2], this subgroup corresponds to a gauge freedom in post-1-Newtonian theory analogous to that of electromagnetism.[12]

III. DEFINITIONS OF AN OBJECT'S MASS AND CURRENT MULTIPOLE MOMENTS AND TIDAL MOMENTS

A. Overview

As discussed in the introduction, a crucial part of the derivation of the equations of motion for strongly self-gravitating bodies is the definition of an object's mass and current multipole moments and also tidal moments in terms of the behavior of the metric in a weak field region surrounding the object. In this section we discuss the definitions of these quantities.

For orientation, we start by reviewing the definition and status of multipole and tidal moments in Newtonian gravity. Suppose that in some reference frame (t, x^i), there exists a region $\mathcal{D}$ between two spheres of the form

$$r_- \le |\mathbf{x}| \le r_+,$$

for some radii r_- and r_+, in which there are no sources. Then the Newtonian potential Φ satisfies the Laplace Eq. (2.9a) in $\mathcal{D}$. The general solution for Φ in $\mathcal{D}$ can then be written in terms of a multipole expansion as

$$\Phi(t, x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!}\, {}^{\rm n}M_L(t)\partial_L\frac{1}{|\mathbf{x}|} - \frac{1}{l!}\, {}^{\rm n}G_L(t)x^L. \tag{3.1}$$

Here L denotes the multi-index $a_1a_2\ldots a_l$ and x^L denotes the tensor $x^{a_1}x^{a_2}\ldots x^{a_l}$, cf. Sec. I F above. The quantity ${}^{\rm n}M_L(t)$ is the lth Newtonian mass multipole moment associated with the reference frame (t, x^j) of the object or objects in the region $|\mathbf{x}| < r_-$. [The superscript "n" in ${}^{\rm n}M_L$ denotes "Newtonian."] Similarly the quantity ${}^{\rm n}G_L(t)$ is the lth Newtonian tidal moment associated with the reference frame (t, x^j) that acts on the region $r < r_-$ due to sources outside $r = r_+$, where $r = |\mathbf{x}|$. The moments ${}^{\rm n}M_L$ and ${}^{\rm n}G_L$ are both STF tensors.

The expansion (3.1) can be taken as the definition of the moments ${}^{\rm n}M_L$ and ${}^{\rm n}G_L$; it is possible to invert Eq. (3.1) to obtain explicit expressions for these moments in terms of surface integrals of Φ in the domain $\mathcal{D}$ (see Appendix E). If we additionally assume that the Newtonian Poisson Eq. (2.8a) is valid everywhere in $r < r_-$, then we obtain the conventional formula for the mass multipole moments as an integral of the Newtonian mass density ${}^{\rm n}T^{00}$:

[12]If one requires only that the coordinate systems be conformally Cartesian and not harmonic, then the most general coordinate transformation is still given by Eqs. (2.17) and (2.18a)–(2.18c), but with the modification that the function β_h can be arbitrary rather than being harmonic [2].

$$ {}^{\rm n}M_L(t) = \int_{r<r_-} {}^{\rm n}T^{00}(t, x^j) x^{\langle L \rangle} d^3x. \quad (3.2) $$

Here the angular brackets denote a STF projection, cf. Sec. I F above. As is well known, the field-based definition (3.1) of the multipole moments is of greater generality than the integral-based definition (3.2), since the former is applicable to strong field sources that possess an asymptotic region in which the Newtonian description is a good approximation.

Next, we recall that the moments ${}^{\rm n}M_L(t)$ and ${}^{\rm n}G_L(t)$ depend on the choice of reference frame or coordinate system (t, x^j). They change when one switches from one reference frame to another according to $\boldsymbol{x} \to \boldsymbol{x} + \mathbf{z}(t)$ [cf. Eq. (2.17) above, to Newtonian order]. This ambiguity is conventionally resolved in Newtonian physics by specializing to the reference frame in which the origin of coordinates coincides with the center of mass of the object, or, equivalently, in which the mass dipole moment ${}^{\rm n}M_i(t)$ vanishes.

Consider now the corresponding situation at post-1-Newtonian order. A suitable generalization of the integral-based definition (3.2) of multipole moments for an isolated system was given by Blanchet and Damour [48]. That definition was generalized to the case of several interacting bodies by DSX [2]. DSX also gave a field-based definition of multipole moments analogous to the definition (3.1) of ${}^{\rm n}M_L$, and gave a different type of definition of tidal moments. In this section we will review, simplify,[13] and generalize the definitions of DSX [2]. Our analysis will be more general than theirs because we will consider spacetimes where the post-1-Newtonian field equations are not satisfied everywhere, whereas DSX assumed the global validity of those field equations.

As is the case at Newtonian order, the post-Newtonian multipole and tidal moments associated with a body are not uniquely defined but depend on the choice of reference frame or coordinate system. This ambiguity can be resolved, as in Newtonian theory, by making a specific choice of canonical reference frame adapted to a given body. However, the freedom in choice of reference frame is much larger at post-Newtonian order than at Newtonian order, cf. the discussion in Sec. II B above. Therefore the specialization to a body-adapted frame is more involved.

The remainder of this section is organized as follows. In Sec. III B we give a form of multipolar expansion of the general solution of the post-1-Newtonian field equations that serves to define the multipole and tidal moments associated with a given body and with a given coordinate system. Section III C discusses the gauge transformation properties of the moments, and Sec. III D defines the body-adapted gauge that fixes the moments uniquely. Finally in Sec. III E we generalize our definition of multipole moments to define moments associated with a given coordinate system about a given specified world line.

B. Definition of mass and current multipole moments and tidal moments

To define the multipole moments of a body, we start by assuming the existence of a local coordinate system (t, x^j) with following properties. (i) The range of the coordinates contains the product of the open ball

$$ |\boldsymbol{x}| < r_+, \quad (3.3) $$

where r_+ is some radius, with some open interval (t_0, t_1) of time. (ii) The vacuum post-1-Newtonian field Eqs. (2.9a)–(2.9c) are valid in a spatial region $\mathcal{D}$ of the form

$$ r_- < |\boldsymbol{x}| < r_+, \quad (3.4) $$

for some nonzero radius r_-.

These assumptions allow us to define the multipole and tidal moments of a body or bodies in the region $r < r_-$, where $r = |\boldsymbol{x}|$. In parallel with the Newtonian case discussed above, the second assumption allows for the possibility of strong field sources for which the post-Newtonian approximation is not valid in the region $r < r_-$. When applying this definition to systems of several bodies, we will choose both r_- and r_+ to be of order the distance between the bodies.

The first assumption, that the coordinates in the region $\mathcal{D}$ can be extended into the interior to cover the body, is not actually necessary for the definition of multipole moments. However it will be used later in the derivation of equations of motion so we include it here. Note that this first assumption does not exclude the possibility that one or more black holes could reside in the region $r < r_-$. As pointed out by Thorne [45], one merely needs to choose as the $t =$ constant surfaces an appropriate set of time slices that pass though the interior of the collapsing object(s) that form the black hole(s).

Given these assumptions, the general solution of the vacuum field Eqs. (2.9a)–(2.9c) in the region $\mathcal{D}$ can be expanded in terms of STF tensors as

[13]The DSX definitions, given in Eqs. (6.9a)–(6.10b) of Ref. [2], involve a splitting of all the post-Newtonian potentials into pieces associated with the individual objects. Our simplified version of their definitions do not require any such splitting.

$$\Phi(t,x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!}\,{}^{\rm n}M_L(t)\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}\,{}^{\rm n}G_L(t)x^L, \tag{3.5a}$$

$$\psi(t,x^j) = \sum_{l=0}^{\infty}\Big\{\frac{(-1)^{l+1}}{l!}\Big[{}^{\rm pn}M_L(t) + \frac{(2l+1)}{(l+1)(2l+3)}\dot{\mu}_L(t)\Big]\partial_L\frac{1}{|\boldsymbol{x}|} + \frac{(-1)^{l+1}}{l!}\,{}^{\rm n}\ddot{M}_L(t)\partial_L\frac{|\boldsymbol{x}|}{2} - \frac{1}{l!}[{}^{\rm pn}G_L(t) - \dot{\nu}_L(t)]x^L - \frac{1}{l!}\frac{|\boldsymbol{x}|^2}{2(2l+3)}\,{}^{\rm n}\ddot{G}_L(t)x^L\Big\}, \tag{3.5b}$$

$$\zeta_i(t,x^j) = \sum_{l=0}^{\infty}\Big\{\frac{(-1)^{l+1}}{l!}\Big[\frac{4}{l+1}\,{}^{\rm n}\dot{M}_{iL}(t) - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S_{L-1\rangle j}(t) + \frac{2l-1}{2l+1}\delta_{i\langle a_l}\mu_{L-1\rangle}(t)\Big]\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}\Big[\nu_{iL}(t) + \frac{l}{l+1}\epsilon_{ji\langle a_l}H_{L-1\rangle j}(t) - \frac{4(2l-1)}{2l+1}\,{}^{\rm n}\dot{G}_{\langle L-1}(t)\delta_{a_l\rangle i}\Big]x^L\Big\}. \tag{3.5c}$$

The quantities that appear in these equations are the following. First, there are the Newtonian mass multipole moments ${}^{\rm n}M_L(t)$ and tidal moments ${}^{\rm n}G_L(t)$ that were discussed in Sec. III A. Second, there are post-Newtonian corrections ${}^{\rm pn}M_L(t)$ and ${}^{\rm pn}G_L(t)$ to these quantities. [The superscripts "pn" denote post-Newtonian.] We shall call the quantities

$$M_L(t) \equiv {}^{\rm n}M_L(t) + \varepsilon^2\,{}^{\rm pn}M_L(t) \tag{3.6}$$

and

$$G_L(t) \equiv {}^{\rm n}G_L(t) + \varepsilon^2\,{}^{\rm pn}G_L(t) \tag{3.7}$$

the total mass multipole and tidal moments, respectively. Third, there are current multipole moments $S_L(t)$, and a new set of tidal moments $H_L(t)$ related to gravitomagnetic forces. Following DSX, we will refer to $H_L(t)$ and $G_L(t)$ as the gravitomagnetic and gravitoelectric tidal moments, respectively. Fourth, there are moments $\mu_L(t)$ and $\nu_L(t)$ that contain information about the coordinate system being used, but do not contain any gauge-invariant information about the body. We shall show below that it is always possible to find a gauge in which $\mu_L(t) = \nu_L(t) = 0$. We shall call these quantities "gauge moments."

All of the quantities ${}^{\rm n}M_L$, ${}^{\rm n}G_L$, ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, μ_L, ν_L, S_L, and H_L are STF on all their indices. As in the Newtonian case, the expansions (3.5a)–(3.5c) can be taken as the definition of all of these moments for a given coordinate system; one can invert these expansions to obtain explicit expressions for all the moments in terms of surface integrals of various combinations of derivatives of the potentials [cf. Appendix E]. The moments ${}^{\rm n}M_L$, ${}^{\rm n}G_L$, ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, μ_L are defined for all $l \geq 0$, while ν_L, S_L, and H_L are defined only for $l \geq 1$. In Eq. (3.5b) and throughout this paper, it is understood that $\nu_L \equiv 0$ for $l = 0$.

We shall call the pieces of the potentials that would diverge as $|\boldsymbol{x}| \to \infty$ the *tidal pieces* of the potentials. Correspondingly, we will use the phrase "tidal moments" to refer to any of the moments ${}^{\rm n}G_L$, ${}^{\rm pn}G_L$, H_L, and ν_L that appear in the coefficients of the growing terms in Eqs. (3.5a)–(3.5c). These moments encode the gravitational influence of other bodies on the body in the region $|\boldsymbol{x}| < r_-$. We shall call the remaining pieces of the potentials, which involve the moments ${}^{\rm n}M_L$, ${}^{\rm pn}M_L$, S_L, and μ_L, the *intrinsic pieces*.

We now turn to the derivation of the expansions (3.5b) and (3.5c). We start by writing the general solution in the region $\mathcal{D}$ of the Laplace Eq. (2.9c) for the gravitomagnetic potential as

$$\zeta_i(t,x^j) = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}Z_{iL}(t)\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}Y_{iL}(t)x^L. \tag{3.8}$$

Here the quantities Z_{iL} and Y_{iL} are STF on their L indices only, and not on the i index. Next, we insert the expressions (3.5a) and (3.8) for the gravitomagnetic and Newtonian potentials into the harmonic gauge condition (2.6). This gives the relations[14]

$${}^{\rm n}\dot{M} = 0, \tag{3.9}$$

$$Z_{\langle iL\rangle} = \frac{4}{l+1}\,{}^{\rm n}\dot{M}_{iL}, \tag{3.10}$$

and

$$Y_{jjL} = -4\,{}^{\rm n}\dot{G}_L. \tag{3.11}$$

Next, we use the identity given in Eq. (6.21) of Ref. [2] that allows one to express in terms of irreducible, STF tensors any tensor that is STF on all its indices except its first index. Specifically, if T_{iL} is any tensor satisfying $T_{iL} = T_{i\langle L\rangle}$, then we have

$$T_{iL} = T^{(+1)}_{iL} + \epsilon_{ji\langle a_l}T^{(0)}_{L-1\rangle j} + \delta_{i\langle a_l}T^{(-1)}_{L-1\rangle}, \tag{3.12}$$

where

[14]Thus, the time independence of the Newtonian mass monopole ${}^{\rm n}M$ can be derived either from stress-energy conservation at Newtonian order in the interior of the body (for a weakly self-gravitating system), or from the validity of the post-Newtonian vacuum field equations and harmonic gauge condition in the far field of the body. This type of phenomenon, where one can avoid dealing with the interior physics by going to one higher post-Newtonian order in the far field, is well known in the literature on equations of motion, and will be encountered again in Sec. IV.

$$T^{(+1)}_{iL} \equiv T_{\langle iL\rangle}, \tag{3.13a}$$

$$T^{(0)}_{L} \equiv \frac{l}{l+1} T_{jk\langle L-1}\epsilon_{a_l\rangle jk}, \tag{3.13b}$$

$$T^{(-1)}_{L-1} \equiv \frac{2l-1}{2l+1} T_{jjL-1}. \tag{3.13c}$$

In order to apply this identity to the tensors Z_{iL} and Y_{iL}, we define

$$S_L \equiv -\frac{1}{4} Z_{jk\langle L-1}\epsilon_{a_l\rangle jk}, \tag{3.14a}$$

$$H_L \equiv Y_{jk\langle L-1}\epsilon_{a_l\rangle jk}, \tag{3.14b}$$

$$\nu_L \equiv Y_{\langle L\rangle}, \tag{3.14c}$$

$$\mu_L \equiv Z_{jjL}. \tag{3.14d}$$

We now insert the relations (3.10) and (3.11) and the definitions (3.14a)–(3.14d) into the general identity (3.12) to obtain

$$Z_{iL} = \frac{4}{l+1}\,{}^{\rm n}\dot{M}_{iL} - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S_{L-1\rangle j} + \frac{2l-1}{2l+1}\delta_{i\langle a_l}\mu_{L-1\rangle} \tag{3.15}$$

and

$$Y_{iL} = \nu_{iL} + \frac{l}{l+1}\epsilon_{ji\langle a_l}H_{L-1\rangle j} - \frac{4(2l-1)}{2l+1}\,{}^{\rm n}\dot{G}_{\langle L-1}\delta_{a_l\rangle i}. \tag{3.16}$$

Inserting these formulas into the expansion (3.8) yields Eq. (3.5c).

We remark that the parametrization (3.8) of the gravitomagnetic potential in terms of the tensors Y_{iL} and Z_{iL} will frequently be more convenient to use in our computations below than the fully STF parametrization (3.5c).

Consider now the post-Newtonian potential ψ. We can write the general solution in the region $\mathcal{D}$ of the vacuum field Eq. (2.9b) as the sum $\psi = \psi_{\rm p} + \psi_{\rm h}$ of a particular solution $\psi_{\rm p}$ of the inhomogeneous equation and a general solution $\psi_{\rm h}$ of the homogeneous equation $\nabla^2\psi = 0$. A particular solution can be obtained by inspection, using the expansion (3.5a) of the Newtonian potential:

$$\psi_{\rm p} = \sum_{l=0}^{\infty}\left[\frac{(-1)^{l+1}}{l!}\,{}^{\rm n}\ddot{M}_L\partial_L\frac{|\boldsymbol{x}|}{2} - \frac{1}{l!}\frac{|\boldsymbol{x}|^2}{2(2l+3)}\,{}^{\rm n}\ddot{G}_L x^L\right]. \tag{3.17}$$

The homogeneous solution can be written in a form paralleling the expansion (3.5a) of the Newtonian potential:

$$\psi_{\rm h} = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\,{}^{\rm pn}\tilde{M}_L\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}\,{}^{\rm pn}\tilde{G}_L x^L. \tag{3.18}$$

Next we define

$${}^{\rm pn}M_L(t) = {}^{\rm pn}\tilde{M}_L(t) - \frac{(2l+1)}{(l+1)(2l+3)}\dot{\mu}_L(t) \tag{3.19}$$

and

$${}^{\rm pn}G_L(t) = {}^{\rm pn}\tilde{G}_L(t) + \dot{\nu}_L(t). \tag{3.20}$$

Inserting the definitions (3.19) and (3.20) into the homogeneous solution (3.18) and adding the particular solution (3.17) yields Eq. (3.5b). The reason for choosing the particular parametrization given by Eqs. (3.19) and (3.20) is so that the moments ${}^{\rm pn}M_L(t)$ and ${}^{\rm pn}G_L(t)$ be invariant[15] under the subclass (2.22) of gauge transformations; see Sec. III C below for more details.

We now specialize to the situation, considered by DSX, where the post-1-Newtonian field equations with sources (2.8a)–(2.8c) are assumed to hold for all $r < r_+$, i.e., in the interior of the body. In this special case, we now show that our definitions of the quantities ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, H_L, and S_L are equivalent to the DSX definitions of these quantities.

Consider first the multipole moments ${}^{\rm pn}M_L$ and S_L. Equations (6.9a) and (6.9b) of DSX [2] define the moments $M_L(t)$ and $S_L(t)$ in terms of an expansion of "locally generated" pieces of the potentials. Their splitting of the potentials into "locally generated" and "external" pieces is defined by their Eq. (4.5), and is easily seen to be equivalent to the splitting which we discussed above of our expansions (3.5a)–(3.5c) into intrinsic terms and tidal terms. Therefore it is sufficient to show that the DSX expansions (6.9a) and (6.9b) coincide with the tidal terms in our Eqs. (3.5a)–(3.5c). This follows from the definition (3.6) and the relations $W = -\Phi - \varepsilon^2\psi$ and $W_i = -\zeta_i/4$ between the DSX potentials (W, W^i) and our potentials Φ, ψ, and ζ^i.

Consider next the gravitoelectric and gravitomagnetic tidal moments G_L and H_L. These are defined by DSX in terms of STF projections of gradients of the external (or tidal) pieces of the gravitoelectric and gravitomagnetic fields evaluated at the origin of spatial coordinates, cf. Eq. (6.13) of Ref. [2]. Inserting our expansions (3.5a)–(3.5c) into the definitions (2.14) and (2.15) of the gravitomagnetic and gravitoelectric fields yields for the tidal pieces (denoted by a superscript T) of these quantities

$$B^T_i = \sum_{l=0}^{\infty}\frac{1}{l!}\left[H_{iL}x^{\langle L\rangle} + \frac{4l}{l+1}\epsilon_{ija_l}\,{}^{\rm n}\dot{G}_{jL-1}x^{\langle L\rangle}\right] \tag{3.21}$$

and

$$E^T_i = \sum_{l=0}^{\infty}\frac{1}{l!}\Bigg\{({}^{\rm n}G_{iL} + \varepsilon^2\,{}^{\rm pn}G_{iL})x^{\langle L\rangle} + \varepsilon^2\left[\frac{|\boldsymbol{x}|^2}{2(2l+3)}x^{\langle L\rangle}\,{}^{\rm n}\ddot{G}_{iL} - \frac{7l-4}{2l+1}x^{\langle iL-1\rangle}\,{}^{\rm n}\ddot{G}_{L-1} + \frac{l}{l+1}\epsilon_{ijk}x^{\langle jL-1\rangle}\dot{H}_{kL-1}\right]\Bigg\}. \tag{3.22}$$

These expressions agree to $O(\varepsilon^2)$ with corresponding ex-

[15] Except for ${}^{\rm pn}G_L$ for $l = 0$ which is not invariant.

pansions derived by DSX [Eqs. (6.23) of Ref. [2]], when the definition (3.7) is used. It follows that the two definitions of tidal moments are equivalent.

Thus, our definition of both the multipole moments ${}^{\rm n}M_L$, ${}^{\rm pn}M_L$, S_L and the tidal moments ${}^{\rm n}G_L$, ${}^{\rm pn}G_L$, and H_L in terms of the general solution (3.5a)–(3.5c) of the post-1-Newtonian vacuum field equations unifies and simplifies the two different types of definition given by DSX. Our definitions also generalize the DSX definitions to strong field sources.

Finally, we note that in the case considered by DSX, one can alternatively define the post-Newtonian multipole moments by integrals over the source that are analogous to the Newtonian integral (3.2) [2]. Translating Eqs. (6.11) of Ref. [2] into our notation gives for these integrals

$$S_L = \int_{r<r_-} \epsilon^{jk\langle a_l} x^{L-1\rangle j}\, {}^{\rm n}T^{0k} d^3x \tag{3.23}$$

and

$$\begin{aligned} {}^{\rm pn}M_L = \int_{r<r_-} \bigg\{ \bigg[{}^{\rm pn}T^{00} + {}^{\rm n}T^{jj} + \frac{x^j x^j}{2(2l+3)}\, {}^{\rm n}T^{00}{}_{,00} \bigg] x^{\langle L\rangle} \\ - \frac{4(2l+1)}{(l+1)(2l+3)}\, {}^{\rm n}T^{0j}{}_{,0} x^{\langle jL\rangle} \bigg\} d^3x. \end{aligned} \tag{3.24}$$

C. Gauge transformation properties of the moments

In this section we compute how the various moments transform under the general transformation (2.17) from an original coordinate system (t, x^j) and a new coordinate system $(\bar t, \bar x^j)$. Under that transformation, the spatial domain $\mathcal{D}$ defined by Eq. (3.4) is mapped onto the domain

$$r_- < |\bar{\boldsymbol x} + \boldsymbol z(\bar t)| < r_+, \tag{3.25}$$

to zeroth order in ε. We therefore restrict the set of coordinate transformations to those that satisfy

$$\max_{\bar t} |\boldsymbol z(\bar t)| < \frac{r_+ - r_-}{2}. \tag{3.26}$$

This restriction ensures that the image $\bar{\mathcal{D}}$ of the domain $\mathcal{D}$ contains a nonempty region of the form $\bar r_- < |\bar{\boldsymbol x}| < \bar r_+$ for some radii $\bar r_-$, $\bar r_+$ with $\bar r_- > 0$, and thus allows us to define multipole and tidal moments in the new coordinate system. We also parametrize the freely specifiable harmonic function $\beta_{\rm h}$ as

$$\beta_{\rm h} = \begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} \lambda_L(\bar t) \partial_L \frac{1}{|\bar{\boldsymbol x}|} - \frac{1}{l!} \tau_L(\bar t) \bar x^L, & \text{in } \bar{\mathcal{D}} \\ \text{arbitrary smooth function} & \text{in } \bar{\mathcal{D}}_{\rm int}, \end{cases} \tag{3.27}$$

where $\bar{\mathcal{D}}_{\rm int}$ is the image of the domain $0 \le r < r_-$. Here the quantities $\lambda_L(\bar t)$ and $\tau_L(\bar t)$ are STF on all their indices. This choice guarantees that the coordinate transformation (2.17) is defined and smooth on the entire domain $0 \le r < r_+$, while maintaining the harmonic property of the coordinates in the domain $r_- < r < r_+$ where the post-1-Newtonian equations are valid.

In the barred coordinate system we define the transformed moments ${}^{\rm n}\bar M_L(\bar t)$, ${}^{\rm n}\bar G_L(\bar t)$, ${}^{\rm pn}\bar M_L(\bar t)$, ${}^{\rm pn}\bar G_L(\bar t)$, $\bar Y_{iL}(\bar t)$, $\bar Z_{iL}(\bar t)$, $\bar S_L(\bar t)$, $\bar H_L(\bar t)$, $\bar\nu_L(\bar t)$, and $\bar\mu_L(\bar t)$ by the following barred versions of Eqs. (3.5a), (3.5b), (3.8), (3.15), and (3.16):

$$\bar\Phi(\bar t, \bar x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!}\, {}^{\rm n}\bar M_L(\bar t) \partial_L \frac{1}{|\bar{\boldsymbol x}|} - \frac{1}{l!}\, {}^{\rm n}\bar G_L(\bar t) \bar x^L, \tag{3.28a}$$

$$\begin{aligned} \bar\psi(\bar t, \bar x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} \bigg[{}^{\rm pn}\bar M_L(\bar t) + \frac{(2l+1)}{(l+1)(2l+3)} \dot{\bar\mu}_L(\bar t) \bigg] \partial_L \frac{1}{|\bar{\boldsymbol x}|} + \frac{(-1)^{l+1}}{l!}\, {}^{\rm n}\ddot{\bar M}_L(\bar t) \partial_L \frac{|\bar{\boldsymbol x}|}{2} - \frac{1}{l!} [{}^{\rm pn}\bar G_L(\bar t) - \dot{\bar\nu}_L(\bar t)] \bar x^L \\ - \frac{1}{l!} \frac{|\bar{\boldsymbol x}|^2}{2(2l+3)}\, {}^{\rm n}\ddot{\bar G}_L(\bar t) \bar x^L, \end{aligned} \tag{3.28b}$$

$$\bar\zeta_i(\bar t, \bar x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} \bar Z_{iL}(\bar t) \partial_L \frac{1}{|\bar{\boldsymbol x}|} - \frac{1}{l!} \bar Y_{iL}(\bar t) \bar x^{\langle L\rangle}, \tag{3.28c}$$

$$\bar Z_{iL}(\bar t) = \frac{4}{l+1}\, {}^{\rm n}\dot{\bar M}_{iL}(\bar t) - \frac{4l}{l+1} \epsilon_{ji\langle a_l} \bar S_{L-1\rangle j}(\bar t) + \frac{2l-1}{2l+1} \delta_{i\langle a_l} \bar\mu_{L-1\rangle}(\bar t), \tag{3.28d}$$

$$\bar Y_{iL}(\bar t) = \bar\nu_{iL}(\bar t) + \frac{l}{l+1} \epsilon_{ji\langle a_l} \bar H_{L-1\rangle j}(\bar t) - \frac{4(2l-1)}{2l+1}\, {}^{\rm n}\dot{\bar G}_{L-1\langle}(\bar t) \delta_{a_l\rangle i}. \tag{3.28e}$$

As before dots denote derivatives with respect to the time argument. By substituting the multipole expansions (3.5a)–(3.5c) of the original potentials into the transformation formulas (2.20a)–(2.20c), and comparing the results with the multipole expansions (3.28a)–(3.28c), we can derive transformation laws for the various moments.

