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Quantum Hall states of gluons in dense quark matter

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We have recently shown that dense quark matter possesses a color ferromagnetic phase in which a stable color-magnetic field arises spontaneously. This ferromagnetic state has been known to be Savvidy vacuum in the vacuum sector. Although the Savvidy vacuum is unstable, the state is stabilized in the quark matter. The stabilization is achieved by the formation of quantum Hall states of gluons, that is, by the condensation of the gluon's color charges transmitted from the quark matter. The phase is realized between the hadronic phase and the color superconducting phase. After a review of quantum Hall states of electrons in semiconductors, we discuss the properties of quantum Hall states of gluons in quark matter in detail. Especially, we evaluate the energy of the states as a function of the coupling constant. We also analyze solutions of vortex excitations in the states and evaluate their energies. We find that the states become unstable as the gauge coupling constant becomes large, or the chemical potential of the quarks becomes small, as expected. On the other hand, with the increase of the chemical potential, the color superconducting state arises instead of the ferromagnetic state. We show the region of the chemical potential of the quarks in which the color ferromagnetic phase is realized. We also show that the quark matter produced by heavy ion collisions generates observable strong magnetic field $\sim 10^{14}$ G when it is in the ferromagnetic phase.

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I. INTRODUCTION

Quark matter is known or expected to have several phases, hadronic phase, quark-gluon plasma phase and color superconducting phase [1]. When the density of the quarks is small, the hadronic phase arises in low temperature owing to very strong gluonic interactions. Thus, the quarks in the phase are confined [2,3] in the hadrons. On the other hand, when the density of the quarks is sufficiently large, the color superconducting phase is expected to arise. In such a case, gluonic interactions are small so that attractive forces operate perturbatively between antitriplet pair of the quarks. Thus, the condensation of the pairs arises to make the superconducting phase realized. When the temperature is sufficiently high, quark matter in both phases melts and forms the quark-gluon plasma. Among the phases only the hadronic phase is observed. Although the color superconducting phase is very intriguing, present experiments could not produce the phase because large chemical potentials of the quarks such as 1 GeV is needed for the production of the phase.

We have recently discussed [4] a possibility of the stable color ferromagnetic states in dense quark matter. The ferromagnetic state is caused by the condensation of the color-magnetic field, not by the alignment of the quark's magnetic moments. The states are realized between the hadronic state and the color superconducting state when the chemical potential is varied. Thus, the phase could be observed in the present experiments. The ferromagnetic states possess a spontaneously generated color-magnetic field in maximal Abelian subalgebra and also involve a quantum Hall state of off-diagonal gluons. The gluons have been known to have unstable modes [5] in the color-magnetic field $\mathcal{B}$. They occupy the lowest Landau level with their spins pointed to the magnetic field and with their energies being imaginary. The existence of these unstable modes implies that the naive ferromagnetic state (Savvidy vacuum [6]) is unstable. We have recently shown [4] that the formation of the gluon's quantum Hall state (QHS) caused by the condensation of the unstable modes stabilizes the ferromagnetic state.

In this paper we review QHS [7] of electrons in semiconductor and Chern-Simons gauge theory [8] for describing the QHS in the next section. This is because the phenomena and its theory are not popular in hadron physicists. In Sec. III, applying the theory to the unstable gluons, we discuss the properties of QHS of the gluons; incompressibility, Laughlin quasiparticle, etc. Especially, we numerically show that the energy of the Laughlin quasiparticle becomes smaller as the gauge coupling constant becomes larger. Since it may vanish at the infinite coupling constant, a bound state of the quasiparticle and antiquasiparticle is expected to a state with zero energy even at finite coupling constant. This implies that the QHS of the gluons becomes unstable against the creation of the bound states at the coupling constant. These excitations destroy Laughlin state of the gluons. As a result, the QHS decays and the ferromagnetic state also decays. Instead, quark confining state would appear at such large coupling constant. In other word, at such small chemical potentials the hadronic phase arises instead of the ferromagnetic phase. In the Sec. IV we show that the color superconducting state is more favored than the ferromagnetic state when the number density of the quarks is sufficiently large. In Sec. V we determine the critical chemical potential beyond which the color ferromagnetic phase arises. In Sec. VI we consider phenomenological implications of the color ferromagnetic states of the quark matter. Finally in Sec. VII we discuss that the existence of the ferromagnetic phase is a very natural consequence in the gluon and quark dynamics.

II. QUANTUM HALL STATE OF ELECTRONS

A. Integer quantum Hall state

QHS of electrons was discovered [9] in 1980 by von Klitzing. He has observed quantized Hall conductivities σ_{xy} with the unit of the fundamental constant $e^2/2\pi\hbar$ in a two-dimensional quantum well fabricated of a semiconductor. (In the notation of σ_{xy}, the direction of x is that of the electric current flowing and the direction of y is that of the Hall voltage arising.) It is called quantum Hall effect. The observation indicated the existence of a specific state of two-dimensional electrons in the well under the strong magnetic field, B, typically 10^5 G. In these experiments, electrons are trapped in two-dimensional quantum well with its width ~ 10 nm so that their motions are restricted in two-dimensional space. In order to move in a direction perpendicular to the space, electrons need to gain energies ~ 100 eV. Thus, in experiments with low temperature ~ 1 K, electrons move only in the two-dimensional space.

The two-dimensional electrons in the magnetic field make cyclotron motions with their radius $\ell_B = 1/\sqrt{eB}$ and their states are specified by Landau levels. Each of them has a large number of degenerate states; the degeneracy per unit area is given by $eB/2\pi$. The original QHS was called integer quantum Hall state since the state is observed at filling factor being integer; the filling factor is defined as $\rho_e/(eB/2\pi)$ (ρ_e is the two-dimensional number density of electrons, typically $10^{11}/\mathrm{cm}^2$.) Thus, the filling factor means a fraction of electron occupation in a Landau level. For example, the filling factor $\nu = 1/3$ implies that electrons occupy a third of the lowest Landau level. The integer filling factor implies that some of Landau levels are completely occupied.

Integer quantum Hall effect can be understood as a localization property of each two-dimensional electrons; some of them are localized due to impurities and some are not localized in the magnetic field in spite of the impurities. In general, all two-dimensional electrons must be localized around impurities. This is well known as Anderson localization. Then, the system is an insulator because there are no carriers of electric currents; localized electrons do not carry the currents. But the localization theorem does not hold when the magnetic field is present. We note that the effect of the impurities splits the degeneracy of the states in a Landau level. Thus, the density of the states is not of the delta function of the electron's energy, but has a finite width. Under this circumstance, almost all of the states are still localized. But, electrons occupying the states around the center of the Landau level are extended all over the system so that they can carry electric currents. This fact generates plateaus around $\nu =$ integers in Hall conductivity vs magnetic field diagram. In this way, integer quantum Hall effects are caused by the interplay of the impurities and each electron, and many-body correlations among electrons are not important.

B. Fractional quantum Hall state

Fractional quantum Hall effects were observed [10] in 1982 by Tsui at the filling factor being fractional numbers, e.g., $1/3, 2/3$. (He observed plateaus at such fractional filling factors.) Electrons occupy a fraction in the lowest Landau level. The QHSs have been understood to be caused by many-body effects of electrons, just like superconductivity. Impurities do not play important roles in these QHSs. Laughlin [11] proposed a wave function for this QHS, called Laughlin wave function at the filling factor being $1/n$,

$$\Psi = \prod_{i,j}(z_i - z_j)^n \exp\left(-eB\sum_i |z_i|^2/4\right), \qquad (1)$$

with $z = x + iy$ denoting complex coordinate of electrons with charge $-e$, where we have used a symmetric gauge potential $\vec{A}^B = (yB/2, -xB/2, 0)$ for the magnetic field. Here, n is an odd integer for the Fermi statistics of electrons. Numerical simulations show that the ground states of the electrons at the fractional filling factors are well described by the Laughlin wave functions even if the repulsive Coulomb interaction is replaced by a delta function; the precise form of the interaction between electrons is not important for the realization of QHS.

