Wormholes supported by a phantom energy

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We extend the notion of phantom energy, which is generally accepted for homogeneously distributed matter with $w < -1$ in the universe, on inhomogeneous spherically symmetric spacetime configurations. A spherically symmetric distribution of phantom energy is shown to be able to support the existence of static wormholes. We find an exact solution describing a static spherically symmetric wormhole with phantom energy and show that a spatial distribution of the phantom energy is mainly restricted by the vicinity of the wormhole's throat. The maximal size of the spherical region, surrounding the throat and containing the most part of the phantom energy, depends on the equation-of-state parameter w and cannot exceed some upper limit.

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I. INTRODUCTION

Recent astrophysical observations [1,2] related to distant supernovas, cosmic microwave background, and galaxy clustering all together essentially changed our view on the evolution of the Universe. Now it is generally accepted that the Universe at present is expanding with acceleration. The explanation of such the ''unexpected'' cosmological behavior in the framework of general relativity requires the supposition that a considerable part ($\sim 70\%$) of the Universe consists of a hypothetical *dark energy*: the exotic matter with a positive energy density $\rho > 0$ and a negative pressure $p = w\rho$ with $w < -1/3$. In last few years intensive efforts have been undertaken in modelling the dark energy (see the reviews, [3–8]). A variety of theoretical ideas and models concerning dark energy includes the cosmological constant, quintessence, the Chaplygin gas, modified gravity and scalar-tensor theories, braneworld models, dark energy driven by quantum effects, k-essence, dark energy models with negative potentials, tachyonic scalar fields, scalar fields with a negative kinetic energy, etc.

The most exotic form of dark energy is a *phantom energy* with $w < -1$ [9], for which the weak energy condition is violated. It is worth to note that values $w < -1$ not only are not excluded but even are favored by recent observations [10–13]. The exotic nature of phantom energy reveals itself in a number of unusual cosmological consequences. One of them is a big rip [14], i.e., a final cosmological singularity to which the universe evolves during a finite interval of time. The related thermodynamical properties of a phantom universe are also strange. Such the universe has a negative entropy diverging near the big rip [15] (an important role of quantum effects near the big rip is discussed in [16]), and a negative temperature [17]. Another interesting phenomenon is that all black holes in the phantom universe lose there masses to vanish exactly in the big rip [18].

If one takes seriously the phantom energy existence, one should expect that its exotic nature would reveal itself not only on cosmological scales. In particular, it is well known that the violation of the weak energy condition is a necessary condition for existence of wormholes [19,20]. Morris and Thorne in their seminal paper [19] named the matter being able to support wormholes ''exotic''. Therefore, one can consider the phantom energy as a possible candidate for exotic matter. However, it is necessary to notice that if one tries to realize such the consideration in practice, one will be faced with a serious problem. The point is that the generally accepted notion of dark/phantom energy applies to an homogeneous distribution of matter in a universe. Such the matter is characterized by the only two values: the energy density ρ and the negative homogeneous pressure p related with each other by the equation of state $p = w\rho$ with $w < -1/3$ (notice that the equation-of-state parameter w could generally speaking be variable). At the same time, a wormhole spacetime is inhomogeneous, and so it demands a nonhomogeneously distributed matter. For example a spherically symmetric wormhole needs a material characterizing by two different pressures: radial and transverse. Thus, the question which should be answered is: Can we extend the notion of dark/phantom energy on inhomogeneous spacetime configurations?

In the recent paper [21] we discussed a model including a scalar field with a negative kinetic term (a ghost scalar field), which is often considered as a simple example of phantom energy [22]. In the framework of the model we have obtained an exact time-dependent solution describing a spherically symmetric wormhole in cosmological setting. The wormhole was shown to connect two asymptotically homogeneous, spatially flat universes expanding with acceleration. It is important that the equation of state of the ghost scalar field can be effectively presented as $p = w\rho$, where p is a *radial* pressure, while a transverse pressure p_{tr} is only indirectly connected with the energy density via field equations. Our analysis has revealed that the radial pressure is everywhere and everywhen negative, and out of the wormhole's throat the radial and transverse pressures

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tend quickly to be equal. This feature indicates that out of the throat (i.e., in regions representing two homogeneous spatially flat universes expanding with acceleration) the ghost scalar field behaves effectively as dark energy providing the accelerated expansion of the Universe.