To illustrate this procedure, we begin with the transformation of the Newtonian potential. We obtain

$$\bar{\Phi}(\bar{t}, \bar{x}^k) = \hat{\Phi}(\bar{t}, \bar{x}^k) + \ddot{z}_i(\bar{t})\bar{x}^i - \frac{1}{2}\dot{z}_i(\bar{t})\dot{z}^i(\bar{t}) + \dot{\alpha}_c(\bar{t})$$
$$= \dot{\alpha}_c(\bar{t}) + \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} {}^{n}M_L(\bar{t})\partial_L \frac{1}{|\bar{\boldsymbol{x}} + \boldsymbol{z}(\bar{t})|}$$
$$- \frac{1}{l!} {}^{n}G_L(\bar{t})[\bar{x} + z(\bar{t})]^L + \ddot{z}_i(\bar{t})\bar{x}^i - \frac{1}{2}\dot{z}^i(\bar{t})\dot{z}_i(\bar{t}). \tag{3.29}$$

Next we use the expansion

$$\frac{1}{|\bar{\boldsymbol{x}} + \boldsymbol{z}|} = \sum_{n=0}^{\infty} \frac{1}{n!} z^N \partial_N \frac{1}{|\bar{\boldsymbol{x}}|} \tag{3.30}$$

which is valid for $|\bar{\boldsymbol{x}}| > |\boldsymbol{z}|$ [cf. Eq. (3.26) above], and we compare with the multipole expansion (3.28a). This gives the well-known transformation laws for the monopole and dipole

$${}^{n}\bar{M} = {}^{n}M, \tag{3.31}$$

and

$${}^{n}\bar{M}_i = {}^{n}M_i - {}^{n}M z_i, \tag{3.32}$$

together with the general result

$${}^{n}\bar{M}_L = \sum_{n=0}^{l} (-1)^{n+l} \frac{l!}{n!(l-n)!} {}^{n}M_{\langle N} z_{L-N\rangle}. \tag{3.33}$$

Here ${}^{n}M_{\langle N} z_{L-N\rangle}$ denotes ${}^{n}M_{\langle a_1 \ldots a_n} z_{a_{n+1}} z_{a_{n+2}} \ldots z_{a_l\rangle}$, cf. Sec. I F above. Similarly we obtain from Eq. (3.29) the transformation law for the Newtonian gravitoelectric tidal moments

$${}^{n}\bar{G}_L = \sum_{k=0}^{\infty} \frac{1}{k!} {}^{n}G_{LK} z^K - l! \Lambda_L^{\Phi}. \tag{3.34}$$

Here

$$\Lambda^{\Phi} = -\frac{1}{2}\dot{z}_j \dot{z}_j + \dot{\alpha}_c, \tag{3.35a}$$
$$\Lambda_i^{\Phi} = \ddot{z}_i, \tag{3.35b}$$

and $\Lambda_L^{\Phi} = 0$ for $l \geq 1$. Following DSX [2], we call the quantities Λ_L^{Φ} *inertial moments.*

We similarly compute the transformation laws for the multipole and tidal moments Z_{iL} and Y_{iL}. We insert the expansion (3.8) into the transformation law (2.20b) and use the formulas (2.18a)–(2.18c) for the quantities α, h^i, and β. We also use the parametrization (3.27) of the harmonic function β_h. We then equate the result to the multipole expansion (3.28c) and obtain the transformation laws

$$\bar{Z}_{iL} = \sum_{n=0}^{l} (-1)^{n+l} \frac{l!}{n!(l-n)!} (Z_{i\langle N} z_{L-N\rangle} - 4\dot{z}_i {}^{n}M_{\langle N} z_{L-N\rangle}) + l\delta_{i\langle a_l} \lambda_{L-1\rangle}, \tag{3.36a}$$

$$\bar{Y}_{iL} = \sum_{k=0}^{\infty} \frac{1}{k!} (Y_{iLK} - 4\dot{z}_i {}^{n}G_{LK}) z^K - \tau_{iL} - l! \Lambda_{iL}^{\zeta}. \tag{3.36b}$$

Here

$$\Lambda_i^{\zeta} = \dot{h}_c^i + \dot{z}_i \dot{z}_j \dot{z}_j - \epsilon_{ijk} \dot{z}_j R_k - 2\dot{\alpha}_c \dot{z}_i, \tag{3.37a}$$
$$\Lambda_{ij}^{\zeta} = -\dot{z}_i \ddot{z}_j - \dot{z}_{(i} \ddot{z}_{j)} + \epsilon_{ijk} \dot{R}_k + 2\delta_{ij} \dot{z}_k \ddot{z}_k - \frac{4}{3}\delta_{ij} \ddot{\alpha}_c, \tag{3.37b}$$
$$\Lambda_{ijk}^{\zeta} = -\frac{6}{5}\delta_{i\langle j} \dddot{z}_{k\rangle}, \tag{3.37c}$$

and all the other gravitomagnetic inertial moments Λ_{iL}^{ζ} are zero.

The transformation laws for the fully STF moments S_L, H_L, μ_L, and ν_L parametrizing the gravitomagnetic potential [cf. Eq. (3.5c) above] are obtained by splitting the expressions (3.36a) and (3.36b) for $\bar{Z}_{iL}$ and $\bar{Y}_{iL}$ into their fully STF pieces using the identities (3.12), (A5), (A18), and (A19). The results are

$$\bar{S}_L = \sum_{n=0}^{l} (-1)^{n+l} \frac{l!}{n!(l-n)!} \left[\frac{n(l+1)}{l(n+1)} S_{\langle N} z_{L-N \rangle} - \frac{(l-n)}{l(n+1)} z_s \, {}^{\mathrm{n}}\dot{M}_{r\langle N} z_{(L-1)-N} \epsilon_{a_l \rangle rs} + \frac{n}{l} \dot{z}_r \, {}^{\mathrm{n}}M_{s\langle N-1} z_{(L-1)-(N-1)} \epsilon_{a_l\rangle rs} + \frac{l-n}{l} \dot{z}_r z_s \, {}^{\mathrm{n}}M_{\langle N} z_{(L-1)-N} \epsilon_{a_l \rangle rs} \right], \tag{3.38a}$$

$$\begin{aligned} \bar{\mu}_L = {} & \sum_{n=0}^{l+1} (-1)^{n+l+1} \frac{(l+1)!}{n!(l+1-n)!} \left[\left(\frac{n}{l+1} + \frac{l+1-n}{(l+1)(2n+1)} \left(2n - 1 - \frac{2}{2l+1} \right) \right) \mu_{\langle N-1} z_{L-(N-1)\rangle} \right. \\ & + \frac{4(l+1-n)(2l-2n+1)}{(l+1)(2l+1)(n+1)} z_j \, {}^{\mathrm{n}}\dot{M}_{j\langle N} z_{L-N\rangle} - \frac{4n(l+1-n)(2l+3)}{(l+1)(n+1)(2l+1)} z_j S_{p\langle N-1} z_{(L-1)-(N-1)} \epsilon_{a_l\rangle pj} \\ & - \frac{4n}{l+1} \dot{z}_j \, {}^{\mathrm{n}}M_{j\langle N-1} z_{L-(N-1)\rangle} - \frac{4(l+1-n)}{l+1} z_j \dot{z}_j \, {}^{\mathrm{n}}M_{\langle N} z_{L-N\rangle} + \frac{8n(l+1-n)}{(l+1)(2l+1)} z_j \, {}^{\mathrm{n}}M_{j\langle N-1} z_{(L-1)-(N-1)} \dot{z}_{a_l\rangle} \\ & \left. + \frac{4(l+1-n)(l-n)}{(l+1)(2l+1)} z_j z_j \, {}^{\mathrm{n}}M_{\langle N} z_{(L-1)-N} \dot{z}_{a_l\rangle} - \frac{4(l+1-n)(l-n)}{(l+1)(2l+1)(n+1)} z_j z_j \, {}^{\mathrm{n}}\dot{M}_{\langle N+1} z_{L-(N+1)\rangle} \right] + \frac{(l+1)(2l+3)}{2l+1} \lambda_L, \end{aligned} \tag{3.38b}$$

$$\bar{H}_L = \sum_{k=0}^{\infty} \frac{1}{k!} \left[H_{LK} z^K + 4 \, {}^{\mathrm{n}}G_{iK\langle L-1} \epsilon_{a_l\rangle ij} \dot{z}^j z^K + \frac{8k}{2l+2k+1} \, {}^{\mathrm{n}}\dot{G}_{iK-1\langle L-1} \epsilon_{a_l\rangle ij} z^j z^{K-1} \right] - l! \Lambda^{\zeta}_{ij\langle L-1} \epsilon_{a_l\rangle ij}, \tag{3.38c}$$

$$\begin{aligned} \bar{\nu}_{iL} = {} & -\tau_{iL} - l! \Lambda^{\zeta}_{\langle iL\rangle} + \sum_{k=0}^{\infty} \left[\nu_{iLK} z^K - 4 \dot{z}_{\langle i} \, {}^{\mathrm{n}}G_{L\rangle K} z^K + \frac{k}{l+k+1} z^j \epsilon_{jm\langle i} H_{L\rangle mK-1} z^{K-1} \right. \\ & - \frac{4k(2l+2k-1)}{(l+k)(2l+2k+1)} z_{i\langle} \, {}^{\mathrm{n}}\dot{G}_{L\rangle K-1} z^{K-1} + \frac{4k(k-1)}{(l+k)(2l+2k+1)} \, {}^{\mathrm{n}}\dot{G}_{iLK-2} z^{K-2} (z_j z_j) \\ & \left. + \frac{4kl}{(l+k)(2l+2k+1)} \, {}^{\mathrm{n}}\dot{G}_{K-1\langle iL-1} z_{a_l\rangle} z^{K-1} \right]. \end{aligned} \tag{3.38d}$$

It is also possible to derive from the expressions (3.36a) and (3.36b) transformation laws for the moments ${}^{\mathrm{n}}\dot{M}_{iL}$ and ${}^{\mathrm{n}}\dot{G}_{iL}$ that enter into the fully STF parametrizations (3.15) and (3.16) of Z_{iL} and Y_{iL}. The resulting transformation laws are consistent with the transformation laws (3.33) and (3.34) derived earlier for ${}^{\mathrm{n}}M_L$ and ${}^{\mathrm{n}}G_L$; this consistency is an important check of the formalism.

Finally we turn to the moments ${}^{\mathrm{pn}}M_L$ and ${}^{\mathrm{pn}}G_L$ parametrizing the post-Newtonian potential ψ. The transformation laws for these moments are by far the most tedious to compute. The results are

$${}^{\mathrm{pn}}\bar{M}_L = -\frac{2l+1}{(l+1)(2l+3)} \ddot{\mu}_L + \dot{\lambda}_L + \sum_{n=0}^{l} (-1)^{n+l} \frac{l!}{n!(l-n)!} \sigma_{\langle N} z_{L-N\rangle}, \tag{3.39}$$

where

$$\begin{aligned} \sigma_L = {} & {}^{\mathrm{pn}}M_L + \frac{2l+1}{(l+1)(2l+3)} \dot{\mu}_L - \dot{z}_j Z_{jL} + 2 \dot{z}_j \dot{z}_j \, {}^{\mathrm{n}}M_L + (\alpha_{\mathrm{c}} - z_j \dot{z}_j) \, {}^{\mathrm{n}}\dot{M}_L - l U_{\langle a_l} \, {}^{\mathrm{n}}M_{L-1\rangle} + \left(\frac{2l+1}{2l+3} \right) \left[(l+2) \ddot{z}_j \, {}^{\mathrm{n}}M_{jL} \right. \\ & \left. - U_{jj} \, {}^{\mathrm{n}}M_L - l \, {}^{\mathrm{n}}M_{j\langle L-1} U_{a_l\rangle j} + \frac{2l}{2l+1} U_{j\langle a_l} \, {}^{\mathrm{n}}M_{L-1\rangle j} + \dot{z}_j \, {}^{\mathrm{n}}\dot{M}_{jL} \right], \end{aligned} \tag{3.40a}$$

$$U_i = h^i_{\mathrm{c}} - z_j V_{ij} - z_j z_k V_{ijk}, \tag{3.40b}$$

$$U_{ij} = V_{ij} - 2 z_k V_{ijk}, \tag{3.40c}$$

$$V_{ij} = \epsilon_{ijk} R^k - \delta_{ij} \dot{\alpha}_{\mathrm{c}} + \frac{1}{2} \delta_{ij} \dot{z}_k \dot{z}_k + \frac{1}{2} \dot{z}_i \dot{z}_j, \tag{3.40d}$$

$$V_{ijk} = -\frac{1}{2} \delta_{ij} \ddot{z}_k + \ddot{z}_{[i} \delta_{j]k}, \tag{3.40e}$$

and

$$ {}^{\rm pn}\bar{G}_L = \dot{\bar{\nu}}_L + \sum_{k=0}^{\infty}\frac{1}{k!}(\rho_{LK} + l z_{\langle a_l}\rho'_{L-1\rangle K})z^K + \dot{\tau}_L - l!\Lambda_L^{\psi_{\rm h}}, \tag{3.41} $$

where

$$ \rho_L = {}^{\rm pn}G_L - \dot{\nu}_L - \dot{z}_i Y_{iL} + 2\dot{z}_j\dot{z}_j {}^{\rm n}G_L + U_j {}^{\rm n}G_{jL} + l U_{j\langle a_l} {}^{\rm n}G_{L-1\rangle j} - l(l-1)\ddot{z}_{\langle a_l} {}^{\rm n}G_{L-1\rangle} + l\dot{z}_{\langle a_l} {}^{\rm n}\dot{G}_{L-1\rangle} + (\alpha_{\rm c} - \dot{z}_j z^j) {}^{\rm n}\dot{G}_L + \frac{z_j z^j}{2(2l+3)}\rho'_L, \tag{3.42a} $$

$$ \rho'_L = {}^{\rm n}\ddot{G}_L + 2\dot{z}_j {}^{\rm n}\dot{G}_{jL} + \dot{z}_j\dot{z}_k {}^{\rm n}G_L + \ddot{z}_j {}^{\rm n}G_{jL}, \tag{3.42b} $$

$$ \Lambda^{\psi_{\rm h}} = -\dot{z}_j \dot{h}^j_{\rm c} - \frac{1}{4}(\dot{z}_j\dot{z}_j)^2 - \frac{1}{2}(\dot{\alpha}_{\rm c})^2 + \dot{\alpha}_{\rm c}\dot{z}_j\dot{z}_j, \tag{3.42c} $$

$$ \Lambda_i^{\psi_{\rm h}} = \epsilon_{ijk}\dot{z}_j\dot{R}_k + \frac{1}{2}\ddot{z}_i\dot{z}_j\dot{z}_j - \frac{3}{2}\dot{z}_i\ddot{z}_j\dot{z}_j - \dot{\alpha}_{\rm c}\ddot{z}_i + \ddot{\alpha}_{\rm c}\dot{z}_i, \tag{3.42d} $$

$$ \Lambda_{jk}^{\psi_{\rm h}} = -\frac{1}{2}\ddot{z}_{\langle j}\ddot{z}_{k\rangle} + \dot{z}_{\langle j}\dddot{z}_{k\rangle}, \tag{3.42e} $$

with all the other $\Lambda_L^{\psi_{\rm h}}$ being zero.

Note that the expression (3.39) for the transformed moment ${}^{\rm pn}\bar{M}_L$ depends on the transformed moment $\dot{\bar{\mu}}_L$. However, using the transformation law (3.38b) for μ_L we can write the expression (3.39) entirely in terms of the untransformed moments, and the dependence on μ_L cancels out. Similarly the expression (3.41) for the transformed moment ${}^{\rm pn}\bar{G}_L$ depends on the transformed moment $\dot{\bar{\nu}}_L$, but that dependence can be eliminated using the transformation law (3.38d) for ν_L.

Finally we note that the left-hand sides of Eqs. (3.33), (3.34), (3.36a), (3.36b), (3.38a)–(3.38d), (3.39), and (3.41), are functions of the new time coordinate $\bar{t}$. The right-hand sides are expressed as functions of $\bar{t}$ by evaluating the untransformed (unbarred) moments, which are functions of t, at $t = \bar{t}$. This replacement of t by $\bar{t}$ in the arguments of the untransformed moments is implicit in the transformation laws (2.20a)–(2.20c) for the potentials.[16]

D. Specialization to body-adapted gauge

The construction in Sec. III B above defined the mass multipole moments ${}^{\rm n}M_L(t)$, ${}^{\rm pn}M_L(t)$, current multipole moments $S_L(t)$, gravitoelectric tidal moments ${}^{\rm n}G_L(t)$, ${}^{\rm pn}G_L(t)$, and gravitomagnetic tidal moments $H_L(t)$ that are associated with a given body and with a given choice of coordinate system. In this section we discuss how to obtain a unique set of multipole and tidal moments associated with a given body by specializing to a coordinate system that is adapted to the body in a certain way. The particular coordinate system defined here is relatively well known; see, for example, Ref. [2]. We shall call it the "body-adapted gauge."

We start by reviewing the well-known construction at Newtonian order. From the transformation laws (3.32) and (3.34) for ${}^{\rm n}M_i$ and ${}^{\rm n}G_L$, we see that the freedom carried by the Newtonian-order world line $z(\bar{t})$ could be used in either of two ways. One can set to zero either the mass dipole moment ${}^{\rm n}\bar{M}_i(\bar{t})$ or the $l = 1$ gravitoelectric tidal moment $\bar{G}_i(\bar{t})$. The second choice is not very useful, as it makes the world line of the origin of spatial coordinates follow the tidal field instead of following the body. The first choice is the conventional and useful choice; choosing $z_i(\bar{t}) = {}^{\rm n}M_i(\bar{t})/{}^{\rm n}M$ achieves

$$ {}^{\rm n}\bar{M}_i = 0 \tag{3.43} $$

and the coordinates are then mass-centered to Newtonian accuracy. Normally such mass-centering would mean that the origin of coordinates coincides with the body's center of mass. In the present context, however, this conclusion is not valid, since the origin of coordinates is in the strong field region $r < r_-$ of space where the Newtonian equations are not necessarily valid. Moreover the definition of the center of mass in the present context is somewhat subtle; we defer to Sec. V C the discussion of this definition.

The second gauge specialization we make at Newtonian order is to set

$$ {}^{\rm n}\bar{G} = 0 \tag{3.44} $$

using the choice

$$ \alpha_{\rm c}(\bar{t}) = \int d\bar{t}\left[\frac{1}{2}\dot{z}_j\dot{z}_j + \sum_{l=0}^{\infty} {}^{\rm n}G_L z^L\right]; \tag{3.45} $$

cf. Eqs. (3.34) and (3.35a) above. In Newtonian physics, this choice simply corresponds to adjusting the zero of the gravitational potential, which does not affect the dynamics.

We now turn to a discussion of the gauge specialization at post-Newtonian order. Without loss of generality we can assume that we have already achieved the Newtonian conditions ${}^{\rm n}M_i = {}^{\rm n}G = 0$, and so we can specialize to the purely post-Newtonian subgroup of the coordinate transformations which is characterized by $z^i(\bar{t}) = \alpha_{\rm c}(\bar{t}) = 0$. That subgroup is parametrized by the functions $h^i_{\rm c}(\bar{t})$, $R^k(\bar{t})$ and by the STF tensors $\lambda_L(\bar{t})$ and $\tau_L(\bar{t})$ that define the harmonic function $\beta_{\rm h}$ via Eq. (3.27).

We first discuss the gravitoelectric sector. For the purely post-Newtonian coordinate transformations the transformation law (3.39) for the post-Newtonian mass multipole simplifies considerably to

$$ {}^{\rm pn}\bar{M}_L = {}^{\rm pn}M_L - l\,{}^{\rm n}M_{\langle L-1}h^{\rm c}_{a_l\rangle} - l\,{}^{\rm n}M_{j\langle L-1}\epsilon_{a_l\rangle jk}R^k. \tag{3.46} $$

[16] Note, in particular, that we do *not* evaluate the untransformed moments at $t = t(\bar{t})$, where $t(\bar{t})$ is the function $t(\bar{t}, \bar{x}^j)$ of the coordinate transformation (2.17) evaluated at $\bar{x}^j = 0$. The corresponding correction terms have already been included in the derivation of Eqs. (2.20a)–(2.20c).

If we specialize to the mass dipole, the last term in Eq. (3.46) vanishes since the coordinates are already mass-centered to Newtonian order. This gives

$$ {}^{\rm pn}\bar{M}_i = {}^{\rm pn}M_i - {}^{\rm n}M h^{\rm c}_i. \tag{3.47}$$

Therefore, as in the Newtonian case, we can use the translational freedom encoded in the function $h^{\rm c}_i$ to mass-center the coordinates by making

$$ {}^{\rm pn}\bar{M}_i = 0. \tag{3.48}$$

The transformation law (3.41) for the gravitoelectric tidal moments similarly simplifies to

$$ {}^{\rm pn}\bar{G}_L = {}^{\rm pn}G_L + h^{\rm c}_j\, {}^{\rm n}G_{jL} - l\, {}^{\rm n}G_{j\langle L-1}\epsilon_{a_l\rangle jk}R^k + \Lambda^G_L, \tag{3.49}$$

where the quantities Λ^G_L are given by $\Lambda^G = \dot{\tau}$, $\Lambda^G_i = -\ddot{h}^{\rm c}_i$, and $\Lambda^G_L = 0$ for $l \geq 2$. By choosing $\tau(\bar{t})$ suitably we can make

$$ {}^{\rm pn}\bar{G} = 0, \tag{3.50}$$

which is analogous to the Newtonian condition (3.44).

Consider next the gravitomagnetic sector. For purely post-Newtonian coordinate transformations, the transformation laws (3.38a) and (3.38c) for the current multipole moments S_L and gravitomagnetic tidal moments H_L simplify to

$$ \bar{S}_L = S_L, \tag{3.51}$$

$$ \bar{H}_L = H_L + \Lambda^H_L, \tag{3.52}$$

where $\Lambda^H_i = -2\dot{R}_i$ and $\Lambda^H_L = 0$ for $l \geq 2$. Therefore we can choose to make

$$ \bar{H}_i = 0 \tag{3.53}$$

by an appropriate choice of the angular velocity $\dot{R}^k$, as noted by DSX [2]. This gauge specialization makes the $l = 0$ part of the tidal piece $\boldsymbol{B}^T$ of the gravitomagnetic field vanish. The resulting coordinate system slowly rotates relative to distant stars, in such a way that the leading order Coriolis acceleration due to the tidal gravitomagnetic field is effaced.

At this stage, the remaining coordinate freedom is parametrized by the STF tensors λ_L for $l \geq 0$ and τ_L for $l \geq 1$, which appear in the formula (3.27) for the harmonic function $\beta_{\rm h}$. Note that these tensors do not enter into the transformation laws (3.46), (3.49), (3.51), and (3.52) for the moments ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, S_L, and H_L.[17] Therefore the values of these moments will be the same in all coordinate systems that satisfy the conditions specified so far, Eqs. (3.43), (3.44), (3.48), (3.50), and (3.53). In other words, these moments (as well as the Newtonian moments ${}^{\rm n}M_L$ and ${}^{\rm n}G_L$) are already uniquely defined by the conditions we have specified so far. Nevertheless, it is useful for some purposes to fix the remaining gauge freedom. To do this we consider the gauge moments μ_L and ν_L. For purely post-Newtonian transformations the transformation laws (3.38b) and (3.38d) for these moments reduce to

$$ \bar{\mu}_L = \mu_L + \frac{(l+1)(2l+3)}{2l+1}\lambda_L, \tag{3.54}$$

$$ \bar{\nu}_L = \nu_L - \tau_L + \Lambda^\nu_L, \qquad l \geq 1, \tag{3.55}$$

where $\Lambda^\nu_i = -\dot{h}^{\rm c}_i$ and $\Lambda^\nu_L = 0$ for $l \geq 2$. It follows that we can make

$$ \bar{\nu}_L = 0 \tag{3.56}$$

by choosing τ_L suitably, for all $l \geq 1$. Similarly we can make

$$ \bar{\mu}_L = 0 \tag{3.57}$$

by choosing λ_L suitably, for all $l \geq 0$. We shall call the unique coordinate system[18] that achieves all of the conditions (3.43), (3.44), (3.48), (3.50), (3.53), (3.56), and (3.57), the *body-adapted* coordinate system.[19]

To summarize, we have demonstrated that there is enough coordinate freedom to accomplish the following:

(1) Mass-center the coordinate system to post-1-Newtonian accuracy by setting ${}^{\rm n}M_i = {}^{\rm pn}M_i = 0$.
(2) Set to zero the $l = 0$ pieces ${}^{\rm n}G$, ${}^{\rm pn}G$ of the tidal pieces of the potentials Φ and ψ.
(3) Set to zero the gravitomagnetic tidal moment H_i.
(4) Set to zero the all the gauge moments ν_L and μ_L.

The role of each free function in the coordinate transformation (2.17) in the derivation of the above conditions is recapitulated in Table I.

We note that our definition of the body-frame coordinates or local asymptotic rest frame differs slightly from that of Thorne and Hartle [34]. We require that the mass dipole M_i should vanish for all time, whereas Thorne and Hartle instead require that the gravitoelectric tidal moment G_i should vanish for all time. At a given initial instant, they also demand that M_i and $\dot{M}_i$ vanish, while $\ddot{M}_i$ can be nonvanishing.

We next consider the special case treated by DSX, where the post-1-Newtonian field Eqs. (2.8a)–(2.8c) are assumed to hold all the way down to $r = 0$. In this case the gauge freedom is somewhat reduced, since the function $\beta_{\rm h}$ must now be both harmonic and smooth for all $r < r_+$. This requirement eliminates the terms parametrized by λ_L in Eq. (3.27), as those terms diverge at $r = 0$. Therefore we

[17] Achieving this gauge invariance was the reason for picking the particular choices of parametrization of Eqs. (3.19) and (3.20) above.

[18] Up to constant displacements in time and time-independent spatial rotations, cf. Sec. II B above.

[19] This gauge is called the skeletonized-body harmonic gauge by DSX [2].

TABLE I. Free functions in the coordinate transformation and their role in defining the body-adapted harmonic gauge.

Free function	Role
$z^i(\bar{t})$	sets ${}^{\rm n}\bar{M}_i = 0$
$\alpha_{\rm c}(\bar{t})$	sets ${}^{\rm n}\bar{G} = 0$
$h^i_{\rm c}(\bar{t})$	sets ${}^{\rm pn}\bar{M}_i = 0$
$\tau(\bar{t})$	sets ${}^{\rm pn}\bar{G} = 0$
$R_k(\bar{t})$	sets $\bar{H}_i = 0$
$\tau_L(\bar{t})$, $l \geq 1$	sets $\bar{\nu}_L = 0$
$\lambda_L(\bar{t})$, $l \geq 0$	sets $\bar{\mu}_L = 0$

no longer have sufficient gauge freedom to set to zero the gauge moments μ_L via Eq. (3.54). We still obtain a unique coordinate system by imposing the remaining requirements (3.43), (3.44), (3.48), (3.50), (3.53), and (3.56), but the μ_L moments will now in general be nonvanishing. This modified version of the body-adapted gauge will be important in Sec. IV below.

E. Definition of multipole and tidal moments about a given worldline

In Sec. III B above we defined the multipole and tidal moments of a body associated with a given coordinate system. Those moments can be interpreted as being moments about the origin $\boldsymbol{x} = 0$ of that coordinate system. In this section, we generalize that definition to define tidal and multipole moments about a specified world line[20] $x^i = z^i(t)$ and associated with a given coordinate system (t, x^i). This more general definition will be used in Sec. V D below.

