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Large-scale magnetic fields from density perturbations

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We derive the *minimal* seed magnetic field which unavoidably arises in the radiation and matter eras, prior to recombination, by the rotational velocity of ions and electrons, gravitationally induced by the nonlinear evolution of primordial density perturbations. The resulting magnetic field power spectrum is fully determined by the amplitude and spectral index of density perturbations. The *rms* amplitude of the seed field at recombination is $B \approx 10^{-23}(\lambda/\mathrm{Mpc})^{-2}$ G, on comoving scales $\lambda \gtrsim 1$ Mpc.

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I. INTRODUCTION

Magnetic fields are present in most astrophysical systems [1–5], but their origin is still unknown. Spiral galaxies typically contain magnetic fields of about 10^{-6} G that are aligned with the spiral density waves [1]. A plausible explanation is that these fields have been produced from the exponential amplification of an initially weak field by a mean-field dynamo [6,7], in which a seed field was amplified by the differential rotation of the galaxy in conjunction with magnetohydrodynamic turbulence. Magnetic fields of comparable amplitude are also found on cluster scales [8].

The seed-field strength required at the time of completed galaxy formation for a galactic dynamo to produce the present magnetic field strength of a few μG is usually quoted in the range $\sim(10^{-23}$–$10^{-19})$ G [1–5]. These estimates, though, are obtained by considering the dynamo amplification in a flat Universe with zero cosmological constant for "typical" values of the parameters of the $\alpha\omega$ dynamo. According to Ref. [9] these lower bounds can be relaxed to about 10^{-30} G, for a Universe with a dark energy component (e.g. a cosmological constant or quintessence), which appears to be favored by recent results from high-redshift supernovae observations, cosmic microwave background experiments and large-scale structure data.

Most proposed models of primordial seed-field generation either fail to meet these requirements or invoke nonstandard coupling between the electromagnetic field and the inflaton scalar sector responsible for a primordial period of accelerated Universe expansion (see, e.g., Ref. [4] and references therein).

The most economic and conservative physical mechanism for producing such a seed field was proposed by Harrison [10]. The mechanism relies on the fact that weak magnetic fields are generated during the radiation era in regions that have nonvanishing vorticity. The main problem with Harrison's mechanism is that the required vorticity of the plasma does not have any dynamical origin, but has to be put in by hand, as an initial condition. In this paper we show that rotational velocity of the plasma is unavoidably induced gravitationally by the nonlinear mode-mode coupling of primordial density perturbations generated during inflation, arising at second order in perturbation theory. Therefore, we predict that a minimal seed magnetic field is generated by the differential rotational velocity of ions and electrons. The resulting magnetic field power spectrum is fully determined by the amplitude and the spectral index n_S of primordial scalar perturbations. The rms amplitude of the seed field at recombination is $B \approx 10^{-23}(\lambda/\mathrm{Mpc})^{-2}G$, on comoving scales $\lambda \gtrsim$ Mpc.

The plan of the paper is as follows. In Sec. II we briefly recall Harrison's mechanism and we show how the seed magnetic field is produced. In Sec. III we compute the power spectrum of such a seed field in terms of the power spectrum of primordial scalar perturbations generated during inflation. Finally, in Sec. IV we draw our conclusions.

II. GENERATION OF SEED MAGNETIC FIELD

Let us first recall how Harrison's mechanism [10] works. Consider a rotating region in the expanding early Universe (with scale factor a), consisting of matter (mostly protons) with average energy density ρ_m, electrons and photons with energy density ρ_γ. Electrons and photons are tightly coupled and are considered as a single fluid in Ref. [10]. Let ω_m and ω_γ be their angular velocities. In the absence of interactions, their angular momenta, proportional, respectively, to $\rho_m\omega_m a^5$, and $\rho_\gamma\omega_\gamma a^5$, are separately conserved (see also the Appendix). As the Universe expands, they scale differently, $\omega_m \propto a^{-2}$ and $\omega_\gamma \propto a^{-1}$. Ions and the electron-photon fluid spin down at different rates and then currents and magnetic fields are generated.