The preceding analysis prompts us a way how to extend the notion of phantom energy on the case of spherically symmetric spacetime configurations. By analogy we will suppose that a spherically symmetric distribution of phantom energy is characterized by the equation of state $p = w\rho$ with $w < -1$, where p is the negative radial pressure, while the transverse pressure p_{tr} is found from field equations. In this paper we will use this approach to study static spherically symmetric wormholes with phantom energy.

The paper is organized as follows. In Sec. II we briefly consider general properties of static spherically symmetric wormholes. In Sec. III we discuss a spherically symmetric distribution of phantom energy and demonstrate that it provides the flareout conditions in the wormhole's throat. An exact solution describing a static spherically symmetric wormhole supported by the phantom energy is constructed and analyzed in detail in Sec. IV. The Sec. V summarizes the results obtained.

II. STATIC SPHERICALLY SYMMETRIC WORMHOLES: BASIC RESULTS

The general metric of a static spherically symmetric Lorentzian wormhole can be written down in Schwarzschild coordinates (t, r, θ, φ) as follows [19,20]:

$$ds^2 = -e^{2f(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2[d\theta^2 + \sin^2\theta d\varphi^2]. \tag{1}$$

The properties of the wormhole geometry dictate some additional requirements for the metric (1), which was in great detail discussed in [19,20]. In particular, we note that (i) the coordinate r runs between $r_0 \le r < +\infty$, where r_0 is the throat radius. In order to cover the whole spacetime one have to use two copies of the coordinate system (1). (ii) The redshift function $f(r)$ must be everywhere finite; this guarantees that no horizons exist in the spacetime. (iii) The shape function $b(r)$ must obey the flareout conditions at the throat $r = r_0$:

$$b(r_0) = r_0, \tag{2}$$

and

$$b'(r_0) < 1. \tag{3}$$

(iv) Out of the throat, i.e., at $r > r_0$, $b(r)$ should satisfy the following inequality:

$$b(r) < r. \tag{4}$$

(v) If one needs an asymptotical flatness of the spacetime geometry one should require the limit

$$b(r)/r \to 0 \quad \text{as} \quad |r| \to \infty. \tag{5}$$

Because of the spherical symmetry the only nonzero components of the stress-energy tensor are $T_0^0 = -\rho(r)$, $T_1^1 = p(r)$, and $T_2^2 = T_3^3 = p_{tr}(r)$, where ρ is the energy density, p is the radial pressure, and p_{tr} is the transverse pressure. The Einstein equations, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, now yield

$$\rho(r) = \frac{b'}{8\pi r^2}; \tag{6}$$

$$p(r) = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\frac{f'}{r}\left(1 - \frac{b}{r}\right)\right]; \tag{7}$$

$$p_{tr}(r) = \frac{1}{8\pi}\left(1 - \frac{b}{r}\right)\left[f'' - \frac{b'r - b}{2r(r-b)}f' + f'^2 + \frac{f'}{r} - \frac{b'r - b}{2r^2(r-b)}\right]. \tag{8}$$

Since the Einstein tensor obeys the identity $G^\alpha_{\beta;\alpha} = 0$ and values of ρ, p and p_{tr} are connected by the conservation law $T^\alpha_{\beta;\alpha} = 0$:

$$p' + f'\rho + \left(f' + \frac{2}{r}\right)p - \frac{2}{r}p_{tr} = 0, \tag{9}$$

the only two equations of the system (6)–(8) are independent. It is convenient to represent them as follows:

$$b' = 8\pi\rho r^2; \tag{10}$$

$$f' = \frac{8\pi p r^3 + b}{2r(r-b)}. \tag{11}$$

The Einstein equations are connecting the geometrical flareout conditions (2) and (3) with a distribution of matter in the wormhole throat. In particular, supposing $b(r_0) = r_0$ in Eq. (7) we find

$$p_0 \equiv p(r_0) = -\frac{1}{8\pi r_0^2}. \tag{12}$$

Thus, the radial pressure at the throat should be negative to prevent it from collapsing. Supposing $b'(r_0) < 1$ in the Eq. (6) we obtain

$$\rho_0 \equiv \rho(r_0) < \frac{1}{8\pi r_0^2}. \tag{13}$$

This inequality imposes a restriction for the value of energy density at the throat.

