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Generalized monopoles in six-dimensional non-Abelian gauge theory

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A spherically symmetric monopole solution is found in $SO(5)$ gauge theory with Higgs scalar fields in the vector representation in six-dimensional Minkowski spacetime. The action of the Yang-Mills fields is quartic in field strengths. The solution saturates the Bogomolny bound and is stable.

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Long time ago Dirac showed that quantum mechanics admits a magnetic monopole of quantized magnetic charge despite the presence of a singular Dirac string [1,2]. A quantized Dirac string is an unphysical entity in the sense that it yields no physical, observable effect. Much later 't Hooft and Polyakov showed that such magnetic monopoles emerge as regular configurations in $SO(3)$ gauge theory with spontaneous symmetry breaking triggered by triplet Higgs scalar fields [3–6]. 't Hooft-Polyakov monopoles emerge in grand unified theory of electromagnetic, weak, and strong interactions as well. Although a monopole has not been found experimentally as a single particle, the existence of such objects has far reaching consequences. In the early universe, monopoles might have beed copiously produced, significantly affecting the history of the universe since then. In strong interactions, monopole configurations are believed vital for color and quark confinement.

In the superstring theory all matter and interactions including gravity are truely unified in ten spacetime dimensions. Six extra dimensions may be compactified in a small size, or the observed four-dimensional spacetime can be a brane immersed in ten-dimensional spacetime. It is important in this scenario to explore solitonic objects in higher dimensional spacetime, which may play an important role in compactfying extra dimensions, or in producing and stabilizing brane structures. Recent extensive study of domain walls in supersymmetric theories, for instance, may have a direct link to the brane world scenario [7]. In this paper we explore and establish solitons with finite energies in higher dimensional spacetime.

The energy of 't Hooft-Polyakov monopoles is bound from below by a topological charge. Monopole solutions saturate such bound, thereby the stability of the solutions being guaranteed by topology [8]. This observation prompts a question if there can be a monopole solution in higher dimensions. Kalb and Ramond introduced Abelian tensor gauge fields coupled to closed strings [9]. Nepomechie showed that a new type of monopole solutions appear in those Kalb-Ramond antisymmetric tensor gauge fields [10]. Their implications to the confinement [11] and to ten-dimensional Weyl invariant spacetime [12] has been explored. Topological defects in six-dimensional Minkowski space-time as generalization of Dirac's monopoles were also found [13]. Tchrakian has investigated monopoles in non-Abelian gauge theory in higher dimensions whose action involves polynomials of field strengths of high degrees [14,15]. Further, it has been known that magnetic monopoles appear in the matrix model in the gauge connections describing Berry's phases on fermi states. In particular, in the USp matrix model they are described by $SU(2)$-valued anti-self-dual connections [16].

The purpose of this paper is to present regular monopole configurations with saturated Bogomolny bound in $SO(5)$ gauge theory in six dimensions. Although the existence of such solutions has been suspected by Tchrakian for a long time, the explicit construction of solutions has not been given. We stress that the monopole solution presented below is the first example of a soliton in non-Abelian gauge theory in higher dimensions which is regular everywhere and has a finte energy.

Let us recall that in 't Hooft-Polyakov monopoles in four dimensions, both $SO(3)$ gauge fields and scalar fields are in the vector representation. In three space dimensions the Bogomolny equations for those fields match both in space indices and internal $SO(3)$ indices. This correspondence seemingly becomes obscure when space dimensions are greater than three. A key to find correct Bogomolny equations is facilitated with the use of the Dirac or Clifford algebra.