A formal derivation of the resulting magnetic field strength in terms of the vorticity in the proton fluid including the electron-proton coupling can be found in [4]. It is necessary to consider the evolution of the multicomponent fluid formed by photons, electrons and protons during the radiation era. At temperatures $T \gtrsim m_e$ (where m_e is the electron mass), the interactions between ions and electrons are strong due to copious lepton-pair production, and they are locked together. The interaction between electrons and photons is also very strong. This means that all the plasma has the same angular velocity at $T \approx m_e$ and no magnetic field can be generated. Below this temperature, electrons and photons are still tightly coupled through Thomson scattering. Protons and electrons are tightly coupled through Coulomb scattering (scattering between photons and protons can be neglected as the coupling is weaker) and so the photon fluid drags the protons in its motion. The difference in the strength of these interactions and in the mass of electrons and protons leads however to a small difference in the electron and proton fluid rotational velocities that gives rise to nonvanishing currents and magnetic fields. In our case the rotational velocity of the plasma is sourced by the nonlinear evolution of scalar (density) perturbations. The generation of the magnetic field ends at recombination, when electrons and protons combine to form neutral hydrogen and radiation decouples from matter. This means that the generation of the magnetic field starts around $T \approx m_e$ and ends when $T \approx T_{\rm rec}$. Dark energy and neutrinos can be ignored throughout the evolution. Moreover, during the radiation era we can safely neglect the role of the cold dark matter (CDM) component, that is, we consider a plasma with only baryons (protons), electrons and photons, and electromagnetic (EM) fields. After matter-radiation equality, CDM plays the dominant gravitational role and it enters our equations determining the evolution of perturbations.

Let us now go into the details of this mechanism. The momentum equation for the interacting components can be written using the total energy-momentum-conservation equation $T^{\nu}_{\mu;\nu} = 0$ The energy-momentum tensor of each component is not conserved independently and its divergence has a source term that takes into account the energy and momentum transfer among the components, $T^{(A)\nu}_{\mu;\nu} = Q^{(A)}_{\mu}$, with $A = \gamma, p, e, \mathrm{EM}$. We will describe the proton, electron and photon components as approximately perfect fluids, thus their energy-momentum tensor can be written as

$$T^{(\alpha)\mu}_{\nu} = (p^{(\alpha)} + \rho^{(\alpha)}) u^{(\alpha)\mu} u_{(\alpha)\nu} + p^{(\alpha)} \delta^{\mu}_{\nu}, \tag{1}$$

with $\alpha = e, p, \gamma$ and $p^{(\alpha)} = w^{(\alpha)} \rho^{(\alpha)}$. We can expand the energy density and four velocity of each component as

$$\rho^{(\alpha)} = \rho^{(\alpha)}_0 (1 + \delta^{(\alpha)}), \qquad u^{(\alpha)\mu} = \frac{1}{a}(\delta^{\mu}_0 + v^{(\alpha)\mu}). \tag{2}$$

In what follows we will neglect the electron and proton pressure, i.e., we set $w^{(e)} \approx w^{(p)} \approx 0$, while for the photons we have $w^{(\gamma)} = 1/3$.

The crucial step in our computation is that we have to include second-order terms in the metric and the matter perturbations [11–13], since the rotational component of the velocity, and hence magnetic field, are only generated at second order. Indeed linear vector modes are not generated during inflation and, by Kelvin's circulation theorem vorticity is conserved for a perfect fluid. Of course, this conservation holds at second order as well, and it applies to the vorticity of the fluid which generates the gravitational field. As explained in detail in the Appendix, this conservation law does not prevent the occurrence of a rotational velocity component of the charged particles, thus giving rise to a nonzero magnetic field. A similar point can be found also in Ref. [14].

Let us expand $\delta^{(\alpha)} = \delta^{(\alpha)}_{(1)} + \delta^{(\alpha)}_{(2)}$ and $v^{(\alpha)\mu} = v^{(\alpha)\mu}_{(1)} + v^{(\alpha)\mu}_{(2)}$, where the superscripts (1) and (2) indicate first- and second-order perturbation quantities. We will work in the Poisson gauge, which is the generalization of the so-called longitudinal (or Newtonian) gauge when vector and tensor modes are allowed. The perturbed metric reads

$$ds^2 = a^2(\eta)\{-(1 + 2\phi)d\eta^2 + 2\chi_i d\eta dx^i + [(1 - 2\psi)\delta_{ij} + \chi_{ij}]dx^i dx^j\}. \tag{3}$$

In this gauge, the first-order scalar perturbations are equal if there are no anisotropic stresses, $\phi^{(1)} = \psi^{(1)} = \varphi$. The vector perturbation is χ_i, and we will assume that it is not generated at first order $\chi^{(1)}_i = 0$. We will also neglect primordial tensor modes.