Out of the throat, Eq. (10) is easily integrated:

$$b(r) = r_0 + \int_{r_0}^{r} 8\pi\rho(\tilde{r})\tilde{r}^2 d\tilde{r}. \tag{14}$$

Here the constant of integration is chosen to provide the condition $b(r_0) = r_0$. Instead of $b(r)$ one may consider the function $m(r) = b(r)/2$ which is the effective mass inside the radius r. The limit $\lim_{r\to\infty} m(r) = M$, if exists, represents the asymptotical wormhole mass seen by an distant observer.

III. SPHERICALLY SYMMETRIC DISTRIBUTION OF PHANTOM ENERGY

In addition to the Einstein equations one has also to specify an equation of state for the matter being a source of gravity. The equation of state describing phantom energy in cosmology is usually taken as $p = w\rho$ where $w < -1$, and p is a negative *spatially homogeneous* pressure. By analogy we will suppose that a spherically symmetric distribution of phantom energy is characterized by the equation of state in the same form, $p = w\rho$, but now p is the negative *radial* pressure, while the transverse pressure p_{tr} is defined by Eq. (9). Denoting, for convenience, $\kappa \equiv -w$ we have hereinafter

$$p = -\kappa\rho \tag{15}$$

with $\kappa > 1$.

An important feature of the spherically symmetric distributed phantom energy is that it is able to provide the flareout conditions in the wormhole's throat. Really, if at the throat it is fulfilled $p_0 = -(8\pi r_0^2)^{-1}$, then

$$\rho_0 = -\frac{p_0}{\kappa} = \frac{1}{8\pi\kappa r_0^2} < \frac{1}{8\pi r_0^2}, \tag{16}$$

and hence both conditions (12) and (13) are satisfied.

In the next section we will demonstrate that the phantom energy actually can support wormholes and present two explicit solutions describing a phantom energy wormhole.

IV. PHANTOM ENERGY WORMHOLES

The system of four Eqs. (9)–(11) and (15) remains still to be incomplete because we have five functions to be determined: f, b, ρ, p, and p_{tr}. To solve this problem one must define one of the functions "by hand". It seems reasonably to specify a certain spatial distribution of the phantom energy density $\rho(r)$. The form of $\rho(r)$ is only restricted by the relation (6) and the conditions (3)–(5). Below we will analyze two examples illustrating a various choice of $\rho(r)$.

A. Phantom energy confined in a bounded spherical region

First consider the simple model

$$\rho(r) = \begin{cases} \rho_0, & r_0 \le r \le r_1 \\ 0, & r > r_1 \end{cases} \tag{17}$$

where ρ_0 is a constant. In this model the phantom energy is confined in the bounded spherical region $r \le r_1$ (the *region I*) including the wormhole's throat, while the region $r > r_1$ (the *region II*) is supposed to be empty, so that $\rho = p = p_{tr} = 0$ there.

Obtain a solution in the region I. For this aim we take into account that $\rho_0 = (8\pi\kappa r_0^2)^{-1}$ (see Eq. (16)). Substituting this value of ρ_0 into (14) and integrating we can find the shape function $b(r)$ in the region I in the following form:

$$\begin{aligned} b_I(r) &= r_0 + \frac{1}{3\kappa r_0^2}(r - r_0)(r^2 + rr_0 + r_0^2) \\ &= r - \frac{1}{3\kappa r_0^2}(r - r_0)(r - r_-)(r_+ - r), \end{aligned} \tag{18}$$

where

$$r_\pm = \frac{r_0}{2}(\pm\sqrt{12\kappa - 3} - 1).$$

Note that the inequality $b_I(r) < r$ would be satisfied for all $r \in [r_0, r_1]$ only if

$$r_1 < r_+ = \frac{r_0}{2}(\sqrt{12\kappa - 3} - 1). \tag{19}$$

It is worth to emphasize that the condition (19) means that the region containing the phantom energy *cannot* be arbitrarily large. Its maximal size does not exceed r_+, which in turn depends on the equation-of-state parameter κ.[1]