We assume that the vacuum post-1-Newtonian field equations are satisfied in a region of the form $r_- < |\boldsymbol{x} - \boldsymbol{z}(t)| < r_+$. Our definition of the moments is given by the following multipole expansions of the potentials Φ, ζ^i, and ψ in this region:

$$\Phi(t, x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} {}^{\rm n}M_L(t)\partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(t)|} - \frac{1}{l!} {}^{\rm n}G_L(t)[x - z(t)]^L, \tag{3.58a}$$

$$\begin{aligned}\psi(t, x^j) = \sum_{l=0}^{\infty}\bigg[&\frac{(-1)^{l+1}}{l!}\bigg\{{}^{\rm pn}M_L(t)\partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(t)|} + \frac{(2l+1)}{(l+1)(2l+3)}\frac{\partial}{\partial t}\bigg[\mu_L(t)\partial_L\frac{1}{|\boldsymbol{x} - \boldsymbol{z}(t)|}\bigg]\bigg\}\\ &+ \frac{(-1)^{l+1}}{l!}\frac{\partial^2}{\partial t^2}\bigg[{}^{\rm n}M_L(t)\partial_L\frac{|\boldsymbol{x} - \boldsymbol{z}(t)|}{2}\bigg] - \frac{1}{l!}{}^{\rm pn}G_L(t)(x - z(t))^L + \frac{1}{l!}\frac{\partial}{\partial t}\{\nu_L(t)[x - z(t)]^L\}\\ &- \frac{1}{2l!(2l+3)}\frac{\partial^2}{\partial t^2}\{{}^{\rm n}G_L(t)[x - z(t)]^L|\boldsymbol{x} - \boldsymbol{z}(t)|^2\}\bigg],\end{aligned} \tag{3.58b}$$

$$\zeta_i(t, x^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} Z_{iL}\partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(t)|} - \frac{1}{l!} Y_{iL}[x - z(t)]^L, \tag{3.58c}$$

where

$$Z_{iL} = \frac{4}{l+1}{}^{\rm n}\dot{M}_{iL} - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S_{L-1\rangle j} + \frac{2l-1}{2l+1}\delta_{i\langle a_l}\mu_{L-1\rangle} + 4\dot{z}_{\langle i}{}^{\rm n}M_{L\rangle}, \tag{3.58d}$$

and

$$Y_{iL} = \nu_{iL} + \frac{l}{l+1}\epsilon_{ji\langle a_l}H_{L-1\rangle j} - \frac{4(2l-1)}{2l+1}{}^{\rm n}\dot{G}_{\langle L-1}\delta_{a_l\rangle i} + \frac{4(2l+1)}{2l+3}{}^{\rm n}G_{\langle L}\delta_{k\rangle i}\dot{z}^k. \tag{3.58e}$$

The expansions (3.58a) and (3.58b) have the same form as Eqs. (3.5a) and (3.5b) except that $\boldsymbol{x}$ is replaced everywhere by $\boldsymbol{x} - \boldsymbol{z}(t)$, and the various time-derivative operators are allowed to act on the factors of $\boldsymbol{x} - \boldsymbol{z}(t)$ as well as on the moments. Note that the formulas (3.58d) and (3.58e) for Z_{iL} and Y_{iL} contain extra terms involving $\dot{z}^i$ compared to the original formulas (3.15) and (3.16).

We now discuss the derivation of the expansions (3.58a)–(3.58c). First, one can verify that these expansions satisfy the field Eqs. (2.9a)–(2.9c). Next, inserting the expansions (3.58a) and (3.58c) into the gauge condition (2.6) yields the Newtonian conservation of mass Eq. (3.9) as before, and also the following replacements for Eqs. (3.10) and (3.11):

$$Z_{\langle iL\rangle} = \frac{4}{l+1}{}^{\rm n}\dot{M}_{iL} + 4\dot{z}_{\langle i}{}^{\rm n}M_{L\rangle} \tag{3.59}$$

and

$$Y_{jjL} = -4{}^{\rm n}\dot{G}_L + 4\dot{z}_j{}^{\rm n}G_{jL}. \tag{3.60}$$

If we now define the moments S_L, H_L, μ_L, and ν_L in terms of Z_{iL} and Y_{iL} using the same formulas (3.14a)–(3.14d) as before and use the decomposition identity (3.12), we obtain

[20]By a world line we mean simply a function $x^i = z^i(t)$ which transforms appropriately under the group (2.17) of coordinate transformations. If $z^i(t)$ lies outside of the domain of definition of the coordinates then it does not correspond to an actual world line in spacetime. See Sec. V C below for further discussion of this point.

Eqs. (3.58d) and (3.58e). Finally, the expression (3.58b) for ψ is chosen so that the moments ${}^{\rm n}M_L$, ${}^{\rm pn}M_L$, ${}^{\rm n}G_L$, ${}^{\rm pn}G_L$, H_L, and S_L defined by these expansions are invariant under the group (2.22) of gauge transformations. This invariance can be verified by using a parametrization of $\beta_{\rm h}$ of the form (3.27) with $\bar{\boldsymbol{x}}$ replaced by $\bar{\boldsymbol{x}} - z(\bar{t})$.

For later computations it is useful to expand the time derivatives in Eq. (3.58b) and express results in terms of STF tensors. This computation gives

$$\psi = \sum_{l=0}^{\infty}\left\{\frac{(-1)^{l+1}}{l!}\left[N_L\partial_L\frac{1}{|\boldsymbol{x}-z(t)|} + P_L\partial_L\frac{|\boldsymbol{x}-z(t)|}{2}\right] - \frac{1}{l!}\left[F_L(x-z(t))^L + J_L\frac{|\boldsymbol{x}-z(t)|^2[x-z(t)]^L}{2(2l+3)}\right]\right\}, \tag{3.61a}$$

where

$$N_L = {}^{\rm pn}M_L + \frac{2l+1}{(l+1)(2l+3)}\dot{\mu}_L + \frac{2l-1}{2l+1}\mu_{\langle L-1}\dot{z}_{a_l\rangle} + \frac{1}{2l+3}[\ddot{z}_j\,{}^{\rm n}M_{jL} + \dot{z}_j\dot{z}_j\,{}^{\rm n}M_L + 2\dot{z}_j\,{}^{\rm n}\dot{M}_{jL} + 2l\dot{z}_j\dot{z}_{\langle a_l}\,{}^{\rm n}M_{L-1\rangle j}], \tag{3.61b}$$

$$P_L = {}^{\rm n}\ddot{M}_L + 2l\dot{z}_{\langle a_l}\,{}^{\rm n}\dot{M}_{L-1\rangle} + l(l-1)\dot{z}_{\langle a_l}\dot{z}_{a_{l-1}}\,{}^{\rm n}M_{L-2\rangle} + l\ddot{z}_{\langle a_l}\,{}^{\rm n}M_{L-1\rangle}, \tag{3.61c}$$

$$F_L = {}^{\rm pn}G_L - \dot{\nu}_L + \nu_{Lj}\dot{z}^j - \frac{2l}{2l+1}\,{}^{\rm n}\dot{G}_{\langle L-1}\dot{z}_{a_l\rangle} + \frac{2l}{2l+3}\dot{z}^k\,{}^{\rm n}G_{k\langle L-1}\dot{z}_{a_l\rangle} + \frac{1}{2l+3}\,{}^{\rm n}G_L\dot{z}^j\dot{z}^j - \frac{l}{2l+1}\,{}^{\rm n}G_{\langle L-1}\ddot{z}_{a_l\rangle}, \tag{3.61d}$$

and

$$J_L = {}^{\rm n}\ddot{G}_L - 2\,{}^{\rm n}\dot{G}_{Lj}\dot{z}^j + {}^{\rm n}G_{Ljk}\dot{z}^j\dot{z}^k - {}^{\rm n}G_{Lk}\ddot{z}^k. \tag{3.61e}$$

All of the tensors N_L, P_L, F_L, and J_L are STF, while the tensors Z_{iL} and Y_{iL} are STF only on their last l indices, as before.

Finally, we note that there is no natural, unique definition of multipole and tidal moments about a given world line associated with a given coordinate system. We have chosen a particular definition, but there are other definitions that are equally valid. For example, if we replace Eq. (3.58d) with

$$Z_{iL} = \frac{4}{l+1}\,{}^{\rm n}\dot{M}_{iL} - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S_{L-1\rangle j} + 4\dot{z}_i\,{}^{\rm n}M\delta_{l0} + \frac{2l-1}{2l+1}\delta_{i\langle a_l}\mu_{L-1\rangle} + 4\,{}^{\rm n}M_{i\langle L-1}\dot{z}_{a_l\rangle}, \tag{3.62}$$

then field equations and harmonic gauge condition are still satisfied. However, while the moments defined by this equation are still invariant under the group (2.22) of gauge transformations, they differ from the moments defined by Eq. (3.58d). By contrast, the definitions of multipole and tidal moments about the origin of coordinates discussed in Sec. III B above are essentially unique. However, this lack of uniqueness will be unimportant for our purposes, since multipole and tidal moments about a given world line will appear only in intermediate steps in our computations and not in our final results.

F. Comparison with other definitions of multipole and tidal moments in the literature

As we have discussed, our multipole moments coincide with those of Blanchet and Damour [48] and of DSX [2,16,17] for weakly self-gravitating bodies. For isolated systems these moments agree to leading order with the unique moments that can be defined for stationary systems in general relativity [49], and with the asymptotic radiative multipole moments of Thorne [38], in the appropriate limits, as noted by DSX [2].

For nonisolated systems, there is another notion of mass and current multipole moments, defined in terms of the metric in a buffer region surrounding a body, due to Thorne and Hartle [34]. These moments are defined in full general relativity by the same type of surface integrals as used here [cf. Appendix E], but applied to the full metric rather than to the post-Newtonian potentials. These moments depend on the choice of coordinate system used to evaluate the surface integrals, but the magnitude of the resulting ambiguities can be estimated and in many applications are small enough to be unimportant [34]. By contrast, the multipole moments used here are defined only the context of post-1-Newtonian theory, but are unique.

Our tidal moments also coincide with those of DSX for weakly self-gravitating bodies. They also appear to coincide with the tidal moments defined by Suen in the context of stationary systems in full general relativity [50].

IV. POST-1-NEWTONIAN LAWS OF MOTION: A SINGLE BODY

A. Overview

As discussed in the introduction, the derivation of equations of motion for several interacting bodies can be divided into two pieces: (i) A derivation of a formula,[21] for any given body, of the second time derivative of its mass dipole moment in terms of its other multipole and tidal moments and their time derivatives. (ii) A derivation of the relation between the tidal moments acting on each body and the multipole moments and center-of-mass world lines of all the other bodies. In this section we will carry out the first of these tasks. The second task will be the subject of Secs. V and VI below.

[21] DSX call this formula the "law of motion" [2].

We start by describing the assumptions and the result. As in Sec. III B above, we assume existence of a local coordinate system (t, x^j) with the following properties: (i) The range of the coordinates contains the product of the open ball

$$|\mathbf{x}| < r_+, \tag{4.1}$$

where r_+ is some radius, with some open interval (t_0, t_1) of time. (ii) On the spatial region $\mathcal{D}$ given by $r_- < r < r_+$, for some $r_- > 0$, the coordinates are conformally Cartesian and harmonic. Also the vacuum Einstein equations are valid on $\mathcal{D}$ for a one-parameter family of metrics of the form of Eq. (4.11) below. Essentially this says that the Newtonian, post-1-Newtonian and post-2-Newtonian vacuum field equations are valid on $\mathcal{D}$. The reason for imposing the post-2-Newtonian field equations in addition to the post-1-Newtonian field equations is discussed below.

These assumptions allow us to define the multipole moments ${}^{\rm n}M_L(t)$, ${}^{\rm pn}M_L(t)$, $S_L(t)$ which characterize the sources in the region $r < r_-$, as well as the tidal moments ${}^{\rm n}G_L(t)$, ${}^{\rm pn}G_L(t)$, and $H_L(t)$, as discussed in the previous section. They also imply some formulas relating time derivatives of the multipole moments. At Newtonian order these formulas are

$${}^{\rm n}\dot{M}(t) = 0, \tag{4.2a}$$

$${}^{\rm n}\ddot{M}_i(t) = \sum_{l=0}^{\infty} \frac{1}{l!}\, {}^{\rm n}M_L(t)\, {}^{\rm n}G_{iL}(t). \tag{4.2b}$$

Here the first formula is just the Newtonian conservation of mass derived earlier [Eq. (3.9) above]. The second formula equates the acceleration of the center of mass to the acceleration produced by the external tidal moments coupling with the multipole moments of the system. This formula is usually derived by integrating the Newtonian stress-energy conservation equations over the interior of the body. Here, however, we do not assume the validity of the Newtonian equations in the interior. Equation (4.2b) contains all the information one needs in order to derive the explicit coupled equations of motion for the center-of-mass world lines of each body in an N-body system. We will describe this derivation later in Sec. VI.

At post-1-Newtonian order, the formulas analogous to Eqs. (4.2a) and (4.2b) are

$${}^{\rm pn}\dot{M} = -\sum_{l=0}^{\infty} \frac{1}{l!}[(l+1)\,{}^{\rm n}M_L\,{}^{\rm n}\dot{G}_L + l\,{}^{\rm n}\dot{M}_L\,{}^{\rm n}G_L], \tag{4.3a}$$

$$\begin{aligned}{}^{\rm pn}\ddot{M}_i = \sum_{l=0}^{\infty} \frac{1}{l!}\bigg[&{}^{\rm pn}M_L\,{}^{\rm n}G_{iL} + {}^{\rm n}M_L\,{}^{\rm pn}G_{iL} + \frac{l}{l+1}S_L H_{iL} + \frac{1}{l+2}\epsilon_{ijk}\,{}^{\rm n}M_{jL}\dot{H}_{kL} + \frac{1}{l+1}\epsilon_{ijk}\,{}^{\rm n}\dot{M}_{jL}H_{kL} - \frac{4(l+1)}{(l+2)^2}\epsilon_{ijk}S_{jL}\,{}^{\rm n}\dot{G}_{kL} \\ &- \frac{4}{l+2}\epsilon_{ijk}\dot{S}_{jL}\,{}^{\rm n}G_{kL} - \frac{2l^3+7l^2+15l+6}{(l+1)(2l+3)}\,{}^{\rm n}M_{iL}\,{}^{\rm n}\ddot{G}_L - \frac{2l^3+5l^2+12l+5}{(l+1)^2}\,{}^{\rm n}\dot{M}_{iL}\,{}^{\rm n}\dot{G}_L - \frac{l^2+l+4}{l+1}\,{}^{\rm n}\ddot{M}_{iL}\,{}^{\rm n}G_L\bigg],\end{aligned} \tag{4.3b}$$

$$\dot{S}_i = \sum_{l=0}^{\infty} \frac{1}{l!}\epsilon_{ijk}\,{}^{\rm n}M_{jL}\,{}^{\rm n}G_{kL}. \tag{4.3c}$$

For bodies in which the post-Newtonian field equations are valid everywhere, DSX derived the formulas (4.3a)–(4.3c) by using Newtonian[22] and post-Newtonian stress-energy conservation in the interior of the body [16]. In this section we will derive Eq. (4.3b) from the assumptions listed above. The formulas (4.3a) and (4.3c) giving the time evolution of the mass monopole and the spin will be derived from the same assumptions in the second paper in this series [37].

Note that the formulas (4.3a)–(4.3c) are valid for all coordinate systems satisfying the assumptions listed above, not just for the body-adapted coordinate system discussed in Sec. III D above.

[22]As is well known, the derivation of the spin evolution Eq. (4.3c) requires only Newtonian-order stress-energy conservation.

B. Method of derivation

We now turn to a description of the surface-integral method of derivation that we use. For this description we return, temporarily, to the context of the full, nonlinear equations of general relativity. The method is well known and is described in Landau and Lifshitz [51] and in Misner, Thorne, and Wheeler [46]. It has been previously applied to the derivation of laws of motion by Thorne and Hartle [34].

The method starts by fixing a coordinate system $x^\mu = (t, x^j)$, and by writing Einstein's equations in that coordinate system in a form involving pseudotensors and partial derivatives as

$$\mathcal{H}^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi[(-g)T^{\mu\nu} + \mathcal{T}^{\mu\nu}]. \tag{4.4}$$

Here $g = \det g_{\mu\nu}$ and the tensor density $\mathcal{H}^{\mu\alpha\nu\beta}$ is given by

$$\mathcal{H}^{\mu\alpha\nu\beta} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \tag{4.5}$$

where

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}. \tag{4.6}$$

In Eq. (4.4), $T^{\mu\nu}$ is the stress-energy tensor and the pseudotensor $\mathcal{T}^{\mu\nu}$ is given by

$$\begin{aligned}\mathcal{T}^{\alpha\beta} = \frac{1}{16\pi}\bigg[&\mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda}\mathfrak{g}^{\beta\mu}{}_{,\mu} \\ &+ \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}{}_{,\rho}\mathfrak{g}^{\rho\mu}{}_{,\nu} + g_{\lambda\mu}g^{\nu\rho}\mathfrak{g}^{\alpha\lambda}{}_{,\nu}\mathfrak{g}^{\beta\mu}{}_{,\rho} \\ &- g_{\mu\nu}(g^{\alpha\lambda}\mathfrak{g}^{\beta\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda}\mathfrak{g}^{\alpha\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda}) \\ &+ \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} \\ &- g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\mu}\bigg]. \end{aligned} \tag{4.7}$$

This formulation of Einstein's equations is due to Landau and Lifshitz [51].

Next, in a surface of constant t within the domain of the coordinates, we consider a three-dimensional region V whose topology is that of the interior of a sphere, and whose boundary $\Sigma = \partial V$ has spherical topology. We assume that the stress-energy tensor $T^{\mu\nu}$ vanishes in a neighborhood of Σ. We define a quantity P^i_Σ associated with Σ and with the choice of coordinate system by

$$P^i_\Sigma(t) \equiv \frac{1}{16\pi}\oint_\Sigma \mathcal{H}^{i\alpha 0j}{}_{,\alpha}d^2\Sigma_j. \tag{4.8}$$

Here $d^2\Sigma_j$ is the natural surface element determined by the flat metric $(dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Using the flat-space Gauss's theorem, the field Eq. (4.4), the symmetry properties of $\mathcal{H}^{\mu\alpha\nu\beta}$, and the assumption (4.1), the definition (4.8) can also be written as

$$P^i_\Sigma(t) \equiv \int_V d^3x[(-g)T^{0i} + \mathcal{T}^{0i}]. \tag{4.9}$$

Here d^3x is the volume element associated with the flat metric $(dx^1)^2 + (dx^2)^2 + (dx^3)^2$. As is well known, for an asymptotically flat spacetime the quantity P^i_Σ yields the ADM 3-momentum in the limit where Σ tends to infinity. For finite Σ, however, P^i_Σ does not have any invariant physical or geometric meaning. Nevertheless, we can still use this quantity as an intermediate tool in mathematical calculations in deriving relations between quantities whose transformation properties and meaning are well understood, such as the multipole and tidal moments discussed in the previous section. For convenience, we will refer to P^i_Σ as the "enclosed 3-momentum" in Σ, even though it is not an invariant quantity.

It follows from the formula (4.9) and from the form $[(-g)T^{\mu\nu} + \mathcal{T}^{\mu\nu}]_{,\nu} = 0$ of stress-energy conservation that the time derivative of the enclosed 3-momentum is[23]

$$\dot{P}^i_\Sigma = -\oint_\Sigma \mathcal{T}^{ij}d^2\Sigma_j. \tag{4.10}$$

The core of the surface-integral method is to compute both sides of Eq. (4.10) explicitly. Namely, we compute the left-hand side by performing the surface integral (4.8) and taking a time derivative of the result, and we compute the right-hand side using Eq. (4.7). These surface integrals are evaluated using the general solutions (3.5a)–(3.5c) to the vacuum post-1-Newtonian field equations. The dependence on the surface Σ drops out and one obtains in this way relations between the moments and their time derivatives that reduce to Eqs. (4.2b) and (4.3b) at Newtonian and post-Newtonian order, respectively.

C. Post-2-Newtonian field equations

In applying the surface-integral method to derive the laws of motion to some post-Newtonian order m, one needs to compute some pieces of the metric to post-Newtonian order $m + 1$. This was emphasized in the original work of Einstein, Infeld, and Hoffmann [7]. For example, the Newtonian mass currents, whose conservation law can be used to derive laws of motion at Newtonian order, are the source for the gravitomagnetic potential ζ [cf. Eq. (2.9c)]. Therefore, if one wants to avoid dealing with the matter distribution itself and to use instead only the far field metric to derive Newtonian laws of motion, one needs to use the post-1-Newtonian gravitomagnetic potential. Note however that knowledge of the post-1-Newtonian scalar potential ψ is not required for this purpose.

Therefore, for our goal of computing the post-1-Newtonian laws of motion, we need to compute some pieces of the metric to post-2-Newtonian order. In this subsection we derive the harmonic-gauge vacuum field equations satisfied by these post-2-Newtonian fields.

It turns out that we need the post-2-Newtonian corrections only to g_{ij} and to g_{0i}. We can therefore parametrize the metric as the post-1-Newtonian metric (2.4) together

[23]Note that the derivation of this conservation law does not require the assumption that Einstein's equations are valid for $r < r_-$, since we can take the point of view that the stress-tensor $T^{\mu\nu}$ is defined by Eq. (4.4). In other words, a different theory of gravity could be applicable in the strong field region $r < r_-$, with the correction to the field equations being incorporated into the definition of $T^{\mu\nu}$. Our application of the conservation law (4.10) to derive the equation of motion (4.3b) will therefore apply to any theory of gravity for which the vacuum field equations coincide with those of general relativity.

with the appropriate correction terms:

$$\begin{aligned} ds^2 = &-\frac{1}{\varepsilon^2}[1 + 2\varepsilon^2\Phi + 2\varepsilon^4(\Phi^2 + \psi) + O(\varepsilon^6)]dt^2 \\ &+ [2\varepsilon^2\zeta_i + 2\varepsilon^4(2\Phi\zeta_i + \xi_i) + O(\varepsilon^5)]dx^i dt \\ &+ [\delta_{ij} - 2\varepsilon^2\Phi\delta_{ij} + \varepsilon^4((2\Phi^2 - 2\psi + \chi_{kk})\delta_{ij} \\ &- \chi_{ij}) + O(\varepsilon^5)]dx^i dx^j. \end{aligned} \tag{4.11}$$

Here the post-2-Newtonian fields are the vector field ξ_i and the spatial symmetric tensor χ_{ij}. We have chosen the particular parametrization of the post-2-Newtonian pieces of the metric (4.11) to simplify the gothic metric and the field equations. From the definition (4.6) of the gothic metric we obtain

$$\begin{aligned} \mathfrak{g}^{00} &= -\varepsilon + 4\varepsilon^3\Phi - \varepsilon^5(8\Phi^2 - 4\psi + \chi_{kk}) + O(\varepsilon^6), \\ \mathfrak{g}^{0i} &= \varepsilon^3\zeta^i + \varepsilon^5\xi^i + O(\varepsilon^6), \\ \mathfrak{g}^{ij} &= \frac{1}{\varepsilon}\delta^{ij} + \varepsilon^3\chi_{ij} + O(\varepsilon^4). \end{aligned} \tag{4.12}$$

Inserting these expressions into the harmonic gauge condition (2.5) yields the equations

$$\dot{\zeta}^i + \chi^{ij}{}_{,j} = 0 \tag{4.13}$$

and

$$\xi^i_{,i} - \dot{\chi}_{kk} + 4\dot{\psi} - 16\Phi\dot{\Phi} = 0. \tag{4.14}$$

The harmonic-gauge vacuum field equations for the fields ξ^i and χ^{ij} can be derived by substituting the expansion (4.12) of the gothic metric into Eq. (4.4). The results are

$$\nabla^2\xi^i = \ddot{\zeta}^i + 12\dot{\Phi}\partial_i\Phi + 8\partial_{[i}\zeta_{k]}\partial_k\Phi \tag{4.15}$$

and

$$\nabla^2\chi_{ij} = 4\partial_i\Phi\partial_j\Phi - 2\delta_{ij}\partial_k\Phi\partial_k\Phi. \tag{4.16}$$

We next compute the expansions of the enclosed momentum P^i_Σ and the spatial components $\mathcal{T}^{ij}$ of the Landau-Lifshitz pseudotensor that appear in conservation law (4.10). Substituting the gothic metric components (4.12) into the definition (4.8) of enclosed momentum yields

$$P^i_\Sigma = \varepsilon^2\,{}^{\rm n}P^i_\Sigma + \varepsilon^4\,{}^{\rm pn}P^i_\Sigma + O(\varepsilon^5), \tag{4.17}$$

where the Newtonian piece is

$${}^{\rm n}P^i_\Sigma = \frac{1}{16\pi}\oint_\Sigma \partial_j\zeta^i d^2\Sigma_j \tag{4.18}$$

and the post-Newtonian piece is

$${}^{\rm pn}P^i_\Sigma = \frac{1}{16\pi}\oint_\Sigma [\partial_j\xi^i + \dot{\chi}_{ij}]d^2\Sigma_j. \tag{4.19}$$

The corresponding expansion of $\mathcal{T}^{ij}$ is, from Eqs. (4.7), (4.11), and (4.12),

$$\mathcal{T}^{ij} = \varepsilon^2\,{}^{\rm n}\mathcal{T}^{ij} + \varepsilon^4\,{}^{\rm pn}\mathcal{T}^{ij} + O(\varepsilon^5), \tag{4.20}$$

where the Newtonian piece is

$${}^{\rm n}\mathcal{T}^{ij} = \frac{1}{4\pi}\left(\partial_i\Phi\partial_j\Phi - \frac{1}{2}\delta_{ij}\partial_k\Phi\partial_k\Phi\right) \tag{4.21}$$

and the post-Newtonian piece is

$$\begin{aligned} {}^{\rm pn}\mathcal{T}^{ij} = \frac{1}{4\pi}\bigg[&\partial_{[i}\zeta_{k]}\partial_{[k}\zeta_{j]} + 2\partial_{(i}\Phi\dot{\zeta}_{j)} + 2\partial_{(i}\Phi\partial_{j)}\psi \\ &- \frac{1}{2}\delta_{ij}\bigg(\frac{1}{2}\partial_{[l}\zeta_{k]}\partial_{[k}\zeta_{l]} + 2\partial_k\Phi\dot{\zeta}_k + 2\partial_k\Phi\partial_k\psi \\ &+ 3\dot{\Phi}^2\bigg)\bigg]. \end{aligned} \tag{4.22}$$

D. Newtonian order derivation

To illustrate the method of computation, we first derive the Newtonian law of motion (4.2b) from the conservation law (4.10). A similar derivation has been given by Futamase [21]. We choose the two-surface Σ to be the coordinate sphere $r = R$, for some R with $r_- < R < r_+$, and we henceforth drop the subscript Σ in P^i_Σ for simplicity. The formula (4.18) then reduces to

$${}^{\rm n}P^i = \frac{R^2}{16\pi}\oint n^j\partial_j\zeta^i(R\mathbf{n})d\Omega, \tag{4.23}$$

where the integral is over the unit sphere and $n^j = x^j/|\mathbf{x}|$. Plugging in the general solution (3.8) for the gravitomagnetic potential and using the identities (A12) and (A13) yields

$$\begin{aligned} {}^{\rm n}P^i &= \frac{R^2}{16\pi}\oint n^j\bigg[\sum_{l=0}^\infty \frac{(-1)^{l+1}}{l!}Z_{iL}\partial_{jL}\frac{1}{|\mathbf{x}|} - \frac{l}{l!}Y_{ijL-1}x^{L-1}\bigg]_{r=R}d\Omega \\ &= \frac{R^2}{16\pi}\oint\bigg[\sum_{l=0}^\infty \frac{(-1)^{l+1}}{l!}\frac{1}{R^{l+2}}(a_l Z_{ijL-1}n^j n^{L-1} - b_l Z_{iL}n_j n^j n^L) - \frac{l}{l!}R^{l-1}Y_{ijL-1}n^j n^{L-1}\bigg]d\Omega \\ &= \frac{R^2}{16\pi}\oint\bigg[\sum_{l=0}^\infty \frac{(-1)^{l+1}}{l!}\frac{1}{R^{l+2}}(a_l - b_l)Z_{iL}n^L - \frac{l}{l!}R^{l-1}Y_{iL}n^L\bigg]d\Omega. \end{aligned} \tag{4.24}$$

Since Z^A_{iL} and Y^A_{iL} are STF on L, only the $l = 0$ terms can contribute [cf. Eqs. (A14) and (A15)]. This implies that

$$ {}^{\rm n}P^i = \frac{R^2}{16\pi}\frac{4\pi b_0}{R^2} Z_i = {}^{\rm n}\dot{M}_i, \tag{4.25}$$

where we have used the relation (3.10). Note that the contribution from the gravitomagnetic tidal terms vanishes identically. Thus, the Newtonian enclosed momentum is the time derivative of the Newtonian mass dipole.