In general, a system of electrons partially occupying the lowest Landau level is compressible, namely, the system has no gap; excitation energies are distributed continuously above the ground-state energy. On the other hand, the QHS has a gap just like the BCS state. Hence, the QHS is characterized as a state with gap. We should remember that free electron gas has no gap so that even with the Coulomb interaction taken into account the gas does not gain the gap in general. But the BCS states are gapped states formed from Fermi gas with a small attractive force among electrons around Fermi surface. Similarly, the QHSs are gapped states formed from the gas of two-dimensional electrons interacting repulsively with each other under the strong magnetic field. These gapped states arise at the fractional filling factors.

If we add an electron to the QHS with the filling factor $\nu = 1/n$ there appear n Laughlin's quasiparticles each of which possesses a fractional charge of $-e/n$. On the other hand, if we extract an electron from the state, there appear n Laughlin's antiquasiparticles, each of which possesses a fractional charge of e/n. They carry electric currents. Therefore, in the fractional QHS only these quasiparticles carry electric currents so that the Hall conductivity is given by $e/2\pi \times e/n$. The plateaus at the fractional filling factor are understood as the localization properties of these quasiparticles, not of electrons. The excitations on the QHS are given by bound states of the quasiparticle and antiquasiparticle; they attract each other because of the Coulomb interaction. The gap of the QHS is given by the energy of this bound state of the Laughlin's quasiparticles.

C. Chern-Simons gauge theory of quantum Hall state

We have the BCS theory for understanding superconducting states. But we do not have a similar theory for the QHS. Namely, there is no appropriate theory of fermionic electrons producing Laughlin wave functions. We understand simply that numerical simulations confirm the validity of the wave functions and the gap in the state. In the case of the superconductivity we have a bosonic theory well known as Landau-Ginzburg theory. Similarly, we have Chern-Simons gauge theory of bosonized electrons for understanding the QHS. In this section, we wish to explain it.

It is well known that BCS states are described by Landau-Ginzburg effective Lagrangian,

$$L_{\mathrm{BCS}} = |(i\partial_\mu + 2eA_\mu)\phi|^2 + m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2}$$

where ϕ and A_μ denote Cooper pair of electrons and electromagnetic fields, respectively. The ground state is given by $\langle\phi\rangle = \sqrt{m^2/2\lambda}$. Namely, it is a condensed state of the Cooper pairs. Since the gauge symmetry, $\phi \to \phi e^{i\Lambda(x)}$ and $A_\mu \to A_\mu + \partial_\mu\Lambda/2e$, is spontaneously broken, we have vortex excitations with the magnetic flux $2\pi/2e$. These are magnetic vortices penetrating superconductors. When we switch off the gauge interaction, we have an effective model of superfluids.

Similarly, there is a bosonic theory for the QHS of electrons. It is a theory of composite electrons described by Chern-Simons gauge theory [8,12],

$$L_{\mathrm{QHS}} = \phi_{\mathrm{e}}^\dagger(i\partial_0 - a_0)\phi_{\mathrm{e}} + \mathrm{c.c.} - \frac{1}{2m_{\mathrm{e}}}|(i\partial_i + eA_i^B - a_i)\phi_{\mathrm{e}}|^2 - V_{\mathrm{Coulomb}} + \frac{1}{4\alpha}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\nu \tag{3}$$

where m_{e} denotes mass of electrons and $V_{\mathrm{Coulomb}} = \int d^2x d^2y[e|\phi_{\mathrm{e}}(x)|^2 - \bar{\rho}](1/2|x-y|)[e|\phi_{\mathrm{e}}(y)|^2 - \bar{\rho}]$ describes the Coulomb interaction between electrons with background positive charges $\bar{\rho}$. The term A_μ^B describes the external magnetic field imposed for the realization of QHS and has no kinetic term. The factor of α should be taken as $\pi \times$ odd integer for the boson field ϕ_{e} describing real electrons.

The boson field ϕ_{e} describes composite electrons; boson ϕ_{e} attached with flux of a_i. That is, particles with Fermi statistics can be described in two-dimensional space by bosonic particles attached with a fictitious flux 2α of Chern-Simons gauge field a_i. Owing to this flux, the exchange of the bosonic particles induces a phase $e^{i\alpha}$ in their wave function. Thus, with the choice of $\alpha = \pi \times$ odd integer, the wave functions describe particles with Fermi statistics. This situation is mathematically described by L_{QHS}.

Using the Hamiltonian derived from the Lagrangian, we can obtain the Schrödinger equation for electrons with Fermi statistics. In that sense, the Lagrangian correctly describes the two-dimensional system of electrons in the magnetic field. We should note that if Chern-Simons gauge fields are absent in the above Lagrangian and ϕ_{e} obeys Fermi statistics, the Lagrangian describes ordinary electron system.

Using equations of motions derived from L_{QHS}, we can see that QHSs are obtained as ground-state solutions such as $\langle\phi_{\mathrm{e}}\rangle \neq 0$ similar to the case of BCS states. These solutions can be obtained only for the case that the relation, $eA_i^B = a_i$, holds. Namely, the magnetic field eA_i^B is canceled by the Chern-Simons gauge field a_i; the field can be represented by the density ρ_{e} of electrons $\phi_{\mathrm{e}}^\dagger\phi_{\mathrm{e}}$ such as $\phi_{\mathrm{e}}^\dagger\phi_{\mathrm{e}} = -\epsilon_{ij}\partial_i a_j/2\alpha$, an equation derived by taking variational derivative of L_{QHS} in a_0. Hence, the solutions can be obtained only when the filling factor $\nu = 2\pi\rho_{\mathrm{e}}/eB$ is given by π/α. In this way we can understand that QHSs are condensed states of bosonized electrons ϕ_{e} and are realized only at $\nu = 1/3, 1/5$, etc., for $\alpha = 3\pi, 5\pi$, etc.

In order to see [12] that the states are really QHSs with appropriate Hall conductivities, σ_{xy}, we derive σ_{xy} in the following. We introduce a gauge potential A_μ of electric field $\vec{E} = -\partial_0\vec{A} - \vec{\partial}A_0$ in L_{QHS}; $\phi_{\mathrm{e}}^\dagger(i\partial_0 - a_0 + eA_0)\phi_{\mathrm{e}} + \cdots$. Shifting the integration variable of a_μ in the functional integral $Z(A_\mu) = \int D\phi_{\mathrm{e}}Da_\mu \times \exp(i\int d^2x dt L_{\mathrm{QHS}})$ such as $a_\mu \to a_\mu + eA_\mu$, we calculate electric current $j_x = -i\partial_{A_x}\log Z$, which is given by

$e^2E_y/2\alpha$ in the state of $\langle\phi_e\rangle = \sqrt{\rho_e}$ and $\langle -\partial_0 a_y - \partial_y a_0\rangle = 0$. Thus, the Hall conductivity is correctly given by $\nu e^2/2\pi$. In this way we find that the condensed state of the bosonized electrons is the QHS. [One (A. I.) of the authors has previously shown [13] that Laughlin wave functions can be derived from the condensed states of the field ϕ_e.]