Although first-order primordial vector and tensor modes are absent, the nonlinear evolution of the primordial scalar perturbations gives rise to nonvanishing vector and tensor perturbations (and corrections to the scalar perturbations) at second order, $\chi_i = \chi^{(2)}_i$ and $\chi_{ij} = \chi^{(2)}_{ij}$ [11,12].

Let us consider the momentum continuity equation for each fluid component (whose general expression up to second order in perturbation theory is reported in the Appendix). The contribution of the electromagnetic field can be included in the source of the charged fluid components, as $T^{(\mathrm{EM})\nu}_{\mu;\nu} = F_{\mu\beta} j^{\beta}$, with $j^{\beta} \equiv en(u^{(p)\beta} - u^{(e)\beta})$, where we have assumed charge neutrality ($n = n^{(e)} \simeq n^{(p)}$). The momentum equation for photons up to second order can be written as

$$\frac{4\rho^{(\gamma)}_0}{3}\Big[(v^{(\gamma)}_i + \chi_i)' + \frac{1}{4}\partial_i \delta^{(\gamma)} + \partial_i \phi + \frac{1}{4}(3\varphi - \delta^{(\gamma)})\partial_i \delta^{(\gamma)} - \varphi' v^{(\gamma)}_i - \frac{1}{3} v^{(\gamma),j}_j v^{(\gamma)}_i + \frac{1}{2}(v^{(\gamma)2})_{,i} + \frac{1}{2}(\varphi^2)_{,i}\Big] = \kappa^{e\gamma}_i, \tag{4}$$

and for protons

$$\rho_0^{(p)}\left[(v_i^{(p)}+\chi_i)'+\mathcal{H}(v_i^{(p)}+\chi_i)+\partial_i\phi-2\varphi' v_i^{(p)}+\frac{1}{2}(v^{(p)2})_{,i}+\frac{1}{2}(\varphi^2)_{,i}\right]=en(E_i+\epsilon_{ijk}v^{(p)i}B^k)-\kappa_i^{ep}, \tag{5}$$

where $\mathcal{H}=a'/a$ is the expansion rate in conformal time. The momentum equation for electrons is similar to that for protons, but it has the opposite sign in the right-hand side terms and an additional source, $-\kappa_i^{e\gamma}$, due to the momentum transfer between the electron and photon fluids. This momentum transfer is due to Thomson scattering and is given by $\kappa_i^{e\gamma}=-\frac{4}{3}\rho_0^{(\gamma)}\tau'(v_i^{(\gamma)}-v_i^{(e)})$, where the differential optical depth is $\tau'=an\sigma_T$, and σ_T is the Thomson cross section. The momentum transfer between electrons and protons is due to Coulomb scattering and can be written as $\kappa_i^{ep}=-n(v_i^{(p)}-v_i^{(e)})/\tau_e$, where the collision time between electrons and protons is $\tau_e=m_e\sigma/ne^2$ in terms of the conductivity of the plasma σ.

During the whole period we are interested in, up to the time of hydrogen recombination, the collision times between electrons and photons ($\propto \tau'^{-1}$) and the electron-proton one are much shorter than the Universe expansion time, thus momentum transfer is very efficient. A tight coupling expansion of the momentum equations gives, to lowest order, $v_i^{(\gamma)}\simeq v_i^{(e)}\simeq v_i^{(p)}$.

We can obtain an equation for the vorticity of the proton fluid by taking the curl of Eq. (5) and combining it with Maxwell's equations

$$\frac{d}{d\eta}\left[a^2\left(\zeta_i^{(p)}+\Omega_i+\frac{e}{m_p}B_i\right)\right]=2\epsilon_{ijk}a\varphi'^{,j}v^{(p)k}+\frac{ea^2}{\sigma m_p}\nabla^2B_i, \tag{6}$$

where we have kept terms up to second order in the metric perturbation and defined $\zeta_i^{(p)}\equiv\epsilon_{ijk}\partial^j v^{(p)k}/a$ and $\Omega_i\equiv\epsilon_{ijk}\partial^j\chi^k/a$. In the last equation the diffusion term can be dropped in the highly conductive protogalactic medium.