We turn next to the surface integral of the Landau-Lifshitz pseudotensor on the right-hand side of Eq. (4.10). To Newtonian order it is given by

$$\oint {}^{\rm n}\mathcal{T}^{ij} d^2\Sigma_j = \oint \frac{1}{4\pi}\left(\partial_i\Phi\partial_j\Phi - \frac{1}{2}\delta_{ij}\partial_k\Phi\partial_k\Phi\right) d^2\Sigma_j, \tag{4.26}$$

from Eq. (4.21). This surface integral can be computed using the expansion (3.5a) of the Newtonian potential and the integrals (A14)–(A17). First we write the derivative of the expansion (3.5a) of the Newtonian potential as

$$\partial_i\Phi = \sum_{l=0}^{\infty}\frac{1}{l!}(V_L^\Phi n^i - \bar{V}_{iL}^\Phi)n^L, \tag{4.27}$$

where

$$V_L^\Phi = \frac{(2l+1)!!}{r^{l+2}}\,{}^{\rm n}M_L, \tag{4.28}$$

and

$$\bar{V}_{iL}^\Phi = \frac{(2l+1)!!}{r^{l+3}}\,{}^{\rm n}M_{iL} + r^l\,{}^{\rm n}G_{iL}. \tag{4.29}$$

Substituting this into the right-hand side of Eq. (4.26) and using the integrals (A16) and (A17) gives

$$\begin{aligned}\oint {}^{\rm n}\mathcal{T}^{ij} d^2\Sigma_j &= \frac{R^2}{4\pi}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{p!q!}\oint n^P n^Q\Big[(V_P^\Phi n^i - \bar{V}_{iP}^\Phi)(V_Q^\Phi - q\bar{V}_Q^\Phi) - \frac{n^i}{2}(V_P^\Phi n^k - \bar{V}_{kP}^\Phi)(V_Q^\Phi n^k - \bar{V}_{kQ}^\Phi)\Big]d\Omega \\ &= \frac{R^2}{4\pi}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{p!q!}\oint\Big[\frac{1}{2}(V_P^\Phi V_Q^\Phi - \bar{V}_{kP}^\Phi\bar{V}_{kQ}^\Phi)n^i n^P n^Q + (q\bar{V}_{iP}^\Phi\bar{V}_Q^\Phi - \bar{V}_{iP}^\Phi V_Q^\Phi)n^P n^Q\Big]d\Omega \\ &= R^2\sum_{l=0}^{\infty}\frac{1}{l!}\Big[\frac{1}{(2l+3)!!}(V_{iL}^\Phi V_L^\Phi - \bar{V}_{ikL}^\Phi\bar{V}_{kL}^\Phi) + \frac{1}{(2l+1)!!}(l\bar{V}_{iL}^\Phi\bar{V}_L^\Phi - \bar{V}_{iL}^\Phi V_L^\Phi)\Big] \\ &= R^2\sum_{l=0}^{\infty}\frac{1}{l!(2l+1)!!}V_L^\Phi\left(\frac{V_{iL}^\Phi}{2l+3} - \bar{V}_{iL}^\Phi\right) = -\sum_{l=0}^{\infty}\frac{1}{l!}\,{}^{\rm n}M_L^A\,{}^{\rm n}G_{iL}^A.\end{aligned} \tag{4.30}$$

Here the last line has been obtained using the expressions (4.28) and (4.29) for V_L^Φ and $\bar{V}_{iL}^\Phi$ in terms of ${}^{\rm n}M_L$, ${}^{\rm n}G_L$, and r, and performing some easy algebra. We thus have the key result:

$$-\oint {}^{\rm n}\mathcal{T}^{ij} d^2\Sigma_j = \sum_{l=0}^{\infty}\frac{1}{l!}\,{}^{\rm n}M_L^A\,{}^{\rm n}G_{iL}^A. \tag{4.31}$$

Substituting Eqs. (4.25) and (4.31) into the expansions (4.17) and (4.20) and then into the conservation law (4.10) finally gives the Newtonian law of motion (4.2b).

E. Post-Newtonian order derivation

We now proceed with the derivation of the post-1-Newtonian law of motion (4.3b) from the conservation law (4.10). Our method of derivation will be somewhat different from that used above in the Newtonian case. We start by describing the differences.

The differences are related to the pieces of the computation that go to one higher post-Newtonian order (post-1-Newtonian order in the last subsection, and post-2-Newtonian order here). In the Newtonian case, we had available the explicit parametrization (3.5b) and (3.5c) of the post-1-Newtonian potentials in terms of (i) the Newtonian order moments ${}^{\rm n}M_L$, ${}^{\rm n}G_L$; (ii) the post-1-Newtonian- order moments ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, H_L, and S_L, and (iii) the gauge moments μ_L and ν_L. The computation of the Newtonian enclosed momentum (4.18) did involve the post-1-Newtonian potentials, but using the parametrization (3.5c) we found that the dependencies on the post-Newtonian multipole, tidal, and gauge moments dropped out. Thus, we obtained equations of motion in terms of the purely Newtonian variables. Similarly, in the post-1-Newtonian case, the computation of the post-1-Newtonian momentum (4.19) involves the post-2-Newtonian potentials ξ^i and χ_{ij}. Those potentials can presumably be parametrized in terms of Newtonian and post-1-Newtonian moments, a set of post-2-Newtonian moments, and gauge degrees of freedom, via expansions analogous to (3.5b) and (3.5c). Using such expansions we could in principle proceed as in the Newtonian computation. However, this approach turns out to be extremely tedious,[24] and we will proceed instead as follows.

[24]The surface integrals encountered in the derivation of the evolution laws (4.3a) and (4.3c) for the mass monopole and spin are significantly simpler than those for the dipole evolution law (4.3b). We have derived explicit expressions for one particular solution of the post-2-Newtonian field equations for ξ^i and χ^{ij}, and in a later paper [37] those expressions will be used to derive the mass monopole and spin evolution laws (4.3a) and (4.3c) using the same explicit method as used here in the Newtonian case.

Our strategy will be to argue indirectly that there is no dependence on post-2-Newtonian degrees of freedom in the post-1-Newtonian momentum (4.19). The space of post-2-Newtonian potentials (ξ^i, χ_{ij}) that satisfy in $\mathcal{D}$ the field Eqs. (4.15) and (4.16) and the gauge conditions (4.13) and (4.14) has the structure of an affine space. Given a particular solution $(\xi^i_{\rm p}, \chi^{ij}_{\rm p})$, any other solution can be written as the sum

$$\xi^i = \xi^i_{\rm p} + \xi^i_{\rm h} \tag{4.32a}$$

$$\chi^{ij} = \chi^{ij}_{\rm p} + \chi^{ij}_{\rm h} \tag{4.32b}$$

of the particular solution and a homogeneous solution $(\xi^i_{\rm h}, \chi^{ij}_{\rm h})$, where the homogeneous solution satisfies homogeneous versions of the field Eqs. (4.15) and (4.16) and gauge conditions (4.13) and (4.14):

$$\nabla^2 \xi^i_{\rm h} = 0, \qquad \nabla^2 \chi^{ij}_{\rm h} = 0, \tag{4.33a}$$

$$\partial_j \chi^{ij}_{\rm h} = 0, \qquad \partial_i \xi^i_{\rm h} - \dot{\chi}^{kk}_{\rm h} = 0. \tag{4.33b}$$

The general solutions to Eqs. (4.33a) can be expanded as

$$\xi^i_{\rm h} = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} X_{iL} \partial_L \frac{1}{|\boldsymbol{x}|} - \frac{1}{l!} W_{iL} x^L, \tag{4.34a}$$

$$\chi^{ij}_{\rm h} = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} C_{ijL} \partial_L \frac{1}{|\boldsymbol{x}|} - \frac{1}{l!} B_{ijL} x^L. \tag{4.34b}$$

These multipole expansions define the multipole and tidal moments X_{iL}, W_{iL}, C_{ijL}, and B_{ijL}, all of which are STF on the indices L. Inserting these expansions into the gauge conditions (4.33b) yields the constraints

$$C_{i\langle L\rangle} = 0, \qquad B_{ijjL} = 0, \tag{4.35a}$$

$$lX_{\langle L\rangle} + \dot{C}_{jjL} = 0, \qquad W_{jjL} - \dot{B}_{jjL} = 0. \tag{4.35b}$$

Next, we insert the decompositions (4.32a) and (4.32b) and the expansions (4.34a) and (4.34b) into the formula (4.19) for the post-1-Newtonian momentum. Using the integrals (A14)–(A17) then gives

$$ {}^{\rm pn}P^i = \frac{1}{16\pi} \oint_\Sigma [\partial_j \xi^i_{\rm p} + \dot{\chi}^{ij}_{\rm p}] d^2\Sigma_j + \frac{1}{4} X_i - \frac{1}{12} \dot{C}_{ijj} - \frac{R^3}{12} \dot{B}_{ijj}. \tag{4.36}$$

Here as before we have chosen the two-surface Σ to be the sphere $|\boldsymbol{x}| = R$. The last three terms in this expression give the dependence of ${}^{\rm pn}P^i$ on the homogeneous solution. From the constraints (4.35a) and (4.35b) it follows that the sum of these three terms vanishes, so that

$$ {}^{\rm pn}P^i = \frac{1}{16\pi} \oint [\partial_j \xi^i_{\rm p} + \dot{\chi}^{ij}_{\rm p}] d^2\Sigma_j. \tag{4.37}$$

Thus, the enclosed momentum ${}^{\rm pn}P_i$ is independent of which solution of the post-2-Newtonian field equations is chosen. Therefore, it must be a function only of the Newtonian and post-1-Newtonian fields, or equivalently of the Newtonian and post-1-Newtonian multipole and tidal moments, as well as of the radius R.

Next, we show that the enclosed momentum can be written as the sum of the time derivative of the post-Newtonian mass dipole ${}^{\rm pn}\dot{M}_i$ and terms that are independent of ${}^{\rm pn}M_i$. To see this, note that any solution (Φ, ζ^i, ψ) of the post-1-Newtonian field Eqs. (2.9a)–(2.9c) and gauge condition (2.6) can be decomposed as

$$\Phi = \Phi_0, \qquad \zeta^i = \zeta^i_0, \tag{4.38}$$

$$\psi = \psi_0 + {}^{\rm pn}M_i \partial_i \frac{1}{|\boldsymbol{x}|}. \tag{4.39}$$

Here $(\Phi_0, \zeta^i_0, \psi_0)$ is another solution with vanishing post-Newtonian mass dipole. Inserting this decomposition into the post-2-Newtonian field equations and gauge conditions (4.13), (4.14), (4.15), and (4.16) yields a corresponding decomposition of the post-2-Newtonian potentials:

$$\chi^{ij} = \chi^{ij}_0, \tag{4.40}$$

$$\xi^i = \xi^i_0 - 4\, {}^{\rm pn}\dot{M}_i \frac{1}{|\boldsymbol{x}|}. \tag{4.41}$$

Here the potentials $(\Phi_0, \zeta^i_0, \psi_0, \chi^{ij}_0, \xi^i_0)$ are a solution of the field equations and gauge conditions and are independent of ${}^{\rm pn}M_i$. Inserting the decompositions (4.40) and (4.41) into Eq. (4.19) gives

$$ {}^{\rm pn}P^i = {}^{\rm pn}\dot{M}_i + \frac{1}{16\pi} \oint [\partial_j \xi^i_0 + \dot{\chi}^{ij}_0] d^2\Sigma_j. \tag{4.42}$$

Inserting this into the expansions (4.17) and (4.20) and then into the conservation law (4.10) now yields

$$ {}^{\rm pn}\ddot{M}_i = - \oint_\Sigma \left\{ \frac{1}{16\pi} \partial_j \dot{\xi}^i_0 + \frac{1}{16\pi} \ddot{\chi}^{ij}_0 + {}^{\rm pn}\mathcal{T}^{ij} \right\} d^2\Sigma^j. \tag{4.43}$$

Equation (4.43) is the key result of this subsection. It shows that the second time derivative of the post-Newtonian mass dipole can be expressed purely in terms of the Newtonian and post-1-Newtonian fields, and is independent of the post-2-Newtonian degrees of freedom. This independence was derived above for the first two terms on the right-hand side of Eq. (4.43), and for the third term it follows from the fact that the expression (4.22) for ${}^{\rm pn}\mathcal{T}^{ij}$ depends only on the Newtonian and the post-1-Newtonian fields. Therefore the right-hand side of Eq. (4.43) is some function of the surface Σ as well as of the multipole and tidal moments of the solution (Φ, ζ^i, ψ) [since the multipole and tidal moments of the solution $(\Phi_0, \zeta^i_0, \psi_0)$ coincide with those of the original solution (Φ, ζ^i, ψ) except for the post-1-Newtonian mass dipole].

We can deduce some properties of the functional dependence of ${}^{\rm pn}\ddot{M}_i$ on the moments as follows. From the post-2-Newtonian field equations and gauge conditions (4.13), (4.14), (4.15), and (4.16) it follows that the potentials ξ_0^i and $\dot{\chi}_0^{ij}$ depend linearly on $\ddot{\zeta}_0^i$ and $\dot{\psi}_0$, and quadratically on Φ_0, $\dot{\Phi}_0$, and $\partial^{[i}\zeta_0^{j]}$. Also from Eq. (4.22) it follows that ${}^{\rm pn}\mathcal{T}^{ij}$ depends quadratically on Φ, $\dot{\Phi}$, $\partial^{[i}\zeta^{j]}$, and $\dot{\zeta}^i + \partial^i\psi$, all of which are independent of the gauge moments μ_L and ν_L. From Eq. (4.43) and the expansions (3.5a)–(3.5c) it now follows that

$$ {}^{\rm pn}\ddot{M}_i = \mathcal{F}_i({}^{\rm n}M_L, {}^{\rm n}\dot{M}_L, {}^{\rm n}\ddot{M}_L, {}^{\rm n}G_L, {}^{\rm n}\dot{G}_L, {}^{\rm n}\ddot{G}_L, H_L, \dot{H}_L, S_L, \dot{S}_L, {}^{\rm pn}M_L, {}^{\rm pn}G_L; R) + \mathcal{G}_i({}^{\rm n}\dddot{M}_L, {}^{\rm n}\dddot{G}_L, \ddot{H}_L, \ddot{S}_L, \ddot{\mu}_L, \dddot{\nu}_L; R). \quad (4.44) $$

Here $\mathcal{G}_i$ is a linear function of all of the moments that appear as its arguments, and can be an arbitrary function of the radius R that defines the two-surface Σ. Similarly the function $\mathcal{F}_i$ is a quadratic function of all the moments that appear as its arguments, and can be an arbitrary function of R.[25]

In Appendix C we compute explicitly the linear term and show that it vanishes identically[26]:

$$ \mathcal{G}_i = 0. \quad (4.45) $$

Next, the left-hand side of Eq. (4.44) is independent of the radius R, as are the definitions of all the moments that appear as the arguments of the function $\mathcal{F}_i$. It follows that $\mathcal{F}_i$ is independent of R, as one would expect. Using these simplifications we can rewrite Eq. (4.44) as

$$ {}^{\rm pn}\ddot{M}_i = \mathcal{F}_i({}^{\rm n}M_L, {}^{\rm n}\dot{M}_L, {}^{\rm n}\ddot{M}_L, {}^{\rm n}G_L, {}^{\rm n}\dot{G}_L, {}^{\rm n}\ddot{G}_L, H_L, \dot{H}_L, S_L, \dot{S}_L, {}^{\rm pn}M_L, {}^{\rm pn}G_L), \quad (4.46) $$

where the function $\mathcal{F}_i$ is now a quadratic function of all of its arguments. This quadratic function could in principle be computed from the expression (4.43). However, it is simpler to appeal to a special case from which the functional form of $\mathcal{F}_i$ can be deduced, as suggested in a different context by Thorne and Hartle [34].

Specifically, we now specialize to the case considered by DSX where the post-1-Newtonian field equations are assumed to hold throughout $r < r_-$. Our analysis applies to that special case, and therefore the functional $\mathcal{F}_i$ coincides with that computed by DSX, given in Eq. (4.21b) of Ref. [16] and in Eq. (4.3b) above. The derivation of the form of $\mathcal{F}_i$ in this case is reviewed in Appendix D. The key point here is that the general argument of this subsection establishes the result (4.3b) up to the values of the coefficients of the terms on the right-hand side, and our argument shows that those coefficients are universal, applying both to the case of weakly self-gravitating bodies analyzed by DSX and to the case of strongly self-gravitating bodies considered here.

This argument, which enables us to avoid doing the surface integral (4.43) explicitly, could of course also be used to avoid doing the surface integrals in Eq. (4.26) when computing the Newtonian laws of motion. In that case, however, we were able to check explicitly that the surface integral method gives the correct answer.

Similarly, we could deduce that the linear term $\mathcal{G}_i$ must vanish by comparison with the case of weakly self-gravitating bodies. Therefore the explicit verification of this result in Appendix C is not really necessary. That verification is useful, however, as a consistency check of our argument and formalism.

Thus, we have established that the law of motion (4.3b) is valid not just for the class of weakly self-gravitating bodies considered by DSX, but also for the more general class of strongly self-gravitating bodies considered here, subject to the assumptions outlined in Sec. IV A.

V. AN N-BODY SYSTEM: FOUNDATIONS

We now turn to an analysis of a system consisting of N bodies with arbitrary internal structure. The bodies' masses can be comparable, but their typical separations must be large compared to their masses in order that their gravitational interactions are well described by the post-1-Newtonian approximation.[27] In this section we lay the foundations for our analysis by defining local coordinate systems associated with each body, and an overall global coordinate system. We also derive relations between the

[25] It is easy to see that the right-hand side of Eq. (4.44) must be independent of the radii r_- and r_+ that define the domain $\mathcal{D}$.

[26] The fact that the right-hand side of Eq. (4.44) is independent of the gauge moments μ_L and ν_L can alternatively be derived as follows. As discussed in Sec. III D above, by making a gauge transformation of the type (2.22) we can alter the values of the gauge moments μ_L and ν_L without altering any of the tidal and multipole moments ${}^{\rm n}M_L$, ${}^{\rm n}G_L$, ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$, S_L, H_L or their time derivatives. Under such a transformation, the left-hand side of Eq. (4.44) is invariant. It follows that the right-hand side does not depend on the gauge moments or their time derivatives.

[27] Our formalism and derivation does not require that the bodies' separations be large compared to their typical sizes. However, that requirement is in practice necessary if one wants to achieve good accuracy using a truncated version of the equation of motion (6.11) containing only a small number of multipoles.

moments that characterize the potentials in each of these coordinate systems. In Sec. VI below we will combine the results derived here with the single-body equation of motion (4.3b) derived in the previous section to obtain the explicit form of the N-body equation of motion.

A. Assumptions

We start by describing our assumptions. We consider a system of N bodies, labeled by the index A with $1 \le A \le N$. We associate with each body a world tube $\mathcal{W}_A$ containing the region where the stress-energy tensor is nonzero, and also containing the strong field region associated with the body. Our key assumption is that the post-Newtonian equations are satisfied everywhere outside all of the world tubes $\mathcal{W}_A$.

More precisely, we make the following assumptions: (i) For each A there exists a coordinate system (s_A, y_A^i) of the type discussed in Sec. III D above which covers the Ath body. Thus, there exist radii $r_{-,A}$ and $r_{+,A}$ such that the range of the coordinates includes the product of the ball $|\mathbf{y}_A| < r_{+,A}$ with an open interval of time; and the coordinates are harmonic, conformally Cartesian and body adapted in the buffer region

$$\mathcal{B}_A \equiv \{(s_A, y_A^i) | r_{-,A} < |\mathbf{y}_A| < r_{+,A}\}. \tag{5.1}$$

(ii) The various buffer regions $\mathcal{B}_A$ are nonintersecting (see Fig. 1 above). (iii) We define the world tube associated with the Ath body to be $\mathcal{W}_A \equiv \{(s_A, y_A^i) | |\mathbf{y}_A| < r_{-,A}\}$, and we define the spacetime region $\mathcal{D}$ to be the complement of the union of all the world tubes,

$$\mathcal{D} = \mathcal{M} \setminus \bigcup_{A=1}^{N} \mathcal{W}_A,$$

where $\mathcal{M}$ is the entire manifold. The vacuum Einstein equations are satisfied for the one-parameter family of metrics (4.11) on the spacetime region $\mathcal{D}$. (iv) There exists a conformally Cartesian and harmonic coordinate system (t, x^i) which covers all of $\mathcal{D}$. We will call the coordinate system (t, x^i) the *global frame*, even though it does not cover the entire manifold. We will call the coordinate systems $(s_A, \mathbf{y}_A)$ the *body frames*.

Note that in the context of the spatially noncompact domain $\mathcal{D}$, the meaning of the $O(\varepsilon^n)$ symbols that appear in Eqs. (2.2) and (4.11) corresponds to pointwise convergence, not uniform convergence. As is well known, solutions of the post-1-Newtonian field equations are not good approximations to exact solutions at distances $\gtrsim 1/\varepsilon$. That is, although they work well in the near zone they break down in the local wave zone [38]. In order to obtain solutions which are good approximations everywhere one has to perform a matching of post-Newtonian solutions onto radiation zone post-Minkowskian solutions; see, for example, Blanchet [3] and references therein. However, the corresponding corrections to the near zone gravitational fields and to the dynamics of the bodies arises at post-2.5-Newtonian order, and can thus be neglected for the post-1-Newtonian analysis of this paper.

B. Body-frame multipole and tidal moments

In each local coordinate system (s_A, y_A^i) we define multipole and tidal moments ${}^{\rm n}M_L^A(s_A)$, ${}^{\rm n}G_L^A(s_A)$, ${}^{\rm pn}M_L^A(s_A)$, ${}^{\rm pn}G_L^A(s_A)$, $S_L^A(s_A)$, and $H_L^A(s_A)$ according to the prescription described in Sec. III B above. We have added superscripts A to these moments to denote the Ath body. The corresponding expansions of the potentials are

$$\Phi^A(s_A, y_A^j) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!}\, {}^{\rm n}M_L^A(s_A)\partial_L \frac{1}{|\mathbf{y}_A|} - \frac{1}{l!}\, {}^{\rm n}G_L^A(s_A) y_A^L, \tag{5.2a}$$

$$\begin{aligned}\psi^A(s_A, y_A^j) = \sum_{l=0}^{\infty}\bigg\{&\frac{(-1)^{l+1}}{l!}\, {}^{\rm pn}M_L^A(s_A)\partial_L \frac{1}{|\mathbf{y}_A|} \\ &+ \frac{(-1)^{l+1}}{l!}\, {}^{\rm n}\ddot{M}_L^A(s_A)\partial_L \frac{|\mathbf{y}_A|}{2} - \frac{1}{l!}\, {}^{\rm pn}G_L^A(s_A) y_A^L \\ &- \frac{1}{l!}\frac{|\mathbf{y}_A|^2}{2(2l+3)}\, {}^{\rm n}\ddot{G}_L^A(s_A) y_A^L\bigg\},\end{aligned} \tag{5.2b}$$

$$\begin{aligned}\zeta_i^A(s_A, y_A^j) = \sum_{l=0}^{\infty}\bigg\{&\frac{(-1)^{l+1}}{l!}\bigg[\frac{4}{l+1}\, {}^{\rm n}\dot{M}_{iL}^A(s_A) \\ &- \frac{4l}{l+1}\epsilon_{ji\langle a_l} S_{L-1\rangle j}^A(s_A)\bigg]\partial_L \frac{1}{|\mathbf{y}_A|} \\ &- \frac{1}{l!}\bigg[\frac{l}{l+1}\epsilon_{ji\langle a_l} H_{L-1\rangle j}^A(s_A) \\ &- \frac{4(2l-1)}{2l+1}\, {}^{\rm n}\dot{G}_{\langle L-1}^A(s_A)\delta_{a_l\rangle i}\bigg] y_A^L\bigg\}\end{aligned} \tag{5.2c}$$

$$\equiv \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{l!} Z_{iL}^A(s_A)\partial_L \frac{1}{|\mathbf{y}_A|} - \frac{1}{l!} Y_{iL}^A(s_A) y_A^L, \tag{5.2d}$$

where overdots mean derivatives with respect to the time argument. These expansions are obtained from the expansions (3.28a)–(3.28c) by replacing x^j with y_A^j and t with s_A, by adding superscripts A to the potentials and the various moments to denote the Ath body, and by omitting the gauge moments μ_L and ν_L which vanish since we have specialized to body-adapted gauge. The specialization to body-adapted gauge also implies that

$${}^{\rm n}M_i^A = {}^{\rm pn}M_i^A = {}^{\rm n}G^A = {}^{\rm pn}G^A = H_i^A = 0, \tag{5.3}$$

cf. Sec. III D above.

