Tilted ghost inflation

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In a ghost inflationary scenario, we study the observational consequences of a tilt in the potential of the ghost condensate. We show how the presence of a tilt tends to make contact between the natural predictions of ghost inflation and the ones of slow roll inflation. In the case of positive tilt, we are able to build an inflationary model in which the Hubble constant H is growing with time. We compute the amplitude and the tilt of the two-point function, as well as the three-point function, for both cases of positive and negative tilt. We find that a good fraction of the parameter space of the model is within experimental reach.

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I. INTRODUCTION

Inflation is a very attractive paradigm for the early stage of the universe, being able to solve the flatness, horizon, monopole problems, and providing a mechanism to generate the metric perturbations that we see today in the cosmic microwave background (CMB) [1].

Recently, ghost inflation has been proposed as a new way for producing an epoch of inflation, through a mechanism different from that of slow roll inflation [2,3]. It can be thought of as arising from a derivatively coupled ghost scalar field ϕ which condenses in a background where it has a nonzero velocity:

$$\langle \dot{\phi} \rangle = M^2 \rightarrow \langle \phi \rangle = M^2 t, \quad (1)$$

where we take M^2 to be positive.

Unlike other scalar fields, the velocity $\langle \dot{\phi} \rangle$ does not redshift to zero as the universe expands, but it stays constant, and indeed the energy momentum tensor is identical to that of a cosmological constant. However, the ghost condensate is a physical fluid, and so it has physical fluctuations which can be defined as

$$\phi = M^2 t + \pi. \quad (2)$$

The ghost condensate then gives an alternative way of realizing de Sitter phases in the universe. The symmetries of the theory allow us to construct a systematic and reliable effective Lagrangian for π and gravity at energies lower than the ghost cutoff M. Neglecting the interactions with gravity, around flat space, the effective Lagrangian for π has the form

$$S = \int d^4x \frac{1}{2} \dot{\pi}^2 - \frac{\alpha}{2M^2} (\nabla^2 \pi)^2 - \frac{\beta}{2M^2} \dot{\pi} (\nabla \pi)^2 + \cdots, \quad (3)$$

where α and β are order one coefficients. In [2], it was shown that, in order for the ghost condensate to be able to implement inflation, the shift symmetry of the ghost field ϕ had to be broken. This could be realized adding a potential to the ghost. The observational consequences of the theory were no tilt in the power spectrum, a relevant amount of non-Gaussianities, and the absence of gravitational waves. The non-Gaussianities appeared to be the aspect closest to a possible detection by experiments such as WMAP. Also, the shape of the three-point function of the curvature perturbation ζ was different from the one predicted in standard inflation. In the same paper [2], the authors studied the possibility of adding a small tilt to the ghost potential, and they did some order of magnitude estimate of the consequences in the case that the potential decreases while ϕ increases.

In this paper, we perform a more precise analysis of the observational consequences of a ghost inflation with a tilt in the potential. We study the two-point and three-point functions. In particular, we also imagine that the potential is tilted in such a way that actually the potential increases as the value of ϕ increases with time. This configuration still allows inflation, since the main contribution to the motion of the ghost comes from the condensation of the ghost, which is only slightly affected by the presence of a small tilt in the potential. This provides an inflationary model in which H is growing with time. We study the two-point and three-point functions also in this case.

The paper is organized as follows. In Sec. II, we introduce the concept of the tilt in the ghost potential; in Sec. III we study the case of negative tilt, we compute the two-point and three-point functions, and we determine the region of the parameter space which is not ruled out by observations; in Sec. IV we do the same as we did in Sec. III for the case of positive tilt; in Sec. V we summarize our conclusions.

II. DENSITY PERTURBATIONS

In an inflationary scenario, we are interested in the quantum fluctuations of the π field, which, out of the horizon, become classical fluctuations. In [3], it was shown that, in the case of ghost inflation, in longitudinal gauge, the gravitational potential Φ decays to zero out of the horizon. So, the Bardeen variable is simply

$$\zeta = -\frac{H}{\dot{\phi}}\pi, \tag{4}$$

and is constant on superhorizon scales. It was also shown that the presence of a ghost condensate modifies gravity on a time scale Γ^{-1}, with $\Gamma \sim M^3/M_{Pl}^2$, and on a length scale m^{-1}, with $m \sim M^2/M_{Pl}$. The fact that these two scales are different is not a surprise since the ghost condensate breaks Lorentz symmetry.

Requiring that gravity is not modified today on scales smaller than the present Hubble horizon, we have to impose $\Gamma < H_0$, which implies that gravity is not modified during inflation:

$$\Gamma \ll m \ll H. \tag{5}$$

This is equivalent to the decoupling limit $M_{Pl} \to \infty$, keeping H fixed, which implies that we can study the Lagrangian for π neglecting the metric perturbations.