Consider $SO(5)$ gauge theory in six dimensions. Gauge fields $A_\mu^{ab} = -A_\mu^{ba}$ are in the adjoint representation, whereas scalar fields ϕ^a are in the vector representation ($a, b = 1 \sim 5$). To interrelate these two, we introduce a basis $\{\gamma_a\}$ of the Clifford algebra; $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ ($a, b = 1 \sim 5$). We write $\phi \equiv \phi^a \gamma_a$ and $A = 1/2 A_\mu^{ab} \gamma_{ab} dx^\mu$ where $\gamma_{ab} = 1/2[\gamma_a, \gamma_b]$. The field strength 2-form is given by $F = F(A) \equiv dA + gA^2$ where g is the gauge coupling constant. Similarly, a covariant derivative 1-form of ϕ is given by $D_A\phi \equiv d\phi + g[A, \phi]$. Under a gauge transformation, $A \rightarrow \Omega A \Omega^{-1} + (1/g)\Omega d\Omega^{-1}$, $F \rightarrow \Omega F \Omega^{-1}$, and $D_A\phi \rightarrow \Omega D_A \phi \Omega^{-1}$, where $\Omega = \exp\{\varepsilon_{ab}(x)\gamma^{ab}\}$

The action is given by

$$\begin{aligned} I &\equiv \int\Big[-\frac{1}{8\cdot 4!}\mathrm{Tr}F^2 * F^2 - \frac{1}{8}\mathrm{Tr}D_A\phi * D_A\phi \\ &\quad - \frac{\lambda}{4!}(\phi^a\phi_a - H_0^2)^2 d^6x\Big] \\ &= \int d^6x\Big\{-\frac{1}{8\cdot 4!}\mathrm{Tr}(F^2)_{\mu\nu\rho\sigma}(F^2)^{\mu\nu\rho\sigma} \\ &\quad -\frac{1}{2}D_\mu\phi^a D^\mu\phi_a + \lambda(\phi^a\phi_a - H_0^2)^2\Big\} \end{aligned} \tag{1}$$

Here the components of $F^2 = \frac{1}{8}\{F_{\mu\nu}, F_{\rho\sigma}\}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$ are given by

$$\begin{aligned} (F^2)_{\mu\nu\rho\sigma} &= T^e_{\mu\nu\rho\sigma}\gamma_e - S_{\mu\nu\rho\sigma}, \\ T^e_{\mu\nu\rho\sigma}(A) &= \frac{1}{2\cdot 4!}\epsilon^{abcde}(F^{ab}_{\mu\nu}F^{cd}_{\rho\sigma} + F^{ab}_{\mu\rho}F^{cd}_{\sigma\nu} + F^{ab}_{\mu\sigma}F^{cd}_{\nu\rho}), \\ S_{\mu\nu\rho\sigma}(A) &= \frac{1}{4!}(F^{ab}_{\mu\nu}F^{ab}_{\rho\sigma} + F^{ab}_{\mu\rho}F^{ab}_{\sigma\nu} + F^{ab}_{\mu\sigma}F^{ab}_{\nu\rho}), \end{aligned} \tag{2}$$

so that in the action $\frac{1}{4}\mathrm{Tr}(F^2)_{\mu\nu\rho\sigma}(F^2)^{\mu\nu\rho\sigma} = T^e_{\mu\nu\rho\sigma}T_e^{\mu\nu\rho\sigma} + 4S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma}$. The action of this type has been considered in Ref. [14]. The relations in (2) are special to $SO(5)$ gauge theory.

The action is quartic in $F_{\mu\nu}$, but is quadratic in F_{0k}. The Hamiltonian is positive semidefinite and is bounded from below by a topological charge. To see it, first notice that

$$\begin{aligned} T^e_{0jkl} &= F^{ab}_{0i}M^{ab,e}_{i,jkl}, \quad M^{ab,e}_{i,jkl} = \frac{1}{2\cdot 4!}\epsilon^{abcde}L^{cd}_{i,jkl}, \\ S_{0jkl} &= F^{ab}_{0i}N^{ab}_{i,jkl}, \quad N^{ab}_{i,jkl} = \frac{1}{4!}L^{ab}_{i,jkl}, \\ L^{cd}_{i,jkl} &= \delta_{ij}F^{cd}_{kl} + \delta_{ik}F^{cd}_{lj} + \delta_{il}F^{cd}_{jk}. \end{aligned} \tag{3}$$