C. Configuration variables for the Ath body

For each body, there is a nonempty region of overlap between the domain of the body-frame coordinates (s_A, y_A^i)

and the domain of the global-frame coordinates (t, x^i), namely, the buffer region $\mathcal{B}_A$ defined by Eq. (5.1). Both of these coordinate systems are harmonic and conformally Cartesian, and therefore the mapping between the two coordinate systems can be parametrized using the general analysis of Sec. II B above. From Eqs. (2.17) and (2.18a)–(2.18c) it follows that there exist functions $z_i^A(s_A)$, $\alpha_c^A(s_A)$, $h_{ci}^A(s_A)$, and $R_k^A(s_A)$ and a harmonic function $\beta_h^A(s_A, y_A^j)$ such that in $\mathcal{B}_A$

$$\begin{aligned} x^i &= y_A^i + z_i^A(s_A) + \varepsilon^2 h_i^A(s_A, y_A^j) + O(\varepsilon^4), \\ t &= s_A + \varepsilon^2 \alpha^A(s_A, y_A^j) + \varepsilon^4 \beta^A(s_A, y_A^j) + O(\varepsilon^6), \end{aligned} \tag{5.4}$$

where

$$\alpha^A = \alpha_c^A(s_A) + y_A^j \dot{z}_j^A(s_A), \tag{5.5}$$

$$\begin{aligned} h_i^A = {} & h_{ci}^A(s_A) + \epsilon_{ijk} y_A^j R_k^A(s_A) + \frac{1}{2}\ddot{z}_i^A(s_A) y_A^j y_A^j - y_A^i \dot{\alpha}_c^A(s_A) \\ & - y_A^i y_A^j \ddot{z}_j^A(s_A) + \frac{1}{2} y_A^i \dot{z}_j^A(s_A)\dot{z}_j^A(s_A) \\ & + \frac{1}{2}\dot{z}_i^A(s_A)\dot{z}_j^A(s_A) y_A^j, \end{aligned} \tag{5.6}$$

and

$$\beta^A = y_A^j y_A^j \left[\frac{1}{10}\dddot{z}_k^A(s_A) y_A^k + \frac{1}{6}\ddot{\alpha}_c^A(s_A)\right] + \beta_h^A(s_A, y_A^j). \tag{5.7}$$

Because the body-frame coordinates are uniquely determined, the functions $z_i^A(s_A)$, $\alpha_c^A(s_A)$, $h_{ci}^A(s_A)$, and $R_k^A(s_A)$ acquire the role of configuration variables that specify the location, orientation, etc., in the global coordinates (t, x^i) of the local rest frame attached to body A. This role is in contrast to the role of the corresponding variables in Sec.I above, which were freely specifiable functions. The task of determining the motion of the N different bodies reduces to solving for the time evolution of these configuration variables. Below we will show that some of these variables can be obtained by solving differential equations, and the remainder are obtained from algebraic relations.

The particular combination of these configuration variables that enters into the equation of motion which we derive below is the *center-of-mass world line*. In the special case where the post-1-Newtonian equations are assumed to hold inside each body, this center-of-mass world line is defined simply as the origin of spatial coordinates of the body-adapted[28] coordinate system (s_A, y_A^i). This world line can be expressed in terms of the global-frame coordinates in parametric form as

$$x^i(s_A) = z_i^A(s_A) + \varepsilon^2 h_{ci}^A(s_A) + O(\varepsilon^4), \tag{5.8}$$

$$t(s_A) = s_A + \varepsilon^2 \alpha_c^A(s_A) + O(\varepsilon^4), \tag{5.9}$$

from the coordinate transformation (5.4), (5.5), (5.6), and (5.7). Eliminating s_A gives $x^i = {}^{\rm cm}z_i^A(t)$, where

$${}^{\rm cm}z_i^A(t) = z_i^A(t) + \varepsilon^2[h_{ci}^A(t) - \dot{z}_i^A(t)\alpha_c^A(t)] + O(\varepsilon^4). \tag{5.10}$$

Here the superscript "cm" means "center of mass."

Consider now the more general context where the post-1-Newtonian equations are not assumed to hold inside each body. Then, the body-frame coordinates (s_A, y_A^i) can be arbitrary for $|\boldsymbol{y}_A| < r_{-,A}$, so the world line in spacetime of the origin $\boldsymbol{y}_A = 0$ of these coordinates has no special significance. Nevertheless, we can still use Eqs. (5.8), (5.9), and (5.10) to define the function ${}^{\rm cm}z_i^A(t)$. That is, we define ${}^{\rm cm}z_i^A(t)$ to be the image of the origin $\boldsymbol{y}_A = 0$ not under the true coordinate transformation, but under the extension to $|\boldsymbol{y}_A| < r_{-,A}$ of the formulas (5.4), (5.5), (5.6), and (5.7) which *a priori* are only valid for $r_{-,A} < |\boldsymbol{y}_A| < r_{+,A}$. The resulting function ${}^{\rm cm}z_i^A(t)$ continues to characterize the location of the local rest frame attached to body A, even though it no longer corresponds to a world line in spacetime,[29] and even though the location $x^i = {}^{\rm cm}z_i^A(t)$ will in general be outside the domain of definition of the global coordinates. We will continue to call this function the center-of-mass world line, in a slight but conventional abuse of terminology.

D. Global-frame multipole moments

In this section we define, for each body A, multipole moments associated with the global coordinate system (t, x^i). We define the global-frame multipole moments ${}^{\rm n}M_L^{{\rm g},A}(t)$, ${}^{\rm pn}M_L^{{\rm g},A}(t)$, $S_L^{{\rm g},A}(t)$, and $\mu_L^{{\rm g},A}(t)$ to be the moments about the Newtonian-order center-of-mass world line $\boldsymbol{x} = \boldsymbol{z}^A(t)$ of body A [cf. Eq. (5.10) above], using the prescription discussed in Sec. III E. Using these multipole moments we can write down multipole expansions of the global-frame potentials, which we denote by $(\Phi^{\rm g}, \zeta_i^{\rm g}, \psi^{\rm g})$, that are valid on the entire domain $\mathcal{D}$:

[28]More precisely, of the slightly modified body-adapted coordinate system discussed in the last paragraph of Sec. III C above, whose domain of definition includes the interior of the body. The difference between this coordinate system and the body-adapted coordinate system arises only at order $O(\varepsilon^4)$ in the time coordinate, which does not affect the definition of center-of-mass world line to post-1-Newtonian accuracy.

[29]The function ${}^{\rm cm}z_i^A(t)$ does however transform like a world line under the group (2.17) of post-1-Newtonian coordinate transformations.

$$\Phi^{\rm g}(t,x^j) = \sum_{A=1}^{N}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\,{}^{\rm n}M_L^{{\rm g},A}(t)\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A(t)|}, \quad (5.11a)$$

$$\psi^{\rm g}(t,x^j) = \sum_{A=1}^{N}\sum_{l=0}^{\infty}\biggl(\frac{(-1)^{l+1}}{l!}\biggl\{{}^{\rm pn}M_L^{{\rm g},A}(t)\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A(t)|} + \frac{(2l+1)}{(l+1)(2l+3)}\frac{\partial}{\partial t}\biggl[\mu_L^{{\rm g},A}(t)\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A(t)|}\biggr]\biggr\} + \frac{(-1)^{l+1}}{l!}\frac{\partial^2}{\partial t^2}\biggl[{}^{\rm n}M_L^{{\rm g},A}(t)\partial_L\frac{|\boldsymbol{x}-\boldsymbol{z}^A(t)|}{2}\biggr]\biggr), \quad (5.11b)$$

$$\zeta_i^{\rm g}(t,x^j) = \sum_{A=1}^{N}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}Z_{iL}^{{\rm g},A}(t)\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A(t)|}, \quad (5.11c)$$

where

$$Z_{iL}^{{\rm g},A} = \frac{4}{l+1}\,{}^{\rm n}\dot{M}_{iL}^{{\rm g},A} - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S_{L-1\rangle j}^{{\rm g},A} + \frac{2l-1}{2l+1}\delta_{i\langle a_l}\mu_{L-1\rangle}^{{\rm g},A} + 4\dot{z}_{\langle i}^{A}\,{}^{\rm n}M_{L\rangle}^{{\rm g},A}. \quad (5.11d)$$

Here the superscript "g" on the potentials and on the moments stands for "global."

The form of the expansions (5.11a)–(5.11c) is dictated by the following considerations: (i) The expansions take the form of a linear superposition of solutions, one for each body A. This follows from the linearity of the vacuum field Eqs. (2.9a)–(2.9c). (ii) We choose to use a gauge for the global coordinates in which all the potentials go to zero as $|\boldsymbol{x}| \to \infty$. This eliminates any tidal terms associated with acceleration of the reference frame, cf. the discussion in Sec. II B above. (iii) There are no other tidal terms, since the terms in the sum over B with $B \neq A$ play the role of tidal terms for body A. With this identification, the expansions (5.11a)–(5.11c) agree with the formulas (3.58a)–(3.58c) of Sec. III E above that define the multipole and gauge moments, in the buffer region $\mathcal{B}_A$ about the Ath body.

We will derive transformation laws relating the global-frame multipole moments ${}^{\rm n}M_L^{{\rm g},A}$, ${}^{\rm pn}M_L^{{\rm g},A}$, and $S_L^{{\rm g},A}$ to the body-frame multipole moments ${}^{\rm n}M_L^{A}$, ${}^{\rm pn}M_L^{A}$, and S_L^{A} in the next subsection. Note that the moments that would be measured by observers residing in the buffer region $\mathcal{B}_A$ about the Ath body are the body-frame moments and not the global-frame moments.

We next discuss the gauge freedom in the global coordinate system. If we make a gauge transformation of the form (2.22) with the harmonic function $\beta_{\rm h}$ chosen to be

$$\beta_{\rm h}(\bar{t},\bar{x}^j) = \sum_{A=1}^{N}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\lambda_L^A(\bar{t})\partial_L\frac{1}{|\bar{\boldsymbol{x}}-\boldsymbol{z}^A(\bar{t})|}, \quad (5.12)$$

then the gauge moments $\mu_L^{{\rm g},A}(t)$ transform according to

$$\bar{\mu}_L^{{\rm g},A} = \mu_L^{{\rm g},A} + \frac{(l+1)(2l+3)}{2l+1}\lambda_L^A, \quad (5.13)$$

cf. Eq. (3.54) above. Therefore there is enough freedom to set

$$\mu_L^{{\rm g},A} = 0 \quad (5.14)$$

for all A and for all $l \geq 0$. This requirement, together with the requirement that the potentials go to zero as $|\boldsymbol{x}| \to \infty$, reduces the residual gauge freedom to the post-Galilean transformation group discussed in Sec. II B.

E. Computation of body-frame tidal moments

Our goal is to deduce equations of motion for the N-body system from the single-body equation of motion (4.3b). To this end, we would like to compute the body-frame tidal moments ${}^{\rm n}G_L^A$, ${}^{\rm pn}G_L^A$, and H_L^A felt by body A in terms of the body-frame multipole moments ${}^{\rm n}M_L^B$, ${}^{\rm pn}M_L^B$, and S_L^B and also the configuration variables of the other bodies B with $B \neq A$. We shall perform this computation in stages, by relating both sets of quantities to the global-frame moments.

We start by expanding the time derivatives that appear in the expansion (5.11b) of the global-frame potential $\psi^{\rm g}$ and by expressing the results in terms of STF tensors. Using the gauge specialization (5.14) this computation gives

$$\psi^{\rm g} = \sum_{A=1}^{N}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\biggl[N_L^{{\rm g},A}\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A|} + P_L^{{\rm g},A}\partial_L\frac{|\boldsymbol{x}-\boldsymbol{z}^A|}{2}\biggr], \quad (5.15)$$

where the STF tensors $N_L^{{\rm g},A}$ and $P_L^{{\rm g},A}$ are given by [cf. Eqs. (3.61b) and (3.61c) above]

$$N_L^{{\rm g},A} = {}^{\rm pn}M_L^{{\rm g},A} + \frac{1}{2l+3}[\ddot{z}_j^A\,{}^{\rm n}M_{jL}^{{\rm g},A} + \dot{z}_j^A\dot{z}_j^A\,{}^{\rm n}M_L^{{\rm g},A} + 2\dot{z}_j^A\,{}^{\rm n}\dot{M}_{jL}^{{\rm g},A} + 2l\dot{z}_j^A\dot{z}_{\langle a_l}^A\,{}^{\rm n}M_{L-1\rangle j}^{{\rm g},A}], \quad (5.16)$$

and

$$P_L^{{\rm g},A} = {}^{\rm n}\ddot{M}_L^{{\rm g},A} + 2l\dot{z}_{\langle a_l}^A\,{}^{\rm n}\dot{M}_{L-1\rangle}^{{\rm g},A} + l\ddot{z}_{\langle a_l}^A\,{}^{\rm n}M_{L-1\rangle}^{{\rm g},A} + l(l-1)\dot{z}_{\langle a_l}^A\dot{z}_{a_{l-1}}^A\,{}^{\rm n}M_{L-2\rangle}^{{\rm g},A}. \quad (5.17)$$

Next, we expand the global potentials in the buffer region $\mathcal{B}_A$ of body A using the Taylor series

$$|\boldsymbol{x}-\boldsymbol{z}^B|^p = |(\boldsymbol{z}^B-\boldsymbol{z}^A)-(\boldsymbol{x}-\boldsymbol{z}^A)|^p = \sum_{k=0}^{\infty}\frac{1}{k!}(x-z^A)^K\mathcal{T}_K^p(\boldsymbol{z}^{BA}), \quad (5.18)$$

where p is any integer, $\boldsymbol{z}^{BA} = \boldsymbol{z}^B - \boldsymbol{z}^A$ and

$$\mathcal{T}_K^p(\boldsymbol{z}) \equiv (\partial_K|\boldsymbol{z}-\boldsymbol{x}|^p)_{\boldsymbol{x}=0}. \quad (5.19)$$

For $p = -1$ we have

$$\mathcal{T}_K^{-1}(\boldsymbol{z}) = (2k-1)!!\frac{z^{\langle K\rangle}}{|\boldsymbol{z}|^{2k+1}}. \quad (5.20)$$

Substituting Eq. (5.18) into Eqs. (5.11a), (5.11c), and (5.15) and using the identity

$$\mathcal{T}^{+1}_{KL} = \mathcal{T}^{+1}_{K\langle L\rangle} + \frac{l(l-1)}{2l-1}\delta_{(a_{l-1}a_l}\mathcal{T}^{-1}_{L-2)K} \tag{5.21}$$

yields

$$\Phi^{\rm g} = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}{}^{\rm n}M^{{\rm g},A}_L\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A|} - \frac{1}{l!}{}^{\rm n}G^{{\rm g},A}_L(x-z^A)^L, \tag{5.22a}$$

$$\psi^{\rm g} = \sum_{l=0}^{\infty}\bigg\{\frac{(-1)^{l+1}}{l!}\bigg[N^{{\rm g},A}_L\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A|} + P^{{\rm g},A}_L\partial_L\frac{|\boldsymbol{x}-\boldsymbol{z}^A|}{2}\bigg] - \frac{1}{l!}\bigg[F^{{\rm g},A}_L(x-z^A)^L + J^{{\rm g},A}_L\frac{|\boldsymbol{x}-\boldsymbol{z}^A|^2(x-z^A)^L}{2(2l+3)}\bigg]\bigg\}, \tag{5.22b}$$

$$\zeta^{\rm g}_i = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}Z^{{\rm g},A}_{iL}\partial_L\frac{1}{|\boldsymbol{x}-\boldsymbol{z}^A|} - \frac{1}{l!}Y^{{\rm g},A}_{iL}(x-z^A)^L, \tag{5.22c}$$

where

$${}^{\rm n}G^{{\rm g},A}_L = \sum_{B\neq A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}{}^{\rm n}M^{{\rm g},B}_K\mathcal{T}^{-1}_{KL}(z^{BA}), \tag{5.23a}$$

$$Y^{{\rm g},A}_{iL} = \sum_{B\neq A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}Z^{{\rm g},B}_{iK}\mathcal{T}^{-1}_{KL}(z^{BA}), \tag{5.23b}$$

$$J^{{\rm g},A}_L = \sum_{B\neq A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}P^{{\rm g},B}_K\mathcal{T}^{-1}_{KL}(z^{BA}), \tag{5.23c}$$

$$F^{{\rm g},A}_L = \sum_{B\neq A}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\bigg[N^{{\rm g},B}_K\mathcal{T}^{-1}_{KL}(z^{BA}) + \frac{1}{2}P^{{\rm g},B}_K\mathcal{T}^{+1}_{K\langle L\rangle}(z^{BA})\bigg]. \tag{5.23d}$$

Here ${}^{\rm n}G^{{\rm g},A}_L$, $F^{{\rm g},A}_L$, $Y^{{\rm g},A}_{iL}$, and $J^{{\rm g},A}_L$ are global-frame tidal moments. The post-Newtonian moments $F^{{\rm g},A}_L$, $Y^{{\rm g},A}_{iL}$, and $J^{{\rm g},A}_L$ could be parametrized in terms of the irreducible global-frame tidal moments ${}^{\rm pn}G^{{\rm g},A}_L$, $H^{{\rm g},A}_L$, and $\nu^{{\rm g},A}_L$ if desired via equations analogous to Eqs. (3.58e), (3.61d), and (3.61e). Here, however, it will be more convenient to work directly with the moments $F^{{\rm g},A}_L$, $Y^{{\rm g},A}_{iL}$, and $J^{{\rm g},A}_L$. The function $z^B(s_B)$ that appears on the right-hand sides of Eqs. (5.23a)–(5.23d) is evaluated at $s_B = t$.

Next, we apply the coordinate transformation (5.4) to the global potentials (5.22a)–(5.22c) using the formulas (2.20a)–(2.20c). We parametrize the harmonic function $\beta^A_{\rm h}$ which appears in Eq. (5.7) as

$$\beta^A_{\rm h}(s_A, y^j_A) = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\lambda^A_L(s_A)\partial_L\frac{1}{|\boldsymbol{y}_A|} - \sum_{l=0}^{\infty}\frac{1}{l!}\tau^A_L(s_A)y^L_A, \tag{5.24}$$

where the tensors λ^A_L and τ^A_L are STF. Comparing the result with the expansions (5.2a)–(5.2c) allows us to compute the body-frame moments in terms of the global-frame moments, as in Sec. III C above. At Newtonian order we obtain [cf. Eqs. (3.33), (3.34), (3.35a), and (3.35b) above]

$${}^{\rm n}M^A_L = {}^{\rm n}M^{{\rm g},A}_L, \tag{5.25a}$$

$${}^{\rm n}G^A_L = {}^{\rm n}G^{{\rm g},A}_L - l!\Lambda^{\Phi}_L, \tag{5.25b}$$

where the nonzero inertial moments Λ^{Φ}_L are given by

$$\Lambda^{\Phi} = -\frac{1}{2}\dot{z}^A_j\dot{z}^A_j + \dot{\alpha}^A_c, \tag{5.26a}$$

$$\Lambda^{\Phi}_i = \ddot{z}^A_i. \tag{5.26b}$$

Transforming the gravitomagnetic potential gives [cf. Eqs. (3.36a) and (3.36b) above]

$$Z^A_{iL} = Z^{{\rm g},A}_{iL} - 4\dot{z}^A_i\,{}^{\rm n}M^{{\rm g},A}_L + l\delta_{i\langle a_l}\lambda^A_{L-1\rangle}, \tag{5.27a}$$

$$Y^A_{iL} = Y^{{\rm g},A}_{iL} - 4\dot{z}^A_i\,{}^{\rm n}G^{{\rm g},A}_L - \tau^A_{iL} - l!\Lambda^{\zeta}_{iL}, \tag{5.27b}$$

where the nonzero inertial moments Λ^{ζ}_{iL} are given by

$$\Lambda^{\zeta}_i = \dot{h}^A_{ci} + \dot{z}^A_i\dot{z}^A_j\dot{z}^A_j - \epsilon_{ijk}\dot{z}^A_jR^A_k - 2\dot{\alpha}^A_c\dot{z}^A_i, \tag{5.28a}$$

$$\Lambda^{\zeta}_{ij} = -\dot{z}^A_i\ddot{z}^A_j - \dot{z}^A_{(i}\ddot{z}^A_{j)} + \epsilon_{ijk}\dot{R}^A_k + 2\delta_{ij}\dot{z}^A_k\ddot{z}^A_k - \frac{4}{3}\delta_{ij}\ddot{\alpha}^A_c, \tag{5.28b}$$

$$\Lambda^{\zeta}_{ijk} = -\frac{6}{5}\delta_{i\langle j}\dddot{z}^A_{k\rangle}. \tag{5.28c}$$

Next, by combining the transformation laws (5.27a) and (5.27b) with the decompositions of Z^A_{iL} and Y^A_{iL} given by Eqs. (5.2c) and (5.2d), the decomposition (5.11d) of $Z^{{\rm g},A}_{iL}$, and the gauge condition (5.14), we can solve for the coordinate transformation functions λ^A_L for $l \geq 0$ and τ^A_L for $l \geq 1$. The result is

$$\lambda^A_L = \frac{4(2l+1)}{(l+1)(2l+3)}\dot{z}^A_j\,{}^{\rm n}M^{{\rm g},A}_{jL}, \tag{5.29a}$$

$$\tau^A_{iL} = Y^{{\rm g},A}_{\langle iL\rangle} - 4\dot{z}^A_{\langle i}\,{}^{\rm n}G^{{\rm g},A}_{L\rangle} - l!\Lambda^{\zeta}_{\langle iL\rangle}. \tag{5.29b}$$

Finally, matching the post-Newtonian potentials and using the definition (5.16) of $N^{{\rm g},A}_L$, the formulas (5.29a) and (5.29b) for λ^A_L and τ^A_{iL}, and the identities (A8)–(A11) gives

$$
\begin{aligned}
{}^{\rm pn}M_L^A = {}& {}^{\rm pn}M_L^{{\rm g},A} - \frac{4l}{l+1}\dot z_j^A \epsilon_{jk\langle a_l} S^{{\rm g},A}_{L-1\rangle k} - l h^A_{{\rm c}\langle a_l}\, {}^{\rm n}M^{{\rm g},A}_{L-1\rangle} - l\, {}^{\rm n}M^{{\rm g},A}_{j\langle L-1}\epsilon_{a_l\rangle jk} R_k^A + \alpha_c^A\, {}^{\rm n}\dot M_L^{{\rm g},A} + (l+1)\dot\alpha_c^A\, {}^{\rm n}M_L^{{\rm g},A} \\
& + l\alpha_c^A \dot z^A_{\langle a_l}\, {}^{\rm n}M^{{\rm g},A}_{L-1\rangle} - \frac{l^2-3l+4}{2(l+1)}\dot z_j^A \dot z_j^A\, {}^{\rm n}M_L^{{\rm g},A} + \frac{l(2l^2-13l+9)}{2(l+1)(2l+1)}\dot z_j^A \dot z^A_{\langle a_l}\, {}^{\rm n}M^{{\rm g},A}_{L-1\rangle j} \\
& + \left[(l+1) + \frac{4(2l+1)}{(l+1)(2l+3)}\right]\ddot z_j^A\, {}^{\rm n}M^{{\rm g},A}_{jL} + \left[1 - \frac{8}{(l+1)(2l+3)}\right]\dot z_j^A\, {}^{\rm n}\dot M^{{\rm g},A}_{jL},
\end{aligned}
\tag{5.30a}
$$

$$
\begin{aligned}
{}^{\rm pn}G_L^A = {}& F_L^{{\rm g},A} + \dot Y^{{\rm g},A}_{\langle L\rangle} - \dot z_j^A Y^{{\rm g},A}_{jL} + (h^A_{{\rm c}j} - \alpha^A_{\rm c}\dot z_j^A)\, {}^{\rm n}G^{{\rm g},A}_{jL} - l\, {}^{\rm n}G^{{\rm g},A}_{j\langle L-1}\epsilon_{a_l\rangle jk}R_k^A + \alpha_{\rm c}^A\, {}^{\rm n}\dot G_L^{{\rm g},A} - l\dot\alpha_{\rm c}^A\, {}^{\rm n}G_L^{{\rm g},A} + \frac{l+4}{2}\dot z_j^A\dot z_j^A\, {}^{\rm n}G_L^{{\rm g},A} \\
& - \frac{l}{2}\dot z_j^A \dot z^A_{\langle a_l}\, {}^{\rm n}G^{{\rm g},A}_{L-1\rangle j} - (l^2-l+4)\ddot z^A_{\langle a_l}\, {}^{\rm n}G^{{\rm g},A}_{L-1\rangle} + (l-4)\dot z^A_{\langle a_l}\, {}^{\rm n}\dot G^{{\rm g},A}_{L-1\rangle} - l!\Lambda_L^{\psi_{\rm h}} - (l-1)!\dot\Lambda^{\zeta}_{\langle L\rangle} + \delta_{l0}\dot\tau^A.
\end{aligned}
\tag{5.30b}
$$

Here then nonzero inertial moments $\Lambda_L^{\psi_{\rm h}}$ are given by

$$\Lambda^{\psi_{\rm h}} = -\dot z_j^A \dot h^A_{{\rm c}j} - \frac{1}{4}(\dot z_j^A \dot z_j^A)^2 - \frac{1}{2}(\dot\alpha_c^A)^2 + \dot\alpha_c^A \dot z_j^A \dot z_j^A, \tag{5.31a}$$

$$\Lambda_i^{\psi_{\rm h}} = \epsilon_{ijk}\dot z_j^A \dot R_k^A + \frac{1}{2}\ddot z_i^A \dot z_j^A \dot z_j^A - \frac{3}{2}\dot z_i^A \ddot z_j^A \dot z_j^A - \dot\alpha_c^A \ddot z_i^A + \ddot\alpha_c^A \dot z_i^A, \tag{5.31b}$$

$$\Lambda_{jk}^{\psi_{\rm h}} = -\frac{1}{2}\ddot z^A_{\langle j}\ddot z^A_{k\rangle} + \dot z^A_{\langle j}\dddot z^A_{k\rangle}. \tag{5.31c}$$

In Eq. (5.30b) it is understood that the moments $Y_L^{{\rm g},A}$ and Λ_L^{ζ} are zero for $l = 0$.

The left-hand sides of Eqs. (5.25a), (5.25b), (5.27a), (5.27b), (5.30a), and (5.30b) are functions of the time coordinate s_A of the body-adapted coordinate system for the Ath body, cf. Eqs. (5.2a)–(5.2c) above. The right-hand sides are expressed as functions of s_A by evaluating the global moments, which are functions of the global time coordinate t, at $t = s_A$, cf. the discussion in the last paragraph of Sec. III C above.

Finally, by combining the transformation laws (5.25b), (5.27b), and (5.30b) with the gauge specializations (5.3) of the body-adapted coordinates we can deduce the values of some of the configuration variables of the Ath body. We obtain

$$\alpha_{\rm c}^A = \int ds_A\left[{}^{\rm n}G^{{\rm g},A} + \frac{1}{2}\dot z_j^A \dot z_j^A\right], \tag{5.32}$$

$$
\begin{aligned}
\tau^A = \int ds_A[& -F^{{\rm g},A} + \dot z_j^A Y_j^{{\rm g},A} - 2\dot z_j^A \dot z_j^A\, {}^{\rm n}G^{{\rm g},A} \\
& - (h^A_{{\rm c}j} - \alpha_c^A \dot z_j^A)\, {}^{\rm n}G_j^{{\rm g},A} - \alpha_{\rm c}^A\, {}^{\rm n}\dot G^{{\rm g},A} + \Lambda^{\psi_{\rm h}}],
\end{aligned}
\tag{5.33}
$$

and

$$R_k^A = \frac{1}{2}\epsilon_{ijk}\int ds_A[\dot z_i^A \ddot z_j^A + Y_{ij}^{{\rm g},A} - 4\dot z_i^A\, {}^{\rm n}G_j^{{\rm g},A}]. \tag{5.34}$$

The only remaining configuration variables that are undetermined are the variables $h^A_{{\rm c}i}$ and z_i^A that determine the center-of-mass world line (5.10).