Now, let us consider the case in which we have a tilt in the potential. Then, the zero mode equation for π becomes

$$\ddot{\pi} + 3H\dot{\pi} + V' = 0, \tag{6}$$

which leads to the solution

$$\dot{\pi} = -\frac{V'}{3H}. \tag{7}$$

We see that this is equivalent to changing the velocity of the ghost field.

In order for the effective field theory to be valid, we need the velocity of π to be much smaller than M^2, so, in agreement with [2], we define the parameter

$$\delta^2 = -\frac{V'}{3HM^2} \quad \text{for } V' < 0, \qquad \delta^2 = +\frac{V'}{3HM^2} \quad \text{for } V' > 0, \tag{8}$$

to be $\delta^2 \ll 1$. We perform the analysis for small tilt, and so at first order in δ^2.

At this point, it is useful to write the 0-0 component of the stress energy tensor for the model of [3]:

$$T_{00} = -M^4 P(X) + 2M^4 P'(X)\dot{\phi}^2 + V(\phi), \tag{9}$$

where $X = \partial_\mu \phi \partial^\mu \phi$. The authors show that the field, with no tilted potential, is attracted to the minimum of the function $P(X)$, such that $P(X_{\min}) = M^2$. So, adding a tilt to the potential can be seen as shifting the ghost field away from the minimum of $P(X)$.

Now, we proceed to study the two-point function and the three-point function for both cases of a positive tilt and a negative tilt.

III. NEGATIVE TILT

Let us analyze the case $V' < 0$.

A. Two-point function

To calculate the spectrum of the π fluctuations, we quantize the field as usual:

$$\pi_k(t) = w_k(t)\hat{a}_k + w_k^*(t)\hat{a}_{-k}^\dagger. \tag{10}$$

The dispersion relation for w_k is

$$\omega_k^2 = \alpha\frac{k^4}{M^2} + \beta\delta^2 k^2. \tag{11}$$

Note, as in [3], that the sign of β is the same as the sign of $\langle\dot{\phi}\rangle = M^2$. In all of this paper, we shall restrict to $\beta \geq 0$, and so the sign of β is fixed.

We see that the tilt introduces a piece proportional to k^2 in the dispersion relation. This is a sign that the role of the tilt is to transform ghost inflation to the standard slow roll inflation. In fact, $\omega^2 \sim k^2$ is the usual dispersion relation for a light field.

Defining $w_k(t) = u_k(t)/a$, and going to conformal time $d\eta = dt/a$, we get the following equation of motion:

$$u_k'' + \left(\beta\delta^2 k^2 + \alpha\frac{k^4 H^2 \eta^2}{M^2} - \frac{2}{\eta^2}\right)u_k = 0. \tag{12}$$

If we were able to solve this differential equation, then we could deduce the power spectrum. But, unfortunately, we are not able to find an exact analytical solution. Anyway, from (12), we can identify two regimes: one in which the term $\sim k^4$ dominates at freezing out, $\omega \sim H$, and one in which it is the term in $\sim k^2$ that dominates at that time. Physically, we know that most of the contribution to the shape of the wave function comes from the time around horizon crossing. So, in order for the tilt to leave a signature on the wave function, we need it to dominate before freezing out. There will be an intermediate regime in which both terms in k^2 and k^4 will be important around horizon crossing, but we decide not to analyze that case as it is not too relevant to our discussion. So, we restrict to

$$\delta^2 \gg \delta_{cr}^2 \equiv \frac{\alpha^{1/2}}{\beta}\frac{H}{M}, \tag{13}$$

where cr stands for *crossing*. In that case, the term in k^2 dominates before freezing out, and we can approximate the differential Eq. (12) to

$$u_k'' + \left(\bar{k}^2 - \frac{2}{\eta^2}\right)u_k = 0, \tag{14}$$

where $\bar{k} = \beta^{1/2}\delta k$. Notice that this is the same differential equation we would get for the slow roll inflation upon replacing k with $\bar{k}$.

Solving with the usual vacuum initial condition, we get

$$w_k = -H\eta\frac{e^{-i\bar{k}\eta}}{2^{1/2}\bar{k}^{1/2}\eta}\left(1 - \frac{i}{\bar{k}\eta}\right), \tag{15}$$

which leads to the power spectrum:

$$P_\pi = \frac{k^3}{2\pi^2}|w_k(\eta \to 0)|^2 = \frac{H^2}{4\pi^2\beta^{3/2}\delta^3}, \tag{16}$$

and, using $\zeta = -\frac{H}{\dot\phi}\pi$,

$$P_\zeta = \frac{H^4}{4\pi^2\beta^{3/2}\delta^3 M^4}. \tag{17}$$

This is the same result as in slow roll inflation, replacing k with $\bar{k}$. Notice that, contrary to standard slow roll inflation, the denominator is not suppressed by slow roll parameters, but by the δ^2 term.