The canonical conjugate momentum fields are given by

$$\begin{aligned} \Pi^{ab}_i &= \frac{\delta I}{\delta \dot{A}^{ab}_i} = \frac{1}{3!}T^e_{0jkl}\frac{\delta T^e_{0jkl}}{\delta \dot{F}^{ab}_{0i}} + \frac{4}{3!}S_{0jkl}\frac{\delta S_{0jkl}}{\delta \dot{F}^{ab}_{0i}} \\ &= \frac{1}{3}(M^{ab,e}_{i,jkl}M^{cd,e}_{m,jkl} + N^{ab}_{i,jkl}N^{cd}_{m,jkl})F^{cd}_{0m} \\ &\equiv U^{ab,cd}_{i,m}F^{cd}_{0m}. \end{aligned} \tag{4}$$

U is a symmetric, positive-definite matrix. To confirm the positivity of the Hamiltonian, we take the $A_0 = 0$ gauge in which $F^{ab}_{0i} = \dot{A}^{ab}_i$. It immediately follows that

$$\begin{aligned} E &= \int d^5x\Big[\frac{1}{2}\Pi U^{-1}\Pi + \frac{1}{2\cdot 4!}\{(T^e_{ijkl})^2 + (S_{ijkl})^2\} \\ &\quad + \mathcal{H}_\phi\Big] \\ &\geq 0, \end{aligned} \tag{5}$$

where $\mathcal{H}_\phi$ is the scalar field part of the Hamiltonian density.

In the $A_0 = 0$ gauge the energy becomes lowest for static configurations $\dot{A}^{ab}_i = \dot{\phi}_a = 0$. It is given by

$$\begin{aligned} E &= \int d^5x\frac{1}{4!}\Big[\frac{1}{2}(T^e_{ijkl} \mp \epsilon^{ijklm}D_m\phi^e)^2 + \frac{1}{2}S^2_{ijkl} \\ &\quad \pm \epsilon^{ijklm}T^e_{ijkl}D_m\phi^e + \lambda(\phi^a\phi_a - H_0^2)^2\Big] \\ &\geq \pm\int d^5x\frac{1}{4!}\epsilon_{ijklm}T^e_{ijkl}D_m\phi^e \\ &= \pm\int \mathrm{Tr}D_A\phi F^2 \\ &\equiv \frac{16\pi^2}{g^2}H_0\mathcal{Q}. \end{aligned} \tag{6}$$

As $D_AF = 0$ and therefore $\mathrm{Tr}D_A\phi F^2 = d(\mathrm{Tr}\phi F^2)$, $\mathcal{Q}$ can be expressed as a surface integral

$$\mathcal{Q} = \pm\frac{g^2}{16\pi^2 H_0}\int_{S^4}\mathrm{Tr}\phi F^2, \tag{7}$$

where S^4 is a space infinity of R^5.

$\mathcal{Q}$ is a charge $\int d^5xk^0$ of a 6-dimensional current k^μ defined by $k = k_\mu dx^\mu = \pm * (g^2/16\pi^2H_0)\mathrm{Tr}D_A\phi F^2$, which is conserved, $d * k = 0$. $\mathcal{Q}$ can also be viewed as a topological charge associated with Abelian Kalb-Ramond 3-form gauge fields whose 4-form field strength $\mathcal{G}$ is given by [14]

$$\begin{aligned} \mathcal{G} &= \mathrm{Tr}\Big\{\hat{\phi}F^2 + \frac{1}{2g}\hat{\phi}(D_A\hat{\phi})^2F + \frac{1}{16g^2}\hat{\phi}(D_A\hat{\phi})^4\Big\} \\ &= \mathrm{Tr}\hat{\phi}\Big\{F + \frac{1}{4g}(D_A\hat{\phi})^2\Big\}^2. \end{aligned} \tag{8}$$

Here $\hat{\phi} = \phi/|\phi|$, $|\phi| = \sqrt{\phi^a\phi^a}$ and $D_A\hat{\phi} = d\hat{\phi} + g[A, \hat{\phi}]$.