To summarize, the main results of this subsection are the explicit expressions (5.25b), (5.27b), and (5.30b) for the body-frame tidal moments ${}^{\rm n}G_L^A$, ${}^{\rm pn}G_L^A$, and Y_{iL}^A which act on body A in terms of the configuration variables of all the bodies, as well as the global-frame mass and current moments ${}^{\rm n}M_L^{{\rm g},B}$, ${}^{\rm pn}M_L^{{\rm g},B}$, and $S_L^{{\rm g},B}$ of the other bodies. These expressions are given by combining (5.25b), (5.27b), and (5.30b) with Eqs. (5.23a)–(5.23d). Also, the global-frame multipole moments ${}^{\rm n}M_L^{{\rm g},B}$, ${}^{\rm pn}M_L^{{\rm g},B}$, and $S_L^{{\rm g},B}$ of body B can be reexpressed in terms of the body-frame multipole moments ${}^{\rm n}M_L^B$, ${}^{\rm pn}M_L^B$, and S_L^B of that body using relations (5.25a), (5.27a), and (5.30a) between the body-frame and global-frame mass and current moments.

F. Definition of body-frame multipole moments $\mathcal{M}_L^A$ and S_L^A

As discussed in the introduction, it is useful to use instead of the body-frame multipole moments M_L^A and S_L^A a modified set of body-frame moments defined as follows. We define for each body A a coordinate system $(\tilde s^A, \tilde y_i^A)$ which is identical to the body-frame coordinate system (s^A, y_i^A) except that it is nonrotating with respect to the global-frame coordinates (t, x^i) (i.e., nonrotating with respect to fixed stars). We define the moments $\mathcal{M}_L^A(t)$ and $S_L^A(t)$ to be the multipole moments of body A in this nonrotating coordinate system, expressed as functions of the global time coordinate t. These are given by the equations

$$\mathcal{M}^A_{a_1\ldots a_l}(t) = U_{a_1}^{Aa_1'}(t)\ldots U_{a_l}^{Aa_l'}(t) M^A_{a_1'\ldots a_l'}[s_A(t)] \tag{5.35}$$

and

$$S^A_{a_1\ldots a_l}(t) = U_{a_1}^{Aa_1'}(t)\ldots U_{a_l}^{Aa_l'}(t) S^A_{a_1'\ldots a_l'}[s_A(t)], \tag{5.36}$$

where $s_A(t)$ is the value of the body-frame time coordinate s_A evaluated at what would be the intersection of the world line of body A with the spacelike hypersurface of constant t. From Eq. (5.4) this function is given by

$$s_A(t) = t - \varepsilon^2\alpha_{\rm c}^A(t) + O(\varepsilon^4). \tag{5.37}$$

Also the rotation matrices $U_a^{Aa'}$ are defined by the formula

$$U_a^{Aa'} = \delta_{a'a} + \varepsilon^2 \epsilon_{aa'j} R_j^A. \tag{5.38}$$

From Eqs. (3.6), (3.7), (5.35), (5.36), (5.37), and (5.38) we can write these moments as

$$\mathcal{M}_L^A = {}^{\rm n}M_L^A + \varepsilon^2[{}^{\rm pn}M_L^A - \alpha_c^A\, {}^{\rm n}\dot{M}_L^A + l\epsilon_{jk\langle a_l}\, {}^{\rm n}M_{L-1\rangle j}^A R_k^A] + O(\varepsilon^4), \tag{5.39}$$

$$S_L^A = S_L^A + O(\varepsilon^2). \tag{5.40}$$

As indicated by Eq. (5.40), when working to post-1-Newtonian order we can identify the moments S_L^A and S_L^A. Nevertheless it might be useful in some circumstances to use the more accurate relation (5.36), for example, for systems which evolve for sufficiently long times that the rotation matrices $U_a^{Aa'}$ become significantly different from unity.

All the tools are now set up to compute explicit equations of motion for the center-of-mass world lines.

VI. EXPLICIT EQUATIONS OF MOTION FOR AN *N*-BODY SYSTEM

In this section we derive explicit equations of motion for the center-of-mass world lines ${}^{\rm cm}z_i^A(t)$ of each body as seen from the global coordinate system, by combining the single-body equations of motion (4.2b) and (4.3b) with the moment transformation formula derived in Sec. V above.

We start by deriving the well-known Newtonian equations of motion, in order to illustrate the computational method. The Newtonian single-body equation of motion (4.2b) applied to body A implies that

$$\sum_{l=0}^{\infty} \frac{1}{l!}\, {}^{\rm n}M_L^A(s_A)\, {}^{\rm n}G_{iL}^A(s_A) = 0, \tag{6.1}$$

since the body-adapted coordinates are mass-centered, i.e., ${}^{\rm n}M_i^A = 0$ for all A. Using the relation (5.25b) between the body-frame tidal moments ${}^{\rm n}G_L^A$ and the global-frame tidal moments ${}^{\rm n}G_L^{{\rm g},A}$ we can rewrite this as

$$\ddot{z}_i^A = {}^{\rm n}G_i^{{\rm g},A} + \sum_{l=2}^{\infty} \frac{1}{l!} \frac{{}^{\rm n}M_L^A}{{}^{\rm n}M^A}\, {}^{\rm n}G_{iL}^{{\rm g},A} = \sum_{l=0}^{\infty} \frac{1}{l!} \frac{{}^{\rm n}M_L^A}{{}^{\rm n}M^A}\, {}^{\rm n}G_{iL}^{{\rm g},A}. \tag{6.2}$$

Here the acceleration $\ddot{z}_i^A$ of the Newtonian-order center-of-mass world line has appeared via the transformation law for ${}^{\rm n}G_i^A$. Next, we substitute the expression (5.23a) for the global-frame tidal moments ${}^{\rm n}G_{iL}^{{\rm g},A}$ in terms of mass multipole moments ${}^{\rm n}M_L^{{\rm g},B}$ of the other bodies, and use Eq. (5.25a). This gives

$$\ddot{z}_i^A = \sum_{B\neq A} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^k}{k!l!} \frac{{}^{\rm n}M_L^A}{{}^{\rm n}M^A}\, {}^{\rm n}M_K^B \mathcal{T}_{iKL}^{-1}(z^B - z^A). \tag{6.3}$$

Here the quantities z^A and ${}^{\rm n}M_L^A$ are functions of s_A, while the quantities z^B and ${}^{\rm n}M_K^B$ are functions of s_B, evaluated at $s_B = s_A$. Writing the dependent variable as t instead of s_A and using the definition (5.19) of $\mathcal{T}_K^p$ we can rewrite Eq. (6.3) in the more explicit, well-known form [2]

$$\ddot{z}_i^A(t) = \sum_{B\neq A} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^k}{k!l!} \frac{{}^{\rm n}M_L^A(t)}{{}^{\rm n}M^A}\, {}^{\rm n}M_K^B(t) \times \left[\partial_{iKL}^{(\boldsymbol{u})} \frac{1}{|\boldsymbol{u}|}\right]_{\boldsymbol{u}=z^A(t)-z^B(t)}. \tag{6.4}$$

The analogous computation carried to post-1-Newtonian order is similar but much more involved. We start by focusing on the first two terms on the right-hand side of the single-body equation of motion (4.3b), and evaluating explicitly the $l = 0$ pieces using Eqs. (5.25b) and (5.30b). [As before the $l = 1$ pieces vanish since the body-frame coordinates are mass-centered, by Eq. (5.3).] Using the Newtonian equation of motion (6.3), the definition (5.39) of $\mathcal{M}_L^A$ and the definition (5.10) of ${}^{\rm cm}z_i^A$, the result can be written in the form

$$\mathcal{M}^A\, {}^{\rm cm}\ddot{z}_i^A = \mathcal{M}^A({}^{\rm n}G_i^{{\rm g},A} + \varepsilon^2 f_i^A) + \sum_{l=2}^{\infty} \mathcal{M}_L^A \mathcal{G}_{iL}^A + \varepsilon^2 g_i^A + O(\varepsilon^4). \tag{6.5}$$

Here by analogy with Eq. (5.39) we have defined

$$\mathcal{G}_L^A = {}^{\rm n}G_L^A + \varepsilon^2[{}^{\rm pn}G_L^A - \alpha_c^A\, {}^{\rm n}\dot{G}_L^A + l\epsilon_{jk\langle a_l}\, {}^{\rm n}G_{L-1\rangle j}^A R_k^A], \tag{6.6}$$

and g_i^A is defined to be all of the terms on the right-hand side of Eq. (4.3b) except for the first two terms (with superscripts A added to all the moments). Also we define

$$\begin{aligned} f_i^A = {} & F_i^{{\rm g},A} + \dot{Y}_i^{{\rm g},A} - \dot{z}_i^A Y_{ji}^{{\rm g},A} + h_{{\rm c}j}^A\, {}^{\rm n}G_{ji}^{{\rm g},A} - \alpha_{\rm c}^A \dot{z}_j^A\, {}^{\rm n}G_{ji}^{{\rm g},A} \\ & + \dot{z}_j^A \dot{z}_j^A\, {}^{\rm n}G_i^{{\rm g},A} - \dot{z}_i^A \dot{z}_j^A\, {}^{\rm n}G_j^{{\rm g},A} - 4\, {}^{\rm n}G_i^{{\rm g},A}\, {}^{\rm n}G^{{\rm g},A} \\ & - 3\dot{z}_i^A\, {}^{\rm n}\dot{G}^{{\rm g},A} + \epsilon_{ijk} \bar{f}_j^A R_k^A + \dot{\alpha}_{\rm c}^A \bar{f}_i^A - \frac{3}{2} \dot{z}_j^A \dot{z}_j^A \bar{f}_i^A \\ & - \frac{1}{2} \dot{z}_i^A \dot{z}_j^A \bar{f}_j^A - 4\bar{f}_i^A\, {}^{\rm n}G^{{\rm g},A}, \end{aligned} \tag{6.7}$$

where

$$\bar{f}_i^A = \ddot{z}_i^A - {}^{\rm n}G_i^{{\rm g},A} = \sum_{l=2}^{\infty} \frac{1}{l!} \frac{{}^{\rm n}M_L^A}{{}^{\rm n}M^A}\, {}^{\rm n}G_{iL}^{{\rm g},A}. \tag{6.8}$$

In order to explicitly evaluate the tidal moments that appear on the right-hand side of Eq. (6.5), we perform the following sequence of moment transformations: (i) Start with the body-frame multipole moments ${}^{\rm n}M_L^B$, ${}^{\rm pn}M_L^B$, andS_L^B of body B. (ii) Compute from these the global-frame multipole moments ${}^{\rm n}M^{{\rm g},B}$, $N_L^{{\rm g},B}$, $P_L^{{\rm g},B}$, and $Z_L^{{\rm g},B}$ of body B using Eqs. (3.14a), (3.61b), (3.61c), (5.11d), (5.27a), (5.29a), and (5.30a). The results are

$$ {}^{\rm n}M_L^{{\rm g},B} = {}^{\rm n}M_L^B, \tag{6.9a}$$

$$ Z_{iL}^{{\rm g},B} = \frac{4}{l+1}\,{}^{\rm n}\dot M_{iL}^B - \frac{4l}{l+1}\epsilon_{ji\langle a_l}S^B_{L-1\rangle j} + 4\dot z_i^B\,{}^{\rm n}M_L^B - \frac{4(2l-1)}{2l+1}\dot z_j^B\,{}^{\rm n}M^B_{j\langle L-1}\delta_{a_l\rangle i}, \tag{6.9b}$$

$$ P_L^{{\rm g},B} = {}^{\rm n}\ddot M_L^B + 2l\dot z^B_{\langle a_l}\,{}^{\rm n}\dot M^B_{L-1\rangle} + l\ddot z^B_{\langle a_l}\,{}^{\rm n}M^B_{L-1\rangle} + l(l-1)\dot z^B_{\langle a_l}\dot z^B_{a_{l-1}}\,{}^{\rm n}M^B_{L-2\rangle}, \tag{6.9c}$$

$$\begin{aligned} N_L^{{\rm g},B} = {}& {}^{\rm pn}M_L^B - \alpha_{\rm c}^B\,{}^{\rm n}\dot M_L^B + l\epsilon_{jk\langle a_l}\,{}^{\rm n}M^B_{L-1\rangle j}R_k^B + lh^B_{{\rm c}\langle a_l}\,{}^{\rm n}M^B_{L-1\rangle} - l\alpha_{\rm c}^B\dot z^B_{\langle a_l}\,{}^{\rm n}M^B_{L-1\rangle} - (l+1)\dot\alpha_{\rm c}^B\,{}^{\rm n}M_L^B + \frac{4l}{l+1}\dot z_j^B\epsilon_{jk\langle a_l}S^B_{L-1\rangle k} \\ & + \frac{(l+2)(2l+7)}{2(2l+3)}\dot z_j^B\dot z_j^B\,{}^{\rm n}M_L^B - \left[\frac{l}{2}+3-\frac{10l+21}{(2l+1)(2l+3)}\right]\dot z_j^B\dot z^B_{\langle a_l}\,{}^{\rm n}M^B_{L-1\rangle j} \\ & - \left[l+1+\frac{7l+3}{(l+1)(2l+3)}\right]\ddot z_j^B\,{}^{\rm n}M^B_{jL} - \left[1-\frac{2(l+5)}{(l+1)(2l+3)}\right]\dot z_j^B\,{}^{\rm n}\dot M^B_{jL}. \end{aligned}\tag{6.9d}$$

(iii) Compute the global-frame tidal moments ${}^{\rm n}G_L^{{\rm g},A}$, $Y_L^{{\rm g},A}$, $J_L^{{\rm g},A}$, and $F_L^{{\rm g},A}$ of body A in terms of the global-frame multipole moments of body B using Eqs. (5.23a)–(5.23d). (iv) Compute the body-frame tidal moments H_L^A, ${}^{\rm n}G_L^A$, and ${}^{\rm pn}G_L^A$ of body A in terms of its global-frame tidal moments ${}^{\rm n}G_L^{{\rm g},A}$, $Y_L^{{\rm g},A}$, $J_L^{{\rm g},A}$, and $F_L^{{\rm g},A}$. Here the results are given by Eq. (5.25b) for ${}^{\rm n}G_L^A$, and by Eq. (5.30b) for ${}^{\rm pn}G_L^A$; note that for the required values of l ($l \ge 3$) the last three terms in Eq. (5.30b) do not contribute. For H_L^A we have $H_i^A = 0$ by the gauge condition (5.3), while for $l \ge 2$ we obtain from Eq. (5.27b) that

$$ H_L^A = Y^{{\rm g},A}_{jk\langle L-1}\epsilon_{a_l\rangle jk} - 4\dot z_j^A\,{}^{\rm n}G^{{\rm g},A}_{k\langle L-1}\epsilon_{a_l\rangle jk}. \tag{6.10}$$

(v) By combining the preceding steps, all the moments can be expressed in terms of the multipole moments ${}^{\rm n}M_L^B$, ${}^{\rm pn}M_L^B$, and S_L^B of body B.

In the resulting expression, we eliminate the variables z_i^C in favor of ${}^{\rm cm}z_i^C$ for all C using the definition (5.10), and we eliminate the moments ${}^{\rm n}M_L^C$, S_L^C in favor of $\mathcal{M}_L^C$, S_L^A for all C using the definitions (5.39) and (5.40). These substitutions generate correction terms only in the $O(\varepsilon^0)$, Newtonian terms in Eq. (6.5), and not in the $O(\varepsilon^2)$, post-Newtonian terms,[30] since we drop all terms of order $O(\varepsilon^4)$. We also eliminate $\dot\alpha_{\rm c}^C$ using Eq. (5.32). The resulting expression then depends only on the variables ${}^{\rm cm}z_i^C$, $\mathcal{M}_L^C$, and S_L^C; the dependencies on the variables R_k^C, $\alpha_{\rm c}^C$, and h_{ci}^C cancel out. Lastly, we set to one the formal expansion parameter ε. The result of this tedious computation is

$$\begin{aligned} {}^{\rm cm}\ddot z_i^A(t) = {}& \sum_{B\neq A} a_i^{AB}(t) + \sum_{B\neq A}\sum_{C\neq A} a_i^{ABC}(t) \\ & + \sum_{B\neq A}\sum_{C\neq B}\tilde a_i^{ABC}(t) + O(\varepsilon^4), \end{aligned}\tag{6.11}$$

where

$$\begin{aligned} a_i^{AB} = {}& \sum_{k=0}^\infty\sum_{l=0}^\infty\Bigg[{}^{(1)}\mathcal{D}^{AB}_{KL}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l+2}} + {}^{(2)}\mathcal{D}^{AB}_{iKL}\frac{n^{BA}_{\langle KL\rangle}}{r_{BA}^{k+l+1}} + {}^{(3)}\mathcal{D}^{AB}_{ijKL}\frac{n^{BA}_{\langle jKL\rangle}}{r_{BA}^{k+l+2}} + {}^{(4)}\mathcal{D}^{AB}_{jKL}\frac{n^{BA}_{\langle ijKL\rangle}}{r_{BA}^{k+l+3}} + {}^{(5)}\mathcal{D}^{AB}_{ijmKL}\frac{n^{BA}_{\langle jmKL\rangle}}{r_{BA}^{k+l+3}} \\ & + {}^{(6)}\mathcal{D}^{AB}_{injmKL}\frac{n^{BA}_{\langle njmKL\rangle}}{r_{BA}^{k+l+4}} + {}^{(7)}\mathcal{D}^{AB}_{KL}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l}}\Bigg], \end{aligned}\tag{6.12a}$$

$$\begin{aligned} a_i^{ABC} = {}& \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{p=0}^\infty {}^{(1)}\mathcal{D}^{ABC}_{KLP}\frac{n^{CA}_{\langle P\rangle}}{r_{CA}^{p+1}}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l+2}} + \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{p=0}^\infty\sum_{q=0}^\infty\Bigg[{}^{(2)}\mathcal{D}^{ABC}_{KLPQ}\frac{n^{CA}_{\langle iPQ\rangle}}{r_{CA}^{p+q+2}}\frac{n^{BA}_{\langle KL\rangle}}{r_{BA}^{k+l+1}} + {}^{(3)}\mathcal{D}^{ABC}_{jKLPQ}\frac{n^{CA}_{\langle jPQ\rangle}}{r_{CA}^{p+q+2}}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l+2}} \\ & + {}^{(4)}\mathcal{D}^{ABC}_{iKLPQ}\frac{n^{CA}_{\langle jPQ\rangle}}{r_{CA}^{p+q+2}}\frac{n^{BA}_{\langle jKL\rangle}}{r_{BA}^{k+l+2}} + {}^{(5)}\mathcal{D}^{ABC}_{ijKLPQ}\frac{n^{CA}_{\langle mPQ\rangle}}{r_{CA}^{p+q+2}}\frac{n^{BA}_{\langle jmKL\rangle}}{r_{BA}^{k+l+3}}\Bigg], \end{aligned}\tag{6.12b}$$

$$\begin{aligned} \tilde a_i^{ABC} = {}& \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{p=0}^\infty {}^{(1)}\tilde{\mathcal{D}}^{ABC}_{KLP}\frac{n^{CB}_{\langle P\rangle}}{r_{CB}^{p+1}}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l+2}} + \sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{p=0}^\infty\sum_{q=0}^\infty\Bigg[{}^{(2)}\tilde{\mathcal{D}}^{ABC}_{ijKLPQ}\frac{n^{CB}_{\langle jPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle KL\rangle}}{r_{BA}^{k+l+1}} + {}^{(3)}\tilde{\mathcal{D}}^{ABC}_{KLPQ}\frac{n^{CB}_{\langle iPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle KL\rangle}}{r_{BA}^{k+l+1}} \\ & + {}^{(4)}\tilde{\mathcal{D}}^{ABC}_{jKLPQ}\frac{n^{CB}_{\langle jPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle iKL\rangle}}{r_{BA}^{k+l+2}} + {}^{(5)}\tilde{\mathcal{D}}^{ABC}_{iKLPQ}\frac{n^{CB}_{\langle jPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle jKL\rangle}}{r_{BA}^{k+l+2}} + {}^{(6)}\tilde{\mathcal{D}}^{ABC}_{ijKLPQ}\frac{n^{CB}_{\langle mPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle jmKL\rangle}}{r_{BA}^{k+l+3}}\Bigg]. \end{aligned}\tag{6.12c}$$

[30] In particular since the current moments S_L^A do not enter at Newtonian order, there are no correction terms generated when one eliminates S_L^A in favor of S_L^A. Thus, we are free to use either S_L^A or S_L^A in the equations of motion, to post-1-Newtonian order, as noted in Sec. V F above.

Here the coefficients are given by

$$
\begin{aligned}
{}^{(1)}\mathcal{D}^{AB}_{KL} = \frac{(-1)^k(2k+2l+1)!!}{k!l!\mathcal{M}^A}\Big\{&\mathcal{M}^A_L\mathcal{M}^B_K\Big[1+v^A_jv^A_j+\frac{6k+11}{2(2k+3)}v^B_jv^B_j-4v^A_jv^B_j\Big]+v^A_j\mathcal{M}^A_{jL}\dot{\mathcal{M}}^B_K\\
&+\mathcal{M}^A_L\dot{\mathcal{M}}^B_{jK}\Big[\frac{4}{k+1}v^{BA}_j-\frac{2k+1}{2k+3}v^B_j\Big]-\frac{1}{2k+2l+5}\mathcal{M}^A_{jL}\hat{P}^B_{jK}-\frac{4}{l+1}\dot{\mathcal{M}}^A_{jL}\Big(v^{BA}_j\mathcal{M}^B_K+\frac{1}{k+1}\dot{\mathcal{M}}^B_{jK}\Big)\Big\},
\end{aligned} \tag{6.13a}
$$

$$
\begin{aligned}
{}^{(2)}\mathcal{D}^{AB}_{iKL} = \frac{(-1)^k(2k+2l+1)!!}{k!l!(2k+2l+1)\mathcal{M}^A}\Big\{&\mathcal{M}^A_L\Big[4\dot{\mathcal{M}}^B_Kv^{BA}_i+\dot{\mathcal{M}}^B_Kv^A_i-\frac{1}{2k+2l+3}\hat{P}^B_{iK}+\frac{4}{k+1}\ddot{\mathcal{M}}^B_{iK}\Big]\\
&+4\dot{\mathcal{M}}^A_L\Big[\frac{1}{k+1}\dot{\mathcal{M}}^B_{iK}+\mathcal{M}^B_Kv^{BA}_i\Big]-\frac{(2l^2+3l+5)}{(l+1)}\dot{\mathcal{M}}^A_{iL}\dot{\mathcal{M}}^B_K+\frac{1}{2l+3}\mathcal{M}^A_{iL}\Big[-(l+2)(2l+1)\ddot{\mathcal{M}}^B_K\\
&-\frac{2k}{(2k+2l+3)}\hat{P}^B_K\Big]-\frac{(l^2+l+4)}{(l+1)}\ddot{\mathcal{M}}^A_{iL}\mathcal{M}^B_K\Big\},
\end{aligned} \tag{6.13b}
$$

$$
\begin{aligned}
{}^{(3)}\mathcal{D}^{AB}_{ijKL} = \frac{(-1)^k(2k+2l+1)!!}{k!l!\mathcal{M}^A}\Big\{&\mathcal{M}^A_L\Big[-\frac{4}{k+1}v^{BA}_j\dot{\mathcal{M}}^B_{iK}-\mathcal{M}^B_K(4v^{BA}_iv^{BA}_j+v^A_iv^B_j)+\frac{3}{\mathcal{M}^A}(\mathcal{M}^A_{ij}\ddot{\mathcal{M}}^B_K+2\dot{\mathcal{M}}^A_{ij}\dot{\mathcal{M}}^B_K\\
&+\ddot{\mathcal{M}}^A_{ij}\mathcal{M}^B_K)+\epsilon_{ijm}\Big(\frac{4}{k+2}\dot{S}^B_{mK}-2\dot{S}^A_m\frac{\mathcal{M}^B_K}{\mathcal{M}^A}-S^A_m\frac{\dot{\mathcal{M}}^B_K}{\mathcal{M}^A}\Big)\Big]+2\mathcal{M}^A_{iL}\dot{\mathcal{M}}^B_K\Big[\frac{(l+2)(2l+1)}{(2l+3)}v^B_j-(l+1)v^A_j\Big]\\
&+\dot{\mathcal{M}}^A_L\Big[\epsilon_{ijm}\Big(\frac{4}{k+2}S^B_{mK}-S^A_m\frac{\mathcal{M}^B_K}{\mathcal{M}^A}\Big)+\frac{6}{\mathcal{M}^A}(\mathcal{M}^A_{ij}\dot{\mathcal{M}}^B_K+\dot{\mathcal{M}}^A_{ij}\mathcal{M}^B_K)\Big]+\frac{(2l^2+3l+5)}{(l+1)}v^{BA}_j\dot{\mathcal{M}}^A_{iL}\mathcal{M}^B_K\\
&+\frac{3}{\mathcal{M}^A}\mathcal{M}^A_{ij}\ddot{\mathcal{M}}^A_L\mathcal{M}^B_K+\frac{4}{l+2}\epsilon_{ijm}(S^A_{mL}\dot{\mathcal{M}}^B_K+\dot{S}^A_{mL}\mathcal{M}^B_K)\Big\},
\end{aligned} \tag{6.13c}
$$

$$
\begin{aligned}
{}^{(4)}\mathcal{D}^{AB}_{jKL} = \frac{(-1)^k(2k+2l+3)!!}{k!l!\mathcal{M}^A}\Big\{&\mathcal{M}^A_L\Big[\frac{2k+1}{2(2k+5)}v^B_jv^B_m\mathcal{M}^B_{mK}-\frac{4}{k+2}v^{BA}_m\epsilon_{jmn}S^B_{nK}\Big]-\mathcal{M}^A_{nL}\mathcal{M}^B_K\Big(v^A_nv^{BA}_j+\frac{1}{2}v^A_nv^A_j\Big)\\
&+\frac{4}{(l+1)(k+2)}\epsilon_{jmn}\dot{\mathcal{M}}^A_{mL}S^B_{nK}-\frac{4}{l+2}\epsilon_{jmn}S^A_{nL}\Big(v^{BA}_m\mathcal{M}^B_K+\frac{1}{k+1}\dot{\mathcal{M}}^B_{mK}\Big)\Big\},
\end{aligned} \tag{6.13d}
$$

$$
\begin{aligned}
{}^{(5)}\mathcal{D}^{AB}_{ijmKL} = \frac{(-1)^k(2k+2l+3)!!}{k!l!\mathcal{M}^A}\Big(&\mathcal{M}^A_L\Big[\epsilon_{ijn}S^A_nv^{BA}_m\frac{\mathcal{M}^B_K}{\mathcal{M}^A}-\frac{4}{k+2}\epsilon_{ijn}S^B_{nK}v^{BA}_m-\frac{6}{\mathcal{M}^A}v^{BA}_m(\mathcal{M}^A_{ij}\dot{\mathcal{M}}^B_K+\dot{\mathcal{M}}^A_{ij}\mathcal{M}^B_K)\Big]\\
&+\frac{2}{2l+3}\mathcal{M}^A_{iL}\mathcal{M}^B_K(v^B_jv^A_m)-\mathcal{M}^B_K\Big\{\frac{\mathcal{M}^A_{iL}}{2l+3}[v^A_jv^A_m+(l+2)(2l+1)v^{BA}_jv^{BA}_m]\Big\}\\
&-\mathcal{M}^B_K\Big[\frac{6}{\mathcal{M}^A}v^{BA}_m\dot{\mathcal{M}}^A_L\mathcal{M}^A_{ij}+\frac{4}{l+2}\epsilon_{imn}S^A_{nL}v^{BA}_j\Big]\Big),
\end{aligned} \tag{6.13e}
$$