The tilt is given by

$$\begin{aligned} n_s - 1 &\equiv \frac{d\ln(P_\zeta)}{d\ln(k)} = \left(4\frac{d\ln(H)}{d\ln k} - \frac{3}{2}\frac{d\ln\beta}{d\ln k} - \frac{3}{2}\frac{d\ln\delta^2}{d\ln k} - 2\frac{d\ln\dot\phi}{d\ln k}\right)\Bigg|_{k=(aH)/(\beta^{1/2}\delta)} \\ &= \frac{2M^2V'}{HV} + \frac{V''}{H^2}\left[\frac{1}{2\delta^2} + \frac{2}{9} + \frac{4M^4}{V}(1-2P''M^8)\right], \end{aligned} \tag{18}$$

where $k = \frac{aH}{\beta^{1/2}\delta}$ is the momenta at freezing out, and where P and its derivatives are evaluated at $X_{\min}$.

Notice the appearance of the term $\sim\frac{1}{\delta^2}$, which can easily be the dominant piece. Please be reminded that this is valid only for $\delta^2 \gg \delta_{cr}^2$. Notice also that, for the effective field theory to be valid, we need

$$\frac{V'}{3H} \ll M^2, \tag{19}$$

so $\frac{M^2V'}{HV} \ll \frac{M^4}{V}$. This last piece is in general $\ll 1$ if the ghost condensate is present today. In order to get an estimate of the deviation from scale invariance, we can see that the larger contribution comes from the piece in $\sim\frac{V''}{\delta^2H^2}$. From the validity of the effective field theory, we get

$$\begin{aligned} \delta^2M^2H &\cong |V'| \gtrsim |V''|\Delta\phi = |V''|(M^2/H)N_e \\ &\Rightarrow |V''| < \delta^2\frac{H^2}{N_e}, \end{aligned} \tag{20}$$

where N_e is the number of e-foldings to the end of inflation. So, we deduce that the deviation of the tilt can be as large as

$$|n_s - 1| \le \frac{1}{N_e}. \tag{21}$$

This is a different prediction from the exact $n_s = 1$ in usual ghost inflation.

B. Three-point function

Let us come to the computation of the three-point function. The leading interaction term (or the least irrelevant one) is given by [2]

$$L_{\rm int} = -\beta\frac{e^{Ht}}{2M^2}[\dot\pi(\nabla\pi)^2]. \tag{22}$$

Using the formula in [4],

$$\langle\pi_{k_1}(t)\pi_{k2}(t)\pi_{k3}(t)\rangle = -i\int_{t_0}^{t}dt'\left\langle\left[\pi_{k_1}(t)\pi_{k2}(t)\pi_{k3}(t), \int d^3xH_{\rm int}(t')\right]\right\rangle, \tag{23}$$

we get [2]

$$\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle = \frac{i\beta}{M^2}(2\pi)^3\delta^3\left(\sum k_i\right)w_1(0)w_2(0)w_3(0) \times[(\vec{k}_2.\vec{k}_3)I(1,2,3) + \text{cyclic} + \text{c.c.}], \tag{24}$$

where "cyclic" stands for cyclic permutations of the k, and where

$$I(1,2,3) = \int_{-\infty}^{0}\frac{1}{H\eta}w_1^*(\eta)'w_2^*(\eta)w_3^*(\eta), \tag{25}$$

and the integration is performed with the prescription that the oscillating functions inside the horizon become exponentially decreasing as $\eta \to -\infty$.

We can do the approximation of performing the integral with the wave function (15). In fact, the typical behavior of the wave function will be to oscillate inside the horizon, and to be constant outside of it. Since we are performing the integration on a path which exponentially suppresses the wave function when it oscillates, and since in the integrand there is a time derivative which suppresses the contribution when a wave function is constant, we see that the main contribution to the three-point function comes from when the wave functions are around freezing out. Since, in that case, we are guaranteed that the term in k^2 dominates, then we can reliably approximate the wave functions in the integrand with those in (15). Using $\zeta = -\frac{H}{\dot\phi}\pi$, we get

$$\begin{aligned}\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)^3\delta^3\left(\sum k_i\right)\frac{H^8}{4\beta^3\delta^8M^8}\frac{1}{k_t^3\prod_{i=1}^3k_i^3} \\ \times\{k_1^2(\vec{k}_2.\vec{k}_3)[(k_2+k_3)k_t + k_t^2 + 2k_3k_2] \\ + \text{cyclic}\},\end{aligned} \tag{26}$$

where $k_i = |\vec{k}_i|$. Let us define

$$F(k_1, k_2, k_3) = \frac{1}{k_t^3 \prod k_i^3} \{k_1^2 (\vec{k}_2.\vec{k}_3)[(k_2 + k_3)k_t + k_t^2 + 2k_3k_2] + \text{cyclic}\}, \quad (27)$$

which, apart for the δ function, holds the k dependence of the three-point function.