The seed magnetic field can be written as

$$B_i=-\frac{m_p}{e}\left(\beta_i^{(p)}-\frac{a_I^2}{a^2}\beta_{(I)i}^{(p)}-\frac{2}{a^2}\int_{\eta_I}^{\eta}d\tilde{\eta}a\epsilon_{ijk}\partial^j\varphi' v^{(p)k}\right), \tag{7}$$

where the subscript I denotes the initial time, corresponding to $T\sim m_e$, when $B_I=0$, and where we introduced the useful combination $\beta_i^{(A)}\equiv\zeta_i^{(A)}+\Omega_i$.

Taking now the curl of the photon momentum equation (4) we obtain an equation for the photon vorticity

$$(a\beta_i^{(\gamma)})'=\frac{3}{4}\epsilon_{ijk}\left[-\partial^j\varphi\partial^k\delta^{(\gamma)}-\frac{4}{9}v^j\partial^k\partial^l v_l+\frac{4}{3}\partial^j\varphi' v^k\right], \tag{8}$$

where we have set $v^{(\gamma)}\equiv v$. Solving Eq. (8) we finally get

$$\beta_i(\eta)=\frac{a_I}{a}\beta_{(I)i}+\frac{3}{4a}\int_{\eta_I}^{\eta}d\tilde{\eta}\epsilon_{ijk}\left(-\partial^j\varphi\partial^k\delta^{(\gamma)}-\frac{4}{9}v^j\partial^k\partial^l v_l+\frac{4}{3}\partial^j\varphi' v^k\right), \tag{9}$$

where we have set $\beta^{(p)}\simeq\beta^{(\gamma)}\equiv\beta$.

Equation (7) with the solution Eq. (9) is our main result and provides the resulting magnetic field in terms of the metric and matter perturbations. In order to finalize our computation we have to consider the linearly perturbed Einstein's equations. For a Universe dominated by a fluid with equation of state $p=w\rho$, with $w=$ const, we can express the fluid velocity $v^{(1)i}$ and the energy density perturbations $\delta^{(1)}$ in terms of the gravitational potential φ as

$$v^{(1)i}=-\frac{2}{3(1+w)\mathcal{H}^2}\partial_i(\varphi'+\mathcal{H}\varphi),\qquad \delta^{(1)}=\frac{2}{3\mathcal{H}^2}[\nabla^2\varphi-3\mathcal{H}(\varphi'+\mathcal{H}\varphi)]. \tag{10}$$

The evolution equation for the peculiar gravitational potential φ is given by

$$\varphi''+3\mathcal{H}(1+w)\varphi'-w\nabla^2\varphi=0, \tag{11}$$

whose solution in the radiation dominated era in Fourier space reads

$$\varphi(\mathbf{k},\eta)=\frac{3j_1(x)}{x}\varphi_0(\mathbf{k}), \tag{12}$$

where $x\equiv k\eta/\sqrt{3}$ and j_ℓ denote spherical Bessel functions of order ℓ. The latter expression manifests the well-known stagnation effect for perturbations which crossed the Hubble radius during the radiation era.

We can now evaluate in Eq. (7) the contribution from $\beta(\eta)$, given by Eq. (9). With the solutions Eqs. (10) it reads

$$\beta_i(\eta)=\frac{a_I}{a}\beta_{(I)i}+\frac{1}{a}\int_{\eta_I}^{\eta}d\tilde{\eta}\epsilon_{ijk}\left(\frac{2\partial^j\varphi\partial^k\varphi'}{\mathcal{H}}-\frac{7\partial^j\varphi\nabla^2\partial^k\varphi}{12\mathcal{H}^2}-\frac{\partial^j\varphi\partial^k\nabla^2\varphi'}{12\mathcal{H}^3}-\frac{\partial^j\varphi'\partial^k\nabla^2\varphi}{12\mathcal{H}^3}-\frac{\partial^j\varphi'\partial^k\nabla^2\varphi'}{12\mathcal{H}^4}\right). \tag{13}$$