$$
{}^{(6)}\mathcal{D}^{AB}_{injmKL} = \frac{(-1)^k(2k+2l+5)!!}{k!l!\mathcal{M}^A}\Big\{\frac{3}{\mathcal{M}^A}v^{BA}_jv^{BA}_m\mathcal{M}^A_{in}\mathcal{M}^A_L\mathcal{M}^B_K-\frac{4\delta_{in}}{(k+2)(l+2)}S^A_{jL}S^B_{mK}\Big\}, \tag{6.13f}
$$

$$
{}^{(7)}\mathcal{D}^{AB}_{KL} = \frac{(-1)^k(2k+2l+1)!!}{k!l!(2k+2l+1)\mathcal{M}^A}\Big\{-\frac{1}{2}\mathcal{M}^A_L\hat{P}^B_K\Big\}, \tag{6.13g}
$$

$$
{}^{(1)}\mathcal{D}^{ABC}_{KLP} = \frac{(-1)^{k+p+1}l(2k+2l+1)!!(2p+1)!!}{k!l!p!(2p+1)\mathcal{M}^A}\mathcal{M}^A_L\mathcal{M}^B_K\mathcal{M}^C_P, \tag{6.14a}
$$

$$
{}^{(2)}\mathcal{D}^{ABC}_{KLPQ} = \frac{(-1)^{k+p+1}(l+4)(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!(2k+2l+1)\mathcal{M}^A}\mathcal{M}^A_L\mathcal{M}^B_K\frac{\mathcal{M}^A_Q}{\mathcal{M}^A}\mathcal{M}^C_P, \tag{6.14b}
$$

$$
{}^{(3)}\mathcal{D}^{ABC}_{jKLPQ} = \frac{(-1)^{k+p+1}(l+1)(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}\mathcal{M}^A_{jL}\mathcal{M}^B_K\frac{\mathcal{M}^A_Q}{\mathcal{M}^A}\mathcal{M}^C_P, \tag{6.14c}
$$

$$
{}^{(4)}\mathcal{D}^{ABC}_{iKLPQ} = \frac{(-1)^{k+p+1}l(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}\mathcal{M}^A_{iL}\mathcal{M}^B_K\frac{\mathcal{M}^A_Q}{\mathcal{M}^A}\mathcal{M}^C_P, \tag{6.14d}
$$

$$
{}^{(5)}\mathcal{D}^{ABC}_{ijKLPQ} = \frac{3(-1)^{k+p}(2k+2l+3)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}\frac{\mathcal{M}^A_L}{\mathcal{M}^A}\mathcal{M}^A_{ij}\mathcal{M}^B_K\frac{\mathcal{M}^A_Q}{\mathcal{M}^A}\mathcal{M}^C_P, \tag{6.14e}
$$

$$^{(1)}\tilde{\mathcal{D}}^{ABC}_{KLP} = \frac{(-1)^{k+p+1}(k+1)(2k+2l+1)!!(2p+1)!!}{k!l!p!(2p+1)\mathcal{M}^A}\mathcal{M}^A_L\mathcal{M}^B_K\mathcal{M}^C_P, \tag{6.15a}$$

$$^{(2)}\tilde{\mathcal{D}}^{ABC}_{ijKLPQ} = \frac{(-1)^{k+p+1}(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!(2k+2l+1)(2k+2l+3)\mathcal{M}^A}\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P\bigg[(k+1)\mathcal{M}^A_L\mathcal{M}^B_{\langle K}\delta_{i\rangle j} + \frac{2k^2}{2l+3}\mathcal{M}^A_{iL}\mathcal{M}^B_{\langle K-1}\delta_{b_k\rangle j}\bigg], \tag{6.15b}$$

$$^{(3)}\tilde{\mathcal{D}}^{ABC}_{KLPQ} = \frac{4(-1)^{k+p}(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!(2k+2l+1)\mathcal{M}^A}\mathcal{M}^A_L\mathcal{M}^B_K\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P, \tag{6.15c}$$

$$^{(4)}\tilde{\mathcal{D}}^{ABC}_{jKLPQ} = \frac{(-1)^{k+p+1}(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P\bigg[\frac{(2k+1)(k+2)}{2k+3}\mathcal{M}^A_L\mathcal{M}^B_{jK} + \frac{k+1}{2k+2l+5}\mathcal{M}^A_{Lr}\mathcal{M}^B_{\langle K}\delta_{r\rangle j}\bigg], \tag{6.15d}$$

$$^{(5)}\tilde{\mathcal{D}}^{ABC}_{iKLPQ} = \frac{(-1)^{k+p}(l+2)(2l+1)(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!(2l+3)\mathcal{M}^A}\mathcal{M}^A_{iL}\mathcal{M}^B_K\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P, \tag{6.15e}$$

$$^{(6)}\tilde{\mathcal{D}}^{ABC}_{ijKLPQ} = \frac{3(-1)^{k+p+1}(2k+2l+3)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}\frac{\mathcal{M}^A_L}{\mathcal{M}^A}\mathcal{M}^A_{ij}\mathcal{M}^B_K\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P, \tag{6.15f}$$

where

$$\hat{P}^B_K = \ddot{\mathcal{M}}^B_K + 2k v^B_{\langle b_k}\dot{\mathcal{M}}^B_{K-1\rangle} + k(k-1)v^B_{\langle b_k}v^B_{b_{k-1}}\mathcal{M}^B_{K-2\rangle}. \tag{6.16}$$

Here we have denoted by $\boldsymbol{n}^{BA}$ the unit vector pointing from the center-of-mass world line of body A to that of body B:

$$n^{BA}_i(t) \equiv \frac{{}^{\rm cm}z^B_i(t) - {}^{\rm cm}z^A_i(t)}{r_{BA}(t)}, \tag{6.17}$$

where

$$r_{BA}(t) \equiv |{}^{\rm cm}\boldsymbol{z}^B(t) - {}^{\rm cm}\boldsymbol{z}^A(t)|. \tag{6.18}$$

We also have defined $v^A_i = {}^{\rm cm}\dot{z}^A_i$ and $v^{BA}_i = {}^{\rm cm}\dot{z}^B_i - {}^{\rm cm}\dot{z}^A_i$.

The right-hand sides of Eqs. (6.11) and (6.12a)–(6.12c) depend on the time derivatives $\dot{S}^A_i$, $\dot{\mathcal{M}}^A$, and $\ddot{\mathcal{M}}^A$ of the bodies' spins and mass monopole moments. These dependencies can be eliminated using the single-body laws of motion (4.2a), (4.3a), and (4.3c), the last two of which are derived for strongly self-gravitating bodies in paper II [37]. For the case of the mass monopole moments, this procedure generates terms that are of post-2-Newtonian order which can be neglected. Thus, all time derivatives of mass monopoles appearing in Eqs. (6.11) and (6.12a)–(6.12c) can be neglected. In other words, we can make the following substitutions in Eqs. (6.13), (6.14), and (6.15):

$$\dot{\mathcal{M}}^A_L \to (1-\delta_{l0})\dot{\mathcal{M}}^A_L, \qquad \ddot{\mathcal{M}}^A_L \to (1-\delta_{l0})\ddot{\mathcal{M}}^A_L. \tag{6.19}$$

For the case of the spin time derivative terms, using Eq. (4.3c) together with Eqs. (5.23a) and (5.25b) we obtain modified values of the coefficients (6.13), (6.14), and (6.15) which are listed in Appendix F.

A simple special case of the above equations of motion is the nonspinning point-particle model, or monopole-truncated model. This is obtained by setting to zero all the mass multipole moments M^A_L for $l \geq 1$, and all the current multipoles S^A_L. In this case Eq. (6.11) reduces to the well-known Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion [6,7], which were also reported in the first DSX paper [Eq. (7.20b) of Ref. [2]].

A second special case is the spinning point-particle model or monopole-spin truncated model, obtained by setting to zero all the mass multipoles ${}^{\rm n}M^A_L$ and ${}^{\rm pn}M^A_L$ for $l \geq 1$, all the current multipoles S^A_L for $l \geq 2$, but allowing nonzero spins S^A_i. For this case our general equation of motion (6.11) reduces to the equations of motion obtained for this case by DSX [Eqs. (6.30)–(6.34) of Ref. [16]].

Finally, we can obtain an explicit expression for the angular velocity (5.34) parametrizing the dragging of inertial frames by using the Newtonian equation of motion (6.3), the formulas (5.23a) and (5.23b) for ${}^{\rm n}G^{{\rm g},A}_L$ and $Y^{{\rm g},A}_L$, the formula (6.9b) for $Z^{{\rm g},B}_L$, the formula (5.20) for $\mathcal{T}^{-1}_K$, and the definitions (5.38), (5.39), and (5.40). The result is

$$[\dot{\mathbf{U}}^A \cdot (\mathbf{U}^A)^{-1}]_{ij} = \frac{1}{2}(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) \sum_{B\neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[\sum_{l=0}^{\infty} \frac{(2k+2l+1)!!}{l!} \frac{\mathcal{M}_L^A}{\mathcal{M}^A} \mathcal{M}_K^B v_r^A \frac{n_{\langle sKL\rangle}^{BA}}{r_{BA}^{k+l+2}} + \frac{4(2k+1)!!}{k+1} \dot{\mathcal{M}}_{Kr}^B \frac{n_{\langle sK\rangle}^{BA}}{r_{BA}^{k+2}} - \frac{4k(2k+1)!!}{k+1} S_{mK-1}^B \epsilon_{b_k mr} \frac{n_{\langle sK\rangle}^{BA}}{r_{BA}^{k+2}} \right] + O(\varepsilon^4). \tag{6.20}$$

VII. CONCLUSION

In this paper, we have given a surface integral derivation of the full post-1-Newtonian DSX laws of motion (4.3b). We have shown that these laws of motion apply to a wide class of strongly self-gravitating objects, provided that the mass and current moments are appropriately defined in terms of the asymptotic weak field metric in the buffer regions around each body. We have given an explicit form for the coupled equations of motion of the bodies' center-of-mass world lines including the effects of *all* the post-Newtonian mass and current multipole couplings. To the best of our knowledge this is the first time these equations of motion have been written out explicitly. The second paper in this series will include surface integral derivations of the evolution laws (4.3a) and (4.3c) for the energy (mass monopole) and the spin S_i^A [37].

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APPENDIX A: USEFUL IDENTITIES INVOLVING STF TENSORS

In this appendix we give the general definition of the STF projection of an arbitrary tensor. We also review some identities that are useful for manipulating expressions involving STF tensors.

The STF projection of any tensor T_L is obtained by taking the symmetric part of T_L, and then subtracting out all the partial traces. One obtains in this way a unique symmetric tensor that is trace free on all pairs of indices. The general formula for this projection is

$$T_{\langle L\rangle} \equiv \sum_{k=0}^{[l/2]} c_k^l \delta_{(a_1a_2} \ldots \delta_{a_{2k-1}a_{2k}} S_{a_{2k+1}\ldots a_l)j_1j_1\ldots j_kj_k}, \tag{A1}$$

where $[l/2]$ is the largest integer less than or equal to $l/2$, the coefficients c_k^l are given by

$$c_k^l = (-1)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!!(2k)!!}, \tag{A2}$$

$l!!$ means $l(l-2)(l-4)\ldots(4)(2)$ or $l(l-2)\times(l-4)\ldots(3)(1)$, and

$$S_L \equiv T_{(L)} \tag{A3}$$

is the symmetric part of T. For example,

$$T_{\langle abc\rangle} = S_{abc} - \frac{1}{5}[\delta_{ab}S_{cdd} + \delta_{ac}S_{bdd} + \delta_{bc}S_{add}]. \tag{A4}$$

From the definition of the STF projection one can derive the following "peeling formula" [2] for any STF tensor T_L and vector V_i:

$$V_{\langle j}T_{L\rangle} = \frac{1}{l+1}V_jT_L + \frac{l}{l+1}T_{j\langle L-1}V_{a_l\rangle} - \frac{2l}{(l+1)(2l+1)}V_kT_{k\langle L-1}\delta_{a_l\rangle j}. \tag{A5}$$

From this peeling formula one can obtain the identities

$$T_{i\langle L}\delta_{j\rangle j} = \frac{2l+3}{2l+1}T_{iL} \tag{A6}$$

and

$$T_{j\langle L}\delta_{j\rangle i} = \frac{1}{(l+1)(2l+1)}T_{iL}, \tag{A7}$$

which are valid for any STF tensor T_{L+1}.

Next, some useful formulas involving derivatives are

$$|\boldsymbol{x}|^2\partial_L\frac{1}{|\boldsymbol{x}|} = -(2l-1)\partial_{\langle L\rangle}|\boldsymbol{x}|, \tag{A8}$$

and

$$\partial_{iL}|\boldsymbol{x}| = \partial_{\langle iL\rangle}|\boldsymbol{x}| + \frac{l(l+1)}{2l+1}\delta_{(ia_l}\partial_{L-1)}\frac{1}{|\boldsymbol{x}|}. \tag{A9}$$

Also, given a sequence of STF tensors T_L, one for each l, we have the identities

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}T_Lx^j\partial_L\frac{1}{|\boldsymbol{x}|} = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[\left(\frac{2l+1}{2l+3}\right)T_{jL}\partial_L\frac{1}{|\boldsymbol{x}|} + T_L\partial_{\langle jL\rangle}|\boldsymbol{x}|\right] \tag{A10}$$

and

$$\sum_{l=0}^{\infty}\frac{1}{l!}T_Lx^{jL} = \sum_{l=0}^{\infty}\frac{1}{l!}\left(T_Lx^{\langle jL\rangle} + \frac{1}{2l+3}|\boldsymbol{x}|^2T_{jL}x^L\right). \tag{A11}$$

These identities are used in Sec. III C above in the computation of the transformation laws for the multipole and tidal moments. In Sec. IV we use the identities

$$T_L \partial_{iL} \frac{1}{|\boldsymbol{x}|} = \frac{1}{|\boldsymbol{x}|^{l+2}} (a_l T_{iL-1} n^{L-1} - b_l T_L n^{iL}), \quad \text{(A12)}$$

$$T_L \partial_i x^L = l |\boldsymbol{x}|^{l-1} T_{iL-1} n^{L-1}, \quad \text{(A13)}$$

where $a_l = (-1)^l l(2l-1)!!$, $b_l = (-1)^l (2l+1)!!$, and $n^i = x^i / |\boldsymbol{x}|$. We also use the following integrals over the unit sphere given in Thorne [38]

$$\frac{1}{4\pi} \oint n^{2L+1} d\Omega = 0, \quad \text{(A14)}$$

$$\frac{1}{4\pi} \oint n^{2L} d\Omega = \frac{1}{2l+1} \delta_{(i_1 i_2} \ldots \delta_{i_{2l-1} i_{2l})}, \quad \text{(A15)}$$

$$\frac{1}{4\pi} \oint T_K S_L n^K n^L d\Omega = \frac{l!}{(2l+1)!!} T_L S_L \quad \text{if } k = l$$
$$= 0 \text{ if } k \neq l, \quad \text{(A16)}$$

and

$$\frac{1}{4\pi} \oint T_K S_L n^K n^L n^i d\Omega = \frac{(l+1)!}{(2l+3)!!} T_{iL} S_L \quad \text{if } k = l+1$$
$$= 0 \text{ if } |k-l| \neq 1. \quad \text{(A17)}$$

Another useful peeling identity, valid for a $k+1$ index tensor which is STF on its last k indices, is

$$T_{\langle iK} z_{L-K\rangle} = \frac{k+1}{l+1} T_{i\langle K} z_{L-K\rangle} + \frac{l-k}{l+1} T_{\langle K+1} z_{L-(K+1)\rangle} z_i - \frac{2(k+1)(l-k)}{(l+1)(2l+1)} z_j T_{j\langle K} z_{(L-1)-K} \delta_{a_l\rangle i}$$
$$- \frac{(l-k)(l-k-1)}{(l+1)(2l+1)} z_j z_j T_{\langle K+1} z_{(L-1)-(K+1)} \delta_{a_l\rangle i}. \quad \text{(A18)}$$

Finally, for any STF tensor T and vectors v_i and z_i we have the identity

$$v_{\langle i} T_{LK\rangle} z^K = \frac{l+1}{l+k+1} v_{\langle i} T_{L\rangle K} z^K - \frac{k(l+1)}{(l+k+1)(2l+2k+1)} v_j T_{jK-1\langle iL-1} z_{a_l\rangle} z^{K-1}$$
$$- \frac{k(k-1)}{(l+k+1)(2l+2k+1)} z_m z_m v_j T_{j\langle iL\rangle K-2} z^{K-2} + \frac{k}{l+k+1} v_j z_j T_{\langle iL\rangle K-1} z^{K-1}. \quad \text{(A19)}$$

APPENDIX B: DERIVATION OF GAUGE TRANSFORMATION PARAMETRIZATION

In this appendix we consider harmonic, conformally Cartesian coordinate systems on a spacetime region $\mathcal{D} \times (t_0, t_1)$, where $\mathcal{D}$ is a simply connected spatial region and (t_0, t_1) is an open interval of time. We show that the most general gauge transformation between two such coordinate systems is of the form given by Eqs. (2.17) and (2.18a)–(2.18c), up to constant displacements in time and up to time-independent spatial rotations.

We start by reviewing the well-known argument that gives this result to Newtonian order. Let the coordinate transformation to zeroth order in ε be

$$x^i = x^i(\bar{t}, \bar{x}^j) + O(\varepsilon^2), \qquad t = t(\bar{t}, \bar{x}^j) + O(\varepsilon^2). \quad \text{(B1)}$$

Substituting this into the metric expansion (2.4), we find that the leading order expression for the spatial metric is

$$-\frac{1}{\varepsilon^2} \frac{\partial t}{\partial \bar{x}^i} \frac{\partial t}{\partial \bar{x}^j} d\bar{x}^i d\bar{x}^j + O(1). \quad \text{(B2)}$$

This is in conflict with the expansion (2.16) unless $\partial t / \partial \bar{x}^i = 0$. Similarly, the leading order expression for the time-time piece of the line element is

$$-\frac{1}{\varepsilon^2} \left(\frac{\partial t}{\partial \bar{t}}\right)^2 d\bar{t}^2 + O(1), \quad \text{(B3)}$$

which disagrees with the expansion (2.16) unless $\partial t / \partial \bar{t} = \pm 1$. Assuming that the coordinate transformation preserves the time orientation and neglecting constant displacements in time we obtain $t = \bar{t} + O(\varepsilon^2)$. Therefore we can write

$$t = \bar{t} + \varepsilon^2 \alpha(\bar{t}, \bar{x}^j) + O(\varepsilon^4), \quad \text{(B4)}$$

where the function $\alpha(\bar{t}, \bar{x}^j)$ is as yet undetermined.

The leading order expression for the spatial metric is now

$$\delta_{kl} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} d\bar{x}^i d\bar{x}^j + O(\varepsilon^2) = \delta_{ij} d\bar{x}^i d\bar{x}^j + O(\varepsilon^2), \quad \text{(B5)}$$

where we have used the expansion (2.16). Thus, for each fixed $\bar{t}$, the function $x^i = x^i(\bar{t}, \bar{x}^j)$ is an isometry of three-dimensional Euclidean space. It follows that

$$x^i = R^i{}_j(\bar{t}) \bar{x}^j + z^i(\bar{t}) + O(\varepsilon^2) \quad \text{(B6)}$$

for some time-dependent rotation matrix $R^i_j(\bar{t})$ and some time-dependent displacement $z^i(\bar{t})$. Using Eqs. (2.4), (B4), and (B6) the leading order expression for the spacetime piece of the line element is

$$\left\{2\delta_{ik}R^k{}_l(\bar{t})[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})] - 2\frac{\partial\alpha}{\partial\bar{x}^l}\right\}d\bar{t}d\bar{x}^l + O(\varepsilon^2). \tag{B7}$$

The first term here must vanish in order to be compatible with Eq. (2.16), which gives

$$\delta_{ik}R^k{}_l(\bar{t})[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})] = \frac{\partial\alpha}{\partial\bar{x}^l}. \tag{B8}$$

If $\dot{R}^i{}_j(\bar{t})$ is nonvanishing, it is impossible to find any function $\alpha(\bar{t}, \bar{x}^j)$ which satisfies this equation, since the left-hand side is not a pure gradient. Therefore we conclude that the rotation matrix is time independent, and we choose the new coordinate system $\bar{x}^i$ so that $R^i{}_j = \delta^i_j$. We can now solve Eq. (B8) for the function α, which gives

$$\alpha(\bar{t}, \bar{x}^j) = \alpha_c(\bar{t}) + \dot{z}^i(\bar{t})\bar{x}^i, \tag{B9}$$

where $\alpha_c(\bar{t})$ is an arbitrary function of $\bar{t}$, cf. Eq. (2.18a) above.

To summarize, the coordinate transformation to Newtonian order is given by

$$\begin{aligned} x^i(\bar{t}, \bar{x}^j) &= \bar{x}^i + z^i(\bar{t}) + O(\varepsilon^2), \\ t(\bar{t}, \bar{x}^j) &= \bar{t} + \varepsilon^2\alpha(\bar{t}, \bar{x}^j) + O(\varepsilon^4), \end{aligned} \tag{B10}$$

where α is given by Eq. (B9). The transformation law for the Newtonian potential Φ can now be obtained by substituting Eqs. (B10) into the metric expansion (2.4) and comparing the time-time piece with the metric expansion (2.16); the result is given by Eq. (2.20a).

We now turn to the post-Newtonian extension of this computation. We assume that the coordinate transformation can be written as the Newtonian-order coordinate transformation (B10) plus arbitrary post-Newtonian correction terms:

$$\begin{aligned} x^i &= \bar{x}^i + z^i(\bar{t}) + \varepsilon^2 h^i(\bar{t}, \bar{x}^j) + O(\varepsilon^4), \\ t &= \bar{t} + \varepsilon^2\alpha(\bar{t}, \bar{x}^j) + \varepsilon^4\beta(\bar{t}, \bar{x}^j) + O(\varepsilon^6). \end{aligned} \tag{B11}$$

Here the functions $h^i(\bar{t}, \bar{x}^j)$ and $\beta(\bar{t}, \bar{x}^j)$ are arbitrary. As before we can compute the transformed metric by combining the coordinate transformation (B11) with the metric expansion (2.4). The resulting leading order expression for the spatial metric is

$$\{\delta_{ij} + \varepsilon^2[-2\hat{\Phi}\delta_{ij} - \dot{z}_i\dot{z}_j + h_{i,j} + h_{j,i}]\}d\bar{x}^i d\bar{x}^j, \tag{B12}$$

where we are using the notation (2.21). Comparing this with the metric expansion (2.16) and using Eqs. (2.18a) and (2.20a) gives the following differential equation for $h^i(\bar{t}, \bar{x}^j)$:

$$h_{i,j} + h_{j,i} = -2\delta_{ij}\left[\ddot{z}_k\bar{x}^k + \dot{\alpha}_c - \frac{1}{2}\dot{\mathbf{z}}^2\right] + \dot{z}_i\dot{z}_j. \tag{B13}$$

The general solution to this equation consists of a homogeneous solution plus an inhomogeneous solution. The homogeneous solution is just the general Killing vector of three-dimensional Euclidean space, which is

$$h^i(\bar{t}, \bar{x}^j) = h^i_c(\bar{t}) + \epsilon^{ijk}\bar{x}_j R_k(\bar{t}), \tag{B14}$$

where the functions $h^i_c(\bar{t})$ and $R_k(\bar{t})$ are arbitrary. The full solution that we will use for h^i is the sum of (B14) and the inhomogeneous solution, which can be obtained by inspection. The result is [cf. Eq. (2.18b) above]

$$\begin{aligned} h^i(\bar{t}, \bar{x}^j) = {}& h^i_c(\bar{t}) + \epsilon^{ijk}\bar{x}_j R_k(\bar{t}) + \frac{1}{2}\ddot{z}^i(\bar{t})\bar{x}_j\bar{x}^j - \bar{x}^i\dot{\alpha}_c(\bar{t}) \\ & - \bar{x}^i\bar{x}_j\ddot{z}^j(\bar{t}) + \frac{1}{2}\bar{x}^i\dot{z}_j(\bar{t})\dot{z}^j(\bar{t}) + \frac{1}{2}\dot{z}^i(\bar{t})\dot{z}^j(\bar{t})\bar{x}_j. \end{aligned} \tag{B15}$$

Next, we use Eqs. (2.4), (2.16), (B11), and (B15) to compute the transformed gravitomagnetic potential $\bar{\zeta}^i$. The result is

$$\begin{aligned} \bar{\zeta}_i(\bar{t}, \bar{x}^j) = {}& \hat{\zeta}_i(\bar{t}, \bar{x}^j) - \frac{1}{2}\ddot{z}^i(\bar{t})\dot{z}_j(\bar{t})\bar{x}^j + \frac{1}{2}\dddot{z}^i(\bar{t})\bar{x}_j\bar{x}^j \\ & + \bar{x}^i[2\dot{z}^j(\bar{t})\ddot{z}_j(\bar{t}) - \bar{x}^j\dddot{z}_j(\bar{t}) - \ddot{\alpha}_c(\bar{t})] \\ & - \dot{z}^i(\bar{t})\left[4\hat{\Phi}(\bar{t}, \bar{x}^j) + 2\dot{\alpha}_c(\bar{t}) + \frac{3}{2}x^j\ddot{z}_j(\bar{t})\right. \\ & \left. - \dot{z}^j(\bar{t})\dot{z}_j(\bar{t})\right] - \frac{\partial\beta}{\partial\bar{x}^i}(\bar{t}, \bar{x}^j) + \dot{h}^i_c(\bar{t}) \\ & + \epsilon_{ijk}\bar{x}^j\dot{R}^k(\bar{t}) + \epsilon_{ijk}R^j(\bar{t})\dot{z}^k(\bar{t}). \end{aligned} \tag{B16}$$

Combining this with the expression (2.20a) for the transformed Newtonian potential, and using the harmonic gauge condition (2.6) applied to both the original and barred coordinate systems gives the differential equation

$$\bar{\nabla}^2\beta = \dddot{z}_j(\bar{t})\bar{x}^j + \ddot{\alpha}_c(\bar{t}). \tag{B17}$$

The general solution to this equation is

$$\beta(\bar{t}, \bar{x}^j) = \beta_h(\bar{t}, \bar{x}^j) + \left[\frac{1}{10}\dddot{z}_k(\bar{t})\bar{x}^k + \frac{1}{6}\ddot{\alpha}_c(\bar{t})\right]\bar{x}_j\bar{x}^j, \tag{B18}$$

where β_h is an arbitrary harmonic function, cf. Eq. (2.18c) above. This completes the derivation.