It is the salient feature of $\mathcal{G}$ given in (8) that it can be written as

$$\begin{aligned} \mathcal{G} &= dC + \frac{1}{16g^2}\mathrm{Tr}\hat{\phi}(d\hat{\phi})^4, \\ C &= \frac{1}{2g}\mathrm{Tr}\hat{\phi}\Big\{(d\hat{\phi})^2A + g(d\hat{\phi}A\hat{\phi}A + dAA + AdA) \\ &\quad + g^2\Big(A^3 + \frac{1}{3}A\hat{\phi}A\hat{\phi}A\Big)\Big\}. \end{aligned} \tag{9}$$

C does not have a singularity of the Dirac string type where $|\phi| \neq 0$. $\mathcal{G}$ and C are the 't Hooft 4-form field strengths and the corresponding Kalb-Ramond 3-form fields in six dimensions, respectively. The expression (9) is valid in the entire six-dimensional spacetime. We remark that the Kalb-Ramond 3-form fields C in (9) is almost the same as those in Ref. [15] where A is replaced by the asymptotic one which is valid only at $r \to \infty$ (on S^4). We also note that for configurations with $\hat{\phi} = \gamma^5$, only gauge fields in the unbroken $SO(4)$, $\hat{A} = \frac{1}{2}\sum_{a,b=1}^4 A^{ab}_\mu\gamma_{ab}dx^\mu$, contribute in

(8) and (9). Indeed, $\mathrm{Tr}\gamma^5(dAA + AdA) = \mathrm{Tr}\gamma^5(d\hat{A}\,\hat{A} + \hat{A}d\hat{A})$ and $\mathrm{Tr}[\hat{\phi}A^3 + \frac{1}{3}(\hat{\phi}A)^3] = \frac{1}{6}\mathrm{Tr}\{\hat{\phi}, A\}^3 = \frac{1}{6}\mathrm{Tr}\{\hat{\phi}, \hat{A}\}^3$.

As $D_A\hat{\phi} = 0$ on S^4 at space infinity for any configuration with a finite energy, $\mathcal{G}$ coincides with $\mathrm{Tr}\hat{\phi}F^2$ on S^4. Hence

$$\mathcal{Q} = \frac{g^2}{16\pi^2 H_0}\int_{S^4}|\phi|\mathcal{G} = \frac{1}{256\pi^2}\int_{S^4}\mathrm{Tr}\hat{\phi}(d\hat{\phi})^4. \tag{10}$$

In the second equality we used the fact that C is regular in S^4 as $|\phi| \sim H_0$. The quantity appearing in the last equality in (10) is the winding number. The charge $\mathcal{Q}$ is thus regarded as the magnetic charge associated with Abelian Kalb-Ramond field strengths $\mathcal{G}$.

The Bogomolny bound equation is

$$*_5(F \wedge F) = \pm D_A\phi, \tag{11}$$

where $*_5$ is Hodge dual in five-dimensional space. In components it is given by

$$\epsilon^{ijklm}T^e_{ijkl} = \pm D_m\phi^e, \quad S_{ijkl} = 0. \tag{12}$$

Let us define $e \equiv x^a\gamma_a/r$. We make a hedgehog ansatz [15]

$$\phi = H_0 U(r)e, \quad A = \frac{1 - K(r)}{2g}ede. \tag{13}$$

It follows immediately that

$$D_A\phi = H_0(KUde + U'edr), \quad F = \frac{1 - K^2}{4g}de \wedge de - \frac{K'}{2g}edr \wedge de. \tag{14}$$

Boundary conditions are given by $U(0) = 0$, $K(0) = 1$, $U(\infty) = \pm 1$, and $K(\infty) = 0$.