Laughlin's quasiparticles are excited states of the QHS. In the picture of the bosonized electrons, they are presented by vortex excitations on the state of $\langle\phi_e\rangle = \sqrt{\rho_e} \neq 0$, where U(1) gauge symmetry ($\phi_e \to \phi_e e^{i\Theta}$ and $a_\mu \to a_\mu - \partial_\mu\Theta$) is spontaneously broken and hence there are topological excitations associated with the symmetry. Actually, we can find a vortex soliton such that $\phi_e(x) = f(r)\exp(in\theta)$ with boundary conditions, $f(r) \to \sqrt{\rho_e}$ and $a_i \to -\partial_i(n\theta) + eA_i$ as $r \to \infty$ and $f(r=0) = 0$, where θ is an azimuthal angle and n is an integer. This vortex is similar to the magnetic vortex in the superconductor, but in the case of QHS the vortex has a quantized electron number contrast to the quantized magnetic flux in the case of the superconductivity. This is because the flux quantization $-\int d^2x\epsilon_{ij}\partial_i a_j = 2\pi n$ implies the electron number quantization $N_e = -\int d^2x\epsilon_{ij}\partial_i a_j/2\alpha = \pi n/\alpha$ of the vortex solutions. Thus, Laughlin's quasiparticles have a fractional electric charge such as $e/3$. This fractionality of the electric charges has been observed [14]. In this way the theory of the composite electrons can describe, in the mean field approximation, the QHSs as condensed states of bosons just like Landau-Ginzburg theory of superconducting states. (This similarity can naturally lead to a prediction of the presence of Josephson effects in bilayer quantum Hall systems [15].)

III. QUANTUM HALL STATE OF GLUONS

A. Unstable gluons in color-magnetic field

Up to now, we have given a brief review of the theory of QHSs in ordinary two-dimensional electron system. We now apply the idea to analyze a QHS of gluons, which appears in dense quark matter. We discuss SU(2) gauge theory with two flavors for simplicity.

It has been known [6] in the SU(n_c) gauge theory with n_f flavors that a one-loop effective potential for the color-magnetic field has a nontrivial minimum; $V(g\mathcal{B}) = \mathcal{B}^2/2 + (11N/96\pi^2)g^2\mathcal{B}^2[\log(g\mathcal{B}/\Lambda^2) - \frac{1}{2}] - (i/8\pi)g^2\mathcal{B}^2$, with an appropriate renormalization of the gauge coupling g, where $N = n_c - 2n_f/11$; $N = 18/11$ in the case of SU(2) gauge theory with two flavors. [Beyond the one-loop approximation, the presence of the nontrivial minimum in $g\mathcal{B}$ has been proved [16] in general under the reasonable assumption that the running coupling constant $g(g\mathcal{B})$ becomes infinity at a finite $g\mathcal{B}$.] This apparently seems to imply spontaneous generation of the color-magnetic field, namely, the realization of a ferromagnetic state. But it is not so simple since the imaginary part in $V(g\mathcal{B})$ is present when $g\mathcal{B} \neq 0$. It means that the state with the magnetic field is unstable as well as the perturbative vacuum state with $g\mathcal{B} = 0$. Actually, the unstable modes of gluons are present in the state with the magnetic field (ferromagnetic state). Thus, this naive ferromagnetic state is unstable [16]. The modes are expected to make some stable condensed states. What kind of the stable state is formed of the unstable gluons? We have shown [4] that the state is a QHS of the gluons with the color-magnetic field. In order to explain it, we decompose the gluon's Lagrangian with the use of the variables, "electromagnetic field" $A_\mu = A_\mu^3$ and "charged vector field" $\Phi_\mu = (A_\mu^1 + iA_\mu^2)/\sqrt{2}$, where indices $1 \sim 3$ denote color components,

$$L = -\frac{1}{4}\vec{F}_{\mu\nu}^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}|D_\mu\Phi_\nu - D_\nu\Phi_\mu|^2 + ig(\partial^\mu A^\nu - \partial^\nu A^\mu)\Phi_\mu^\dagger\Phi_\nu + \frac{g^2}{4}(\Phi_\mu\Phi_\nu^\dagger - \Phi_\nu\Phi_\mu^\dagger)^2, \tag{4}$$

with $D_\mu = \partial_\mu + igA_\mu$. We have omitted a gauge term $D_\mu\Phi^\mu = 0$. Using the Lagrangian we can derive that the energy E of the charged vector field $\Phi_\mu \propto e^{iEt}$ in the magnetic field, $A_\mu = A_\mu^B$, is given by $E^2 = k_3^2 + 2g\mathcal{B}(n + 1/2) \pm 2g\mathcal{B}$ with a gauge choice, $A_j^B = (0, x_1\mathcal{B}, 0)$ and $(\partial_\mu + igA_\mu^B)\Phi^\mu = 0$, where we have taken the spatial direction of $\vec{\mathcal{B}}$ being along x_3 axis. $\pm 2g\mathcal{B}$ (the integer $n \geq 0$) denotes the contribution from spin components of Φ_μ (Landau levels) and k_3 denotes momentum in the direction parallel to the magnetic field.

Obviously, the modes with $E^2(n=0) < 0$ are unstable modes occupying the lowest Landau level and with spin parallel to $\vec{\mathcal{B}}$. Among them, the modes with $k_3 = 0$ are the most unstable ones, which means that they have the largest negative value of $E^2(k_3 = 0)$. Thus, they are expected to form a stable state. Here we should remember a simple model of a scalar field with a double-well potential, $-m^2|\phi|^2 + \lambda|\phi|^4/2$. The state $\langle\phi\rangle = 0$ is unstable and unstable modes $\phi(\vec{k})$ with their energies $E^2 = \vec{k}^2 - m^2 < 0$ arise on the state. In this case, the most unstable uniform mode, $\phi(\vec{k} = 0)$, among $\phi(\vec{k})$, condenses to form a stable state $\langle\phi\rangle = \sqrt{m^2/\lambda}$. Therefore, the most unstable modes with $k_3 = 0$ are relevant to the formation of the true ground state also in the gauge theory. Since they have no x_3 dependence, they are two-dimensional objects occupying the lowest Landau level. The situation is quite similar to the case in the two-dimensional electrons forming QHSs just as mentioned above. The only difference is that in the gauge theory gluons are bosons, while electrons are fermions.

In order to find the stable state in the gauge theory, we extract only the most unstable modes from the Lagrangian, Eq. (4), ignoring the other modes coupled with them and obtain a two-dimensional Lagrangian,

$$L_{\text{unstable}} = |(i\partial_\nu - gA^B_\nu)\phi_{\text{u}}|^2 + 2g\mathcal{B}|\phi_{\text{u}}|^2 - \frac{\lambda}{2}|\phi_{\text{u}}|^4, \quad (5)$$

with $\lambda = g^2/\ell$, where the field $\phi_{\text{u}} = (\Phi_1 - i\Phi_2)\sqrt{\ell/2}$ denotes the unstable modes in the lowest Landau level. ℓ denotes the coherent length of the magnetic field, namely, its extension in the direction of the field. Here we are thinking the quark matter with its length scale ℓ. Then, obviously, a condition of $\ell \gg \ell_B = 1/\sqrt{g\mathcal{B}}$ must be satisfied for the consistency. The index ν runs from 0 to 2. We note that the field ϕ_{u} has a color charge associated with τ_3 of SU(2) algebra. This color charge is only a conserved quantity when the spontaneous generation of the color-magnetic field $\propto \tau_3$ occurs in the SU(2) gauge theory.

This Lagrangian is quite similar to the Lagrangian in Eq. (2) of the superconductivity. It apparently seems that the ground state is simply given by $\langle\phi_{\text{u}}\rangle = \sqrt{2g\mathcal{B}/\lambda}$, the condensed state of the field ϕ_{u}. But it is impossible because the term of A^B_μ is present in the kinetic term. If this term vanishes, the term of the negative mass also vanishes so that the solution $\langle\phi_{\text{u}}\rangle \neq 0$ does not exist. Physically, the Lagrangian L_{unstable} describes such a system that the particles of ϕ_{u} move in the magnetic field and interact with each other through a repulsive potential of a delta function. There is a numerical simulation [17] that the nonrelativistic particles with such an interacting potential can form a Laughlin state even if they are bosons. Thus, the gluons represented by ϕ_{u} may form a quantum Hall state.