The obtained result agrees with the order of magnitude estimates given in [2]:

$$\frac{\langle\zeta^3\rangle}{(\langle\zeta^2\rangle)^{3/2}} \sim \frac{1}{\delta^8}\left(\frac{H}{M}\right)^8 \frac{1}{[\frac{1}{\delta^3}(\frac{H}{M})^2]^{3/2}} \sim \frac{1}{\delta^{7/2}}\left(\frac{H}{M}\right)^2. \quad (28)$$

The total amount of non-Gaussianities is decreasing with the tilt. This is in agreement with the fact that the tilt makes the ghost inflation model closer to slow roll inflation, where, usually, the total amount of non-Gaussianities is too low to be detectable.

The three-point function we obtained can be better understood if we do the following observation. This function is made up of the sum of three terms, each one obtained on cyclic permutations of the k. Each of these terms can be split into a part which is typical of the interaction and of scale invariance, and the rest which is due to the wave function. For the first cyclic term, we have

$$\text{Interaction} = \frac{(\vec{k}_2.\vec{k}_3)}{k_1 k_2^3 k_3^3}, \quad (29)$$

while the rest, which I will call wave function, is

$$\text{Wave function} = \frac{[(k_2 + k_3)k_t + k_t^2 + 2k_2k_3]}{k_t^3}. \quad (30)$$

The interaction part appears unmodified also in the untilted ghost inflation case, while the wave function part is characteristic of the wave function and changes in the two cases.

Our three-point function can be approximately considered as a function of only two independent variables. The delta function, in fact, eliminates one of the three momenta, imposing the vectorial sum of the three momenta to form a closed triangle. Because of the symmetry of the de Sitter universe, the three-point function is scale invariant, and so we can choose $|\vec{k}_1| = 1$. Using rotation invariance, we can choose $\vec{k}_1 = \hat{e}_1$, and impose $\vec{k}_2$ to lie in the $\hat{e}_1, \hat{e}_2$ plane. So, we have finally reduced the three-point function from being a function of three vectors, to a function of two variables. From this, we can choose to plot the three-point function in terms of $x_i \equiv \frac{k_i}{k_1}$, $i = 1, 2$. The result is shown in Fig. 1. Note that we chose to plot the three-point function with a measure equal to $x_2^2 x_3^2$. The reason for this is that this results in being the natural measure in the case we wish to represent the ratio between the signal associated to the three-point function with respect to the signal associated to the two-point function [5].

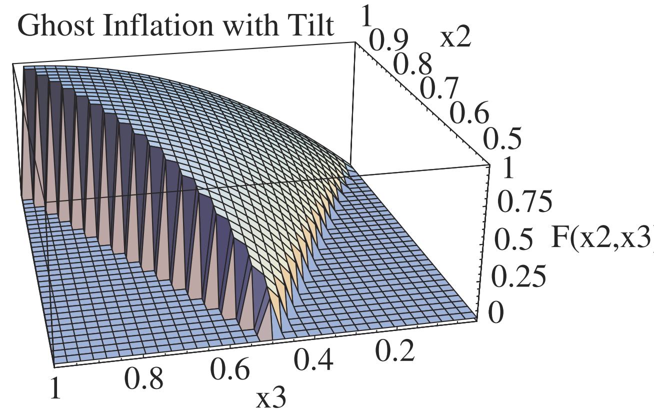

FIG. 1 (color online). Plot of the function $F(1, x_2, x_3)x_2^2x_3^2$ for the tilted ghost inflation three-point function. The function has been normalized to have value 1 for the equilateral configuration $x_2 = x_3 = 1$, and it has been set to zero outside of the region $1 - x_2 \leq x_3 \leq x_2$.

Because of the triangular inequality, which implies $x_3 \leq 1 - x_2$, and in order to avoid doubly representing the same momenta configuration, we set to zero the three-point function outside the triangular region: $1 - x_2 \leq x_3 \leq x_2$. In order to stress the difference with the case of standard ghost inflation, we plot in Fig. 2 the correspondent three-point function for the case of ghost inflation without tilt. Note that, even though the two shapes are quite similar, the three-point function of ghost inflation without tilt changes signs as a function of k, while the three-point function in the tilted case has constant sign.

An important observation is that, in the limit as $x_3 \to 0$ and $x_2 \to 1$, which corresponds to the limit of very long and thin triangles, we find that the three-point function goes to zero as $\sim \frac{1}{x_3}$. This is expected, and in contrast with the usual slow roll inflation result $\sim(1/x_3^3)$. The reason for this is the same as the one which creates the same kind of

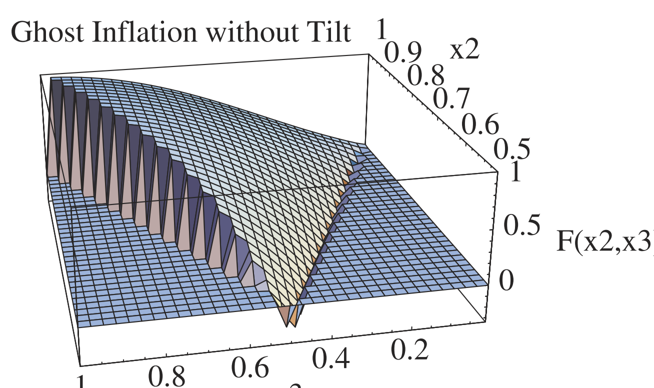

FIG. 2 (color online). Plot of the similarly defined function $F(1, x_2, x_3)x_2^2x_3^2$ for the standard ghost inflation three-point function. The function has been normalized to have value 1 for the equilateral configuration $x_2 = x_3 = 1$, and it has been set to zero outside of the region $1 - x_2 \leq x_3 \leq x_2$ [5].

