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Unified treatment of cosmological perturbations from superhorizon to small scales

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We study the evolution of cosmological perturbations, using a *hybrid* approximation scheme which upgrades the *weak-field* limit of Einstein's field equations to account for *post-Newtonian* scalar and vector metric perturbations and for leading-order source terms of gravitational waves, while including also the first- and second-order perturbative approximations. Our equations, which are derived in the Poisson gauge, provide a unified description of matter inhomogeneities in a Universe filled with a pressureless and irrotational fluid and a cosmological constant, ranging from the linear to the highly nonlinear regime. The derived expressions for scalar, vector and tensor modes may have a wide range of cosmological applications, including secondary CMB anisotropy and polarization effects, cosmographic relations in a inhomogeneous Universe, gravitational lensing and the stochastic gravitational-wave backgrounds generated by nonlinear cosmic structures.

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I. INTRODUCTION

In recent years important results have been obtained in cosmology from several observations of cosmic microwave background (CMB) anisotropy and polarization, weak gravitational lensing effects and large-scale structure by means of galaxy redshift catalogs. The increasing precision that has been, and will be achieved by future experimental determinations requires comparable accuracy in the theoretical estimate of the several contributions to these effects. Accurate study of the evolution of cosmological metric perturbations is therefore crucial for understanding these contributions. Different kinds of techniques have been developed for this analysis, depending on the specific range of scales under consideration. For example, on scales well inside the Hubble horizon, but still much larger than the Schwarzschild radius of collapsing bodies, the study of gravitational instability of collisionless matter is performed using the Newtonian approximation. It consists of inserting in the line element of a Friedmann-Robertson-Walker (FRW) background the lapse perturbation $2\varphi_N/c^2$, where φ_N is related to the matter density fluctuation $\delta\rho$ via the cosmological Poisson equation $\nabla^2\varphi_N = 4\pi G a^2 \delta\rho$. The dynamics of the system is then studied in Eulerian coordinates by accounting for the Newtonian mass and momentum conservation equations, owing to the fact that the peculiar matter flow v never becomes relativistic [1]. The Newtonian limit, according to which the gravitational field φ is always much less than the square of the speed of light, c^2, can be improved by a post-Newtonian (PN) approach to account for the moderately strong gravitational fields generated during collapse. In this case, by considering the expansion of the general relativistic equations in inverse powers of the speed of light, it is possible to neglect terms of order $(v/c)^4$ in the equations of motions, i.e., to perform a first post-Newtonian (1PN) approximation, which in Eulerian coordinates accounts for nonvanishing shift components and for an extra perturbation term $-2\varphi_N/c^2$ in the spatial part of the line element. Calculations using higher and higher orders of $1/c$ would generally lead to a more accurate description of the system, e.g., accounting for the generation of gravitational waves, and possibly allow for an extension of the range of scales to which the formalism can be applied. A PN approach to cosmological perturbations has been followed in Refs. [2–4], using Eulerian coordinates, while Ref. [5] uses Lagrangian coordinates.

On the other hand, the first-order perturbations for nonrelativistic matter, obtained with the Newtonian treatment coincide with the results of linear general relativistic perturbation theory in the so-called longitudinal gauge [6]. The relativistic linear perturbative approach is the one applied to the study of the small inhomogeneities giving rise to large-scale temperature anisotropies of the CMB. However, on small and intermediate scales, linear theory is no longer accurate and a general fully relativistic second-order perturbative technique is required. In fact, second-order metric perturbations determine new contributions to the CMB temperature anisotropy [7,8] and polarization [9]. In particular, second-order scalar, vector and tensor metric perturbations produce secondary anisotropies in the temperature and polarization of the CMB which are in competition with other nonlinear effects, such as that due to weak gravitational lensing produced by matter inhomogeneities, which induces the transformation of *E*-mode into *B*-mode polarization [10].

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Moreover, accounting for second-order effects helps to follow the gravitational instability on a longer time scale and to include new nonlinear and nonlocal phenomena. The pioneering work in this field is due to Tomita [11] who performed a synchronous-gauge calculation of the second-order terms produced by the mildly nonlinear evolution of scalar perturbations in the Einstein-de Sitter Universe. An equivalent result was obtained with a different technique in Ref. [12]. The inclusion of vectors and tensor modes at the linear level, acting as further seeds for the origin of second-order perturbations of any kind was first considered in Ref. [13]; in Ref. [14] the evolution of relativistic perturbations in the Einstein-de Sitter cosmological model was considered and second-order effects were included in two different settings: the synchronous and the Poisson gauge.

As we have stressed, the evolution of cosmological perturbations away from the linear level is rich of several effects as, in particular, mode mixing which not only implies that different Fourier modes influence each other, but also that density perturbations act as a source for curl vector modes and gravitational waves.

The aim of the present paper is therefore to obtain a unified treatment able to follow the evolution of cosmological perturbations from the linear to the highly nonlinear regime. As we will show hereafter, this goal is indeed possible on scales much larger than the Schwarzschild radius of collapsing bodies, by means of a "hybrid approximation" of Einstein's field equations, which mixes PN and second-order perturbative techniques to deal with the perturbations of matter and geometry. In our study we adopt the Poisson gauge [15], which, being the closest to the Eulerian Newtonian gauge, allows a simple physical interpretation of the various perturbation modes. We derive a new set of equations which holds on all cosmologically relevant scales and allows us to describe matter inhomogeneities during all the different stages of their evolution. The new approach gives a more accurate description of the metric perturbations generated by nonlinear structures than the second-order perturbation theory, which can only account for small deviations from the linear regime. For example, on small scales our set of equations can be used to provide a PN description of metric perturbations generated by highly nonlinear structures within dark-matter halos, while describing their motion by means of the standard Newtonian hydrodynamical equations. On large scales our equations converge to the first- and second-order perturbative equations as obtained in Ref. [14] (see also Refs. [16,17]), which implies that they can be applied to every kind of cosmological sources.

The plan of the paper is as follows. In Sec. II we obtain metric perturbations in the Poisson gauge according to our hybrid approximation. In Sec. III we obtain the source terms for scalar, vector and tensor metric perturbations and the evolution equations for the matter density and peculiar velocity. Section IV is devoted to a comparison of our approach with known approximation schemes, such as the standard Newtonian one, the linear and the second-order relativistic perturbative approaches, in the appropriate regimes of applicability. Moreover, we derive a PN approximation to describe the highly nonlinear regime on small scales. In Sec. V we show that the PN expressions for vector and tensor modes actually hold on all cosmological scales. In Sec. VI we sketch some cosmological applications of our approach and we draw our main conclusions. An Appendix is devoted to the solution of the inhomogeneous gravitational-wave equation.

II. SCALAR, VECTOR AND TENSOR METRIC PERTURBATION MODES

Adopting the conformal time η, the perturbed line element around a spatially flat FRW background in the Poisson gauge [6,14] takes the form

$$ds^2 = a^2(\eta)\{-(1+2\phi)d\eta^2 - 2V_\alpha d\eta dx^\alpha + [(1-2\psi)\delta_{\alpha\beta} + h_{\alpha\beta}]dx^\alpha dx^\beta\}. \quad (1)$$

In Eq. (1) the metric includes perturbative terms of any order around the FRW background. In this gauge, V_α are pure vectors, i.e., they have vanishing spatial divergence, $\partial^\alpha V_\alpha = 0$, while $h_{\alpha\beta}$ are traceless and transverse, i.e., pure tensor modes, $h^\alpha{}_\alpha = \partial^\alpha h_{\alpha\beta} = 0$. Here and in what follows spatial indices are raised by the Kronecker symbol $\delta^\alpha{}_\beta$. Unless otherwise stated, we use units with $c = 1$. From the results of the post-Newtonian theory [5,18–20], we deduce that vector and tensor metric modes, to the leading order in powers of $1/c$, are, respectively, of $\mathcal{O}(1/c^3)$ and $\mathcal{O}(1/c^4)$, since to the lowest order it is well known that the line element (1) assumes the weak-field form $ds^2 = a^2[-(1+2\phi_1)d\eta^2 + (1-2\psi_1)\delta_{\alpha\beta}dx^\alpha dx^\beta]$, where the scalars ϕ_1 and ψ_1 are both of $\mathcal{O}(1/c^2)$ and $\phi_1 = \psi_1 \equiv \varphi$ [1].

