PHYSICAL REVIEW D **71,** 044006 (2005)

"Footballs," conical singularities, and the Liouville equation

Michele Redi

Department of Physics and CCPP, New York University, 4 Washington Place, New York, New York 10003, USA

(Received 4 January 2005; published 7 February 2005)

We generalize the football shaped extra dimensions scenario to an arbitrary number of branes. The problem is related to the solution of the Liouville equation with singularities, and explicit solutions are presented for the case of three branes. The tensions of the branes do not need to be tuned with each other but only satisfy mild global constraints.

DOI: 10.1103/PhysRevD.71.044006 PACS numbers: 04.50.+h, 11.25.Mj

I. INTRODUCTION

It is a notoriously difficult problem to find static solutions of Einstein's equations coupled to brane sources. Exact solutions can sometimes be found in supergravity theories in the Bogomol'nyi-Prasad-Sommerfield limit but little is known for nonsupersymmetric compactifications. Codimension two branes are in this regard special. In the simplest cases, as particles in $2+1$ dimensions [1], the branes do not curve the space outside of the source but only create a deficit angle. The simplified dynamics of gravity then allows to determine many interesting solutions [2–4]. Recently codimension two-brane worlds have also drawn a lot of attention especially in relation to the cosmological constant problem.

In this note we study generalizations of the so-called football shaped extra dimensions scenario [2,3] to include several codimension two branes. Our results also can be repeated almost verbatim for the Supersymmetric Large Extra Dimensions scenario [5], which can be considered as a supersymmetric extension of this model and more in general for product compactifications where the internal space is a sphere (warped compactifications in 6D supergravity have also been considered in [6]). In [2,3], the authors considered a compactification of six-dimensional gravity to Minkowski space times a sphere, obtained by tuning the magnetic flux of a U(1) gauge field through the sphere with the bulk cosmological constant. It was found that by placing *equal* tension branes at the antipodal points of the sphere the internal space is deformed into a sphere with a wedge removed (a "football"). A very interesting feature of this scenario is that the large dimensions remain flat even in the presence of the branes. While the tuning between the tensions can be justified assuming a $\mathbb{Z}_2$ symmetry, certainly this solution appears very special. It is the purpose of this paper to show that these types of solutions are quite generic and *no tuning* between the tensions needs to be invoked when several branes are considered. The mathematical problem consists in solving the Liouville equation with singularities, a topic which appears in 2D quantum gravity. Quite remarkably we will be able to find explicit solutions for the case with three branes, but solutions exist in general. The space so constructed describes a sphere with conical singularities at the brane locations.

This paper is organized as follows: In Sec. II we review our model and generalize it to an arbitrary number of branes and curved background. In Sec. III the problem of determining the metric on the internal space is related to the Liouville equation with singularities. Some background material regarding the solution of the Liouville equation is reviewed in the appendix. In Sec. III A we derive exact solutions for the metric with three branes. In Secs. III B and III C we discuss the case where four or more branes are included and consider the scenario where the internal manifold is a Riemann surface. We derive the low-energy effective action of the model in Sec. IV. In Sec. V we summarize the results.

II. THE MODEL

In this section we review and generalize the scenario introduced in [2,3]. For appropriate values of the parameters this is just a truncation of the SLED scenario. The bulk action is 6D gravity with cosmological constant coupled to a U(1) gauge field,

$$S_6 = M_6^4 \int d^6x\sqrt{-G}\Big(\frac{1}{2}R - \frac{1}{4}F^2 - \lambda\Big). \tag{2.1}$$

The branes are assumed to be minimally coupled and infinitesimal so their action is just the Nambu-Goto action,

$$S_{\text{branes}} = -\sum_{i=1}^{N} T_i \int d^4x\sqrt{-g_i}, \tag{2.2}$$

where g_i is the induced metric on each brane. Thick branes have been considered in [7].