Further, substituting the found expression for b_I into (11) we can obtain the following expression for the redshift function $f(r)$:

$$e^{2f_I(r)} = C\Psi(r/r_0), \tag{20}$$

where

$$\Psi(x) = x^{-1}(x_+ - x)^{3\kappa x_-/(1+2x_-)}(x - x_-)^{3\kappa x_+/(1+2x_+)} \tag{21}$$

C is a constant of integration, and $x_\pm = r_\pm/r_0$.

The solution in the region II has the Schwarzschild form:

$$b_{II}(r) = 2M, \quad e^{2f_{II}(r)} = 1 - \frac{2M}{r}, \tag{22}$$

where $M > 0$ is a mass parameter.

The formulas (18), (20), and (22) represent two separate solutions for the regions I and II. To construct a solution in the whole spacetime including both the region I and II we have to suppose the continuity of the metric at the boundary $r = r_1$. Assuming $b_I(r)|_{r_1} = b_{II}(r)|_{r_1}$ yields

$$\begin{aligned} 2M &= r_0 + \frac{1}{3\kappa r_0^2}(r_1 - r_0)(r_1^2 + r_1 r_0 + r_0^2) \\ &= r_1 - \frac{1}{3\kappa r_0^2}(r_1 - r_0)(r_1 - r_-)(r_+ - r_1). \end{aligned} \tag{23}$$

This relation expresses the wormhole mass M via the throat's radius r_0 and the size of region I, r_1. The second condition $f_I(r)|_{r_1} = f_{II}(r)|_{r_1}$ is fixing the value of C as follows

$$C = (1 - \frac{2M}{r_1})\Psi^{-1}(r_1/r_0). \tag{24}$$

[1] This conclusion was made for the radial coordinate r. One may check that the same is true for the proper radial coordinate $l = \pm\int_{r_0}^{r}\frac{d\tilde{r}}{\sqrt{1-b(\tilde{r})/\tilde{r}}}$.

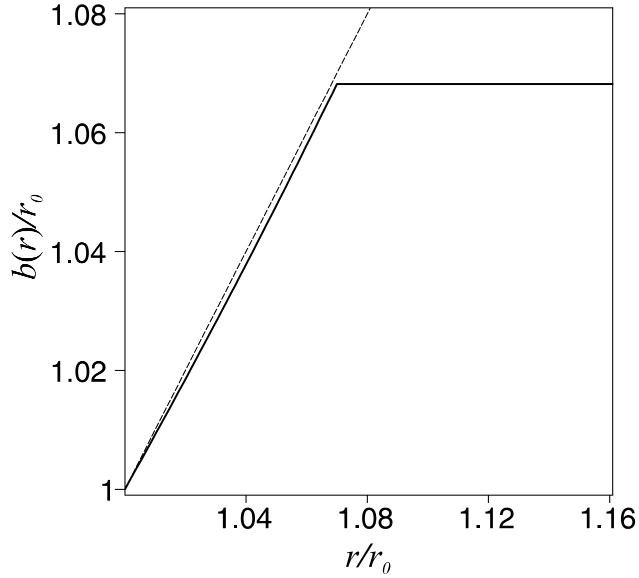

FIG. 1. The diagram represents the solution $b(r)/r_0$ given by the formulas (18), (22), and (23) in case $\kappa = 1.1$. The dashed straight line corresponds to r/r_0.

Now, the functions $b_{I,II}$ and $f_{I,II}$ given by Eqs. (18), (20), and (22) together with the matching conditions (23) and (24) form a solution describing a static spherically symmetric wormhole supported by the phantom energy. In the Figs. 1 and 2 we give the graphical representation of the obtained solution.

B. The smooth phantom energy density distribution

Now let us discuss a smooth distribution of the phantom energy density. For this aim we will model the function $\rho(r)$ using the normal Gaussian distribution law:

$$\rho(r) = \rho_0 e^{-\alpha(r/r_0 - 1)^2}, \tag{25}$$

where $\rho_0 = (8\pi\kappa r_0^2)^{-1}$ is the value of phantom energy density at the throat, and $\alpha > 0$ is a model parameter.