Let us now write Einstein's equations $G^i{}_j = \kappa^2 T^i{}_j$ in the perturbed form

$${}^{(0)}G^i{}_j + \delta G^i{}_j = \kappa^2({}^{(0)}T^i{}_j + \delta T^i{}_j), \quad (2)$$

where $\kappa^2 = 8\pi G/c^4$ and ${}^{(0)}G^i{}_j = \kappa^2\,{}^{(0)}T^i{}_j$ reduce to the background Friedmann equations. Hereafter, we assume that the Universe is filled with a cosmological constant Λ and a pressureless fluid—made of cold dark matter (CDM) plus baryons—whose stress-energy tensor reads $T^i{}_j = \rho u^i u_j$ ($u^i u_j = -1$). In this case the Friedmann equations read $3\mathcal{H}^2 = a^2(8\pi G\bar{\rho} + \Lambda)$ and $\bar{\rho}' = -3\mathcal{H}\bar{\rho}$, where primes indicate differentiation with respect to η, $\mathcal{H} \equiv a'/a$ and $\bar{\rho}$ is the mean matter energy density.

Since the metric (1) can be expressed in an invariant form both in the PN and in the second-order perturbative approximations, we introduce a hybrid formalism that consists in evaluating Einstein's field equations up to the correct order in powers of $1/c$, while including some "hybrid" correction terms, which cannot be completely absorbed in a "rigid" PN approximation, but are required for consistency with a second-order general relativistic approximation.

Namely, writing $\psi = \psi_1 + \psi_2$ and $\phi = \phi_1 + \phi_2$ (where ψ_2 and ϕ_2 in principle contain all powers of $1/c$), and replacing the metric (1) in Eq. (2), the (0-0) and (0-α) components of the perturbed Einstein equations take the form

$$\begin{aligned}\delta G^0{}_0 = &-\frac{1}{a^2}[-6\mathcal{H}(\mathcal{H}\phi_1 + \psi_1') + 2\nabla^2\psi \\ &+ 3\partial^\nu\psi_1\partial_\nu\psi_1 + 8\psi_1\nabla^2\psi_1 + S_1] \\ &= \kappa^2\delta T^0{}_0, \end{aligned} \tag{3}$$

where $S_1 \equiv 12\mathcal{H}^2\phi_1^2 + 3\psi_1'^2 + 12\mathcal{H}\phi_1\psi_1' - 12\mathcal{H}\psi_1\psi_1'$, and

$$\begin{aligned}\delta G^0{}_\alpha = &-\frac{2}{a^2}\left(\mathcal{H}\partial_\alpha\phi_1 + \partial_\alpha\psi_1' + \frac{1}{4}\nabla^2 V_\alpha + S_{2\alpha}\right) \\ &= \kappa^2\delta T^0{}_\alpha, \end{aligned} \tag{4}$$

where $S_{2\alpha} \equiv -4\mathcal{H}\phi_1\partial_\alpha\phi_1 - 2\phi_1\partial_\alpha\psi_1' - \psi_1'\partial_\alpha\phi_1 + 2\psi_1'\partial_\alpha\psi_1 + 2\psi_1\partial_\alpha\psi_1'$.

The traceless part of the (α-β) Einstein's equations $\delta G^\alpha{}_\beta = \kappa^2\delta T^\alpha{}_\beta$ reads

$$\begin{aligned}&\left[\frac{2}{3}\nabla^2(\phi - \psi) - \frac{8}{3}\psi_1\nabla^2\psi_1 - \frac{4}{3}\phi_1\nabla^2\phi_1 + \frac{4}{3}\psi_1\nabla^2\phi_1 - \frac{2}{3}\partial^\nu\phi_1\partial_\nu\phi_1 + \frac{4}{3}\partial_\nu\phi_1\partial^\nu\psi_1 - 2\partial^\nu\psi_1\partial_\nu\psi_1\right]\delta^\alpha{}_\beta \\ &- 2\partial^\alpha\partial_\beta(\phi - \psi) + 8\psi_1\partial^\alpha\partial_\beta\psi_1 + 6\partial^\alpha\psi_1\partial_\beta\psi_1 - 2\partial^\alpha\phi_1\partial_\beta\psi_1 + 2\partial^\alpha\phi_1\partial_\beta\phi_1 - 2\partial^\alpha\psi_1\partial_\beta\phi_1 + 4\phi_1\partial^\alpha\partial_\beta\phi_1 \\ &- 4\psi_1\partial^\alpha\partial_\beta\phi_1 + \partial^\alpha(2\mathcal{H}V_\beta + V_\beta') + \partial_\beta(2\mathcal{H}V^\alpha + V'^\alpha) + h''^\alpha{}_\beta + 2\mathcal{H}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta \\ &= 2a^2\kappa^2\left(\delta T^\alpha{}_\beta - \frac{1}{3}\delta T^\nu{}_\nu\delta^\alpha{}_\beta\right), \end{aligned} \tag{5}$$

while its trace becomes

$$\begin{aligned}&2\nabla^2(\phi - \psi) + 6\mathcal{H}\phi_1' + 6(\mathcal{H}^2 + 2\mathcal{H}')\phi_1 + \\ &6\psi_1'' + 12\mathcal{H}\psi_1' - 2\partial^\nu\phi_1\partial_\nu\phi_1 - 4\phi_1\nabla^2\phi_1 - \\ &3\partial_\nu\psi_1\partial^\nu\psi_1 - 8\psi_1\nabla^2\psi_1 + 4\psi_1\nabla^2\phi_1 - \\ &2\partial_\nu\phi_1\partial^\nu\psi_1 = a^2\kappa^2\delta T^\nu{}_\nu. \end{aligned} \tag{6}$$

The components of the perturbed stress-energy tensor $\delta T^i{}_j$ will be calculated later to the correct order in powers of $1/c$. Note that Eqs. (3), (5), and (6) are evaluated up to $\mathcal{O}(1/c^4)$, while Eq. (4) is evaluated up to $\mathcal{O}(1/c^3)$. However, the terms S_1 and $S_{2\alpha}$ are at least of $\mathcal{O}(1/c^6)$ and $\mathcal{O}(1/c^5)$, respectively, and come out from our hybrid scheme, which mixes PN and second-order perturbative approaches. Moreover, since our purpose is to calculate the source of gravitational waves to the leading order in powers of $1/c$, in Eq. (5) we do not take into account contributions of order higher than $\mathcal{O}(1/c^4)$, though retaining time derivatives of $h^\alpha{}_\beta$.

Taking the divergence of Eq. (4), to solve for the combination $\mathcal{H}\phi_1 + \psi_1'$, and replacing it in Eq. (3), we obtain

$$\nabla^2(\mathcal{H}\phi_1 + \psi_1') = -\frac{a^2\kappa^2}{2}\partial^\nu\delta T^0{}_\nu - \partial^\nu S_{2\nu}, \tag{7}$$

and

$$\begin{aligned}\nabla^2\nabla^2\psi &= \nabla^2\nabla^2(\psi_1 + \psi_2) \\ &= -\frac{a^2\kappa^2}{2}[\nabla^2\delta T^0{}_0 + 3\mathcal{H}(\partial^\nu\delta T^0{}_\nu)] \\ &\quad - 3\mathcal{H}\partial^\nu S_{2\nu} - \frac{1}{2}\nabla^2 S_1 \\ &\quad - \nabla^2\left(\frac{3}{2}\partial^\nu\psi_1\partial_\nu\psi_1 + 4\psi_1\nabla^2\psi_1\right). \end{aligned} \tag{8}$$

The pure vector part V_α can be isolated by replacing $\nabla^2(\mathcal{H}\phi_1 + \psi_1')$ in Eq. (4), where we now neglect the term $S_{2\alpha}$ since it is at least of $\mathcal{O}(1/c^5)$

$$\nabla^2\nabla^2 V_\alpha = 2a^2\kappa^2(\partial_\alpha\partial^\nu\delta T^0{}_\nu - \nabla^2\delta T^0{}_\alpha). \tag{9}$$

Finally, applying the operator $\partial^\beta\partial_\alpha$ to Eq. (5), we can solve for the combination $\phi - \psi$ and write

$$\begin{aligned}\nabla^2\nabla^2(\phi - \psi) = &-\frac{3}{2}a^2\kappa^2\partial^\beta\partial_\alpha\left(\delta T^\alpha{}_\beta - \frac{1}{3}\delta T^\nu{}_\nu\delta^\alpha_\beta\right) + \frac{3}{2}\partial_\alpha\partial^\beta(\partial_\beta\psi_1\partial^\alpha\phi_1 - \partial_\beta\phi_1\partial^\alpha\psi_1 - \partial_\beta\psi_1\partial^\alpha\psi_1 - \partial_\beta\phi_1\partial^\alpha\phi_1) \\ &+ \nabla^2\left(\frac{9}{2}\partial_\nu\psi_1\partial^\nu\psi_1 - 2\partial_\nu\psi_1\partial^\nu\phi_1 + \frac{5}{2}\partial_\nu\phi_1\partial^\nu\phi_1 + 4\psi_1\nabla^2\psi_1 + 2\phi_1\nabla^2\phi_1 - 2\psi_1\nabla^2\phi_1\right). \end{aligned} \tag{10}$$