With the use of $*_5(de \wedge de \wedge de \wedge de) = 4!edr/r^4$ and $*_5(edr \wedge de \wedge de \wedge de) = 3!de/r^2$, the Bogomolny bound Eq. (11) (with a plus sign) becomes

$$KU = -\frac{1}{\tau^2}(1 - K^2)\frac{dK}{d\tau}, \quad \frac{dU}{d\tau} = \frac{1}{\tau^4}(1 - K^2)^2, \quad \tau = ar, \quad a = \left(\frac{2g^2}{3}H_0\right)^{1/3}. \tag{15}$$

In this case U increases as τ so that $U(\infty) = 1$. A solution in the case $-D_A\phi = *_5(F \wedge F)$ is obtained by replacing U by $-U$. We note that the two equations in (15) can be combined to yield

$$\frac{d}{d\tau}\left(\frac{1 - K^2}{\tau^2 K}\frac{dK}{d\tau}\right) + \frac{(1 - K^2)^2}{\tau^4} = 0, \tag{16}$$

or equivalently, in terms of $s = \ln\tau$ and $f(s) = K^2$,

$$f'' - \left\{3 + \frac{f'}{f(1 - f)}\right\}f' + 2f(1 - f) = 0. \tag{17}$$

The Eq. (16) with the boundary conditions $K(0) = 1$ and $K(\infty) = 0$ is scale invariant, i.e., if $K(\tau)$ is a solution, so is $K(\alpha\tau)$ with arbitrary $\alpha > 0$. However, $U(\tau)$ changes, under this transformation, to $\alpha^{-3}U(\alpha\tau)$ in (15) so that the boundary condition $U(\infty) = 1$ is fulfilled only with a unique value for α.

The behavior of the solution near the origin is given by

$$K = 1 - b\tau^2 + \frac{5}{14}b^2\tau^4 + \cdots, \quad U = 4b^2\tau\left\{1 - \frac{4}{7}b\tau^2 + \frac{20}{63}b^2\tau^4 + \cdots\right\}. \tag{18}$$

The value of the parameter b needs to be determined such that $U(\infty) = 1$ is satisfied. The behavior of the solution at a space infinity $\tau = \infty$ is given by

$$K \sim K_0 e^{-\tau^3/3}, \quad U \sim 1 - \frac{1}{3\tau^3}. \tag{19}$$

Note that $F \sim (4g)^{-1}de \wedge de$ and $D_A\phi \sim H_0\tau^{-4}ed\tau$.

A solution is obtained numerically. We adopted the shooting method to solve Eq. (15) from $\tau = 0$ to $\tau = \infty$. Precisely tuning the value of b in (18), we find a solution with the boundary conditions $U(\infty) = 1$ and $K(\infty) = 0$. It is found that $b = 0.494$ and $K_0 = 1.2$. The solution is displayed in Fig. 1.

The energy, (6), of the solution is given by $\int \mathrm{Tr}D_A\phi F^2$ in the $\lambda \to 0$ limit. The insertion of (14) leads, with the aid of the identities $\mathrm{Tr}edr(de)^4 = (4 \cdot 4!/r^4)d(\text{volume})$ and $\mathrm{Tr}(de)^5 = 0$, to

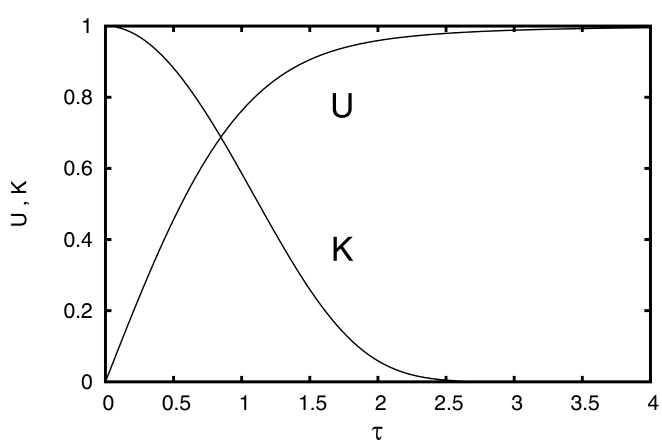

FIG. 1. Solution : $U(\tau)$ and $K(\tau)$ in (13).