APPENDIX C: PIECE OF SURFACE INTEGRAL THAT DEPENDS LINEARLY ON MOMENTS

In this appendix we compute explicitly the piece of the surface integral (4.43) that depends linearly on the multipole, tidal and gauge moments. That linear piece appears on the right-hand side of Eq. (4.44) as the function $\mathcal{G}_i({}^{\rm n}\dddot{M}_L, {}^{\rm n}\ddot{G}_L, \ddot{H}_L, \dddot{S}_L, \dddot{\mu}_L, \dddot{\nu}_L; R)$. We will show that this function vanishes.

We start by noting that the splitting of the surface integral (4.43) into pieces that are linear in the moments and pieces that are quadratic in the moments is unambiguous for all the multipole and tidal moments, except for the Newtonian mass dipole ${}^{\rm n}M_i(t)$. That mass dipole is constrained by the Newtonian equation of motion (4.2b), and therefore a term proportional to the fourth time derivative of ${}^{\rm n}M_i$ could be reexpressed as a quadratic expression in the moments ${}^{\rm n}M_L$ and ${}^{\rm n}G_L$ and their time derivatives up to second order. We resolve this ambiguity by demanding that there be no dependence on ${}^{\rm n}M_i$ in $\mathcal{G}_i$; the relevant term if present can be reexpressed as a quadratic expression and moved into the function $\mathcal{F}_i$.

To compute the linear piece of the surface integral, we simply drop all the quadratic source terms in the post-2-Newtonian field equations and gauge conditions (4.13), (4.14), (4.15), and (4.16). We also drop the term ${}^{\rm pn}\mathcal{T}^{ij}$ in Eq. (4.43), since the expression (4.22) for ${}^{\rm pn}\mathcal{T}^{ij}$ is explicitly quadratic. We also assume without loss of generality that ${}^{\rm n}M_i = 0$, for the reason discussed above. This yields from Eqs. (4.43) and (4.44) the set of equations

$$\mathcal{G}_i = -\frac{1}{16\pi}\oint[\partial_j\dot{\xi}^i_0 + \ddot{\chi}^{ij}_0]d^2\Sigma^j, \qquad \text{(C1)}$$

where

$$\partial_j\chi^{ij}_0 = -\dot{\zeta}^i_0, \qquad \partial_i\xi^i_0 - \dot{\chi}^{kk}_0 = -4\dot{\psi}_0, \qquad \text{(C2)}$$

$$\nabla^2\xi^i_0 = \ddot{\zeta}^i_0, \nabla^2\chi^{ij}_0 = 0. \qquad \text{(C3)}$$

Here the subscripts 0 indicate that the post-1-Newtonian mass dipole associated with the potentials $(\Phi_0, \zeta^i_0, \psi_0)$ vanishes, cf. the discussion in Sec. IV E above.

To compute the function $\mathcal{G}_i$,[31] we can pick any solution of the post-2-Newtonian Eqs. (C2) and (C3), since we showed in Sec.IV E that the result is independent of which solution is chosen. A particular solution (ξ^i_1, χ^{ij}_1) of Eqs. (C3) can be obtained using the expansion (3.5c) of the gravitomagnetic potential ζ^i. This gives

$$\chi^{ij}_1 = 0, \qquad \text{(C4)}$$

$$\xi^i_1 = \sum_{l=0}^{\infty}\left[\frac{(-1)^{l+1}}{2l!}\ddot{Z}_{iL}\partial_L|\boldsymbol{x}| - \frac{|\boldsymbol{x}|^2}{2(2l+3)l!}\ddot{Y}_{iL}x^L\right]. \qquad \text{(C5)}$$

These potentials do not satisfy the gauge conditions (C2), but we can fix this by adding appropriately chosen solutions of Laplace's equation. Thus, we define

$$\xi^i_0 = \xi^i_1 + \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}X_{iL}\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}W_{iL}x^L, \qquad \text{(C6)}$$

$$\chi^{ij}_0 = \chi^{ij}_1 + \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}C_{ijL}\partial_L\frac{1}{|\boldsymbol{x}|} - \frac{1}{l!}B_{ijL}x^L. \qquad \text{(C7)}$$

Inserting Eqs. (C4)–(C7) into the formula (C1) gives

$$\mathcal{G}_i = -\frac{1}{12}R^3\ddot{\nu}_i + \frac{1}{4}X_i - \frac{1}{12}\dot{C}_{ijj} - \frac{R^3}{12}\dot{B}_{ijj}. \qquad \text{(C8)}$$

To obtain the moments X_i, $\dot{C}_{ijj}$, and $\dot{B}_{ijj}$ we substitute Eqs. (C4)–(C7) into the gauge conditions (C2). A useful intermediate result is

$$\partial_i\xi^i_1 = -\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\left[2\ddot{M}^{\rm n}{}_L\partial_L|\boldsymbol{x}| + \frac{\ddot{\mu}_L}{2l+3}\partial_L\frac{1}{|\boldsymbol{x}|}\right] + \sum_{l=0}^{\infty}\frac{x^L}{(2l+3)l!}[2|\boldsymbol{x}|^2\ddot{G}^{\rm n}{}_L - \ddot{\nu}_{iL}x^i]. \qquad \text{(C9)}$$

This gives $B_{ijj} = -\dot{\nu}_i$ and $X_i - \dot{C}_{ijj}/3 = 0$, yielding from Eq. (C8) that $\mathcal{G}_i = 0$.

APPENDIX D: LAW OF MOTION FOR A SINGLE BODY WITH WEAK SELF-GRAVITY

In this appendix we sketch briefly the DSX derivation [16] of Eq. (4.3b), translated into our notation. This equation is obtained by direct computation of the second time derivative of the mass dipole, making use of the stress-energy conservation law in the interior of the body. In our notation, the total mass dipole is [cf. Eqs. (3.2) and (3.24) above]

$${}^{\rm n}M_i + \varepsilon^2\,{}^{\rm pn}M_i \equiv \int_{r<r_-}\left[x^i\,{}^{\rm n}T^{00} + \varepsilon^2\left(x^i\,{}^{\rm pn}T^{00} + x^i\,{}^{\rm n}T^{jj} + \frac{x^i|\boldsymbol{x}|^2}{6}\frac{\partial^2\,{}^{\rm n}T^{00}}{\partial t^2} - \frac{6x^{\langle ij\rangle}}{5}\frac{\partial\,{}^{\rm n}T^{0j}}{\partial t}\right)\right]d^3x. \qquad \text{(D1)}$$

The conservation equations $\nabla_\mu T^{\mu\nu}$ can be written using the expansions (2.2) and (2.4) in the form [16]

$$\frac{\partial}{\partial t}({}^{\rm n}T^{00} + \varepsilon^2\,{}^{\rm pn}T^{00}) + \frac{\partial}{\partial x^j}({}^{\rm n}T^{0j} + \varepsilon^2\,{}^{\rm pn}T^{0j}) = \varepsilon^2\,{}^{\rm n}T^{00}\frac{\partial\Phi}{\partial t} + O(\varepsilon^4), \qquad \text{(D2)}$$

$$\begin{aligned}&\frac{\partial}{\partial t}[(1-4\varepsilon^2\Phi)({}^{\rm n}T^{0i} + \varepsilon^2\,{}^{\rm pn}T^{0i})] + \frac{\partial}{\partial x^j}[(1-4\varepsilon^2\Phi)({}^{\rm n}T^{ij} + \varepsilon^2\,{}^{\rm pn}T^{ij})]\\ &= -[{}^{\rm n}T^{00} + \varepsilon^2({}^{\rm pn}T^{00} + {}^{\rm n}T^{kk})]\left[\varepsilon^2\frac{\partial}{\partial t}\zeta_i + \frac{\partial}{\partial x^i}(\Phi + \varepsilon^2\psi)\right] - \varepsilon^2\,{}^{\rm n}T^{0j}\left(\frac{\partial}{\partial x^j}\zeta_i - \frac{\partial}{\partial x^i}\zeta_j\right) + O(\varepsilon^4).\end{aligned} \qquad \text{(D3)}$$

These conservation equations can be used to evaluate

[31] One might think that the easiest way to evaluate the expression (C1) for $\mathcal{G}_i$ is to use Gauss' theorem to convert the surface integral to a volume integral. However, this strategy does not work: Because the fields are only defined on the domain $r_0 < r < r_1$ one obtains a surface term at $r = r_0$ in addition to the volume term. It is impossible to extend the definitions of the fields smoothly all the way to $r = 0$, so one is always forced to evaluate a surface term.

explicitly the time derivatives appearing in Eq. (D1). Some algebra leads to the following expression:

$$ {}^{\rm n}M_i + \varepsilon^2\, {}^{\rm pn}M_i = \int_{r<r_-} d^3x \left[x^i({}^{\rm n}T^{00} + \varepsilon^2\, {}^{\rm pn}T^{00}) + \varepsilon^2\, {}^{\rm n}T^{00}\left(x^i x^j \frac{\partial \Phi}{\partial x^j} - \frac{|\boldsymbol{x}|^2}{2}\frac{\partial \Phi}{\partial x^i}\right)\right]. \tag{D4}$$

Taking two time derivatives of this expression and using the conservation Eqs. (D2) and (D3) gives

$$ \begin{aligned} \frac{d^2}{dt^2}({}^{\rm n}M_i + \varepsilon^2\, {}^{\rm pn}M_i) = &- \int_{r<r_-}\left\{[{}^{\rm n}T^{00} + \varepsilon^2({}^{\rm pn}T^{00} + {}^{\rm n}T^{kk})]\left[\varepsilon^2 \frac{\partial}{\partial t}\zeta_i + \frac{\partial}{\partial x^i}(\Phi + \varepsilon^2 \psi)\right] + \varepsilon^2\, {}^{\rm n}T^{0j}\left(\frac{\partial}{\partial x^j}\zeta_i - \frac{\partial}{\partial x^i}\zeta_j\right)\right\} d^3x \\ &+ \varepsilon^2 \frac{d}{dt}\int_{r<r_-}\left(4\Phi\, {}^{\rm n}T^{0i} + x^i\, {}^{\rm n}T^{00}\frac{\partial\Phi}{\partial t}\right) d^3x + \varepsilon^2 \frac{d^2}{dt^2}\int_{r<r_-}\left(x^i x^j - \frac{1}{2}x^k x^k \delta_{ij}\right){}^{\rm n}T^{00}\frac{\partial \Phi}{\partial x^j} d^3x. \end{aligned} \tag{D5}$$

We next substitute in explicit expressions for the gravitational potentials in which the intrinsic terms are expressed in terms of integrals over the matter distribution using the field Eqs. (2.8a)–(2.8c):

$$ \begin{aligned} \Phi &= \int_{r<r_-}\frac{{}^{\rm n}T^{00}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' - \sum_{l=0}^{\infty}\frac{1}{l!}\,{}^{\rm n}G_L x^L, \\ \psi &= \int_{r<r_-}\frac{{}^{\rm pn}T^{00}(t,\boldsymbol{x}') + {}^{\rm n}T^{jj}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' + \frac{d^2}{dt^2} \\ &\quad \times \int_{r<r_-}{}^{\rm n}T^{00}(t,\boldsymbol{x}')\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{2}d^3x' - \sum_{l=0}^{\infty}\frac{1}{l!}\left[{}^{\rm pn}G_L \right. \\ &\quad \left. + \frac{|\boldsymbol{x}|^2}{2(2l+3)}\,{}^{\rm n}\ddot{G}_L\right]x^L, \end{aligned} \tag{D6}$$

and

$$ \zeta_i = \int_{r<r_-}\frac{{}^{\rm n}T^{0i}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' - \sum_{l=0}^{\infty}\frac{1}{l!}Y_{iL}x^L. \tag{D7}$$

It is a straightforward exercise to show that all the terms involving double integrals over $\boldsymbol{x}$ and $\boldsymbol{x}'$ in Eq. (D5) cancel out. The laws of motion can thus be obtained by simply substituting the tidal pieces of the gravitational potentials into Eq. (D5). The remaining integrals over $\boldsymbol{x}$ can then be expressed in terms of the moments ${}^{\rm n}M_L$, ${}^{\rm pn}M_L$, and Z_{iL} via the integral definitions (3.2) and (3.24) and

$$ Z_{iL}(t) = 4\int_{r<r_-}{}^{\rm n}T^{0i}(t,x^j)x^{\langle L\rangle}d^3x. \tag{D8}$$

This gives

$$ \begin{aligned} \mathcal{F}_i = \sum_{l=0}^{\infty}\frac{1}{l!}\Big[&{}^{\rm pn}M_L\,{}^{\rm n}G_{iL} + {}^{\rm n}M_L({}^{\rm pn}G_{iL} + \dot{Y}_{iL} - \dot{Y}_{\langle iL\rangle}) \\ &- \frac{(l+2)(2l+1)}{(2l+3)}\,{}^{\rm n}M_{iL}\,{}^{\rm n}\ddot{G}_L - (2l+1)\,{}^{\rm n}\dot{M}_{iL}\,{}^{\rm n}\dot{G}_L \\ &- l\,{}^{\rm n}\ddot{M}_{iL}\,{}^{\rm n}G_L + \frac{(2l+1)}{(l+1)(2l+3)}\dot{\mu}_L\,{}^{\rm n}G_{iL} \\ &+ \frac{1}{2}Z_{jL}Y_{[ij]L} - \dot{Z}_{iL}\,{}^{\rm n}G_L - Z_{iL}\,{}^{\rm n}\dot{G}_L\Big], \end{aligned} \tag{D9}$$

where we have used Eqs. (4.2b), (4.46), and (D5). Using the STF decompositions (3.15) and (3.16) of the moments Z_{iL} and Y_{iL}, it is straightforward to check that Eq. (D9) is equivalent to Eq. (4.3b).

APPENDIX E: FORMULAS FOR MOMENTS IN TERMS OF SURFACE INTEGRALS

In this appendix we show that the various moments are uniquely defined by the expansions (3.5a)–(3.5c), by writing down surface integrals from which the moments can be explicitly computed. From the definitions (3.5a), (3.5b), and (3.8) we obtain

$$ \oint_{\Sigma} n_{\langle L\rangle}\partial_j \Phi d^2\Sigma_j = \frac{1}{(2l+1)R^l}\,{}^{\rm n}M_L - \frac{lR^{l+1}}{(2l+1)!!}\,{}^{\rm n}G_L, \tag{E1}$$

$$ \oint_{\Sigma} n_{\langle L\rangle}\partial_j \zeta_i d^2\Sigma_j = \frac{1}{(2l+1)R^l}Z_{iL} - \frac{lR^{l+1}}{(2l+1)!!}Y_{iL}, \tag{E2}$$

and

$$\oint_\Sigma n_{\langle L\rangle}\partial_j\psi d^2\Sigma_j = \frac{1}{(2l+1)R^l}\,{}^{\rm pn}M_L + \frac{1}{(l+1)(2l+3)R^l}\dot{\mu}_L - \frac{lR^{l+1}}{(2l+1)!!}[{}^{\rm pn}G_L - \dot{\nu}_L] - \frac{l-1}{2(2l-1)(2l+1)R^{l-2}}\,{}^{\rm n}\ddot{M}_L - \frac{(l+2)R^{l+3}}{2(2l+3)!!}\,{}^{\rm n}\ddot{G}_L. \tag{E3}$$

Here $d^2\Sigma_j$ is the natural surface element determined by the flat metric $(dx^1)^2 + (dx^2)^2 + (dx^3)^2$, and the two-surface Σ is the coordinate sphere $r = R$. By evaluating the right-hand sides of these equations at several different values of R, and by using the decompositions (3.15) and (3.16), one can extract explicit expressions for the moments ${}^{\rm n}M_L$, ${}^{\rm n}G_L$ [Eq. (E1)], H_L, S_L, μ_L, ν_L [Eq. (E2)], and ${}^{\rm pn}M_L$, ${}^{\rm pn}G_L$ [Eq. (E3)] in terms of the surface integrals and their time derivatives.

APPENDIX F: COEFFICIENTS OF FINAL EQUATION OF MOTION AFTER SIMPLIFICATION USING SPIN EVOLUTION EQUATION

In paper II [37], the following spin evolution equation is derived

$$\dot{S}_i^A = \sum_{B\neq A}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^k}{k!l!}(2k+2l+1)!!\epsilon_{ijm}\mathcal{M}^A_{jL}\mathcal{M}^B_K \times \frac{n^{BA}_{\langle mKL\rangle}}{r_{BA}^{k+l+2}}. \tag{F1}$$

If we substitute this equation into the full equation of motion (6.11), the following coefficients take the following new values:

$$\begin{aligned}{}^{(3)}\mathcal{D}^{AB}_{ijKL} = \frac{(-1)^k(2k+2l+1)!!}{k!l!\mathcal{M}^A}\Big\{&\mathcal{M}^A_L\Big[-\frac{4}{k+1}v_j^{BA}\dot{\mathcal{M}}^B_{iK} - \mathcal{M}^B_K(4v_i^{BA}v_j^{BA} + v_i^A v_j^B) + \frac{3}{\mathcal{M}^A}(\mathcal{M}^A_{ij}\ddot{\mathcal{M}}^B_K + 2\dot{\mathcal{M}}^A_{ij}\dot{\mathcal{M}}^B_K\\ &+ \ddot{\mathcal{M}}^A_{ij}\mathcal{M}^B_K) + \epsilon_{ijm}\Big(\frac{4(1-\delta_{0k})}{k+2}\dot{S}^B_{mK} - S^A_m\frac{\dot{\mathcal{M}}^B_K}{\mathcal{M}^A}\Big)\Big] + 2\mathcal{M}^A_{iL}\dot{\mathcal{M}}^B_K\Big[\frac{(l+2)(2l+1)}{(2l+3)}v_j^B - (l+1)v_j^A\Big]\\ &+ \dot{\mathcal{M}}^A_L\Big[\epsilon_{ijm}\Big(\frac{4}{k+2}S^B_{mK} - S^A_m\frac{\mathcal{M}^B_K}{\mathcal{M}^A}\Big) + \frac{6}{\mathcal{M}^A}(\mathcal{M}^A_{ij}\dot{\mathcal{M}}^B_K + \dot{\mathcal{M}}^A_{ij}\mathcal{M}^B_K)\Big] + \frac{(2l^2+3l+5)}{(l+1)}v_j^{BA}\dot{\mathcal{M}}^A_{iL}\mathcal{M}^B_K\\ &+ \frac{3}{\mathcal{M}^A}\mathcal{M}^A_{ij}\ddot{\mathcal{M}}^A_L\mathcal{M}^B_K + \frac{4}{l+2}\epsilon_{ijm}[S^A_{mL}\dot{\mathcal{M}}^B_K + (1-\delta_{0l})\dot{S}^A_{mL}\mathcal{M}^B_K]\Big\}\end{aligned} \tag{F2}$$

and

$$\begin{aligned}{}^{(5)}\tilde{\mathcal{D}}^{ABC}_{iKLPQ} = \frac{(-1)^{k+p}(2k+2l+1)!!(2p+2q+1)!!}{k!l!p!q!\mathcal{M}^A}&\times\Big[\frac{(l+2)(2l+1)}{(2l+3)}\mathcal{M}^A_{iL}\mathcal{M}^B_K\frac{\mathcal{M}^B_Q}{\mathcal{M}^B}\mathcal{M}^C_P\\ &+ \frac{4\delta_{0k}}{k+2}\mathcal{M}^A_L\mathcal{M}^B_{iQ}\mathcal{M}^C_P\Big].\end{aligned} \tag{F3}$$

A new three-body term is generated, which contributes to $\tilde{a}_i^{ABC}$. It can be written as

$$\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}{}^{(7)}\tilde{\mathcal{D}}^{ABC}_{jLPQ}\frac{n^{CB}_{\langle iPQ\rangle}}{r_{CB}^{p+q+2}}\frac{n^{BA}_{\langle jL\rangle}}{r_{BA}^{l+2}} \tag{F4}$$

where

$${}^{(7)}\tilde{\mathcal{D}}^{ABC}_{jLPQ} = \frac{(-1)^{p+1}(2l+1)!!(2p+2q+1)!!}{l!p!q!\mathcal{M}^A} \times 2\mathcal{M}^A_L\mathcal{M}^B_{jQ}\mathcal{M}^C_P. \tag{F5}$$

Finally, note that the $\dot{S}^A_m$ terms in the second and last lines of Eq. (6.13c) cancel each other out. Therefore no three-body terms are generated from $\dot{S}^A_m$.

TABLE II. In this table we list, for ease of reference, some of the symbols used in the paper in alphabetical order. We do not list symbols whose meaning is very conventional, or which are used only in the immediate vicinity of where they are introduced. For each item listed, we give a brief description, and also a reference to the equation or section in the text where the symbol first appears, or in the vicinity of which the symbol is first introduced.

Symbol	Meaning	First appears in
g	Superscript appended to a symbol denoting that it is defined with respect to the global coordinate system	(5.11a)
n	Superscript prepended to a symbol denoting the Newtonian piece of a quantity	
pn	Superscript prepended to a symbol denoting the post-1-Newtonian piece of a quantity	
A	Index appended to a symbol indicating that it is associated with the A^{th} body in an N-body system (B and C are used similarly)	Sec. I C
$\alpha(t, x^j)$	Function appearing in gauge transformation which parametrizes Newtonian-order changes in the time variable	(2.17)
$\alpha_c(t)$	Piece of $\alpha(t, x^j)$ that is independent of spatial coordinates	(2.18a)
$\beta(t, x^j)$	Function appearing in gauge transformation which parametrizes post-1-Newtonian changes in the time variable	(2.17)
$\beta_h(t, x^j)$	Piece of $\beta(t, x^j)$ that satisfies Laplace's equation	(2.18c)
χ_{ij}	Symmetric tensor parametrizing the post-2-Newtonian spatial metric	(4.11)
ε	Post-Newtonian dimensionless expansion parameter	(2.4)
$F_L(t)$	Tidal moment of order l parametrizing ψ about a world line	(3.61a)
$\mathfrak{g}^{\mu\nu}$	Tensor density sometimes called the "gothic metric" equal to $\sqrt{-g}g^{\mu\nu}$	(4.6)
${}^{n}G_L(t)$	Newtonian gravitoelectric tidal moment of order l	(3.1)
${}^{pn}G_L(t)$	Post-1-Newtonian gravitoelectric tidal moment of order l	(3.5b)
$G_L(t)$	Total gravitoelectric tidal moment of order l	(3.7)
$h^i(t, x^j)$	Free function in gauge transformation which parametrizes post-1-Newtonian translations	(2.17)
$h_c^i(t)$	Piece of $h^i(t, x^j)$ that is independent of spatial coordinates	(2.18b)
$\mathcal{H}^{\mu\alpha\nu\beta}$	Tensor density appearing in the Landau-Lifshitz formulation of general relativity	(4.4)
$H_L(t)$	Post-1-Newtonian gravitomagnetic tidal moment of order l	(3.5c)
$J_L(t)$	Tidal moment of order l parametrizing ψ about a world line	(3.61a)
K	The multi-index $b_1 b_2 \ldots b_k$	Sec. I F
L	The multi-index $a_1 a_2 \ldots a_l$	Sec. I F
$\lambda_L(t)$	Intrinsic-type multipole moment parametrizing the harmonic gauge-transformation function $\beta_h(t, x^j)$	(3.27)
$\Lambda_L^{\Phi}(t)$	Inertial moments that appear in the transformation law of ${}^{n}G_L(t)$, nonvanishing for $l = 0, 1$ only	(3.34)
$\Lambda_L^{\zeta}(t)$	Inertial moments that appear in the transformation law of $Y_{iL}(t)$, nonvanishing for $l = 1, 2, 3$ only	(3.36b)
$\Lambda_L^{\psi_h}(t)$	Inertial moments that appear in the transformation law of ${}^{pn}G_L(t)$, nonvanishing for $l = 0, 1, 2$ only	(3.41)
${}^{n}M_L(t)$	Newtonian mass multipole moment of order l	(3.1)
${}^{pn}M_L(t)$	Post-1-Newtonian mass multipole moment of order l	(3.5b)
$M_L(t)$	Total mass multipole moment of order l	(3.6)
$\mathcal{M}_L(t)$	Mass multipole moment of order l defined in a body-frame nonrotating with respect to distant stars	(5.35)
$\mu_L(t)$	Intrinsic gauge moment of order l	(3.5b)
N	The multi-index $a_1 a_2 \ldots a_n$	Sec. I F
$N_L(t)$	Intrinsic multipole moment of order l parametrizing ψ about a world line	(3.61a)
$n_j^{BA}(t)$	j^{th} component of a spatial unit vector pointing from the world line of body A at time t to the world line of body B at time t	Sec. I C
$\nu_L(t)$	Tidal gauge moment of order l	(3.5b)
P	The multi-index $c_1 c_2 \ldots c_p$	Sec. I F
P_Σ^i	Momentum enclosed by a surface Σ	(4.8)
$P_L(t)$	Intrinsic multipole moment of order l parametrizing ψ about a world line	(3.61a)
Φ	Newtonian potential	(2.4)
ψ	Post-1-Newtonian correction to the Newtonian potential	(2.4)
Q	The multi-index $d_1 d_2 \ldots d_q$	Sec. I F
$R_k(t)$	Function appearing in gauge transformation which parametrizes post-1-Newtonian rotations	(2.18b)
r_{BA}	Coordinate distance between the center-of-mass world lines of bodies A and B, defined with respect to the flat metric δ_{ij}	Sec. I C
s_A	Time coordinate of a coordinate system adapted to body A	(5.1)
$S_L(t)$	Current multipole moment of order l	(3.5c)
$\mathcal{S}_L(t)$	Current multipole moment of order l defined in a body-frame nonrotating with respect to distant stars	(5.36)
t	Time coordinate of a generic, harmonic, conformally Cartesian coordinate system in Secs. II, III, and IV	(2.4)
t	Time coordinate of a global coordinate system for an N body system in Secs. V and VI	Sec. V A
$T^{\mu\nu}$	Components of the stress-energy tensor	(2.2)
$\mathcal{T}^{\mu\nu}$	Landau-Lifshitz pseudotensor	(4.4)
$\tau_L(t)$	Tidal-type multipole moment parametrizing the harmonic gauge-transformation function $\beta_h(t, x^j)$	(3.27)
$\mathcal{T}_N^p(z)$	Taylor coefficients of the function $\vert z - x\vert$ about $x = 0$	(5.18)
$U_i^j(t)$	Rotation matrix describing the dragging of asymptotic rest frames	(1.3)
v_i^A	Velocity of the Ath body	(6.18)
v_i^{AB}	Relative velocity of bodies A and B	(6.18)
x^i	Spatial coordinates of a generic, harmonic, conformally Cartesian coordinate system in Secs. II, III, and IV	(2.4)
x^i	Spatial coordinates of global coordinate system for an N body system in Secs. V and VI	Sec. V A
ξ_i	i^{th} component of the post-2-Newtonian correction to the gravitomagnetic vector potential	(4.11)
y_A^j	Spatial coordinates of a coordinate system adapted to body A	Sec. I C
$Y_{iL}(t)$	Tidal moments of order l of the gravitomagnetic potential	(3.8)
$z_i(t)$	Free function in gauge transformation which parametrizes Newtonian-order translations	(2.17)
$z_i(t)$	The center-of-mass world line of a body, to Newtonian order	(5.4)
${}^{cm}z_i(t)$	Center-of-mass world line of a body, to post-1-Newtonian order	Sec. V C
Z_{iL}	Intrinsic multipole moments of order l of the gravitomagnetic potential	(3.8)
ζ_i	i^{th} component of the gravitomagnetic vector potential	(2.4)

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