We will be interested in product compactifications of the form $M_4 \times K$ where M_4 is maximally symmetric and K is a compact two-dimensional manifold. The metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \psi(z,\bar z)dzd\bar z, \tag{2.3}$$

where for convenience we have introduced complex coordinates on the internal manifold. The branes are located at points z_i in the internal space. Consistently with the equa-

*Electronic address: redi@physics.nyu.edu

tions of motion it is assumed that the gauge field has a magnetic flux threading the internal space,

$$F = iB_0\psi(z,\bar z)dz \wedge d\bar z, \tag{2.4}$$

where B_0 is a constant. Using the ansatz (2.3) and (2.4) one finds (see [2]),[1]

$$R^4_{\mu\nu} = \frac{1}{2}\left(\lambda - \frac{1}{2}B_0^2\right)g_{\mu\nu}, \tag{2.5}$$

$$\partial_z\partial_{\bar z}\log\psi = -\frac{k}{2}\psi - \sum_{i=1}^{N}\frac{T_i}{M_6^4}\delta(z-z_i,\bar z-\bar z_i), \tag{2.6}$$

where k is the curvature of the internal manifold,

$$k = \frac{\lambda}{2} + \frac{3}{4}B_0^2. \tag{2.7}$$

Looking at (2.5) we note that a very remarkable thing has happened: the four-dimensional metric does not depend on the brane sources. The only effect of the branes in the vacuum is to change the geometry of the internal space without affecting the vacuum energy of the four-dimensional ground state. As pointed out in [8,9], however, this should not lead to easy enthusiasms regarding solutions of the cosmological constant problem. Equation (2.6) is the famous Liouville equation describing a two-dimensional metric of constant curvature k. We will study at length this equation and its solutions in the next section.

Depending on the value of B_0 and λ the four-dimensional ground state will be de Sitter, Anti de Sitter, or Minkowski space,[2]

$$\begin{cases} \lambda > \frac{B_0^2}{2} & \mathrm{dS}_4 \\ \lambda < \frac{B_0^2}{2} & \mathrm{AdS}_4 \\ \lambda = \frac{B_0^2}{2} & \mathrm{M}_4. \end{cases} \tag{2.8}$$

For the Minkowski and de Sitter case one finds that the curvature k of the internal space is positive. In Sec. III C we will also consider the case with negative k where the ground state is AdS. This leads naturally to compactifications on Riemann surfaces.

In [2,3] the authors considered the case of a brane located at $z = 0$. Assuming axial symmetry one readily finds the solution,

$$\psi = \frac{(1-\alpha_1)^2}{k}\frac{4(z\bar z)^{-\alpha_1}}{[1+(z\bar z)^{1-\alpha_1}]^2}, \tag{2.9}$$

where we have defined

$$\alpha_1 = \frac{T_1}{2\pi M_6^4}. \tag{2.10}$$

With a simple change of variables one can see that this is just the metric of a sphere with radius $1/\sqrt{k}$ with a wedge removed, the football. The deficit angle is $2\pi\alpha_1$ so clearly $\alpha_1 < 1$. Physically we will only allow positive tension branes so we also assume $0 < \alpha_1 < 1$.

The solution (2.9) implies the existence of a second brane with exactly the same tension at $z = \infty$ (the north pole of the sphere). In fact, up to reparametrization, this is the only solution (with no warping) with two branes (see appendix). As we shall show the tuning between the tensions can be removed considering three or more branes.

III. LIOUVILLE EQUATION

The mathematical problem of determining the metric on the internal space consists in finding solutions of the Liouville equation with prescribed singularity on the complex plane,

$$\partial_z\partial_{\bar z}\log\psi = -\frac{k}{2}\psi - 2\pi\sum_{i=1}^{N}\alpha_i\delta(z-z_i,\bar z-\bar z_i), \tag{3.1}$$

where the α_i's are related to the tensions as in (2.10). The left-hand side of this equation is proportional to the two-dimensional curvature $\sqrt{\gamma}R_2$ of the internal space. Integrating the Liouville equation and using the Gauss-Bonnet formula for compact surfaces with no boundaries,

$$\frac{1}{4\pi}\int\sqrt{\gamma}R_2 = 2 - 2g \tag{3.2}$$

(where g is the genus of the surface), one derives a simple formula for the volume,

$$V_2 = \frac{2\pi}{k}\left(2 - 2g - \sum_i \alpha_i\right). \tag{3.3}$$

Clearly a compact solution can only exist when $V_2 > 0$.

For the case of negative curvature k this equation has been extensively studied starting with the work of Poincaré and Picard, in particular, in relation to the problem of uniformization of Riemann surfaces. The general result is that a unique solution describing a compact Riemann surface of genus g exists unless it is forbidden by the volume formula (3.3) [11]. Until Sec. III C we will be interested in the positive curvature case which is relevant for the Minkowski background. To the best of our knowledge much less is known in this case. In fact, we will find that an additional constraint on the tensions applies.