behavior in the ghost inflation without tilt [2]. The limit of $x_3 \to 0$ corresponds to the physical situation in which the mode k_3 exits from the horizon, freezes out much before the other two, and acts as a sort of background. In this limit, let us imagine a spatial derivative acting on π_3, which is the background in the interaction Lagrangian. The two-point function $\langle \pi_1 \pi_2 \rangle$ depends on the position on the background wave, and, at linear order, will be proportional to $\partial_i \pi_3$. The variation of the two-point function along the π_3 wave is averaged to zero in calculating the three-point function $\langle \pi_{k_1} \pi_{k_2} \pi_{k_3} \rangle$, because the spatial average $\langle \pi_3 \partial_i \pi_3 \rangle$ vanishes. So, we are forced to go to the second order, and we therefore expect to receive a factor of k_3^2, which accounts for the difference with the standard slow roll inflation case. In the model of ghost inflation, the interaction is given by derivative terms, which favors the correlation of modes freezing roughly at the same time, while the correlation is suppressed for modes of very different wavelength. The same situation occurs in standard slow roll inflation when we study non-Gaussianities generated by higher derivative terms [6].

The result is in fact very similar to the one found in [6]. In that case, in fact, the interaction term could be represented as

$$L_{\text{int}} \sim \dot{\varphi}^2 [-\dot{\varphi}^2 + e^{-2Ht} (\partial_i \varphi)^2], \tag{31}$$

where one of the time derivative fields is contracted with the classical solution. This interaction gives rise to a three-point function, which can be recast as

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \sim \left(\frac{[k_1^2 (\vec{k}_2 . \vec{k}_3)]}{\prod_i (k_i^3) k_t^3} [(k_2 + k_3) k_t + k_t^2 + 2 k_2 k_3] + \text{cyclic} \right) + \frac{12}{\prod_i (k_i^3) k_t^3} (k_1^3 + k_2^3 + k_3^3). \tag{32}$$

We can easily see that the first part has the same k dependence as our tilted ghost inflation. That part is in fact due to the interaction with spatial derivative acting, and it is equal to our interaction. The integrand in the formula for the three-point function is also evaluated with the same wave functions, so it gives necessarily the same result as in our case. The other term is due instead to the term with three time derivatives acting. This term is not present in our model because of the spontaneous breaking of Lorentz symmetry, which makes that term more irrelevant than the one with spatial derivatives, as explained in [2]. This similarity could have been expected, because, adding a tilt to the ghost potential, we are converging towards standard slow roll inflation. Besides, since we have a shift symmetry for the ghost field, the interaction term which will generate the non-Gaussianities will be a higher derivative term, as in [6].

We can give a more quantitative estimate of the similarity in the shape between our three-point function and the three-point functions which appear in other models. Following [5], we can define the cosine between two three-point functions $F_1(k_1, k_2, k_3)$, $F_2(k_1, k_2, k_3)$, as

$$\cos(F_1, F_2) = \frac{F_1 \cdot F_2}{(F_1 \cdot F_1)^{1/2} (F_2 \cdot F_2)^{1/2}}, \tag{33}$$

where the scalar product is defined as

$$F_1(k_1, k_2, k_3) \cdot F_2(k_1, k_2, k_3) = \int_{1/2}^{1} dx_2 \int_{1-x_2}^{x_2} dx_3 x_2^4 x_3^4 F_1(1, x_2, x_3) F_2(1, x_2, x_3), \tag{34}$$

where, as before, $x_i = \frac{k_i}{k_1}$. The result is that the cosine between ghost inflation with tilt and ghost inflation without tilt is approximately 0.96, while the cosine with the distribution from slow roll inflation with higher derivatives is practically one. This means that a distinction between ghost inflation with tilt and slow roll inflation with higher derivative terms, just from the analysis of the shape of the three-point function, sounds very difficult. This is not the case for distinguishing from these two models and ghost inflation without tilt.

Finally, we would like to make contact with the work in [7], on the Dirac-Born-Infeld (DBI) inflation. The leading interaction term in DBI inflation is, in fact, of the same kind as the one in (31), with the only difference being the fact that the relative normalization between the term with time derivatives acting and the one with space derivatives acting is weighted by a factor $\gamma^2 = (1 - v_p^2)^{-1}$, where v_p is the gravity-side proper velocity of the brane whose position is the inflaton. This relative different normalization between the two terms is in reality only apparent, since it is canceled by the fact that the dispersion relation is $w \sim \frac{k}{\gamma}$. This implies the relative magnitude of the term with space derivatives acting, and the one of time derivatives acting, are the same, making the shape of the three-point function in DBI inflation exactly equal to the one in slow roll inflation with higher derivative couplings, as found in [6].