This expression can be integrated analytically exploiting the nontrivial result given by Kelvin's circulation theorem (see the Appendix), which states the conservation of angular momentum to all orders. In fact, it can be seen that at second order

$$\beta_i(\eta) = \frac{1}{a}\epsilon_{ijk}\left(\frac{2}{\mathcal{H}^2}\partial^j\varphi'\partial^k\varphi - \frac{1}{12\mathcal{H}^3}\partial^j\varphi\partial^k\nabla^2\varphi - \frac{1}{12\mathcal{H}^4}\partial^j\varphi'\partial^k\nabla^2\varphi\right), \tag{14}$$

where we simply used Eqs. (10) and (11), with $w = 1/3$, in Eq. (A10), assuming vanishing vorticity of the radiation fluid, $\omega_i = 0$, which fixes the value of our integration constant $\beta_{(I)i}$. One can explicitly check that the time derivative of Eq. (14) gives the integrand in Eq. (13).

III. POWER SPECTRUM OF THE SEED MAGNETIC FIELD

We can now compute the power spectrum of the magnetic field generated up to recombination. The correlation of the Fourier modes of the field can generally be written as [15]

$$\langle B_l(\mathbf{k})B_m^*(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta^3(\mathbf{k}-\mathbf{k}')[(\delta_{lm} - \hat{k}_l\hat{k}_m)S(k) + i\epsilon_{lmj}\hat{k}_jA(k)], \tag{15}$$

where $\hat{\mathbf{k}}$ denotes the unit vector in the direction of $\mathbf{k}$. The term proportional to $A(k)$ represents a "helical" component, that is a nonvanishing field component in the direction of the current $[\mathbf{B}\cdot(\nabla\times\mathbf{B}\neq 0)]$. It can be seen that the seed field obtained in the previous section has no helical component, thus $A(k) = 0$. The relevant component $S(k)$ can be obtained from

$$\langle \mathbf{B}(\mathbf{k})\cdot\mathbf{B}^*(\mathbf{k}')\rangle = (2\pi)^3 S(k)\delta^3(\mathbf{k}-\mathbf{k}'). \tag{16}$$

By replacing Eq. (14) into Eq. (7), the Fourier modes of $\mathbf{B}$ can be written as

$$\mathbf{B}(\mathbf{k},\eta) = -\frac{m_p(1+z)}{e\mathcal{H}^2}\int\frac{d^3k'}{(2\pi)^3}\mathbf{k}\times\mathbf{k}'\left[2\varphi'(|\mathbf{k}-\mathbf{k}'|)\varphi(k') - \frac{k'^2}{12\mathcal{H}^2}\varphi'(|\mathbf{k}-\mathbf{k}'|)\varphi(k') - \frac{k'^2}{12\mathcal{H}}\varphi(|\mathbf{k}-\mathbf{k}'|)\varphi(k')\right], \tag{17}$$

where we have neglected the last term in Eq. (7).

Up to the matter-radiation equality time the gravitational potential and its derivative are given by Eq. (12), and the correlation function of $\mathbf{B}$ modes can be computed in terms of that of $\varphi_0(\mathbf{k})$, which is a Gaussian random field with autocorrelation function

$$\langle\varphi_0(\mathbf{k})\varphi_0(\mathbf{k}')\rangle = (2\pi)^3 P_\varphi(k)\delta^3(\mathbf{k}-\mathbf{k}'), \tag{18}$$

where $P_\varphi(k)$ is the gravitational potential power spectrum, $P_\varphi(k) = P_{0\varphi}k^{-3}(k/k_0)^{n_s-1}$ and k_0 is some pivot wave number. For the normalization of P_φ we can use its relation with the power spectrum of the comoving curvature perturbation (see, e.g., Ref. [16]) $\mathcal{R}$, namely $\Delta^2_{\mathcal{R}}(k) = \Delta^2_{\mathcal{R}}(k_0)(k/k_0)^{n_s-1}$, where $\Delta^2_{\mathcal{R}}(k_0) = (25/9)(P_{0\varphi}/2\pi^2)$ [17]. Then, the resulting magnetic field is a *chi-square* distributed random field with power spectrum

$$S(k,\eta) = \frac{162m_p^2(1+z)^2}{e^2\mathcal{H}}\int\frac{d^3k'}{(2\pi)^3}P_\varphi(|\mathbf{k}-\mathbf{k}'|)P_\varphi(k')|\mathbf{k}\times\mathbf{k}'|^2\left[\frac{x_1}{4x_2}j_1(x_1)j_1(x_2) + j_2(x_2)j_1(x_1)\left(\frac{2}{x_1}-\frac{x_1}{4}\right)\right]^2, \tag{19}$$

where we have defined $x_1 = k'\tilde{\eta}/\sqrt{3}$, $x_2 = |\mathbf{k}-\mathbf{k}'|\tilde{\eta}/\sqrt{3}$.