Since we only allow positive tension branes, the positivity of the volume forces $g = 0$ and[3]

$$\sum_{i=1}^{N}\alpha_i < 2. \tag{3.4}$$

[1] We use normalizations where $\int d^2z\delta(z,\bar z) = 1$.

[2] This has also been discussed long ago in [10].

[3] In the special case $k = 0$ it is possible to compactify the space on the topology of the sphere but the tensions need to be tuned so that $\sum_i\alpha_i = 2$ [12]. The metric in this case is easily found to be given by $\psi = A\Pi_i|z-z_i|^{-2\alpha_i}$ and the volume remains arbitrary.

Away from the singularities the most general solution of the Liouville equation with positive curvature is given by

$$\psi = \frac{1}{k}\frac{4|w'|^2}{[1+|w|^2]^2}, \tag{3.5}$$

where $w(z)$ is an arbitrary holomorphic function. For the simplest case $w = z$ one recognizes (3.5) as the metric of the stereographically projected sphere.[4] In terms of the Kähler potential the metric can be derived from

$$K = \frac{4}{k}\log[1 + w\bar{w}]. \tag{3.6}$$

Given that in two dimensions,

$$\partial_z\partial_{\bar{z}}\log|z|^2 = 2\pi\delta(z, \bar{z}), \tag{3.7}$$

the Liouville Eq. (3.1) implies the following asymptotic behaviors near the singular points,

$$\begin{aligned} \psi &\sim |z - z_i|^{-2\alpha_i} \quad \text{as } z \to z_i, \\ \psi &\sim |z|^{-2(2-\alpha_\infty)} \quad \text{as } z \to \infty. \end{aligned} \tag{3.8}$$

Integrability of the metric around the singularities then requires

$$\alpha_i < 1. \tag{3.9}$$

This is equivalent to the statement that the deficit angle around each singularity cannot exceed 2π. For $\alpha_i \geq 1$ solutions can still be found but they do not describe compact spaces.

Coming to the main point, the function $w(z)$ reproducing the prescribed singularities can be found using the technology of the Fuchsian equations which we review in the appendix. In brief, given N singularities (z_i, α_i) one considers the Fuchsian equation,

$$\frac{d^2u}{dz^2} + \sum_{i=1}^{N}\left[\frac{\alpha_i(2-\alpha_i)}{4(z-z_i)^2} + \frac{c_i}{2(z-z_i)}\right]u = 0, \tag{3.10}$$

where c_i are known as the accessory parameters. The required function w is then given by

$$w(z) = \frac{u_1(z)}{u_2(z)}, \tag{3.11}$$

where u_1 and u_2 are two linearly independent solutions of (3.10) such that their monodromy around the singular points is contained in SU(2), i.e., u_1 and u_2 are multivalued functions on the complex plane and transform with an SU(2) rotation going around the singularities. To see how this formalism works in practice we now turn to the case with three singularities. In the appendix the solution with two singularities is also derived using the technique of the Fuchsian equations.

[4]A simple physical argument suggests the form of the solution (3.5). Since codimension two objects locally do not curve the space, away from the branes the metric must still be the metric of a sphere. In fact, starting with the metric of the Riemann sphere and performing the change of variables $z \to w(z)$ one obtains (3.5).

A. Solution with three branes

With three branes an explicit solution of the Liouville equation can be found in terms of hypergeometric functions. Using reparametrization invariance it is convenient and conventional to choose the singularities at $(0, 1, \infty)$.[5] The relevant Fuchsian equation is given by

$$\frac{d^2u}{dz^2} + \frac{1}{4}\left[\frac{\alpha_1(2-\alpha_1)}{z^2} + \frac{\alpha_2(2-\alpha_2)}{(z-1)^2} + \frac{\alpha_1(2-\alpha_1) + \alpha_2(2-\alpha_2) - \alpha_\infty(2-\alpha_\infty)}{z(1-z)}\right]u = 0. \tag{3.12}$$