C. Observational constraints

We are finally able to find the observational constraints that the negative tilt in the ghost inflation potential implies.

In order to match with the COBE experiment,

$$P_\zeta = \frac{1}{4\pi^2 \beta^{3/2} \delta^3} \left(\frac{H}{M} \right)^4 \cong (4.8 \times 10^{-5})^2 \Rightarrow \frac{H}{M} \cong 0.018 \beta^{3/8} \delta^{3/4}. \tag{35}$$

From this, we can get a condition for the visibility of the tilt. Remembering that $\delta_{cr}^2 = \frac{\alpha^{1/2}}{\beta} (\frac{H}{M})$, we find that, in order for δ to be visible,

$$\delta^2 \gg \delta_{cr}^2 = 0.018 \frac{\alpha^{1/2}\delta^{3/4}}{\beta^{5/8}} \Rightarrow \delta^2 \gg \delta_{\text{visibility}}^2 = 0.0016 \frac{\alpha^{4/5}}{\beta}. \quad (36)$$

In the analysis of the data (see for example [8]), it is usually assumed that the non-Gaussianities come from a field redefinition:

$$\zeta = \zeta_g - \frac{3}{5} f_{NL}(\zeta_g^2 - \langle \zeta_g^2 \rangle), \quad (37)$$

where ζ_g is Gaussian. This pattern of non-Gaussianity, which is local in real space, is characteristic of models in which the nonlinearities develop outside the horizon. This happens for all models in which the fluctuations of an additional light field, different from the inflaton, contribute to the curvature perturbations we observe. In this case the nonlinearities come from the evolution of this field into density perturbations. Both these sources of nonlinearity give non-Gaussianity of the form (37) because they occur outside the horizon. In the data analysis, (37) is taken as an ansatz, and limits are therefore imposed on the scalar variable f_{NL}. The angular dependence of the three-point function in momentum space implied by (37) is given by

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^3 \delta^3\left(\sum_i \vec{k}_i\right)(2\pi)^4\left(-\frac{3}{5} f_{NL} P_R^2\right) \frac{4\sum_i k_i^3}{\prod_i 2k_i^3}. \quad (38)$$

In our case, the angular distribution is much more complicated than in the previous expression, so the comparison is not straightforward. In fact, the cosine between the two distributions is -0.1. We can nevertheless compare the two distributions (26) and (37) for an equilateral configuration, and define in this way an "effective" f_{NL} for $k_1 = k_2 = k_3$. Using COBE normalization, we get

$$f_{NL} = -\frac{0.29}{\delta^2}. \quad (39)$$

The present limit on the non-Gaussianity parameter from the WMAP collaboration [8] gives

$$-58 < f_{NL} < 138 \text{ at } 95\% \text{ C.L.} \quad (40)$$

and it implies

$$\delta^2 > 0.005, \quad (41)$$

which is larger than $\delta_{\text{visibility}}^2$ (which nevertheless depends on the coupling constants α,β).

Since for $\delta^2 \gg \delta_{\text{visibility}}^2$ we do see the effect of the tilt, we conclude that there is a minimum constraint on the tilt: $\delta^2 > 0.005$.

In reality, since the shape of our three-point function is very different from the one which is represented by f_{NL}, it is possible that an analysis done specifically for this shape of non-Gaussianities may lead to an enlargement of the experimental boundaries. As shown in [5], an enlargement of a factor 5–6 can be expected. This would lead to a boundary on δ^2 of the order $\delta^2 \gtrsim 0.001$, which is still in the region of interest for the tilt.

Most important, we can see that future improved measurements of non-Gaussianity in CMB will immediately constrain or verify an important fraction of the parameter space of this model.

Finally, we remind that the tilt can be quite different from the scale invariant result of standard ghost inflation:

$$|n_s - 1| \lesssim \frac{1}{N_e}. \quad (42)$$

IV. POSITIVE TILT

In this section, we study the possibility that the tilt in the potential of the ghost is positive, $V' > 0$. This is quite an unusual condition, if we consider the case of the slow roll inflation. In this case, in fact, the value of H is actually increasing with time. This possibility is allowed by the fact that, on the contrary with respect to what occurs in the slow roll inflation, the motion of the field is not due to an usual potential term, but is due to a spontaneous symmetry breaking of time diffeomorphism, which gives a vacuum expectation value to the velocity of the field. So, if the tilt in the potential is small enough, we expect it to be no big deviance from the ordinary motion of the ghost field, as we already saw in Sec. I.