Replacing the latter expression in Eq. (5), together with the expression for the vector mode V_α obtained by taking the divergence of Eq. (5), we find

$$\nabla^2\nabla^2(h''^\alpha{}_\beta + 2\mathcal{H}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta) = 2\kappa^2 a^2\bigg[\nabla^2(\nabla^2\mathcal{R}^\alpha{}_\beta - \partial^\alpha\partial_\nu\mathcal{R}^\nu{}_\beta - \partial_\beta\partial^\nu\mathcal{R}^\alpha{}_\nu) + \frac{1}{2}(\nabla^2\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu\delta^\alpha{}_\beta + \partial^\alpha\partial_\beta\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu)\bigg], \tag{11}$$

where we have defined the traceless tensor

$$\mathcal{R}^\alpha{}_\beta = \delta T^\alpha{}_\beta - \frac{1}{3}\delta T^\nu{}_\nu\delta^\alpha{}_\beta - \frac{1}{\kappa^2a^2}\Big(\frac{2}{3}\psi_1\nabla^2\phi_1 + \frac{2}{3}\partial_\nu\phi_1\partial^\nu\psi_1 - \frac{4}{3}\psi_1\nabla^2\psi_1 - \frac{2}{3}\phi_1\nabla^2\phi_1 - \frac{1}{3}\partial^\nu\phi_1\partial_\nu\phi_1 - \partial^\nu\psi_1\partial_\nu\psi_1\Big)\delta^\alpha{}_\beta - \frac{1}{\kappa^2a^2}(4\psi_1\partial^\alpha\partial_\beta\psi_1 + 3\partial^\alpha\psi_1\partial_\beta\psi_1 - \partial^\alpha\phi_1\partial_\beta\psi_1 + \partial^\alpha\phi_1\partial_\beta\phi_1 - \partial^\alpha\psi_1\partial_\beta\phi_1 + 2\phi_1\partial^\alpha\partial_\beta\phi_1 - 2\psi_1\partial^\alpha\partial_\beta\phi_1). \tag{12}$$

The form of Eqs. (9) and (11) allows us to directly check that vector sources are transverse while tensor sources are doubly transverse and traceless.

Actually, there is a very simple way of solving the perturbed Einstein equations $\delta G^0{}_\alpha = \kappa^2a^2\delta T^0{}_\alpha$ and $\delta G^\alpha{}_\beta = \kappa^2a^2\delta T^\alpha{}_\beta$ with respect to vectors V_α and tensors $h^\alpha{}_\beta$, respectively. In fact, after retaining only the metric terms which appear linearly on the left-hand side of Eqs. (4) and (5) [21] and defining $\mathcal{R}^\alpha{}_\beta$ the consequently obtained right-hand side, it is possible to apply to both sides the correct combinations of the direction-independent projection operator [22]

$$\mathcal{P}^\alpha{}_\beta = \delta^\alpha{}_\beta - (\nabla^2)^{-1}\partial^\alpha\partial_\beta \tag{13}$$

and automatically obtain the vectors

$$\nabla^2 V_\alpha = -2a^2\kappa^2\mathcal{P}^\nu{}_\alpha\Big(\delta T^0{}_\nu - \frac{2}{a^2\kappa^2}S_{2\nu}\Big), \tag{14}$$

and tensors

$$h''^\alpha{}_\beta + 2\mathcal{H}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta = 2\kappa^2a^2\Big(\mathcal{P}^\alpha{}_\nu\mathcal{P}^\mu{}_\beta - \frac{1}{2}\mathcal{P}^\alpha{}_\beta\mathcal{P}^\mu{}_\nu\Big)\mathcal{R}^\nu{}_\mu. \tag{15}$$

After applying twice the Laplacian operator to Eq. (15) and neglecting as before the term $S_{2\nu}$ in Eq. (14), we recover Eqs. (9) and (11).

III. THE STRESS-ENERGY TENSOR AND THE SOURCE OF METRIC PERTURBATIONS

For the purpose of calculating the components of the perturbed stress-energy tensor $\delta T^i{}_j$ to the correct order in powers of $1/c$, it is convenient to restore the speed of light c in the time coordinate $dx^0 = cd\eta$. From Eq. (1) we obtain the four-velocity $u^i \equiv dx^i/ds$

$$u^0 \simeq \frac{1}{a}\bigg[1 - \frac{1}{2}\Big(2\phi_1 - \frac{v^2}{c^2}\Big)\bigg] + \mathcal{O}\Big(\frac{1}{c^4}\Big), \tag{16}$$

$$u^\alpha = \frac{v^\alpha}{c}u^0 + \mathcal{O}\Big(\frac{1}{c^5}\Big), \tag{17}$$

where $v^2 = v^\nu v_\nu$ and $v^\alpha \equiv dx^\alpha/d\eta$ is the coordinate three-velocity with respect to the FRW background.

The total energy-momentum tensor for our fluid of dust and cosmological constant reads

$$T^i{}_k = {}^{(0)}T^i{}_k + \delta T^i{}_k = [(\rho_\Lambda + \rho)c^2 + p_b]g_{kj}u^iu^j + p_b\delta^i{}_k, \tag{18}$$

where ${}^{(0)}T^i{}_k$ is the background stress-energy tensor and $\rho = \bar\rho + \delta\rho$ is the total mass density. The background density $\rho_b = \bar\rho + \rho_\Lambda$ includes the contribution from the cosmological constant, $\rho_\Lambda = (\Lambda c^2)/(8\pi G)$, while the background pressure $p_b = p_\Lambda = -\rho_\Lambda c^2$ is only due to the latter.

Turning to the components of the perturbed energy-momentum tensor, in terms of the coefficients ϕ_1 and ψ_1 of the metric (1) and up to the correct orders in powers of $1/c$, we find

$$\delta T^0{}_0 = T^0{}_0 - {}^{(0)}T^0{}_0 = -c^2\delta\rho - v^2\rho + \mathcal{O}\Big(\frac{1}{c^2}\Big), \tag{19}$$

$$\delta T^0{}_\alpha = T^0{}_\alpha - {}^{(0)}T^0{}_\alpha = v_\alpha\rho c(1 - 2\phi_1 - 2\psi_1) + \frac{v_\alpha}{c}\rho v^2 + \mathcal{O}\Big(\frac{1}{c^3}\Big), \tag{20}$$

$$\delta T^\alpha{}_0 = T^\alpha{}_0 - {}^{(0)}T^\alpha{}_0 = -v^\alpha\rho c - \frac{v^\alpha}{c}\rho v^2 + \mathcal{O}\Big(\frac{1}{c^3}\Big), \tag{21}$$

$$\delta T^\alpha{}_\beta = T^\alpha{}_\beta - {}^{(0)}T^\alpha{}_\beta = v^\alpha v_\beta\rho\Big(1 - 2\phi_1 - 2\psi_1 + \frac{v^2}{c^2}\Big) + \mathcal{O}\Big(\frac{1}{c^4}\Big). \tag{22}$$

In the hybrid equations which we are about to derive we will keep step by step only the $\delta T^i{}_j$ components we need to let our set of equations hold in the first, second perturbative order and PN regimes.

By substituting $\delta T^\alpha{}_\beta$ in Eq. (10), we obtain up to $\mathcal{O}(1/c^2)$

$$\nabla^2\nabla^2(\phi_1 - \psi_1) = 0, \tag{23}$$

and we can safely assume $\phi_1 = \psi_1 \equiv \varphi$.