To determine solutions with SU(2) monodromies we follow [13] where the same problem for the case of SU(1,1) monodromies was considered (see also [14] for similar work). Two linearly independent solutions of the previous equation are

$$\begin{aligned} u_1 &= K_1 z^{[1-(\alpha_1/2)]}(1-z)^{(\alpha_2/2)}\tilde{F}[a_1, b_1, c_1, z], \\ u_2 &= K_2 z^{(\alpha_1/2)}(1-z)^{(\alpha_2/2)}\tilde{F}[a_2, b_2, c_2, z], \end{aligned} \tag{3.13}$$

where as in [13] we found it convenient to define modified hypergeometric functions,

$$\tilde{F}[a, b, c, z] = \frac{\Gamma[a]\Gamma[b]}{\Gamma[c]}\,{}_2F_1[a, b, c, z], \tag{3.14}$$

and the indexes are

$$\begin{aligned} a_1 &= \frac{(2-\alpha_1+\alpha_2-\alpha_\infty)}{2}, & a_2 &= \frac{\alpha_1+\alpha_2-\alpha_\infty}{2}, \\ b_1 &= -\frac{(\alpha_1-\alpha_2-\alpha_\infty)}{2}, & b_2 &= \frac{-2+\alpha_1+\alpha_2+\alpha_\infty}{2}, \\ c_1 &= 2-\alpha_1, & c_2 &= \alpha_1. \end{aligned} \tag{3.15}$$

Since the hypergeometric functions are regular at the origin (they have a branch cut between 0 and ∞), the monodromy around $z = 0$ is diagonal,

$$M_0 = \hat{M}(\alpha_1) = \begin{pmatrix} e^{-i\pi\alpha_1} & 0 \\ 0 & e^{i\pi\alpha_1} \end{pmatrix}. \tag{3.16}$$

[5]Notice that the physical position of the singularities does not depend on this choice.

Expanding (3.13) around $z = 1$ one finds,

$$u_i \sim a_{i1}(z-1)^{1-(\alpha_2/2)} + a_{i2}(z-1)^{\alpha_2/2}, \tag{3.17}$$

where,

$$a_{ij} = (A)_{ij} = \begin{pmatrix} K_1\Gamma(\alpha_2 - 1) & K_1 \frac{\Gamma(1-\alpha_2)\Gamma(a_1)\Gamma(b_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)} \\ K_2\Gamma(\alpha_2 - 1) & K_2 \frac{\Gamma(1-\alpha_2)\Gamma(a_2)\Gamma(b_2)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} \end{pmatrix}. \tag{3.18}$$

This allows to compute the monodromy around $z = 1$,

$$M_1 = A\hat{M}(\alpha_2)A^{(-1)} = \begin{pmatrix} \cos\pi\alpha_2 - i\frac{a_{11}a_{22}+a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\sin\pi\alpha_2 & 2i\frac{a_{11}a_{12}}{a_{11}a_{22}-a_{12}a_{21}}\sin\pi\alpha_2 \\ -2i\frac{a_{21}a_{22}}{a_{11}a_{22}-a_{12}a_{21}}\sin\pi\alpha_2 & \cos\pi\alpha_2 + i\frac{a_{11}a_{22}+a_{12}a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\sin\pi\alpha_2 \end{pmatrix}. \tag{3.19}$$

In general this is an SL(2, $\mathbb{C}$) matrix. The condition that the monodromy is contained in SU(2) then boils down to

$$(M_1)^+_{12} = -(M_1)_{21}, \tag{3.20}$$

which determines the ratio $|K_1/K_2|$. A short computation shows

$$\left|\frac{K_1}{K_2}\right|^2 = -\frac{\Gamma[a_2]\Gamma[b_2]\Gamma[c_1-a_1]\Gamma[c_1-b_1]}{\Gamma[a_1]\Gamma[b_1]\Gamma[c_2-a_2]\Gamma[c_2-b_2]} = -\frac{\cos\pi(\alpha_1-\alpha_2) - \cos\pi\alpha_\infty}{\cos\pi(\alpha_1+\alpha_2) - \cos\pi\alpha_\infty}. \tag{3.21}$$

The expression above is not positive definite for each set $(\alpha_1, \alpha_2, \alpha_\infty)$ that satisfies $\sum_i \alpha_i < 2$. Assuming without loss of generality that $\alpha_\infty \geq \alpha_{1,2}$, the requirement that the right-hand side be positive implies the non trivial constraint,

$$\alpha_\infty < \alpha_1 + \alpha_2. \tag{3.22}$$

This is an important result as it is independent from the Gauss-Bonnet formula.[6] One also can check using the formulas in [13] that the monodromy at infinity does not give extra constraints. In general this is a consequence of the fact that

$$\Pi_i M_i = 1. \tag{3.23}$$

Having determined the functions (u_1, u_2) with SU(2) monodromies, the Liouville equation is solved by $w = u_1/u_2$.