B. Quantum Hall state of unstable gluons

In order to see explicitly the QHS of the field, we introduce Chern-Simons gauge field to make composite gluons; bosons attached with the Chern-Simons flux. Then, a relevant Lagrangian is given by

$$L_a = |(i\partial_\nu - gA_\nu + a_\nu)\phi_a|^2 + 2g\mathcal{B}|\phi_a|^2 - \frac{\lambda}{2}|\phi_a|^4 + \frac{\epsilon^{\mu\nu\lambda}}{4\alpha}a_\mu\partial_\nu a_\lambda, \quad (6)$$

where the statistical factor α should be taken as $\alpha = 2\pi \times$ integer to keep the equivalence of the system described by L_a to that of L_{unstable}. The field ϕ_a represents the composite gluons attached with the Chern-Simons flux a_i.

The equivalence between L_{unstable} and L_a has been shown [18] in the operator formalism although the equivalence had been known in the path integral formalism using the world lines of the ϕ_a particles. [In the formalism the last term in Eq. (6) produces a phase, $e^{i\alpha/\pi}$, in wave functions when trajectories of two particles are interchanged.] This Lagrangian corresponds to L_{QHS} of composite electrons. Obviously, there is the U(1) gauge symmetry such that $\phi_a \to \phi_a e^{i\Lambda}$ and $a_\mu \to a_\mu + \partial_\mu\Lambda$; a nonvanishing term, $(\epsilon^{\mu\nu\lambda}/4\alpha)\partial_\mu\Lambda\partial_\nu a_\lambda$ in L_a under the gauge transformation vanishes in the action integral $\int d^3xL_a$ with appropriate boundary conditions.

In deriving equations of motion, we need to impose a condition of the modes ϕ_a (or ϕ_{u}) occupying the lowest Landau level, namely, the lowest Landau level condition [19]. The condition was used when L_{unstable} was derived. The condition is given by $(D^a_1 + iD^a_2)\phi_a = 0$ with $iD^a_i = i\partial_i - gA_i + a_i$. [In the case of ϕ_{u} the condition is $(D_1 + iD_2)\phi_{\text{u}} = 0$.] Thus, adding a term $C(D^a_1 + iD^a_2)\phi_a$ to L_a with a Lagrange multiplier C, we derive equations of motion by taking functional derivatives in ϕ_a, a_μ, and C,

$$\phi^\dagger_a i\partial_0\phi_a + \text{c.c.} + 2a_0|\phi_a|^2 = -\frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j, \quad (7)$$

$$-\epsilon_{ij}\partial_j|\phi_a|^2 + \delta_{i1}i(-C\phi_a + C^\dagger\phi^\dagger_a) + \delta_{i2}(C\phi_a + C^\dagger\phi^\dagger_a) = \frac{1}{2\alpha}\epsilon_{ij}(\partial_0 a_j - \partial_i a_0), \quad (8)$$

$$(i\partial_0 + a_0)^2\phi_a + (g\mathcal{B} - \epsilon_{ij}\partial_i a_j)\phi_a - (D^a_1 - iD^a_2)C^\dagger = \lambda|\phi_a|^2\phi_a, \quad (9)$$

$$(D^a_1 + iD^a_2)\phi_a = 0, \quad (10)$$

where we have used a formula [19] of $\int d^2x|D^a_i\phi_a|^2 = \int d^2x(|(D^a_1 + iD^a_2)\phi_a|^2 + (g\mathcal{B} + \epsilon_{ij}\partial_i a_j)|\phi_a|^2)$; surface terms are omitted in this formula.

We find that the solution of the uniform ground state is given such that $C = 0$, $a_i = gA^B_i$, and a_0, ϕ_a are solutions of the equations,

$$2a_0|\phi_a|^2 = \frac{g\mathcal{B}}{2\alpha} \quad \text{and} \quad a_0^2 + 2g\mathcal{B} = \lambda|\phi_a|^2. \quad (11)$$

That is, the QHS represented by the condensed state, $\langle\phi_a\rangle = v \neq 0$, arises only when the magnetic field is canceled by the Chern-Simons field, $-\epsilon_{ij}\partial_i a_j = g\mathcal{B} = 2\alpha\rho_{\text{c}}$; ρ_{c} is given by the left hand side of Eq. (7), i.e., $\rho_{\text{c}} = 2a_0 v^2$. This ρ_{c} represents color-charge density possessed by the gluons ϕ_a. The composite gluons condense to form the QHS only when $\nu = 2\pi\rho_{\text{c}}/g\mathcal{B}$ is equal to π/α. This is quite similar to the case of the ordinary QHS mentioned above. It is easy to show that this state possesses appropriate Hall conductivity $\sigma_{xy} = (\pi/\alpha)g^2/2\pi$. Therefore, we understand that the condensed state $\langle\phi_a\rangle \neq 0$ is a QHS of gluons. It apparently seems that there are infinitely many QHSs with the filling factor $\nu = \pi/\alpha$ because α can take infinitely many values such as $n \times \pi$ with positive even integer n. In the QHSs of electrons, the states with small filling factors have low densities of electrons. In such a case electrons forming Wigner crystal is energetically more stable than electrons forming the QHSs. Actually, such QHSs with small filling factors, e.g., 1/9, have not been observed. We expect that similarly in the gauge theory, Wigner crystal of gluons would be realized when the filling factor is much small $\nu \ll 1$. The analysis is now in progress.

We should mention that the QHS of gluons is realized in a sector with nonzero color charge, not in the vacuum sector; the condensed state of ϕ_a possesses a color charge. Such a state can arise in dense quark matter where the color charge of quarks is transmitted to the condensate. This fact leads to the minimum number density of quarks for realizing a QHS, for example, QHS with $\nu = 1/2$ where the color-charge density of the condensate given by $g\mathcal{B}/4\pi\ell$ must be supplied by the quarks. Since the color charge of the quarks is a half of the gluon's, $g/2$, the number density ρ_q of the quarks for producing the QHS of the gluons must be larger than a critical one given by

$$\rho_q = \rho^{(+)} + \rho^{(-)} = 2\rho^{(+)} = 2n_f \frac{g\mathcal{B}/2}{2\pi} \int \frac{dk_3}{2\pi} = \frac{n_f k_f g\mathcal{B}/2}{\pi^2} = \frac{g\mathcal{B}\sqrt{\mu^2 - m_q^2}}{\pi^2} = \frac{g\mathcal{B}}{2\pi\ell} \tag{12}$$

with number density $\rho^{(\pm)}$ of positive (negative) colored quarks, where $n_f = 2$ is the number of flavors. k_f denotes the Fermi momentum given by $\sqrt{\mu^2 - m_q^2}$ with the constituent quark mass m_q and the chemical potential μ of the quarks at zero temperature. (We discuss mainly the chemical potential of the quarks, which gives the chemical potential of baryons, $\mu_B = 3\mu$.) Here, we have assumed that the quarks occupy only the lowest Landau level, that is, $\mu \le \sqrt{g\mathcal{B} + m_q^2}$.

Therefore, it turns out that the minimum chemical potential μ for realizing the QHS is given by $\sqrt{m_q^2 + (\pi/2\ell)^2}$. We should mention that this quantity does not depend on the unknown value of $g\mathcal{B}$. We also note that this value of μ is necessary, not sufficient for the realization of the state. In this way, the presence of the dense quark matter is necessary for producing the QHS of gluons. On the other hand, when we are concerned with the vacuum sector, such a QHS cannot arise so that the ferromagnetic state is unstable. Probably, the large fluctuation of the unstable modes may form a confining vacuum called a spaghetti vacuum [20].