In reality, there is an important difference with respect to the case of negative tilt: A positive tilt introduces a wrong sign kinetic energy term for π. The dispersion relation, in fact, becomes

$$\omega^2 = \alpha \frac{k^4}{M^2} - \beta\delta^2 k^2. \quad (43)$$

The k^2 term is instable. The situation is not as bad as it may appear, and the reason is the fact that we will consider a de Sitter universe. In fact, deep in the ultraviolet the term in k^4 is going to dominate, giving a stable vacuum well inside the horizon. As momenta are redshifted, the instable term will tend to dominate. However, there is another scale entering the game, which is the freeze-out scale $\omega(k) \sim H$. When this occurs, the evolution of the system is freezed out, and so the presence of the instable term is forgotten.

So, there are two possible situations, which resemble the ones we met for the negative tilt. The first is that the term in k^2 begins to dominate after freezing out. In this situation we would not see the effect of the tilt in the wave function. The second case is when there is a phase between the ultraviolet and the freezing out in which the term in k^2 dominates. In this case, there will be an instable phase, which will make the wave function grow exponentially, until the freezing out time, when this growing will be stopped. We shall explore the phase space allowed for this scenario, which occurs for

$$\delta^2 \gg \delta_{cr}^2 = \frac{\alpha^{1/2}}{\beta} \frac{H}{M}, \tag{44}$$

and we restrict to it.

Before actually beginning the computation, it is worth making an observation. All the computation we are going to do could in principle be obtained from the case of positive tilt, just doing the transformation $\delta^2 \rightarrow -\delta^2$ in all the results we obtained in the former section. Unfortunately, we cannot do this. In fact, in the former case, we imposed that the term in k^2 dominates at freezing out, and then solved the wave equation with the initial ultraviolet vacua defined by the term in k^2, and not by the one in k^4 as, because of adiabaticity, the field remains in the vacua well inside the horizon. On the other hand, in our present case, the term in k^2 does not define a stable vacua inside the horizon, so the proper initial vacua is given by the term in k^4 which dominates well inside the horizon. This leads us to solve the full differential equation:

$$u'' + \left(-\beta\delta^2 k^2 + \alpha \frac{k^4 H^2 \eta^2}{M^2} - \frac{2}{\eta^2}\right) u = 0. \tag{45}$$

Since we are not able to find an analytical solution, we address the problem with the semiclassical WKB approximation. The equation we have is a Schrodinger-like eigenvalue equation, and the effective potential is

$$\tilde{V} = \beta\delta^2 k^2 - \alpha \frac{k^4 H^2 \eta^2}{M^2} + \frac{2}{\eta^2}. \tag{46}$$

Defining

$$\eta_0^2 = \frac{\beta\delta^2 M^2}{\alpha H^2 k^2}, \tag{47}$$

we have the two semiclassical regions: for $\eta \ll \eta_0$, the potential can be approximated to

$$\tilde{V} \cong -\alpha \frac{k^4 H^2 \eta^2}{M^2}, \tag{48}$$

while, for $\eta \gg \eta_0$,

$$\tilde{V} \cong \beta\delta^2 k^2 + \frac{2}{\eta^2}. \tag{49}$$

The semiclassical approximation tells us that the solution, in these regions, is given by, for $\eta \ll \eta_0$,

$$u \cong \frac{A_1}{[p(\eta)]^{1/2}} e^{-i\int_\eta^{\eta_{cr}} p(\eta')d\eta'}, \tag{50}$$

while, for $\eta \gg \eta_0$,

$$u \cong \frac{A_2}{[p(\eta)]^{1/2}} e^{\int_{\eta_{cr}}^{\eta} p(\eta')d\eta'}, \tag{51}$$

where $p(\eta) = [|\tilde{V(\eta)}|]^{1/2}$.

The semiclassical approximation fails for $\eta \sim \eta_0$. In that case, one can match the two solutions using a standard linear approximation for the potential, and get $A_2 = A_1 e^{-i\pi/4}$ [9]. It is easy to see that the semiclassical approximation is valid when $\delta^2 \gg \delta_{cr}^2$.

Let us determine our initial wave function. In the far past, we know that the solution is the one of standard ghost inflation [2]:

$$u = \left(\frac{\pi}{8}\right)^{1/2} (-\eta)^{1/2} H_{3/4}^{(1)}\left(\frac{Hk^2\alpha}{2M}\eta^2\right). \tag{52}$$

We can put this solution, for the remote past, in the semiclassical form, to get

$$u = \frac{1}{[\frac{2H\alpha k^2}{M}(-\eta)]^{1/2}} e^{i[-(5/8)\pi + (\beta\delta^2 M)/(2H)]} e^{i[(Hk^2\alpha)/(2M)]\eta^2}. \tag{53}$$

So, using our relationship between A_1 and A_2, we get, for $\eta \gg \eta_0$, the following wave function for the ghost field:

$$\begin{aligned} w &= u/a \\ &= -\frac{1}{2^{1/2}\bar{k}^{1/2}} e^{i[-(7/8)\pi + (\beta\delta^2 M)/(2H)]} e^{(\beta\delta^2 M)/(\alpha H)} \\ &\quad \times \left(H\eta e^{\bar{k}\eta} + \frac{H}{i\bar{k}} e^{-\bar{k}\eta}\right). \end{aligned} \tag{54}$$

Notice that this is exactly the same wave function we would get if we just rotated $\delta \rightarrow i\delta$ in the solutions we found in the negative tilt case. But the normalization would be very different, in particular, missing the exponential factor, which can be large. It is precisely this exponential factor that reflects the phase of instability in the evolution of the wave function.