In summary we have shown that a solution for the metric of the internal space with three branes exists as long as $\sum_i \alpha_i < 2$ and $\alpha_\infty < \alpha_1 + \alpha_2$. The solution is given in terms of the holomorphic function w,

$$w(z) = \frac{K_1}{K_2}\frac{\tilde{F}[a_1, b_1, c_1, z]}{\tilde{F}[a_2, b_2, c_2, z]} z^{1-\alpha_1}, \tag{3.24}$$

which determines the metric on the Riemann sphere through (3.5). Physically when $\alpha_\infty \to \alpha_1 + \alpha_2$ the proper distance between the point $z = 0$ and $z = 1$ goes to zero. In this limit the solution then reduces to the one with two singularities. In fact the condition (3.22) implies that when only two singularities are present $\alpha_1 = \alpha_\infty$.

B. More branes

When four or more singularities are included the situation becomes immediately much more involved. In principle for N singularities the canonical way to proceed would be to consider the Fuchsian Eq. (3.10). With an SL(2, C) transformation we can again fix the positions of three singularities at (0, 1, ∞) leaving $N - 3$ undetermined. The accessory parameters c_i satisfy three linear equations (see appendix) so one can express c_1, c_2, and c_∞ as linear combinations of $c_3, \ldots, c_{N-1}$. The remaining accessory parameters should then be determined from the requirement that the monodromy of two linearly independent solutions of the Fuchsian Eq. (3.10) belongs to SU(2). Counting the number of equations one sees that the position of $N - 3$ singularities remains unconstrained. In physical terms this means that the physical position of $N > 3$ branes is not fixed; $N - 3$ complex moduli label different vacua. Unfortunately the solution of the Fuchsian equation with more than two singularities (plus the one at infinity) is not known in closed form so we could not find explicit solutions. Some progress in this direction was done in [16] where the problem with three finite singularities and one infinitesimal was solved in the context of SU(1, 1) monodromies. The same methods could be applied here.

Besides the problem of finding exact solutions, it would be important, both from the physical and mathematical point of view, to determine for which values of α_i a solution of the Liouville equation with positive curvature exists and is unique. To the best of our knowledge, contrary to the negative curvature case, this is not known [15]. With no pretense of giving a proof here we notice that from the discussion at the end of the previous paragraph it would seem natural that

[6] This restriction agrees with the result recently found in [15].

$$\alpha_\infty < \sum_{i=1}^{N-1} \alpha_n, \tag{3.25}$$

where we have assumed $\alpha_\infty \geq \alpha_i$. This generalizes the formula with two and three singularities and reduces to it when $N-3$ tensions are taken to zero.

C. Riemann surfaces

We shall now consider compactifications where the internal manifold has negative curvature (similar compactifications of string theory have appeared very recently in [17]). In the model under investigation this corresponds to

$$\lambda < -\frac{3}{2}B_0^2, \tag{3.26}$$

which implies that the four-dimensional ground state is AdS_4. In general, starting from a theory in AdS_{d+3} we could consider compactifications to $AdS_{d+1} \times K$ which might have interest from the point of view of the AdS/conformal field theories correspondence [18].

In absence of singularities, the metric of the internal space is

$$\psi = -\frac{1}{k}\frac{4}{[1-z\bar{z}]^2}, \qquad |z| < 1, \tag{3.27}$$

i.e., the hyperbolic metric on the unit disk D. This manifold is noncompact but we can obtain a compact space considering the coset D/Γ where Γ is an appropriately chosen discrete subgroup of the isometries SU(1,1) that acts without fixed points in D. The space so constructed is a compact Riemann surface of constant negative curvature k and genus g.