This allows us to further simplify Eqs. (6)–(10) and obtain our final set of hybrid equations for cosmological perturbations, namely

$$\nabla^2(\mathcal{H}\varphi + \varphi') = -\frac{a^2\kappa^2c^2}{2}\partial^\nu(\rho v_\nu), \tag{24}$$

$$\begin{aligned}\nabla^2\nabla^2\psi = &-\frac{a^2\kappa^2}{2}[3\mathcal{H}\partial^\nu(\rho v_\nu(1-4\varphi))\\ &-\nabla^2(c^2\delta\rho + \rho v^2)]\\ &+3\frac{\mathcal{H}}{c}\partial^\nu\Big(4\frac{\mathcal{H}}{c}\varphi\partial_\nu\varphi - \frac{1}{c}\varphi'\partial_\nu\varphi\Big)\\ &-\nabla^2\Big(\frac{3}{2}\partial^\nu\varphi\partial_\nu\varphi + 4\varphi\nabla^2\varphi\\ &+6\frac{\mathcal{H}^2}{c^2}\varphi^2 + \frac{3}{2c^2}\varphi'^2\Big),\end{aligned} \tag{25}$$

$$\nabla^2\nabla^2 V_\alpha = 2a^2\kappa^2[\partial_\alpha\partial^\nu(c\rho v_\nu) - \nabla^2(c\rho v_\alpha)], \tag{26}$$

$$\begin{aligned}\nabla^2\nabla^2(\phi - \psi) = &-\frac{3}{2}a^2\kappa^2\partial^\mu\partial_\nu\Big(\rho v^\nu v_\mu - \frac{1}{3}\rho v^2\delta^\nu{}_\mu\Big)\\ &+\nabla^2(5\partial_\nu\varphi\partial^\nu\varphi + 4\varphi\nabla^2\varphi)\\ &-3\partial_\nu\partial^\mu(\partial_\mu\varphi\partial^\nu\varphi),\end{aligned} \tag{27}$$

$$\begin{aligned}\nabla^2\nabla^2\phi = &-\frac{a^2\kappa^2}{2}[3\mathcal{H}\partial^\nu(\rho v_\nu(1-4\varphi))\\ &+3\partial^\mu\partial_\nu(\rho v^\nu v_\mu) - \nabla^2(c^2\delta\rho + 2\rho v^2)]\\ &+\frac{7}{2}\nabla^2(\partial_\nu\varphi\partial^\nu\varphi) - 3\partial_\nu\partial^\mu(\partial_\mu\varphi\partial^\nu\varphi)\\ &+3\frac{\mathcal{H}}{c}\partial^\nu\Big(4\frac{\mathcal{H}}{c}\varphi\partial_\nu\varphi - \frac{1}{c}\varphi'\partial_\nu\varphi\Big)\\ &-\nabla^2\Big(6\frac{\mathcal{H}^2}{c^2}\varphi^2 + \frac{3}{2c^2}\varphi'^2\Big),\end{aligned} \tag{28}$$

$$\begin{aligned}&\nabla^2\nabla^2\Big(\frac{1}{c^2}h''^\alpha{}_\beta + \frac{2\mathcal{H}}{c^2}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta\Big)\\ &\quad= 2\kappa^2a^2\Big[\nabla^2(\nabla^2\mathcal{R}^\alpha{}_\beta - \partial^\alpha\partial_\nu\mathcal{R}^\nu{}_\beta - \partial_\beta\partial^\nu\mathcal{R}^\alpha{}_\nu)\\ &\qquad+\frac{1}{2}(\nabla^2\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu\delta^\alpha{}_\beta + \partial^\alpha\partial_\beta\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu)\Big],\end{aligned} \tag{29}$$

where the traceless tensor $\mathcal{R}^\alpha{}_\beta$, Eq. (12), now reads

$$\begin{aligned}\mathcal{R}^\alpha{}_\beta = &\rho\Big(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\Big)\\ &-\frac{2}{\kappa^2a^2}\Big(\partial^\alpha\varphi\partial_\beta\varphi - \frac{1}{3}\partial^\nu\varphi\partial_\nu\varphi\delta^\alpha{}_\beta\Big)\\ &-\frac{4}{\kappa^2a^2}\Big(\varphi\partial^\alpha\partial_\beta\varphi - \frac{1}{3}\varphi\nabla^2\varphi\delta^\alpha{}_\beta\Big),\end{aligned} \tag{30}$$

while the trace part of the (α-β) component of Eq. (2) becomes

$$\begin{aligned}&2\nabla^2(\phi - \psi) + 18\frac{\mathcal{H}}{c^2}\varphi' +\\ &6\Big(2\frac{\mathcal{H}'}{c^2} + \frac{\mathcal{H}^2}{c^2}\Big)\varphi + \frac{6}{c^2}\varphi'' -\\ &7\partial_\nu\varphi\partial^\nu\varphi - 8\varphi\nabla^2\varphi = \kappa^2a^2\rho v^2.\end{aligned} \tag{31}$$

Using Eqs. (19)–(22), and the expression $\phi_1 = \psi_1 = \varphi$, we can write the stress-energy tensor conservation equations $T^j_{i;j} = 0$ in a form that will give us the equations for our pressureless fluid in the first-, second-order and PN regimes, respectively.

More specifically, the energy conservation equation reads

$$\begin{aligned}&\rho' + 3\mathcal{H}\rho + \partial_\nu(\rho v^\nu) - \frac{3}{c}\rho\varphi' + \frac{1}{c^2}[(\rho v^2)' +\\ &\partial_\nu(v^\nu\rho v^2) + 4\mathcal{H}\rho v^2] - 2\rho v^\nu\partial_\nu\varphi -\\ &\frac{3}{c}\rho\psi_2' - \frac{6}{c}\rho\varphi\varphi' = 0,\end{aligned} \tag{32}$$

while the momentum conservation equation reads

$$\begin{aligned}&(\rho v_\alpha)' + \partial_\nu(\rho v^\nu v_\alpha) + 4\mathcal{H}v_\alpha\rho + \rho c^2\partial_\alpha\varphi -\\ &16\mathcal{H}\rho v_\alpha\varphi + \frac{1}{c^2}\partial_\nu(\rho v^2 v^\nu v_\alpha) - 2\rho c^2\varphi\partial_\alpha\varphi -\\ &\mathcal{H}c\rho V_\alpha + \rho c v_\nu\partial_\alpha V^\nu - 4\partial_\nu(\rho v^\nu v_\alpha\varphi) +\\ &\frac{4}{c^2}\mathcal{H}\rho v^2 v_\alpha - 4\rho' v_\alpha\varphi - 4\rho v_\alpha'\varphi - 6\rho v_\alpha\varphi' +\\ &\frac{1}{c^2}(v_\alpha\rho v^2)' - 2\rho v_\alpha v^\nu\partial_\nu\varphi + 2v^2\rho\partial_\alpha\varphi +\\ &\rho c^2\partial_\alpha\phi_2 = 0.\end{aligned} \tag{33}$$

Equations (25), (26), (28), and (29), together with Eqs. (24), (32), and (33), are the main result of our paper and represent a new set of equations which allow us to describe the evolution of metric perturbation from the linear to the strongly nonlinear stage in terms of the gravitational field φ, the matter density ρ and the peculiar velocity v^α.

IV. LIMITING FORMS OF THE HYBRID APPROXIMATION IN DIFFERENT REGIMES

We now show how our approach accounts for known approximation schemes in different regimes.

A. The linear perturbative regime

Linearly perturbing Eqs. (24), (25), and (28) with respect to the FRW background we deduce that ϕ and ψ coincide and we obtain the linear scalar potential φ in terms of first-order density and velocity fluctuations,

$$\nabla^2\nabla^2\varphi = 4a^2\pi G[\nabla^2\delta\rho - 3\mathcal{H}\bar{\rho}\partial^\nu v_\nu], \tag{34}$$

$$\nabla^2(\mathcal{H}\varphi + \varphi') = -4a^2\pi G\bar{\rho}\partial^\nu v_\nu. \tag{35}$$

Moreover, linearizing Eqs. (26), (29), and (31), using the linearized expressions for $\delta T^i{}_j$ we obtain

$$\varphi'' + 3\mathcal{H}\varphi' + (2\mathcal{H}' + \mathcal{H}^2)\varphi = 0, \tag{36}$$

$$\nabla^2\nabla^2 V_\alpha = 0, \tag{37}$$

$$h''^\alpha_\beta + 2\mathcal{H}h'^\alpha_\beta - \nabla^2 h^\alpha{}_\beta = 0. \tag{38}$$

Perturbing to first order Eqs. (32) and (33), we recover also the linear continuity and momentum equations which read respectively,

$$\delta\rho' + 3\mathcal{H}\delta\rho + \bar{\rho}\partial_\nu v^\nu - 3\bar{\rho}\varphi' = 0, \tag{39}$$

$$v'^\alpha + \mathcal{H}v^\alpha + \partial^\alpha\varphi = 0. \tag{40}$$

In other words, we obtain all the results of linear perturbation theory (see, e.g., Refs. [23,24]), if we interpret φ as the linear scalar potential.

B. The second-order perturbative regime

On the other hand, selecting only the growing-mode solution of Eq. (36) and perturbing up to second order Eqs. (25)–(29), in the limit of a pressureless and irrotational fluid, we recover all the results of second-order perturbation theory [9,14,25].