The main contribution to the power spectrum comes from the term in $\mathbf{B}$ with a time derivative of φ [second term in Eq. (19)]. As φ tends to a constant after equality, the contribution to the generation of $\mathbf{B}$ during the period from equality to recombination is subdominant with respect to that up to equality time.

We show in Fig. 1 the power spectrum of $\mathbf{B}$ at recombination, $S(k)$, obtained from the numerical integration of Eq. (19), as a function of the comoving wave number k. We have used the following values for the model parameters: $\Delta^2_{\mathcal{R}}(k_0) = 2.3\times10^{-9}$, $z_{\rm EQ} = 3454$, $z_{\rm rec} = 1088$, $h = 0.7$ and $n_s = 1$ [16,18]. At small scales diffusion damps the fluctuations in the photon and baryon fluids. This effect can be taken into account by a factor $\exp(-k^2/k_D^2)$ multiplying the velocity perturbations (see e.g. [19]). As shown in the plots, Silk damping affects the produced magnetic field on scales smaller than about two comoving Mpc.

For small k ($k < 0.04$ Mpc^{-1}), $S(k)\propto k^2$, as pointed out in Ref. [15], in order for the magnetic field to be divergenceless. For wave numbers $k > 1$ Mpc^{-1} the magnetic field spectral index approaches instead $n\simeq -1$, i.e., $S(k)\propto k^{-1}$, for scale-invariant ($n_S = 1$) primordial perturbations. For a general scalar spectral index n_S, $S(k)\propto k^{2n_S-3}$.

The mean square value of the field on a given scale λ is obtained by averaging over a volume of size $\propto\lambda^3$ [15,20],

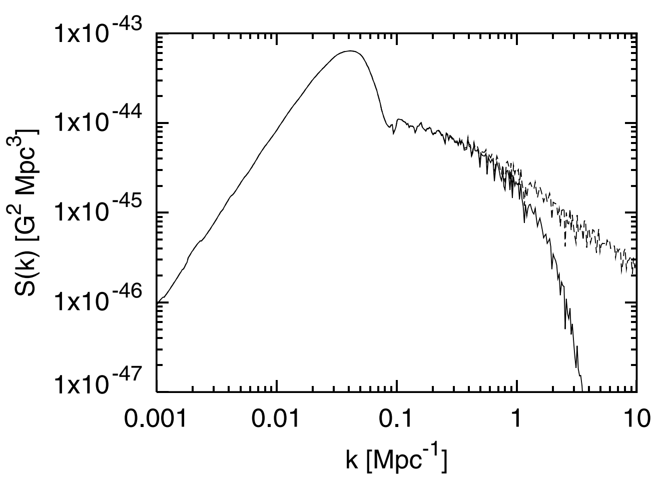

FIG. 1. Power spectrum of the magnetic field at recombination as a function of the comoving wave number. The upper and lower curves refer to the case with or without Silk damping included, respectively.

convolving with a Gaussian window function, namely

$$B_\lambda^2 = \int \frac{d^3k}{(2\pi)^3} S(k) \exp(-\lambda^2 k^2/2). \qquad (20)$$

For comoving scales larger than ~ 1 Mpc, up to 50 Mpc the rms magnetic field at recombination is well approximated by

$$B_\lambda(\eta_{\rm rec}) \approx 10^{-23} (\lambda/{\rm Mpc})^{-2}\ {\rm G}. \qquad (21)$$

After recombination the magnetic field can be considered to be frozen into the plasma and thus it redshifts with the expansion of the Universe as $\mathbf{B}(\eta) = \mathbf{B}(\eta_{\rm rec})(a/a_{\rm rec})^{-2}$. Then, the average field scaled to its value today results

$$B_\lambda(\eta_0) \approx 10^{-29} (\lambda/{\rm Mpc})^{-2}\ {\rm G}. \qquad (22)$$

IV. CONCLUSIONS

In this paper we have discussed a new mechanism for the generation of the cosmic seed magnetic field. It acts during the evolution of the Universe up to the epoch of hydrogen recombination. The underlying physical phenomenon is the vorticity of charged particles driven by gravitational vector modes. The latter, in turn, arise from the nonlinear evolution of purely scalar (density) primordial perturbations. Hence, our mechanism is a generic prediction of the standard hierarchical structure formation scenario and does not require any *ad hoc* assumption.