Including branes leads again to the Liouville Eq. (3.1) but k is now negative. This is the case most commonly studied in the literature and a wealth of results is available (see [19] and references therein). The general solution of the Liouville equation with negative curvature is given by

$$\psi = -\frac{1}{k}\frac{4|w'|^2}{[1-|w|^2]}. \tag{3.28}$$

The holomorphic function $w(z)$ can in principle be found using techniques similar to the ones described in Sec. III A. According to Picard's theorem (and its generalizations [11]), a solution of the Liouville equation with negative curvature exists and is unique provided that the topological constraint (3.3)

$$\sum_{i=1}^{N} \alpha_i > (2-2g), \tag{3.29}$$

is satisfied. Curiously deficit angles increase the volume when the curvature is negative. Notice that the additional condition $\alpha_\infty < \sum_{i=1}^{N-1}\alpha_i$ that appears when k is positive is automatically satisfied. It should be mentioned that in the negative curvature case the singularities $\alpha_i = 1$ are also allowed. These are called parabolic points and play a special role due to their relation to the uniformization of Riemann surfaces. The asymptotic behavior of the metric is

$$\psi \sim \frac{1}{|z-z_i|^2(\log|z-z_i|)^2} \quad \text{as } z \to z_i. \tag{3.30}$$

The singularity is integrable so that the volume remains finite. The proper distance from the singularity to any point at finite z is however infinite so the space constructed with these singularities is noncompact.

As an example we can consider the case $g = 0$, the so-called hyperbolic sphere. This requires at least three singularities such that $\sum_{i=1}^{3}\alpha_i > 2$. The Fuchsian equation is exactly the same as the one studied in Sec. III A but we need to impose that the monodromies belong to SU(1,1). This requires

$$\left|\frac{K_1}{K_2}\right|^2 = \frac{\cos\pi(\alpha_1-\alpha_2) - \cos\pi\alpha_\infty}{\cos\pi(\alpha_1+\alpha_2) - \cos\pi\alpha_\infty}. \tag{3.31}$$

By inspection it is not difficult to show that the right-hand side of this equation is always positive definite for the allowed values of α_i so that a solution always exists. The function w is again given by (3.24).

IV. EFFECTIVE ACTION

In this section we discuss the low-energy effective action valid at energies smaller than the curvature k.

We start by noting that in absence of branes and for positive curvature the internal space is a sphere whose isometry group is SO(3). Upon Kaluza-Klein (KK) reduction one obtains an unbroken SO(3) gauge theory[7] (for the detailed KK reduction see [10]). In addition to this, from the reduction of the 6D gauge field one also obtains an extra U(1) gauge field which however will not play a role in what follows. Placing equal tension branes at the poles has the effect of removing a wedge from the sphere. This breaks SO(3) → U(1) so that a massless U(1) gauge boson survives. The other two gauge bosons are Higgsed by the presence of the branes. From the low-energy point of view we can understand this as follows. Each brane carries two physical degrees of freedom describing the fluctuations of the brane in the internal space. Two of these degrees of freedom are precisely the Goldstone bosons necessary to implement the breaking SO(3) → U(1) spontaneously. These modes correspond to the overall rotation of the system. In this language choosing the singularities at fixed positions (0, ∞) corresponds to the unitary gauge. The remaining 2 degrees of freedom describe the relative motion of the branes. These modes are massive as the branes repel from each other. When the third brane is added the

[7]As is well known Riemann surfaces do not possess any continuous isometry so there are no massless KK gauge bosons from the metric when the curvature is negative.

original SO(3) symmetry is completely broken. Out of the two new degrees, the one describing the rotation around the axis is eaten by the U(1) gauge boson while the other is massive (this is implied by the fact that the distance between the branes is fixed in the vacuum). Adding more branes obviously does not change this picture for the gauge bosons but introduces new massless degrees of freedom. As we have seen in Sec. III B, for $N > 3$ the physical positions of the branes is not determined in the vacuum and they will appear as $N - 3$ complex flat directions of the potential in the low-energy effective theory. An interesting object to consider in this case would be the metric on the moduli space. This is related in a deep way to the accessory parameters of the associated Fuchsian equation [19].

For completeness let us now turn to the effective action for the breathing mode of the internal manifold (see also [8,10,20]). Depending on the values of the parameters this mode might be as heavy as the first KK modes in which case it should be integrated out. It is however important to check that the mass is positive so that the compactification is stable. This is not guaranteed in general. To derive the effective action we consider the following ansatz for the metric

$$ds^2 = \phi^{-2}(x)g_{\mu\nu}(x)dx^\mu dx^\nu + \phi^2(x)\psi(z, \bar{z})dzd\bar{z}. \quad (4.1)$$