More specifically, the first-order vector metric perturbations vanish, while the linear tensor metric perturbations are negligible for every kind of cosmological sources, thus we can safely neglect terms which can be expressed as products of first-order scalar and tensor metric perturbations. Writing $\varphi(\mathbf{x}, \eta) = \varphi_0(\mathbf{x})g(\eta)$, where φ_0 is the peculiar gravitational potential linearly extrapolated to the present time and $g \equiv D_+/a$ is the so-called growth-suppression factor, where $D_+(\eta)$ is the linear growing mode of density fluctuations in the Newtonian limit, and using the results of the previous subsection we obtain

$$\begin{aligned}\nabla^2\nabla^2\psi_s = &-4\pi Ga^2[3\mathcal{H}(\bar{\rho}\partial^\nu v_{s\nu} + \delta\rho\partial^\nu v_\nu + v_\nu\partial^\nu\delta\rho \\ &-4\bar{\rho}\partial^\nu(\varphi v_\nu)) - 3\bar{\rho}\partial^\nu(\varphi' v_\nu) - \nabla^2(\delta\rho_s + \bar{\rho}v^2)] \\ &-\nabla^2\left(\frac{3}{2}\partial^\nu\varphi\partial_\nu\varphi + 4\varphi\nabla^2\varphi\right),\end{aligned} \tag{41}$$

$$\begin{aligned}\nabla^2\nabla^2\phi_s = &-4\pi Ga^2[3\mathcal{H}(\bar{\rho}\partial^\nu v_{s\nu} + \delta\rho\partial^\nu v_\nu + v_\nu\partial^\nu\delta\rho \\ &-4\bar{\rho}\partial^\nu(\varphi v_\nu)) - 3\bar{\rho}\partial^\nu(\varphi' v_\nu) \\ &-\nabla^2(\delta\rho_s + 2\bar{\rho}v^2) + 3\bar{\rho}\partial^\mu\partial_\nu(v^\nu v_\mu)] \\ &+\frac{7}{2}\nabla^2(\partial_\nu\varphi\partial^\nu\varphi) - 3\partial_\nu\partial^\mu(\partial_\mu\varphi\partial^\nu\varphi),\end{aligned} \tag{42}$$

$$\nabla^2\nabla^2 V_\alpha = 16\pi Ga^2\partial^\nu(v_\nu\partial_\alpha\delta\rho - v_\alpha\partial_\nu\delta\rho), \tag{43}$$

$$\begin{aligned}&\nabla^2\nabla^2(h''^\alpha{}_\beta + 2\mathcal{H}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta) \\ &= 16\pi Ga^2\Big[\nabla^2(\nabla^2\mathcal{R}^\alpha{}_\beta - \partial^\alpha\partial_\nu\mathcal{R}^\nu{}_\beta - \partial_\beta\partial^\nu\mathcal{R}^\alpha{}_\nu) \\ &\quad+\frac{1}{2}(\nabla^2\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu\delta^\alpha{}_\beta + \partial^\alpha\partial_\beta\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu)\Big],\end{aligned} \tag{44}$$

where $R^\alpha{}_\beta$ has the same analytic form of Eq. (30), that is

$$\begin{aligned}\mathcal{R}^\alpha{}_\beta = &\rho\left(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\right) \\ &-\frac{1}{4\pi Ga^2}\left(\partial^\alpha\varphi\partial_\beta\varphi - \frac{1}{3}\partial^\nu\varphi\partial_\nu\varphi\delta^\alpha{}_\beta\right) \\ &-\frac{1}{2\pi Ga^2}\left(\varphi\partial^\alpha\partial_\beta\varphi - \frac{1}{3}\varphi\nabla^2\varphi\delta^\alpha{}_\beta\right).\end{aligned} \tag{45}$$

The subscript s indicates quantities evaluated at the second perturbative order, v_s^α is the velocity $dx^\alpha/d\eta$ perturbed at the second order and is related to the second-order spatial part $v^\alpha_{(2)}/a$ of the 4-velocity by the relation $v^\alpha_{(2)} = v^\alpha_s - \varphi v^\alpha$ [26].

We can also find the equations that describe the evolution of $\delta\rho_s$ and v_s^α by perturbing up to second order Eqs. (32) and (33) and taking the divergence of the latter. In this way we recover also the second-order energy continuity equation

$$\begin{aligned}&\delta\rho_s' + 3\mathcal{H}\delta\rho_s + \bar{\rho}\partial_\nu v_s^\nu + \delta\rho\partial_\nu v^\nu + \\ &v^\nu\partial_\nu\delta\rho + \mathcal{H}\bar{\rho}v^2 + \bar{\rho}(v^2)' - 3\delta\rho\varphi' - \\ &3\bar{\rho}\psi_s' - 6\bar{\rho}\varphi\varphi' - 2\bar{\rho}v^\mu\partial_\mu\varphi = 0,\end{aligned} \tag{46}$$

and the divergence of the second-order momentum conservation equation

$$\begin{aligned}&\mathcal{H}\bar{\rho}\partial^\alpha v_{s\alpha} + \bar{\rho}\partial^\alpha v'_{s\alpha} + \bar{\rho}\nabla^2\phi_s + \\ &\partial^\alpha[4\mathcal{H}\delta\rho v_\alpha - 4\mathcal{H}\bar{\rho}\varphi v_\alpha + \delta\rho' v_\alpha + \\ &\delta\rho v'_\alpha - 4\bar{\rho}\varphi v'_\alpha - 6\bar{\rho}\varphi' v_\alpha + \delta\rho\partial_\alpha\varphi - \\ &2\bar{\rho}\varphi\partial_\alpha\varphi + \bar{\rho}\partial_\nu(v^\nu v_\alpha)] = 0.\end{aligned} \tag{47}$$

C. The Newtonian approximation

From Eqs. (23) and (25) up to $\mathcal{O}(1/c^2)$ we deduce

$$\nabla^2\psi_1 = \nabla^2\phi_1 = \nabla^2\varphi = \frac{4\pi Ga^2}{c^2}\delta\rho, \tag{48}$$

and writing $\varphi \equiv \varphi_N/c^2$, we recover the Poisson equation $\nabla^2\varphi_N = 4\pi G a^2 \delta\rho$, where the subscript N stands for *Newtonian*.

Analogously, to leading order in $1/c$, Eqs. (32) and (33) respectively become the usual continuity and Euler equations of Newtonian cosmology which apply in the limit of weak fields and nonrelativistic velocities [6]

$$\rho' + 3\mathcal{H}\rho + \partial_\nu(\rho v^\nu) = 0, \tag{49}$$

$$v'_\alpha + \mathcal{H}v_\alpha + v_\nu \partial^\nu v_\alpha = -\partial_\alpha \varphi_N. \tag{50}$$

The latter equation was obtained taking Eq. (33) up to $\mathcal{O}(1/c^0)$ and inserting Eq. (49).

In the linear limit Eqs. (48)–(50) become

$$\nabla^2\varphi = 4\pi G a^2 \delta\rho, \tag{51}$$

$$\delta\rho' + 3\mathcal{H}\delta\rho + \bar{\rho}\partial_\nu v^\nu = 0, \tag{52}$$

$$v'_\alpha + \mathcal{H}v_\alpha = -\partial_\alpha\varphi_N. \tag{53}$$

As we can observe, the equations which characterize the linearized Newtonian theory differ from the linearized relativistic ones. In particular, while the momentum conservation Eqs. (40) and (53) are identical, the linear energy density conservation Eq. (39) differs from the Newtonian one, Eq. (52), by the extra term $-3\bar{\rho}\varphi'$ which does not vanish, even for the pure growing-mode solution of Eq. (36), owing to the presence of a cosmological constant contribution to the FRW background.

Moreover, Eq. (34) represents the linear relativistic generalization of the Poisson equation, since it includes the contribution of the so-called *longitudinal momentum density* φ_f ($\partial_\nu\varphi_f = -4a^2\pi G\bar{\rho}v_\nu$) which acts as a source term for the linear potential φ. Thus, the Poisson gauge gives the *relativistic* cosmological generalization of *Newtonian* gravity [6].

D. The highly nonlinear regime in the PN approximation

Finally, we consider the case of cosmic structures, in the highly nonlinear regime, whose size is much larger than their Schwarzschild radius (in order to avoid non-Newtonian terms in the expressions of the sources).

Our sources can generate vector and tensor metric perturbations by mode mixing in the nonlinear regime. In particular, this mechanism applies to dark-matter halos around galaxies and galaxy clusters or, more specifically, to the highly condensed substructures by which these halos are characterized.

We obtain the continuity and momentum equations up to $\mathcal{O}(1/c^2)$, the equation describing the evolution of the (0-0) component of the metric (1) up to $\mathcal{O}(1/c^4)$, and the equation for the vector modes V_α up to $\mathcal{O}(1/c^3)$, i.e., their 1PN approximation. Moreover, we describe the scalar mode of the (α-β) component of the metric (1) up to $\mathcal{O}(1/c^4)$, i.e., we consider its second post-Newtonian (2PN) approximation, while we obtain the leading-order terms in powers of $1/c$ for the source of the tensor modes $h^\alpha{}_\beta$.