Maybe the most important feature of the created magnetic field discussed in this paper is that its power spectrum is fully determined by the power spectrum of the primordial density perturbations. As a consequence, a significant signal is expected over cosmological scales much greater than those encountered in other mechanisms acting during the early evolution of the Universe. It may be worth stressing that weak magnetic fields on megaparsec scales may also be generated through large-scale oblique shocks, which are expected as part of galaxy formation in the standard model of cosmic structure formation.

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APPENDIX: VORTICITY CONSERVATION

We want to illustrate here how one can generate a nonzero rotational component of the fluid velocity at second order in the (dominant) radiation fluid without violating the conservation of vorticity, dictated by Kelvin's circulation theorem, which is nonperturbative and applies to a perfect fluid with equation of state $p = p(\rho)$ coupled to gravity (see Ref. [21]).

As we can see from Eq. (8), we were indeed able to generate at second order a nonzero value for $\beta_i^{(\gamma)}$ (and also for $\zeta_i^{(\gamma)}$). And then, through Eq. (7), this $\zeta_i \simeq \zeta_i^{(p)} \simeq \zeta_i^{(\gamma)}$ acts as a source for the magnetic field. The crucial point is that none of these quantities (neither ζ_i nor β_i) coincides at second order with the quantity that is conserved, according to Kelvin's theorem.

We recall what the theorem states [21]. First of all one considers a system with a single perfect fluid (in our case this is the radiation fluid, in the limit in which the other components are negligible), with four-velocity u^μ. Then one constructs the quantity

$$v_{\mu\nu} = h_\mu^\alpha h_\nu^\beta u_{\alpha;\beta}, \qquad (A1)$$

where $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor into the rest frame of an observer moving with four-velocity u^μ. Then we split $v_{\mu\nu}$ into its symmetric and antisymmetric parts, namely

$$v_{\mu\nu} = \theta_{\mu\nu} + \omega_{\mu\nu}, \qquad (A2)$$

where the antisymmetric part $\omega_{\mu\nu}$ is called the *vorticity tensor*. The symmetric part $\theta_{\mu\nu}$ can be further split into its trace (*volume expansion*) and trace-free part (*shear tensor*):

$$\theta_{\mu\nu} \equiv \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu} \qquad \theta = u^\mu{}_{;\mu}. \qquad (A3)$$

Finally, one can construct the vorticity vector as [22]

$$\omega^\mu \equiv \frac{1}{2}\eta^{\mu\nu\rho\sigma} u_\nu \omega_{\rho\sigma}. \qquad (A4)$$

It is possible to show now, using the field equations [21], that this quantity obeys the equation

$$h_\nu^\mu \dot{(l^2\omega^\nu)} = \sigma_\nu^\mu (l^2 \omega^\nu) + \frac{l^2}{2}\eta^{\mu\nu\rho\sigma} u_\nu \dot{u}_{\rho;\sigma}, \qquad (A5)$$

where the overdot stands for the convective time derivative along u^μ and the scalar l is defined by

$$\frac{\dot{l}}{l} = \frac{1}{3}\theta, \qquad (A6)$$

thus representing the local expansion of the fluid in its rest frame (which coincides with the scale factor in an exactly homogeneous and isotropic Friedmann-Robertson-Walker model). Substituting into Eq. (A5) the momentum-conservation equation one can find (see Ref. [23]) that the following equation holds for $\omega \equiv \sqrt{\omega_\mu \omega^\mu}$:

$$\frac{\dot{\omega}}{\omega} + \frac{5}{3}\theta + \frac{\dot{(\rho + p)}}{\rho + p} = \sigma^{\mu\nu}\frac{\omega_\mu \omega_\nu}{\omega^2}, \qquad (A7)$$

for a perfect fluid with equation of state $p = p(\rho)$, immediately leading to the result that if ω is zero at some given initial time, then it will be always zero.