Conservation of the flux requires that F remains at its ground state value (2.4). Plugging the ansatz into the action and using the Liouville equation for the background we obtain

$$S_4 = M_6^4 \int \frac{\psi}{2} dzd\bar{z} \int d^4x\sqrt{-g}\left(\frac{R_4}{2} - 2\frac{\partial^\mu\phi\partial_\mu\phi}{\phi^2} - V\right), \quad (4.2)$$

where,

$$V = \frac{\lambda}{\phi^2} - \left(\frac{\lambda}{2} + \frac{3}{4}B_0^2\right)\frac{1}{\phi^4} + \frac{B_0^2}{2\phi^6}. \quad (4.3)$$

By means of the volume formula (3.3) the four-dimensional Planck mass is

$$M_4^2 = M_6^4 V_2 = M_6^4 \frac{2\pi}{k}\left(2 - 2g - \sum_i \alpha_i\right). \quad (4.4)$$

Notice that from the low-energy point of view the only effect of the branes is to change the normalization of the Planck mass. It should be stressed that, as can be seen from (4.2), the KK reduction is consistent so that no tadpoles corrections arise to the classical effective action.

As required the potential has a stationary point at $\phi = 1$ which corresponds to dS, AdS, or Minkowski space according to (2.8). The mass of ϕ is given by

$$m_\phi^2 = \frac{3}{2}B_0^2 - \lambda. \quad (4.5)$$

We conclude that the compactification is stable unless $\lambda > 3/2B_0^2$ which corresponds to dS space (see also [21]). In this case the system will roll to the other stationary point of the potential at $\phi^2 = 3B_0^2/(2\lambda)$.

V. CONCLUSIONS

Let us summarize what we have achieved in this paper. Starting from the football shaped extra dimensions scenario with two equal tension branes [2,3], we have generalized the model to include an arbitrary number of branes. We have also considered the case where the ground state is dS or AdS space and the internal manifold is a Riemann surface. The internal space has constant curvature with conical singularities at the location of the branes. The problem of determining the metric consists in finding a solution of the Liouville equation with singularities, a topic which goes back to Poincaré and Picard. Explicit solutions have been presented for the case of three branes. Most importantly, contrary to the scenario with two branes, the tensions of the branes do not need to be tuned with each other but only satisfy mild constraints. For the case relevant to the Minkowski background, topologically the internal space is a sphere. For three branes (say $T_3 \geq T_2 \geq T_1$) solutions exist when,

$$T_1 + T_2 + T_3 < 4\pi M_6^4, \qquad T_3 < T_1 + T_2, \quad (5.1)$$

where the first condition is a direct consequence of the Gauss-Bonnet theorem while the second has a more mysterious geometrical origin. We conjectured in (3.25) the generalization of this formula to the scenario with an arbitrary number of branes. Finally we have described the low energy effective action for the model. For more than three branes, the positions of the branes are not fixed in the ground state so $N - 3$ complex moduli appear in the low-energy effective theory.

ACKNOWLEDGMENTS

I am grateful to Massimo Porrati for very helpful discussions about the Liouville equation and Riemann surfaces. I would also like to thank Massimo Porrati and especially José Santiago for comments on the manuscript. This work was supported in part by the NSF Grant No. PHY-0245068.

APPENDIX: RIEMANN-HILBERT PROBLEM

The solution of the Liouville equation is closely related to the Riemann-Hilbert problem of determining functions with prescribed monodromies in the complex plane.

In order to solve this problem one introduces the Fuchsian equation,

$$\frac{d^2u}{dz^2} + \sum_{i=1}^{N-1}\left[\frac{\alpha_i(2-\alpha_i)}{4(z-z_i)^2} + \frac{c_i}{2(z-z_i)}\right]u = 0, \quad (A1)$$

where α_i are directly related to the monodromies and c_i are known as the accessory parameters. The condition that infinity is a regular singular point of the Fuchsian equation implies three linear equations on the c_i's,

$$\begin{cases} \sum_{i=1}^{N-1} c_i = 0 \\ \sum_{i=1}^{N-1}[2c_i z_i + \alpha_i(2-\alpha_i)] = \alpha_\infty(2-\alpha_\infty) \\ \sum_{i=1}^{N-1}[c_i z_i^2 + z_i(\alpha_i(2-\alpha_i))] = c_\infty, \end{cases} \tag{A2}$$

so that the c_i are fully determined for $N = 3$. The double poles singularities in (A1) fix the behavior of the solutions near the singular points,