$$E = \int d^5x \frac{H_0}{16g^2}\frac{4\cdot 4!}{r^4}\left\{(1-K^2)^2\frac{dU}{dr} - 4\frac{dK}{dr}UK(1-K^2)\right\} = \frac{16\pi^2}{g^2}H_0[(1-K^2)^2U]_0^\infty = \frac{16\pi^2}{g^2}H_0. \tag{20}$$

The same result follows from $E = (16\pi^2 H_0/g^2)\mathcal{Q}$ as $\mathcal{Q} = 1$.

As Dirac showed, a monopole configuration in $U(1)$ gauge theory in four dimensions necessarily has a Dirac string, or a singular point in gauge potentials on the space infinity S^2. Quantization of Dirac strings, or monopole charges, corresponds to nontrivial mapping around the singular point, or the hole, on S^2, namely $\pi_1[U(1)]$. The configuration of a 't Hooft-Polyakov monopole in the $SO(3)$ gauge theory is regular everywhere and the monopole charge is related to the winding number of the Higgs fields, which breaks $SO(3)$ to $U(1)$. This fact is summarized in the exact sequence in the homotopy group

$$\mathrm{Ker}\{\pi_1[U(1)] \to \pi_1[SO(3)]\} \simeq \pi_2(S^2). \tag{21}$$

In our case $SO(5)$ gauge symmetry is broken to $SO(4)$ by the Higgs fields ϕ^a. A monopole in $SO(4)$ gauge theory in six dimensions accompanies a singularity in gauge potentials on the space infinity S^4. Quantization of monopole charges is associated with $\pi_3[SO(4)]$. The singularity is lifted by embedding $SO(4)$ into $SO(5)$, and the monopole charge is reduced to the winding number $\pi_4(S^4)$ of the Higgs fields. The relation is summarized in

$$\mathrm{Ker}\{\pi_3[SO(4)] \to \pi_3[SO(5)]\} \simeq \pi_4(S^4). \tag{22}$$

Thus we observe that generalized monopoles in $SO(5)$ gauge theory in six dimensions described in the present paper are completely parallel to 't Hooft-Polyakov monopoles in four dimensions.

As another interesting aspect, the generalized monopole solution presented in this paper may realize the electric-magnetic duality in the M-theory of strings. The 3-form Kalb-Ramond fields C defined in (9) couple to 2-branes in 11 dimensions. Dual of the field strength dC is the 7-form field strength so that the generalized monopole can be regarded as a source to the corresponding 6-form Kalb-Ramond fields, namely, a 5-brane in 11 dimensional space-time. A similar argument applies to 2- and 4-branes in ten dimensions.

In this paper we have shown that there exists a regular, spherically symmetric monopole solution in the six-dimensional $SO(5)$ gauge theory with the action quartic in field strengths. This is the first example of particlelike solitons in space dimensions bigger than four. The energy in the $\lambda \to 0$ limit is given by the monopole charge associated with the Abelian Kalb-Ramond 3-form fields. The connection between $SO(5)$ gauge fields and the Kalb-Ramond fields is given by the generalized 't Hooft tensors $\mathcal{G}$ and C. The solution is stable. Physical consequences of these generalized monopoles are yet to be investigated. They affect the evolution of the universe at the very early stage, should there exist extra dimensions. Their role for the compactification of extra dimensions and their relation to extended objects in the matrix models need to be clarified. Generalization of solutions to multimonopole states is also awaited. We hope to come back to these points in future publications.

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