Equations (25)–(29) in this limit become

$$\begin{aligned}\nabla^2\nabla^2\psi = &-\frac{1}{c^4}\nabla^2\left(\frac{3}{2}\partial^\nu\varphi_N\partial_\nu\varphi_N + 4\varphi_N\nabla^2\varphi_N\right)\\ &+\frac{4\pi G a^2}{c^4}[\nabla^2(\rho v^2) - 3\mathcal{H}\partial^\nu(v_\nu\rho)]\\ &+\frac{4\pi G a^2}{c^2}\nabla^2\delta\rho,\end{aligned} \tag{54}$$

$$\begin{aligned}\nabla^2\nabla^2\phi = &\frac{4\pi G a^2}{c^2}\nabla^2\delta\rho + \frac{4\pi G a^2}{c^4}[2\nabla^2(\rho v^2)\\ &- 3\mathcal{H}\partial^\nu(v_\nu\rho) - 3\partial^\mu\partial_\nu(\rho v^\nu v_\mu)]\\ &+\frac{7}{2c^4}\nabla^2(\partial_\nu\varphi_N\partial^\nu\varphi_N)\\ &-\frac{3}{c^4}\partial_\nu\partial^\mu(\partial_\mu\varphi_N\partial^\nu\varphi_N),\end{aligned} \tag{55}$$

$$\nabla^2\nabla^2 V_\alpha = \frac{16\pi G a^2}{c^3}[\partial_\alpha\partial^\nu(v_\nu\rho) - \nabla^2(v_\alpha\rho)], \tag{56}$$

$$\begin{aligned}&\nabla^2\nabla^2\left(\frac{1}{c^2}h''^\alpha{}_\beta + \frac{2\mathcal{H}}{c^2}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta\right)\\ &= \frac{16\pi G a^2}{c^4}\Big[\nabla^2(\nabla^2\mathcal{R}^\alpha{}_\beta - \partial^\alpha\partial_\nu\mathcal{R}^\nu{}_\beta - \partial_\beta\partial^\nu\mathcal{R}^\alpha{}_\nu)\\ &\quad + \frac{1}{2}(\nabla^2\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu\delta^\alpha{}_\beta + \partial^\alpha\partial_\beta\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu)\Big],\end{aligned} \tag{57}$$

where the post-Newtonian limit of the traceless tensor $\mathcal{R}^\alpha{}_\beta$ is

$$\begin{aligned}\mathcal{R}^\alpha{}_\beta = &\rho\left(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\right)\\ &-\frac{1}{4\pi G a^2}\left(\partial^\alpha\varphi_N\partial_\beta\varphi_N - \frac{1}{3}\partial^\nu\varphi_N\partial_\nu\varphi_N\delta^\alpha{}_\beta\right)\\ &-\frac{1}{2\pi G a^2}\left(\varphi_N\partial^\alpha\partial_\beta\varphi_N - \frac{4\pi G a^2}{3}\varphi_N\delta\rho\delta^\alpha{}_\beta\right).\end{aligned} \tag{58}$$

Since, in order to compute the metric coefficients up to the PN approximation, we only need the terms in $\delta T^i{}_j$ which satisfy the Newtonian equations of motions, in Eq. (58) we have inserted the Poisson equation. The 1PN extensions of the Newtonian continuity and Euler equations, respectively, read

$$\begin{aligned}&\rho' + 3\mathcal{H}\rho + \partial_\nu(\rho v^\nu) +\\ &\frac{1}{c^2}[(\rho v^2)' + \partial_\nu(v^\nu\rho v^2) +\\ &4\mathcal{H}\rho v^2 - 2\rho v^\nu\partial_\nu\varphi_N] = 0,\end{aligned} \tag{59}$$

$$\rho(v'_\alpha + \mathcal{H}v_\alpha + v_\nu\partial^\nu v_\alpha + \partial_\alpha\varphi_N)+ \frac{1}{c^2}[-4\rho' v_\alpha\varphi_N - 4\rho v'_\alpha\varphi_N - 6\rho v_\alpha\varphi'_N + (v_\alpha\rho v^2)' - 2\rho v_\alpha v^\nu\partial_\nu\varphi_N + 2v^2\rho\partial_\alpha\varphi_N + \rho\partial_\alpha\phi_{PN} - 2\rho\varphi_N\partial_\alpha\varphi_N - \mathcal{H}\rho V_\alpha + \rho v_\nu\partial_\alpha V_\nu - 16\mathcal{H}\rho v_\alpha\varphi_N - 4\partial_\nu(\rho v^\nu v_\alpha\varphi_N) + \partial_\nu(\rho v^2 v^\nu v_\alpha) + 4\mathcal{H}\rho v^2 v_\alpha] = 0, \quad (60)$$

where ϕ_{PN} is given by the 1PN part of Eq. (55). It can be worth noting that the sources of the metric coefficients involve only quantities of Newtonian origin, i.e., they do not contain terms defined in higher-order approximations.

To conclude this subsection, let us stress that all the PN expressions derived here are *new*, as they are derived in a different gauge than the usual post-Newtonian [2,3,18,27], or synchronous and comoving one [5].

V. VECTOR AND TENSOR MODES

It can be worth to observe that, in the linear limit, Eq. (31) becomes

$$\varphi''_N + 3\mathcal{H}\varphi'_N + (2\mathcal{H}' + \mathcal{H}^2)\varphi_N = 0. \quad (61)$$

This result is extremely important since it implies that the Newtonian potential φ_N and the linear potential φ evolve in the same way with time. Equation (61) can be also obtained by mixing together the Newtonian continuity, Euler and Poisson equations perturbed at first order. This means that, starting from the same initial conditions, i.e., from the same primordial potential as given, e.g., by inflation, the two linear potentials φ_N/c^2 and φ will assume the same values in each point and at each time. In other words, Eq. (61) implies that, in the case of first-order matter perturbations, it is sufficient to use Newtonian gravity on all scales, provided that we define a "Newtonian" linear density perturbation $\delta\rho_N$ via the Poisson equation applied to the linear relativistic potential φ, even if $\delta\rho_N$ differs from the relativistic density $\delta\rho$, as given by Eq. (34). The previous considerations allow us to conclude that, for pure growing-mode solutions of Eq. (61), in the case of an irrotational and pressureless fluid, Eqs. (56)–(58) apply to all cosmologically relevant scales, i.e., from superhorizon to the smallest ones, even if the density ρ, the velocity v^α and the potential φ are required to follow the usual Newtonian hydrodynamical equations. In the equations that follow, therefore, we will drop the subscript N on the various quantities and write

$$\nabla^2\varphi = 4\pi Ga^2\delta\rho, \quad (62)$$

$$\rho' + 3\mathcal{H}\rho + \partial_\nu(\rho v^\nu) = 0, \quad (63)$$

$$v'_\alpha + \mathcal{H}v_\alpha + v_\nu\partial^\nu v_\alpha = -\partial_\alpha\varphi. \quad (64)$$

Thus, for the vector modes we have

$$\nabla^2\nabla^2 V_\alpha = \frac{16\pi Ga^2}{c^3}[\partial_\alpha\partial^\nu(v_\nu\rho) - \nabla^2(v_\alpha\rho)], \quad (65)$$

and, for the tensor modes,

$$\begin{aligned}\nabla^2\nabla^2\left(\frac{1}{c^2}h''^\alpha{}_\beta + \frac{2\mathcal{H}}{c^2}h'^\alpha{}_\beta - \nabla^2 h^\alpha{}_\beta\right) &= \frac{16\pi Ga^2}{c^4}\Big[\nabla^2(\nabla^2\mathcal{R}^\alpha{}_\beta - \partial^\alpha\partial_\nu\mathcal{R}^\nu{}_\beta - \partial_\beta\partial^\nu\mathcal{R}^\alpha{}_\nu) \\ &\quad + \frac{1}{2}(\nabla^2\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu\delta^\alpha{}_\beta + \partial^\alpha\partial_\beta\partial^\mu\partial_\nu\mathcal{R}^\nu{}_\mu)\Big] \\ &= \frac{16\pi Ga^2}{c^4}\nabla^2\nabla^2\left(\mathcal{P}^\alpha{}_\nu\mathcal{P}^\mu{}_\beta - \frac{1}{2}\mathcal{P}^\alpha{}_\beta\mathcal{P}^\mu{}_\nu\right)\mathcal{R}^\nu{}_\mu,\end{aligned} \quad (66)$$

where

$$\begin{aligned}\mathcal{R}^\alpha{}_\beta &= \rho\left(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\right) - \frac{1}{4\pi Ga^2}\left(\partial^\alpha\varphi\partial_\beta\varphi - \frac{1}{3}\partial^\nu\varphi\partial_\nu\varphi\delta^\alpha{}_\beta\right) - \frac{1}{2\pi Ga^2}\left(\varphi\partial^\alpha\partial_\beta\varphi - \frac{4\pi Ga^2}{3}\varphi\delta\rho\delta^\alpha{}_\beta\right) \\ &= \rho\left(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\right) + \frac{1}{4\pi Ga^2}\left(\partial^\alpha\varphi\partial_\beta\varphi - \frac{1}{3}\partial^\nu\varphi\partial_\nu\varphi\delta^\alpha{}_\beta\right) - \frac{1}{2\pi Ga^2}\left[\partial^\alpha(\varphi\partial_\beta\varphi) - \frac{1}{3}\partial^\nu(\varphi\partial_\nu\varphi)\delta^\alpha{}_\beta\right],\end{aligned} \quad (67)$$

which represent the most important results of our paper, since these equations imply that, in the case of matter perturbations, the Newtonian description of the sources of vector and tensor metric fluctuations can take into account all the effects of the relativistic second-order perturbation theory.