$$u(z) \sim A(z-z_i)^{1-(\alpha_i/2)} + B(z-z_i)^{(\alpha_i/2)}, \tag{A3}$$

from which one can easily derive the monodromies. Given a pair of linearly independent solutions (u_1, u_2), it is easy to see that $w = u_1/u_2$ satisfies the Liouville Eq. (3.1). However, since the monodromy of (u_1, u_2) belongs in general to SL$(2, \mathbb{C})$, the function ψ is not single valued. In order to find a solution well defined on the entire complex plane one needs to require that the monodromies are contained in SU(2) [or SU(1,1) when the curvature is negative]. These conditions determine the accessory parameters c_n as well as (u_1, u_2). Since $w = u_1/u_2$ now transforms as

$$w \to \frac{aw+b}{-\bar{b}w+\bar{a}}, \qquad |a|^2 + |b|^2 = 1, \tag{A4}$$

it leaves (3.5) invariant. Therefore w defines a single valued solution of the Liouville equation on the complex plane.

As a simplest example one can consider the case with two singularities. Using reparametrization invariance these can be chosen at $(0, \infty)$. The constraints (A2) determine the Fuchsian equation to be

$$\frac{d^2u}{dz^2} + \frac{\alpha_1(2-\alpha_1)}{4z^2}u = 0. \tag{A5}$$

Notice that Eqs. (A2) also requires $\alpha_1 = \alpha_\infty$. Two linearly independent solutions are

$$u_1 = z^{1-(\alpha_1/2)} u_2 = z^{\alpha_1/2}. \tag{A6}$$

Since the monodromy of these solutions is obviously contained in SU(2), $w = u_1/u_2$ is a well-defined solution of the Liouville equation. In fact this just reproduces the football solution (2.9). This derivation also shows that there are no other solutions with two branes.

[1] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984); S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).

[2] S. M. Carroll and M. M. Guica, hep-th/0302067.

[3] I. Navarro, J. Cosmol. Astropart. Phys. 09 (2003) 004; Classical Quantum Gravity **20**, 3603 (2003).

[4] M. Gell-Mann and B. Zwiebach, Phys. Lett. **147B**, 111 (1984); V. P. Nair and S. Randjbar-Daemi, hep-th/0408063; S. Randjbar-Daemi and V. Rubakov, J. High Energy Phys. 10 (2004) 054; H. M. Lee and A. Papazoglou, Nucl. Phys. **B705**, 152 (2005); A. Kehagias, Phys. Lett. B **600**, 133 (2004); I. Navarro and J. Santiago, J. High Energy Phys. 04 (2004) 062.

[5] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran, and F. Quevedo, Nucl. Phys. **B680**, 389 (2004); C. P. Burgess, J. Matias, and F. Quevedo, Nucl. Phys. **B706**, 71 (2005).

[6] G. W. Gibbons, R. Guven, and C. N. Pope, Phys. Lett. B **595**, 498 (2004); Y. Aghababaie *et al.*, J. High Energy Phys. 09 (2003) 037; C. P. Burgess, F. Quevedo, G. Tasinato, and I. Zavala, J. High Energy Phys. 11 (2004) 069.

[7] I. Navarro and J. Santiago, hep-th/0411250.

[8] J. Garriga and M. Porrati, J. High Energy Phys. 08 (2004) 028.

[9] J. Vinet and J. M. Cline, Phys. Rev. D **70**, 083514 (2004).

[10] S. Randjbar-Daemi, A. Salam, and J. Strathdee, Nucl. Phys. **B214**, 491 (1983).

[11] M. Troyanov, Trans. Am. Math. Soc. **324**, 793 (1991).

[12] R. Sundrum, Phys. Rev. D **59**, 085010 (1999).

[13] A. Bellini, M. Ciafaloni, and P. Valtancoli, Nucl. Phys. **B462**, 453 (1996).

[14] L. Hadasz and Z. Jaskolski, Nucl. Phys. **B694**, 493 (2004).

[15] A. Eremenko, math.cv/0208025.

[16] P. Menotti and G. Vajente, hep-th/0411003.

[17] A. Saltman and E. Silverstein, hep-th/0411271.

[18] J. M. Maldacena and C. Nunez, Int. J. Mod. Phys. A **16**, 822 (2001).

[19] L. Takhtajan and P. Zograf, math.cv/0112170.

[20] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran, and F. Quevedo, J. High Energy Phys. 03 (2003) 032; P. Wang and X. H. Meng, Phys. Rev. D **70**, 087901 (2004).

[21] I. Navarro and J. Santiago, J. Cosmol. Astropart. Phys. 09 (2004) 005.