In order to calculate the ground-state energy we derive Hamiltonian,

$$H = \int d^2x[a_0^2|\phi_a|^2 + (g\mathcal{B} + \epsilon_{ij}\partial_i a_j)|\phi_a|^2 - 2g\mathcal{B}|\phi_a|^2 + \frac{\lambda}{2}|\phi_a|^4]. \tag{13}$$

Thus, the energy density $E_2(v)$ of the QHS is given by $E_2(v) = a_0^2 v^2 - 2g\mathcal{B}v^2 + \frac{\lambda}{2}v^4$. We should note that $E_2(v)$ represents the energy density in two-dimensional space and that the three-dimensional one is given by $E_2(v)/\ell$.

The behavior of the ground-state solution with respect to the coupling $\lambda = g^2/\ell$ and the filling factor $\nu = \pi/\alpha$ is given by

$$v \to \left(\frac{g\mathcal{B}}{4\alpha\sqrt{\lambda}}\right)^{1/3}, \qquad a_0 \to \left(\frac{g\mathcal{B}\lambda}{4\alpha}\right)^{1/3} \quad \text{for } \lambda \to \infty,$$
$$v \to \sqrt{\frac{2g\mathcal{B}}{\lambda}}, \qquad a_0 \to \frac{\lambda}{8\alpha} \quad \text{for } \frac{\lambda}{\alpha} \ll \ell_B^{-1}. \tag{14}$$

Thus, we find that the ground-state energy density $[= E_2(v)/\ell + \mathrm{Re}V(g\mathcal{B})]$ in three-dimensional space becomes large such as $E_3 \sim 1.5\lambda^{1/3}(g\mathcal{B}/4\alpha)^{4/3}/\ell$ as the gauge coupling constant becomes large, $\lambda \to \infty$ (or as the length scale of the system in the direction of the magnetic field becomes small, $\ell \to 0$). The fact implies that the QHS becomes unstable as the coupling becomes large. This is because the energy of the ferromagnetic state ($\mathcal{B} \neq 0$) involving the QHS becomes larger than the energy of the perturbative ground state with $\mathcal{B} = 0$; we have normalized the energy such that the energy of the perturbative ground state vanishes at $g\mathcal{B} = 0$. This is consistent with naive expectation that at sufficiently large g^2, the hadronic state ($\mathcal{B} = 0$) is realized instead of the ferromagnetic state: the hadronic or confining ground state is more stable than the perturbative ground state for such a large coupling. Therefore, at large coupling constants the ferromagnetic state becomes unstable and the hadronic state would be realized.

On the other hand, $E_3 \sim -0.5(2g\mathcal{B})^2/(\lambda\ell)$ as $\lambda \to 0$. This implies that when the coupling constant is sufficiently small, the ferromagnetic state is stable since it has much small energy. It apparently seems to be unnatural because the perturbative ground state may be realized at the small coupling. But we should mention that the QHS of the gluons is realized only in dense quark matter, not in the vacuum because for the realization of the QHS, the color charge associated with τ_3 must be supplied from somewhere in the neutral system: The condensate of ϕ_a possesses the color charge, which must be supplied from the quark matter. Therefore, even at small coupling constants, the QHS can arise as a stable state in the quark matter. (In the vacuum the perturbative ground state is realized at such small coupling since there are no color charges.)

Analyzing small fluctuations $\delta\phi_a$, etc., around the solution of the ground state, we can see that the energy of the fluctuations has a real positive gap given by $\sqrt{4a_0^2 + 2\lambda v^2}$. The fluctuations represent extended collective motions, while there are individual localized collective motions, namely, Laughlin's quasiparticles. They are vortex topological solitons in the Chern-Simons gauge theory. We find from numerical analysis of such solutions that the energies of the solitons are positive. Therefore, no instability in the ferromagnetic state ($\mathcal{B} \neq 0$) appears as a result of the formation of the QHS of the gluons. In the next subsection we discuss the vortex solitons in the QHS of the gluons in detail.

C. Vortex excitations in the quantum Hall state

The vortex solitons arise owing to the spontaneous breakdown of the U(1) gauge symmetry of the Lagrangian in Eq. (6) describing spatially two-dimensional gluons. The vortex solutions can be obtained in the following. The lowest Landau level condition, $(D_1^a + iD_2^a)\phi_a = 0$ leads us to the form of $\phi_a = f(z)e^{a(x)}$ with an arbitrary function $f(z)$ of $z = x + iy$. Here, $a(x)$ is defined such that $a_i = gA_i + \epsilon_{ij}\partial_j a$. We assume that the solutions are spherically symmetric, namely, $a(r)$ and $a_0(r)$ are functions only of the radial coordinate $r = \sqrt{x^2 + y^2}$. Then, when we take $f(z) = vz^n = vr^n e^{in\theta}$, it represents a solution of a vortex with vorticity being equal to a positive integer n. Hereafter, we consider a solution with $n = 1$, for simplicity. Boundary conditions are imposed such that $re^{a(r)} \to 1$ (or $\phi_a \to v$), $a_0(r) \to a_0$ as $r \to \infty$, and $re^{a(r)} \to 0$ as $r \to 0$ to avoid singularity at $r = 0$. These boundary conditions lead to a quantization of color charges carried by the topological soliton. Namely, the soliton has the flux $\int d^2x(-\epsilon_{ij}\partial_i a_j - g\mathcal{B}) = \int d^2x \partial^2 a = 2\pi$ due to the boundary condition. This means that the color charge of the soliton is given by π/α, because the color-charge density is defined as the left hand side of Eq. (7). In general the charge is given by $n \times \pi/\alpha$ for the soliton with the vorticity of n. Thus, we can see that the color charge of the soliton is quantized.

In order to obtain the vortex solutions, we derive variational equations by inserting $\phi_a = vze^{a(r)}$ into the Hamiltonian, Eq. (13),

$$2a_0(r)v^2r^2e^{2a(r)} = \frac{g\mathcal{B} + \partial^2 a(r)}{2\alpha}, \tag{15}$$

$$\begin{aligned} &a_0(r)^2 + 2g\mathcal{B} + \partial^2 a(r) \\ &\quad - \frac{1}{2v^2r^2e^{2a(r)}}\partial^2\left(\frac{a_0(r)}{2\alpha} - v^2r^2e^{2a(r)}\right) \\ &\quad = \lambda v^2 r^2 e^{2a(r)}. \end{aligned} \tag{16}$$

These are derived also from Eqs. (7)–(9), by taking $C = \bar{z}^2 b(r)$ with $b(r)$ being function only of r. We solved the equations numerically to obtain their configurations and energies for various coupling parameter $\lambda = g^2/\ell$ and α (Figs. 1 and 2).

We can see that the energy of the vortex soliton approaches zero as λ goes to infinity. We note that the typical energy scale of the solution is governed by $v^2(\lambda)$, which goes to zero as $\lambda \to \infty$. That is why the energy of the solution goes to zero as $v^2(\lambda) \to 0$. This indicates a possibility that the QHS becomes unstable at sufficiently large coupling constant of λ since the energy of the bound state of a vortex ($n = 1$) and an antivortex ($n = -1$) can become negative at large coupling constants. They have opposite color charges with each other and their binding energy may become larger than the intrinsic energies of the vortices at sufficiently large coupling constants. Then, such excitations of the bound states are produced unlimitedly and consequently, the QHS decays: The whole space is occupied by such solitons and the condensate melts because the condensate of the gluons vanishes at the center of the vortex; $\phi_a(r = 0) = 0$. This bound state corresponds to the roton excitation in the QHS of electrons.