It is important to stress that the third term on the last line of Eq. (67) does not contribute to the source of gravitational waves since it vanishes after applying the projection operation in Eq. (13); thus we are allowed to drop it and define as effective source of the gravitational wave $h^\alpha{}_\beta$ the traceless tensor

$$\mathcal{R}^\alpha_{\mathrm{eff}\beta} = \rho\left(v^\alpha v_\beta - \frac{1}{3}v^2\delta^\alpha{}_\beta\right) + \frac{1}{4\pi Ga^2}\left(\partial^\alpha\varphi\partial_\beta\varphi - \frac{1}{3}\partial^\nu\varphi\partial_\nu\varphi\delta^\alpha{}_\beta\right). \quad (68)$$

Actually from a post-Newtonian point of view, these equations hold true also for a pressureless fluid with a vorticity contribution to the peculiar velocity v^α, but the reader should not be surprised if curl terms can be produced even by a pressureless and irrotational perfect fluid. In fact, the curl of the quantity ρv^α is still nonvanishing, even if v^α is derived from a scalar potential. The solution of the inhomogeneous gravitational-wave equation Eq. (66) is given in the Appendix.

Comparison with the quadrupole radiation

We want to show how the gravitational-wave source $\mathcal{R}^{\alpha}_{\mathrm{eff}\beta}$ in Eq. (68) includes the contribution by the reduced quadrupole moment [22] of the matter distribution expressed via comoving coordinates

$$\mathcal{Q}^{\alpha}{}_{\beta} = \int d^3\tilde{x}\rho\left(\tilde{x}^{\alpha}\tilde{x}_{\beta} - \frac{1}{3}\tilde{x}^{\nu}\tilde{x}_{\nu}\delta^{\alpha}{}_{\beta}\right). \tag{69}$$

First of all, let us choose the origin of our coordinates O inside the mass-energy distribution described by the stress-energy tensor $\delta T^{i}{}_{j}$. Let $\mathbf{x}$ be the vector from O to the observation point P and $\tilde{\mathbf{x}}$ the vector from O to the volume element $d^3\tilde{x}$. On scales well inside the Hubble horizon, the solution of Eq. (66), augmented by an outgoing-wave boundary condition, is

$$h^{\alpha}{}_{\beta}(\eta, \mathbf{x}) = \frac{4G}{ac^4}\mathcal{P}^{\alpha}{}_{\nu}{}^{\mu}{}_{\beta}\int d^3\tilde{x}\frac{(a^3\mathcal{R}^{\nu}_{\mathrm{eff}\mu})_{\mathrm{ret}}}{|\mathbf{x} - \tilde{\mathbf{x}}|}, \tag{70}$$

where the transverse-traceless operator is $\mathcal{P}^{\alpha}{}_{\nu}{}^{\mu}{}_{\beta} \equiv \mathcal{P}^{\alpha}{}_{\nu}\mathcal{P}^{\mu}{}_{\beta} - \frac{1}{2}\mathcal{P}^{\alpha}{}_{\beta}\mathcal{P}^{\mu}{}_{\nu}$, with $\mathcal{P}^{\alpha}{}_{\beta}$ given by Eq. (13) and $\mathcal{R}^{\alpha}_{\mathrm{eff}\beta}$ by Eq. (68). The subscript "ret" means the quantity is to be evaluated at the retarded space-time point $(\eta - |\mathbf{x} - \tilde{\mathbf{x}}|/c, \tilde{\mathbf{x}})$.

Our purpose is to evaluate $h^{\alpha}{}_{\beta}$ in the wave zone, that is far outside the source region: $|\mathbf{x}| \equiv r \gg |\tilde{\mathbf{x}}|$, thus we expand the retarded integral Eq. (70) in powers of $\tilde{\mathbf{x}}/r$ and take only the first term of the multipole expansion

$$h^{\alpha}{}_{\beta}(\eta, \mathbf{x}) = \frac{4G}{c^4}\frac{1}{ar}\mathcal{P}^{\alpha}{}_{\nu}{}^{\mu}{}_{\beta}\left[a^3\int d^3\tilde{x}\mathcal{R}^{\nu}_{\mathrm{eff}\mu}\right]_{\mathrm{ret}}, \tag{71}$$

where for radially traveling waves $\mathcal{P}^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - x^{\alpha}x_{\beta}/r^2$. Equation (71) expresses the gravitational waves $h^{\alpha}{}_{\beta}$ in terms of integrals over the "stress distribution" $\mathcal{R}^{\alpha}_{\mathrm{eff}\beta}$, while Eq. (69) represents an integral over the source "energy distribution." In order to make the comparison between these two equations, we need to convert the spatial components $T^{\alpha}{}_{\beta}$ of the stress-energy tensor in terms of the time components by means of the conservation equations $T^{j}_{i;j} = 0$. Since $\mathcal{Q}^{\alpha}{}_{\beta}$ in Eq. (69) is the Newtonian quadrupole and the dynamics of the tensor source is also Newtonian, we only need the continuity and Euler equations (63) and (64), by which, after some mathematical manipulations, we obtain

$$\begin{aligned}\int d^3\tilde{x}(\rho v^{\alpha}v_{\beta}) &= \frac{1}{2}\frac{\partial}{\partial\eta}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}v_{\beta} + \tilde{x}_{\beta}v^{\alpha}) \\ &\quad + 2\mathcal{H}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}v_{\beta} + \tilde{x}_{\beta}v^{\alpha}) \\ &\quad + \frac{1}{2}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}\partial_{\beta}\varphi + \tilde{x}_{\beta}\partial^{\alpha}\varphi),\end{aligned} \tag{72}$$

and

$$\begin{aligned}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}v_{\beta} + \tilde{x}_{\beta}v^{\alpha}) &= \frac{\partial}{\partial\eta}\int d^3\tilde{x}(\rho\tilde{x}^{\alpha}\tilde{x}_{\beta}) \\ &\quad + 3\mathcal{H}\int d^3\tilde{x}(\rho\tilde{x}^{\alpha}\tilde{x}_{\beta}),\end{aligned} \tag{73}$$

where we have dropped surface terms at infinity.

By substituting Eq. (73) into Eq. (72), we finally find

$$\begin{aligned}\int d^3\tilde{x}(\rho v^{\alpha}v_{\beta}) &= \frac{1}{2}\frac{\partial^2}{\partial\eta^2}\int d^3\tilde{x}(\rho\tilde{x}^{\alpha}\tilde{x}_{\beta}) \\ &\quad + \frac{3}{2}\frac{\partial}{\partial\eta}\int d^3\tilde{x}\mathcal{H}(\rho\tilde{x}^{\alpha}\tilde{x}_{\beta}) \\ &\quad + 2\mathcal{H}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}v_{\beta} + \tilde{x}_{\beta}v^{\alpha}) \\ &\quad - \frac{1}{2}\int d^3\tilde{x}\rho(\tilde{x}^{\alpha}\partial_{\beta}\varphi + \tilde{x}_{\beta}\partial^{\alpha}\varphi).\end{aligned} \tag{74}$$