Substituting (25) into (14) and integrating yields

$$\frac{b(r)}{r_0} = 1 + \frac{1}{\kappa\alpha} - (x+1)e^{-\alpha(x-1)^2} + \frac{\sqrt{\pi}}{2\kappa\sqrt{\alpha}}\left[1 + \frac{1}{2\kappa\alpha}\right] \times \mathrm{erf}[\sqrt{\alpha}(x-1)], \tag{26}$$

where $x \equiv r/r_0$, and $\mathrm{erf}(z) = 2\pi^{-1/2}\int_0^z e^{-t^2}dt$ is the error function. The function $b(r)$ given by (26) automatically obeys the flareout conditions (2) and (3), at the wormhole's throat. The additional condition (4), restricting a behavior of $b(r)$ out of the throat, imposes a constraint on the parameter α. A typical behavior of $b(r)$ is illustrated in the Fig. 3. One may see that a curve $b(r)/r_0$ is entirely situated under the straight line r/r_0 in case $\alpha > \alpha_*$, where α_* is some critical value of the parameter α. That is the condition $b(r) < r$ is satisfied for all $r > r_0$, i.e., everywhere out of the throat, if and only if $\alpha > \alpha_*$. This imposes a certain restriction on the spatial distribution of the phantom energy density. Really, the value of α in (25) determines how fast $\rho(r)$ is decreasing. Namely, $\rho(r)$ becomes e

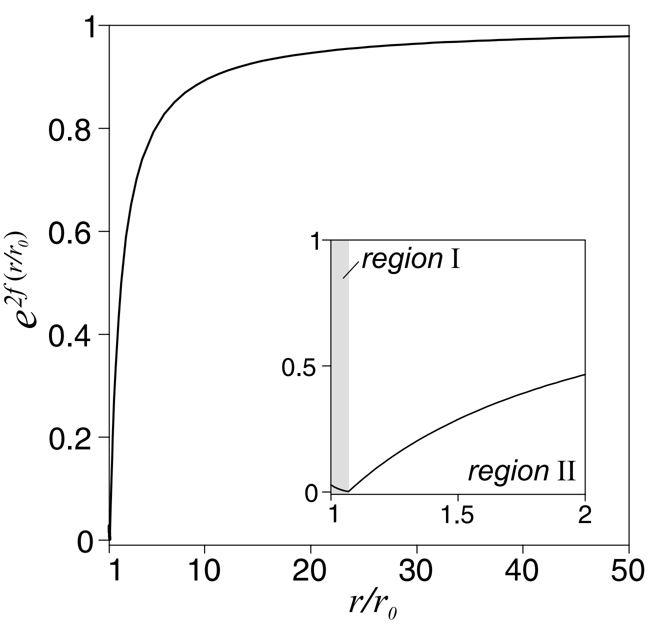

FIG. 2. The graph of $e^{2f(r/r_0)}$ is plotted for $\kappa = 1.1$. The narrow shaded strip marks the region I containing the phantom energy.