Actually, we estimated the critical coupling constant and have found that it is given by $\lambda/\sqrt{g\mathcal{B}} = g^2\ell_B/\ell \simeq 20$ for $\ell = 3$ fm and $\ell_B \sim 0.5$ fm or $\sqrt{g\mathcal{B}} = 400$ MeV. The critical coupling becomes large for the larger magnetic length ℓ_B. When we take the value of the strong coupling constant $g^2/4\pi = 1$ at the energy scale of 1 GeV, the critical baryonic chemical potential is evaluated as $\sim$300 MeV with an assumption of $g(\mu)^2 \propto \mu^{-2}$ for the

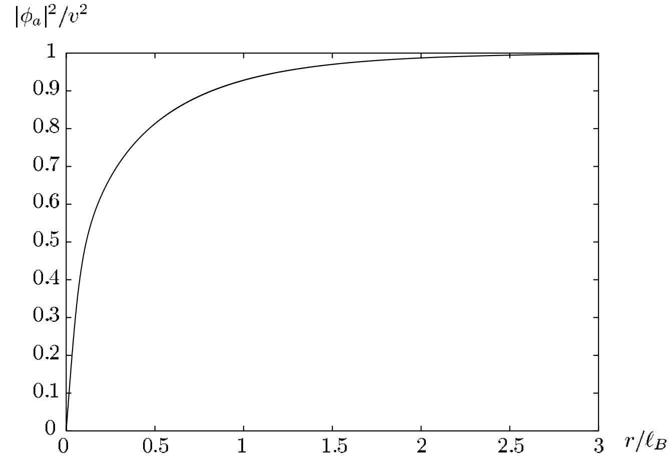

FIG. 1. Profile of the vortex. The size of the vortex is determined by the magnetic length ℓ_B as in QHS of electrons.

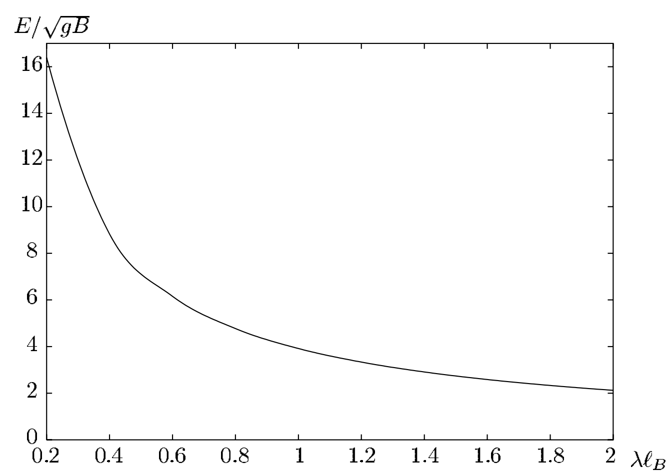

FIG. 2. Coupling dependence of the vortex excitation energy.

small energy scale. [This behavior of the coupling constant comes from the assumption that the linear potential ($\propto r$) among quarks arises such as $g(r)^2/r$.] The result simply indicates that at small chemical potentials the quantum Hall state becomes unstable. The value of the critical chemical potential estimated here is consistent with ones obtained in the latter section, where we discuss a transition between the hadronic phase and the color ferromagnetic phase at small chemical potentials.

We found that the dependence of the energy on α or the filling factor, π/α, is very small. We also mention that the energy of the vortex becomes a nonzero constant as α goes to infinity. The color-charge density of the condensate is very small at small filling factors, but the value of the field $\langle\phi_a\rangle = v$ is never small. The energy of the soliton is governed by $\langle\phi_a\rangle$, so that the energy never becomes small even as the filling factor becomes small. In this respect, we cannot find any instabilities of the QHS at small filling factors π/α. But similarly to the QHS of electrons, the state might be unstable at such a small filling factor because a Wigner crystal of the vortices is energetically more stable than the QHS at such small color-charge density of gluons. We remember that the number difference of positively color-charged quarks and negatively charged quarks is also small in such a case. Thus, for example, the excessive negatively charged quarks form a Wigner crystal. Therefore, by the formation of the vortex's Wigner crystal with positive charges, the system reduces color Coulomb energy; in the QHS the color-charge distribution of the gluons is uniform, while the Wigner crystal is not so. Accordingly, it is natural to expect that the QHS at much small filling factor is unstable.

D. Effects of the third spatial dimension

Up to now, we have considered the ground-state structure of gluons in two-spatial dimension. This is because the unstable modes are two-dimensional objects and they may form a stable ground state with their condensation. The unstable modes, in general, depend on all of the coordinates in three dimensions; $\phi(k_3 < \sqrt{g\mathcal{B}}) \sim \exp[ik_2x_2 + ik_3x_3 - iE(k_3)t]\exp[-(x_1 - k_2\ell_B^2)^2/2\ell_B^2]$ with $E(k_3) = \sqrt{-g\mathcal{B} + k_3^2}$. But, among them the modes with the largest amplitude as $t \to \infty$, $\phi(k_3 = 0)$, depend only on x_1 and x_2. They are two-dimensional objects and form the stable ground state, namely, the QHS of gluons as we have shown. In the derivation of the QHS we have used Chern-Simons gauge theory, which can be used only in two-dimensional space. In this way we have fully used two dimensionality of the problem. We may wonder whether or not unstable modes with small, but nonvanishing k_3 ($\ll \sqrt{g\mathcal{B}}$) contribute to the ground-state structure. We have a symmetry of the rotation around the magnetic field and of the translation along it. Thus, it is natural to expect that the ground state should be uniform in the x_3 direction. Then, the modes should not be important, otherwise their contributions make the ground state nonuniform in the direction. Therefore, it is reasonable to expect that three-dimensional effects on the ground state change our main result; the stable QHS of gluons is realized in the ferromagnetic state (Savvidy vacuum).

We also found the gap energy, Δ, above the ground-state energy based on the two-dimensional theory. The effect of x_3 direction is simply that the corresponding mode propagating in the direction gains the energy, $\sqrt{\Delta^2 + k_3^2}$. This is because the relativistic covariance in the direction still remains at least in the limit of infinitely large quark matter ($\ell \to \infty$). We may also wonder whether or not the gapless mode with $E = |k_3|$ exists. In order to see it, we may assume that the fluctuation $\delta\phi_a$ does not have dependence on any spatial coordinates. We found that there is no such solution; the condition of $\partial_3\delta\phi_3 = 0$ has been taken into account explicitly in our treatment. Therefore, the QHS we have found is really the stable gapped state of the unstable gluons.

IV. COLOR FERROMAGNETISM VS COLOR SUPERCONDUCTIVITY

Until now, quarks do not play any roles for the realization of the ferromagnetic phase except for supplying color charges for the condensate of the gluons. But, quarks play important roles for the realization of the phase. Here, we show the region of the chemical potential μ of the quarks in which the phase is realized between the hadronic phase and the color superconducting phase. Especially, we determine the critical chemical potential at which the color superconducting state arises.

In general, the energy density of the quarks in the magnetic field is smaller than that of the free quarks without the magnetic field. (This fact is favorable to the ferromagnetic state.) The fact is easily understood intuitively in the case of strong magnetic field. When the magnetic field is sufficiently strong, all of the quarks occupy the lowest Landau level; their energy is given by $\sqrt{m_q^2 + k_3^2}$ with the degeneracy of $g\mathcal{B}/4\pi$ per unit area where k_3 denotes the momentum parallel to the magnetic field of the quarks. Hence, the energy density of the quarks in the strong magnetic field is much lower than that of the free quarks without the magnetic field. On the other hand, for sufficiently large number density of the quarks, equivalently, for sufficiently weak magnetic field, the quarks occupy much higher Landau levels. Eventually, both energy densities (with and without $g\mathcal{B}$) approach each other in the limit of ρ or $\mu \to \infty$. We can show that the energy density, $E_{\text{quark}}(g\mathcal{B}, \mu)$, of the quarks in the magnetic field is lower than $E_{\text{quark}}(g\mathcal{B} = 0, \mu)$ for any strength of the magnetic field $g\mathcal{B}$. Furthermore, in the case of massless quarks $E_{\text{quark}}(g\mathcal{B}, \mu)$ behaves such as

$$E_{\mathrm{quark}}(g\mathcal{B},\mu) = E_{\mathrm{quark}}(g\mathcal{B}=0,\mu) \times \left[1 - 0.43\left(\frac{g\mathcal{B}}{\mu^2}\right)^{3/2} + \text{"higher order"}\right] \quad (17)$$

as $\mu^2/g\mathcal{B}$ goes to infinity, where we have numerically obtained the second term.