From this observation, the results for the two-point and three-point functions are immediately deduced from the case of negative tilt, paying attention to the factors coming from the different normalization constants in the wave function. So, we get

$$P_\zeta = \frac{1}{4\pi^2} \frac{e^{(2\beta\delta^2 M)/(\alpha H)}}{\beta^{3/2}\delta^3} \left(\frac{H}{M}\right)^4. \tag{55}$$

Notice the exponential dependence on α, β, H/M, and δ^2.

The tilt gets modified, but the dominating term $\sim \frac{1}{\delta^2}$ is not modified:

$$n_s - 1 = V'\left(\frac{2M^2}{HV} + \frac{2\pi\beta}{\alpha}\frac{\delta^2 M}{H^2 M_{Pl}^2}\right) + \frac{V''}{H^2}\left[\frac{1}{2\delta^2} + \frac{2}{9} + \frac{4M^4}{V}(1 - 2P''M^8) - \frac{2\beta}{3\alpha}\frac{H}{M} + \frac{\pi}{3}\frac{\delta^2 M^3}{H^3 M_{Pl}^2}(2 - 4P''M^8)\right]. \tag{56}$$

For the three-point function, we get

$$\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (2\pi)^3\delta^3\left(\sum k_i\right)\frac{H^8}{4\beta^3\delta^8 M^8}\frac{1}{k_t^3\prod k_i^3}\{k_1^2(\vec{k}_2.\vec{k}_3) \times [(k_2 + k_3)k_t + k_t^2 + 2k_3k_2] + \text{cyclic}\}e^{6[(\beta\delta^2 M)/(\alpha H)]}, \tag{57}$$

which has the same k dependence as in the former case of negative tilt. Estimating the f_{NL} as in the former case, we get

$$f_{NL} \sim -\frac{0.29}{\delta^2}e^{6[(\beta\delta^2 M)/(\alpha H)]}. \tag{58}$$

Notice again the exponential dependence.

Combining the constraints from the two-point and three-point functions, it is easy to see that a relevant fraction of the parameter space is already ruled out. Anyway, because of the exponential dependence on the parameters $\delta^2, \frac{H}{M}$, and the coupling constants α and β, which allows for big differences in the observable predictions, there are many configurations that are still allowed.

V. CONCLUSIONS

We have presented a detailed analysis of the consequences of adding a small tilt to the potential of ghost inflation.

In the case of negative tilt, we see that the model represents a hybrid between ghost inflation and slow roll inflation. When the tilt is big enough to leave some signature, we see that there are some important observable differences with the original case of ghost inflation. In particular, the tilt of the two-point function of ζ is no more exactly scale invariant $n_s = 1$, which was a strong prediction of ghost inflation. The three-point function is different in shape, and is closer to the one due to higher derivative terms in slow roll inflation. Its total magnitude tends to decrease as the tilt increases. It must be emphasized that the size of these effects for a relevant fraction of the parameter space is well within experimental reach.

In the case of a positive tilt to the potential, thanks to the freezing out mechanism, we are able to make sense of a theory with a wrong sign kinetic term for the fluctuations around the condensate, which would lead to an apparent instability. Consequently, we are able to construct an interesting example of an inflationary model in which H is actually increasing with time. Even though a part of the parameter space is already excluded, the model is not completely ruled out, and experiments such as WMAP and Plank will be able to further constrain the model.

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[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981); A. D. Linde, Phys. Lett. **108B**, 389 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[2] N. Arkani-Hamed, P. Creminelli, S. Mukohyama and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 04 (2004) 001.
[3] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, and S. Mukohyama, J. High Energy Phys. 05 (2004) 074.
[4] J. Maldacena, J. High Energy Phys. 05 (2003) 013. See also V. Acquaviva, N. Bartolo, S. Matarrese, and A. Riotto, Nucl. Phys. **B667**, 119 (2003).
[5] D. Babich, P. Creminelli, and M. Zaldarriaga, J. Cosmol. Astropart. Phys. 08 (2004) 009.
[6] P. Creminelli, On nongaussianities in single-field inflation, J. Cosmol. Astropart. Phys. 10 (2003) 003.
[7] M. Alishahiha, E. Silverstein, and D. Tong, Phys. Rev. D **70**, 123505 (2004).
[8] E. Komatsu *et al.*, Astrophys. J. Suppl. Ser. **148**, 119 (2003).
[9] See, for example, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-relativistic Theory)*, Course in Theoretical Physics (Butterworth-Heinemann, Oxford, United Kingdom, 2000), Vol. III.