After substituting Eq. (74) into Eq. (68) and using again the continuity equation, in the wave zone the gravitational wave $h^{\alpha}{}_{\beta}$, to leading order in powers of $1/c$ and $\tilde{\mathbf{x}}/r$, reads

$$\begin{aligned}h^{\alpha}{}_{\beta}(\eta, \mathbf{x}) = \frac{2G}{arc^4}\mathcal{P}^{\alpha}{}_{\nu}{}^{\mu}{}_{\beta}\{a^3[&\frac{\partial^2\mathcal{Q}^{\nu}{}_{\mu}}{\partial\eta^2} + 7\mathcal{H}\frac{\partial\mathcal{Q}^{\nu}{}_{\mu}}{\partial\eta} + (3\mathcal{H}' + 12\mathcal{H}^2)\mathcal{Q}^{\nu}{}_{\mu} \\ &- \int d^3\tilde{x}\rho(\tilde{x}^{\nu}\partial_{\mu}\varphi + \tilde{x}_{\mu}\partial^{\nu}\varphi - \frac{2}{3}\tilde{x}^{\sigma}\partial_{\sigma}\varphi\delta^{\nu}{}_{\mu}) \\ &+ \frac{1}{2\pi Ga^2}\int d^3\tilde{x}(\partial^{\nu}\varphi\partial_{\mu}\varphi - \frac{1}{3}\partial^{\sigma}\varphi\partial_{\sigma}\varphi\delta^{\nu}{}_{\mu})]\}_{\mathrm{ret}}.\end{aligned} \tag{75}$$

Let us observe that the first line of Eq. (75) recovers the known expression of the quadrupole radiation in the limit of a flat and static Universe [22,28], while contributions on the other lines derive from the backreaction of the gravitational potential φ which can act as a source of gravitational waves. Moreover, on scales much smaller than the Hubble horizon, the last two terms on the first line in Eq. (75) can be neglected in comparison to the first one. In fact, the typical free fall time of a mass distribution is proportional to $\rho^{-1/2}$, while the Hubble time goes as $\rho_b^{-1/2}$, where b stands for background; thus, on small scales, where the density contrasts can be very high, the characteristic rate of the structure collapse is much larger than the expansion rate. This allows us to drop the contributions proportional to $\mathcal{H}$ in Eq. (75) and recover the results expected well inside the horizon.

VI. CONCLUDING REMARKS

The main result of this paper is represented by the set of equations (25), (26), (28)–(30), and (68), expressing metric perturbation in terms of the gravitational field φ, where the matter density ρ and the peculiar velocity v^α, satisfy Eqs. (24), (32), and (33). These equations, when applied in a cosmological setting characterized by a pressureless and irrotational fluid and a cosmological constant, provide a unified description of cosmological perturbations during their evolution from the linear to the highly nonlinear regime. On large scales, these equations reduce to the equations of the first- and second-order perturbation theory developed in the Poisson gauge, while, on very small scales, where the perturbative approach is no longer applicable, they describe the evolution of cosmological perturbations by a PN approximation. Indeed, we calculate the (0-0) and (0-α) components of the metric (1) up to the 1PN order, the (α-β) scalar-type component up to the 2PN order, while we find for the (α-β) tensor-type component the leading-order source terms in powers of $1/c$.

We also derive the generalization of the standard Euler-Poisson system of equations of Newtonian hydrodynamics that consistently accounts for all the effects up to order $1/c^2$. The curl term and anisotropic stress, that produce vectors and tensor metric perturbations, arise already at the second perturbative order and at the strongly nonlinear level they are dominated by the contribution of the high-density contrast and the high peculiar velocity typical of small-scale structures. It can be worth to stress that the quantities which source vector and tensor modes are of Newtonian origin on all scales, in the sense that they involve only terms that satisfy the Newtonian Euler-Poisson system. This result is of extreme importance in view of a possible numerical implementation of our set of equations, as it implies that one can compute directly vector and tensor modes starting from the outputs of N-body simulations.

Finally, it should be stressed that our new set of equations has many possible cosmological applications such as, for example, the evaluation of the stochastic gravitational-wave backgrounds produced by CDM halos [29–31] and substructures within halos. It can be also used to improve the estimate of gravitational lensing effects and gravity-induced secondary CMB temperature/polarization anisotropies generated by small-scale structures [7–9].

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APPENDIX: SOLUTION OF THE INHOMOGENEOUS GRAVITATIONAL-WAVE EQUATION

We can formally write Eq. (66) as

$$h''^{\alpha}_{\beta} + 2\mathcal{H} h'^{\alpha}_{\beta} - \nabla^2 h^\alpha_\beta = a^2\kappa^2 S^\alpha_\beta, \tag{A1}$$

and solve it in the two cases of matter domination and Λ domination, when the equation of state of the background fluid is $p = w\rho$ with $w = 0, -1$, respectively. In order to obtain these solutions, we have Fourier expanded the functions $h^\alpha{}_\beta(\eta, \mathbf{x})$ and $S^\alpha{}_\beta(\eta, \mathbf{x})$ and decomposed them in the so-called $\sigma = +, \times$ polarization modes, as follows:

$$h^\alpha{}_\beta(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} h^\alpha{}_\beta(\eta, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A2}$$

$$h^\alpha{}_\beta(\eta, \mathbf{k}) = h_+(\eta, \mathbf{k}) p^{+\alpha}{}_\beta(\hat{k}) + h_\times(\eta, \mathbf{k}) p^{\times\alpha}{}_\beta(\hat{k}), \tag{A3}$$

$$S^{\alpha}{}_\beta(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} S^\alpha{}_\beta(\eta, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A4}$$

$$S^{\alpha}{}_\beta(\eta, \mathbf{k}) = S_+(\eta, \mathbf{k}) p^{+\alpha}{}_\beta(\hat{k}) + S_\times(\eta, \mathbf{k}) p^{\times\alpha}{}_\beta(\hat{k}), \tag{A5}$$

where $p^{+\alpha}{}_\beta(\hat{k})$ and $p^{\times\alpha}{}_\beta(\hat{k})$ are the polarization tensors.

After imposing the initial conditions

$$h_\sigma(\eta_{\rm eq}, \mathbf{k}) = h'_\sigma(\eta_{\rm eq}, \mathbf{k}) = 0, \quad \text{if } w = 0, \tag{A6}$$

$$h_\sigma(\eta_\Lambda, \mathbf{k}) = h_{\sigma,\Lambda}(\mathbf{k}), \quad h'_\sigma(\eta_\Lambda, \mathbf{k}) = h'_{\sigma,\Lambda}(\mathbf{k}) \quad \text{if } w = -1, \tag{A7}$$

($\eta_{\rm eq}$ is the conformal time at matter-radiation equality, while η_Λ corresponds to the time when $\bar{\rho} = \rho_\Lambda$), we obtain for $w = 0$

$$h_\sigma(\eta, \mathbf{k}) = \frac{a_0^2\kappa^2}{\eta_0^4} k^3 \left[\frac{n_1(k\eta)}{k\eta} \int_{\eta_{\rm eq}}^{\eta} \tilde{\eta}^8 \frac{j_1(k\tilde{\eta})}{k\tilde{\eta}} S_\sigma(\tilde{\eta}, \mathbf{k}) d\tilde{\eta} - \frac{j_1(k\eta)}{k\eta} \int_{\eta_{\rm eq}}^{\eta} \tilde{\eta}^8 \frac{n_1(k\tilde{\eta})}{k\tilde{\eta}} S_\sigma(\tilde{\eta}, \mathbf{k}) d\tilde{\eta}\right], \tag{A8}$$

where η_0 represents the present value of the conformal time. For $w = -1$ we have

$$\begin{aligned} h_\sigma(\eta, \mathbf{k}) = {} & (\eta k)^2 j_1(\eta k)\left[h_{\Lambda,\sigma} \frac{2n_1(\eta_\Lambda k) + \eta_\Lambda n_1'(\eta_\Lambda k)}{\eta_\Lambda} - \frac{\kappa^2}{H_0^2 k} \int_{\eta_\Lambda}^{\eta} \frac{n_1(k\tilde{\eta})}{\tilde{\eta}^2} S_\sigma(\tilde{\eta}, \mathbf{k}) d\tilde{\eta}\right] \\ & - (\eta k)^2 n_1(\eta k)\left[h_{\Lambda,\sigma} \frac{2j_1(\eta_\Lambda k) + \eta_\Lambda j_1'(\eta_\Lambda k)}{\eta_\Lambda} - \frac{\kappa^2}{H_0^2 k} \int_{\eta_\Lambda}^{\eta} \frac{j_1(k\tilde{\eta})}{\tilde{\eta}^2} S_\sigma(\tilde{\eta}, \mathbf{k}) d\tilde{\eta}\right], \end{aligned} \tag{A9}$$

where $h_{\sigma,\Lambda}(\mathbf{k})$ is obtained from Eq. (A8) at $\eta = \eta_\Lambda$, while $j_1(\eta k)$ and $n_1(\eta k)$ are, respectively, the spherical Bessel and Neumann functions of order $n = 1$.

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