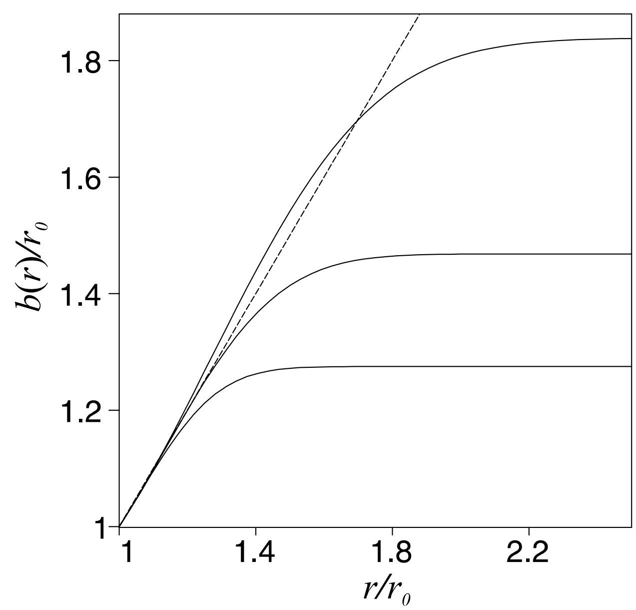

FIG. 3. The diagram represents the solution $b(r)/r_0$ (solid curves) corresponding to the smooth phantom energy density distribution $\rho(r) = \rho_0 e^{-\alpha(r/r_0-1)^2}$ and given by the formula (26) in case $\kappa = 1.1$ and $\alpha = 3$, 6.68, 15 from top to bottom, respectively. The corresponding critical value of α is $\alpha_* \approx 6.68$. The dashed straight line represents r/r_0.

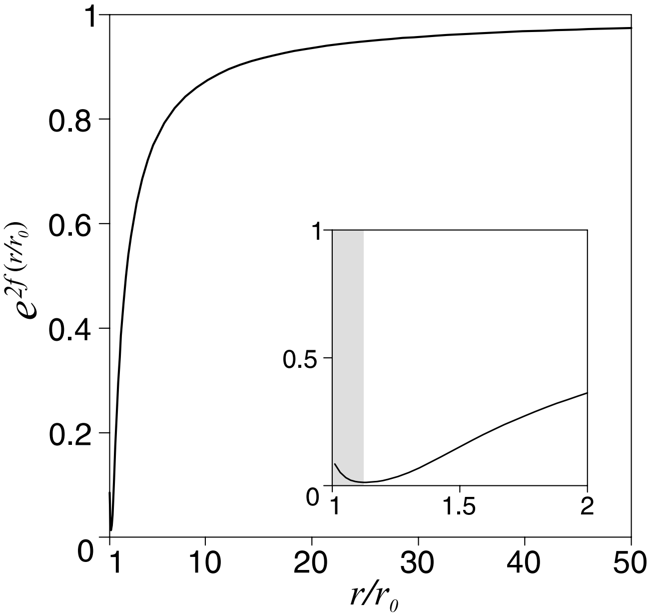

FIG. 4. The graph of $e^{2f(r/r_0)}$ is plotted for $\kappa = 1.1$ and $\alpha = 15$.

times less than ρ_0 at $r_1 = r_0(1 + \alpha^{-1/2})$, and then, for $r > r_1$, the value of $\rho(r)$ is rapidly decreasing. In this sense one can regard the region $r_0 \le r \le r_1$ as that which concentrates the most part of the phantom energy density. The size of this region is proportional to $\alpha^{-1/2}$ and *cannot* exceed some maximal size $\sim\alpha_*^{-1/2}$. (Note that, as follows from numerical analysis, α_* depends ultimately on κ.) The asymptotical wormhole mass M reads

$$M = \lim_{r\to\infty} \frac{1}{2} b(r) = \frac{1}{2} r_0 \left\{ 1 + \frac{1}{\kappa\alpha} + \frac{\sqrt{\pi}}{2\kappa\sqrt{\alpha}} \left[1 + \frac{1}{2\kappa\alpha} \right] \right\}. \tag{27}$$

It is worth noticing that M is positive.

The redshift function $f(r)$ can be found numerically by using Eq. (11). The figure 4 illustrates a typical behavior of $f(r)$.

V. CONCLUDING REMARKS

In this paper we have constructed exact solutions describing a static spherically symmetric wormhole supported by the phantom energy and, thus, explicitly demonstrated that the phantom energy can support the existence of static wormholes. The obtained solutions have revealed an interesting and important feature of the phantom energy wormholes. It turns out that a spatial distribution of the phantom energy is mainly restricted by the vicinity of the wormhole's throat; and the maximal size of the spherical region, surrounding the throat and containing the most part of the phantom energy, cannot exceed some upper limit depending on the equation-of-state parameter κ. Thus, the phantom energy looks like to be confined near the wormhole's throat. In this connection we notice that, since the asymptotical mass M of the phantom energy wormhole is positive, a distant observer could not see a difference (of gravitational nature) between such the wormhole and a compact mass M.

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