In the color ferromagnetic phase we have the gluon condensation forming the quantum Hall state. As we have shown, the condensation energy is approximately given by $-2\mathcal{B}^2 = -2(g\mathcal{B})^2/g^2$ when the size ℓ of the quark matter is larger than $3\ell_B$. On the other hand, the color superconducting phase possesses the condensate of Cooper pair of quarks, which leads to the decrease of the energy in the quark matter. The Cooper pairs are formed by quarks around the Fermi surface with its width approximately given by the gap energy Δ of the state; their binding energy is also approximately given by Δ. Thus, the gain of the energy due to the condensation is approximately given by

$$\Delta \int_{\mu-\Delta/2<|k|<\mu+\Delta/2} \frac{d^3k}{(2\pi)^3} = \frac{\mu^2\Delta^2}{2\pi^2}, \quad (18)$$

where we have assumed the formula of the free quark gas with $g\mathcal{B}=0$ because the quark matter behaves as the free gas in the limit of $\mu^2/g\mathcal{B} \to \infty$. Since the energy density of the quarks in both phases is almost identical in the limit of large μ, we compare this condensation energy of the diquark with the condensation energy of the gluons; $2(g\mathcal{B})^2/g^2$ vs $\Delta^2\mu^2/(2\pi^2)$. Obviously, for much large μ the color superconducting phase is energetically favored as expected. Thus, we can determine the critical chemical potential μ_c which is given by

$$\mu_c \simeq 2.5\ \mathrm{GeV}\frac{g\mathcal{B}}{(200\ \mathrm{MeV})^2}\frac{50\ \mathrm{MeV}}{\Delta}, \quad (19)$$

where we have taken $g^2/4\pi = 1$.

This critical value is large sufficiently for neglecting the effects of the quark mass so that we may use the formula in Eq. (17). We should comment that our result of the color superconducting phase being realized at larger chemical potential than 1 GeV is consistent with the results of others.

Here we make a comment on the case of SU(3) gauge theory. In the gauge theory the color ferromagnetic state can coexist with the color superconducting phase, 2SC. This is possible only when the direction of color-magnetic field generated spontaneously is pointed into λ_3 in the color space. Since the diquark condensate in 2SC is color antitriplet, $(0, 0, v \neq 0)$, the magnetic field does not affect the condensate. Therefore, in the 2SC, the color-magnetic field can still be present as well as the gluon condensation. We also note the chiral symmetry is restored in the 2SC and that the formula Eq. (17) can be used. Thus, the critical chemical potential in SU(3) gauge theory is the one separating the color ferromagnetic state without the 2SC and with the 2SC. Then, we compare the energy decrease of the quark matter $E_{\mathrm{quark}}(g\mathcal{B}=0,\mu) \times \frac{1}{2}(g\mathcal{B}/\mu^2)^{3/2}$ with the energy decrease due to the Cooper pair condensation $\Delta^2\mu^2/(2\pi^2)$,

$$\mu_c = \sqrt{g\mathcal{B}}\frac{g\mathcal{B}}{\Delta^2} = 0.8\ \mathrm{GeV}\sqrt{\frac{g\mathcal{B}}{(200\ \mathrm{MeV})^2}}\frac{g\mathcal{B}/(200\ \mathrm{MeV})^2}{(\Delta/100\ \mathrm{MeV})^2}. \quad (20)$$

As we will show in the next section, the value of the magnetic field is probably larger than $\sim(200\ \mathrm{MeV})^2$, the estimation of $\mu_c = 0.8$ GeV gives a minimum one. Therefore, we find in the SU(3) gauge theory that the color superconducting state arises at the chemical potential larger than $\mu_c \sim 0.8$ GeV.

In this way, the quarks play the role of realizing the color superconducting phase for sufficiently large chemical potential. As we have mentioned before, the quarks also play the important role of realizing the color ferromagnetic state; the quarks supply color charges for the gluon condensation. In the sense the chemical potential necessary for the realization of the phase is given by $\sqrt{m_q^2 + (\pi/2\ell)^2} \simeq 300\ \mathrm{MeV}\sqrt{m_q^2/(300\ \mathrm{MeV})^2 + 0.12(3\ \mathrm{fm}/\ell)^2}$. The value is necessary, but not sufficient. In the next section we discuss more details of the critical point.

V. HADRON PHASE VS COLOR FERROMAGNETIC PHASE

We now discuss the critical chemical potential at which the phase transition from the hadronic phase to the color ferromagnetic phase occurs. In the hadronic phase the gauge symmetry is exact, but $\langle \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\rangle \neq 0$. The chiral condensate also exists which breaks the chiral symmetry. These condensates make the energy of the real vacuum lower than that of the perturbative vacuum. We denote the energy decrease by E_{vac}, which has been estimated as $300\ \mathrm{MeV} \leq E_{\mathrm{vac}}^{1/4} \leq 350$ MeV. We use these values even at finite chemical potential as far as the chemical potential of the nucleon is smaller than the nucleon mass. On the other hand, in the color ferromagnetic phase we have the gluon condensate $\langle A_j^i\rangle \neq 0$ $(ij = 1, 2)$ which forms a quantum Hall state under the spontaneously generated color-magnetic field ($\propto \lambda_3$). This condensate makes the energy of the vacuum in the gluon sector lower than that of the perturbative vacuum by $-2\mathcal{B}^2$. We take account of these vacuum energies as well as the nucleon energy and the quark energy for the determination of the critical chemical potential. In this section, we consider the realistic case, i.e., SU(3) gauge theory with two flavors, and use the observed nucleon mass. In the hadron phase we assume an equal number of free protons and neutrons for simplicity. The energy density of the free nucleons is given by

$$E_N = \frac{1}{4\pi^2}\left[2\mu_N(\mu_N^2 - M^2)^{3/2} + M^2\mu_N(\mu_N^2 - M^2)^{1/2} - M^4 \log\frac{\mu_N + \sqrt{\mu_N^2 - M^2}}{M}\right], \tag{21}$$

with the chemical potential for the nucleon, μ_N, and the nucleon mass, M. In the color ferromagnetic phase the magnetic field is assumed to point to the direction of λ_3 in the SU(3) maximal Abelian subalgebra. Then, only the quarks with colors $(1, 0, 0)$ and $(0, 1, 0)$ are trapped by the magnetic field. The energy density of the quarks interacting with the color-magnetic field is given by

$$E_q = 2n_{\mathrm{f}}\frac{g\mathcal{B}/2}{2\pi}\int\frac{dk}{2\pi}\sqrt{m_q^2 + k^2} = \frac{g\mathcal{B}/2}{\pi^2}\left(\mu\sqrt{\mu^2 - m_q^2} + m_q^2\log\frac{\mu + \sqrt{\mu^2 - m_q^2}}{m_q}\right), \tag{22}$$

with the chemical potential μ and the quark mass m_q where $n_{\mathrm{f}} = 2$ represents relevant number of flavor degrees of freedom. Here we have assumed that the quarks occupy only the lowest Landau level. In addition to E_q, we have a contribution from the quark with color $(0, 0, 1)$, which does not couple with the color-magnetic field $\propto \lambda_3$. Its energy density, E'_q, is given by $E_N/81$ when $\mu_N = 3\mu$ and $M = 3m_q$.