If we deal with ω as a small perturbation, the right-hand side of Eq. (A7) can be neglected (being of higher order due to the presence of $\sigma^{\mu\nu}$), so, we get the usual angular-momentum-conservation law (e.g., Ref. [24])

$$\omega(\rho + p)a^5 = \text{const}, \tag{A8}$$

which means that, even if we start with a small nonzero ω, it will decay with time at a rate which depends on the equation of state $p(\rho)$.

We can show indeed that at first order ω^i is related to the curl of the fluid velocity

$$\omega^i_{(1)} = -\frac{1}{2a^2}\epsilon^{ijk}(\partial_j v_k + \partial_j \chi_k) \tag{A9}$$

and it is well known that this vortical component is not generated by standard perturbation generating processes—such as inflation—so it can be safely set to zero.

At second order, though, the quantity ω^i does not coincide with the curl of the velocity ζ^i, nor with the quantity β^i, but it also contains squared first-order terms [25],

$$\omega^i = -\frac{1}{2a^2}[(a\beta^i) + \epsilon^{ijk}(3v_j\partial_k\varphi + v_j v_k')]. \tag{A10}$$

This shows that in order for ω^i to vanish β^i has to be generated. In other words, the momentum-conservation equation, whose general form for a perfect fluid at second order in the Poisson gauge is

$$(v_i + \chi_i)' + (1-3w)\mathcal{H}(v_i + \chi_i) + \frac{w}{1+w}\partial_i\delta + \partial_i\phi + \frac{w}{1+w}(3\varphi - \delta)\partial_i\delta - (2-3w)\varphi' v_i - w v_j^{\prime j} v_i + \frac{1}{2}(v^2)_{,i} + \frac{1}{2}(\varphi^2)_{,i} = 0, \tag{A11}$$

leads to a conservation equation for ω_i

$$\omega_i' + 3(1-w)\mathcal{H}\omega_i = 0, \tag{A12}$$

which is of course equivalent to Eq. (A8) above. In particular, it can be checked that the direct solution, Eq. (13), of Eq. (A11) (with $w = 1/3$) coincides with the expression of β_i which is found using the fact that ω^i in Eq. (A10) is always zero. This result is given in Eq. (14).

Finally we can understand why the magnetic field is generated by looking at Eq. (5). In fact, from there, we see that the magnetic field is sensitive to the vorticity of the charged matter components like protons. In a Universe with only self-gravitating pressureless matter, this would be conserved, so no magnetic field would be created. Note also that in the case of pure matter it is possible to show that the conserved vorticity vector is $a^3\omega^{(m)i} = -a^2\beta^{(m)i}/2$ (see Ref. [12]). Then, let us consider what happens to a subdominant matter fluid in a radiation dominated Universe. We may consider first, as an illustration, the case of a noninteracting matter component. Since we are in the radiation era the matter components yield a negligible contribution to gravity, and the potentials are driven by radiation. So the quantity $\beta^{(m)i}$ is no longer conserved but has a source (which is nonzero as long as φ' is not zero and its gradient is not parallel with that of φ). In fact it has to satisfy (the curl of) the momentum-conservation equation for a pressureless fluid, which is

$$\frac{(a\beta_i^{(m)})'}{a} + \mathcal{H}(\beta_i^{(m)}) - 2\frac{\epsilon_{ijk}\partial^j(\varphi' v^{k(m)})}{a} = 0. \tag{A13}$$

Finally, accounting for an interacting fluid (and so $v^{(p)} \simeq v^{(\gamma)}$ and $\beta^{(p)} \simeq \beta^{(\gamma)}$), Eq. (A13) is no longer satisfied. However, the left-hand side is compensated by the rotational part of a nonzero electromagnetic field, as one can see in Eq. (6), that we rewrite (neglecting once again the diffusion term and taking $v^{(p)} \simeq v^{(e)}$) as

$$\frac{(a\beta_i^{(m)})'}{a} + \mathcal{H}(\beta_i^{(m)}) - 2\frac{\epsilon_{ijk}\partial^j(\varphi' v^{k(m)})}{a} = -\frac{e}{m}\frac{(a^2 B_i)'}{a^2}. \tag{A14}$$

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