We can easily see that the energy density of the nucleons is smaller than that of the quarks when the chemical potential, $\mu(= \mu_q/3)$, is very small, $\mu/m_q \ll 1$. Thus, at such a small chemical potential, the hadron phase is realized. Increasing the chemical potential the energy density of the nucleons increases more rapidly than that of the quarks. Eventually, at large chemical potential the color-magnetic phase is energetically favored. Inclusion of the vacuum energies in two phases does not change this result.

Consequently, the critical chemical potential, μ_c, can be determined by the condition that the energy density of the hadron phase measured from the perturbative vacuum coincides with that of the color ferromagnetic phase:

$$E_N - E_{\mathrm{vac}} = E_q + E'_q - 2\mathcal{B}^2 + \mathrm{Re}V(g\mathcal{B}). \tag{23}$$

We have included the real part of the potential $V(g\mathcal{B})$ in the SU(3) gauge theory with two flavors; $\mathrm{Re}V(g\mathcal{B}) = \mathcal{B}^2/2 + 29g^2\mathcal{B}^2[\log(g\mathcal{B}/\Lambda^2) - 1/2]/96\pi^2$ and take Λ to be 250 MeV. We assume that the critical chemical potential, μ_c, is not so large and that the chiral symmetry is only partially restored at μ_c. Therefore, in Eq. (23) we take the quark mass to be $50\ \mathrm{MeV} \leq m_q \leq 250\ \mathrm{MeV}$. Furthermore, we assume tentatively $g^2/4\pi = 1$, that is, the gauge coupling constant does not depend on μ and $g\mathcal{B}$. The different value of $g^2/4\pi$ does not seriously affect our final results: The phase transition from the hadronic phase to the color ferromagnetic phase occurs at baryon chemical potentials less than 900 MeV. We have checked that the critical chemical potential, μ_c, obtained from Eq. (23) is consistent with the above assumption of the lowest Landau level, namely, $\mu \leq \sqrt{g\mathcal{B} + m_q^2}$.

In solving Eq. (23) for the chemical potential, we have much ambiguity in the values of the color-magnetic field, $g\mathcal{B}$ and the quark mass, m_q, in dense matter, while not so much ambiguity in E_{vac}. Thus, fixing tentatively $m_q = 250$ MeV and $E_{\mathrm{vac}}^{1/4} = 300$ MeV, we try to find the critical chemical potential, μ_c by changing $\sqrt{g\mathcal{B}}$ from 100 to 600 MeV. (We note that the vacuum energy, E_{vac} in the hadronic phase at finite chemical potential is expected to be smaller than that in the real vacuum so that the choice of $E_{\mathrm{vac}} = 300$ MeV is reasonable.) The results are as follows, The critical chemical potential is a smoothly decreasing function of $g\mathcal{B}$; the dependence on $g\mathcal{B}$ is very weak. Furthermore, we find that when we take the large values of $\sqrt{g\mathcal{B}}$ such as 600 MeV, the color ferromagnetic phase is energetically favored more than the hadronic phase at any chemical potentials. That is, the hadronic phase does not exist in nature. Thus there are no solutions in Eq. (23). This is due to the presence of the large condensation energy, $-2\mathcal{B}^2$ in the color ferromagnetic phase. Accordingly, the value of $\sqrt{g\mathcal{B}}$ is restricted phenomenologically such as $\sqrt{g\mathcal{B}} \leq 550$ MeV. The lowest Landau level condition restricts the value such as $180\ \mathrm{MeV} \leq \sqrt{g\mathcal{B}}$.

We now evaluate μ_c by changing the quark mass with taking $\sqrt{g\mathcal{B}} = 300$ MeV; the different choice of the value, $\sqrt{g\mathcal{B}}$, does not affect the final result so much. Then, it follows that contrary to the case of the magnetic field, the critical chemical potential is a smoothly increasing function of m_q. Thus, it turns out that the critical chemical potential μ_c depends slightly both on the mass of the quarks and the magnetic field. Therefore, we find that the critical baryonic chemical potential is given such that $600\ \mathrm{MeV} \lesssim \mu_B \lesssim 900\ \mathrm{MeV}$, depending on the quark mass, $50\ \mathrm{MeV} \leq m_q \leq 250\ \mathrm{MeV}$, and the magnetic field, $200\ \mathrm{MeV} \leq \sqrt{g\mathcal{B}} \leq 500\ \mathrm{MeV}$, in dense quark matter when the vacuum energy, $E_{\mathrm{vac}}^{1/4} = 300$ MeV, in the hadronic phase is taken.

We find that μ_c depends slightly on the magnetic field. On the other hand, obviously the quark number density, $\rho_{\mathrm{q}} = g\mathcal{B}\sqrt{\mu_c^2 - m_q^2}/\pi^2$, depends heavily on the magnetic field. For example, $\rho_{\mathrm{q}} \simeq 0.34/\mathrm{fm}^3$ with $g\mathcal{B} = (400\ \mathrm{MeV})^2$ and $m_{\mathrm{q}} = 150$ MeV for which $\mu_c = 231$ MeV, while the nucleon number density, $\rho_{\mathrm{N}} = 18(\mu_c^2 - m_q^2)^{3/2}/\pi^2 \simeq 1.26/\mathrm{fm}^3$, does slightly depend on $g\mathcal{B}$. Although we do not know exactly the critical nucleon number density, it is reasonable to suppose it being given approximately by $5 \sim 10$ times the normal nuclear density, i.e., $0.85 \sim 1.7/\text{fm}^3$. Thus, the choice of the parameters such as $g\mathcal{B} = (400\text{ MeV})^2$ and $m_q = 150$ MeV, is physically reasonable since it leads to $\rho_{\rm N} \simeq 1.26/\text{fm}^3$ and $\rho_{\rm q} \simeq 0.34/\text{fm}^3$. Unfortunately, we have no idea about the critical quark number density. Thus, we cannot further limit the physically acceptable range of the parameters.

In order to obtain the critical point more precisely we need to determine precisely the dependence of the coupling constant, $g^2/4\pi$, and the quark mass, m_q, on μ and $g\mathcal{B}$. Furthermore, we need to include interactions of nucleons as well as those of quarks. As we pointed out, the critical nucleon number density is very large so that their short range interactions are important. On the other hand, the critical quark number density is so small that their long range interactions are important. Since including these interactions is a difficult task, we have much ambiguity in the determination of the critical baryonic chemical potential. But we expect that the ambiguity gives rise to a change of a few factors in our result.

In concluding this section we should mention the following: Although there are ambiguities in the values of the parameters such as m_q, $g\mathcal{B}$, $g^2/4\pi$, $E_{\rm vac}$, E_N, and E_q, the color ferromagnetic phase must arise at a baryon chemical potential less than 900 MeV; it never being larger than 2 GeV. This baryon chemical potential is accessible at present experimental apparatus.

VI. OBSERVATIONAL IMPLICATION

As we have shown previously, the color ferromagnetic phase with QHS of gluons is realized in quark matter at baryon chemical potentials μ_B; roughly, $600\text{ MeV} \sim 900\text{ MeV} \le \mu_B \le 2$ GeV. The matter can be produced by heavy ion collisions. The matter produced in the collisions initially has high temperature so that it is in the phase of the quark-gluon plasma. After that, it gradually loses its energy and then enters into the phase of the color ferromagnetic state with the QHS of gluons if the value of the chemical potential is appropriate. (We have shown in the previous paper [4] that the critical temperature is less than $\sqrt{g\mathcal{B}}$ at small chemical potentials such as $\mu \sim 230$ MeV.) How do we detect whether or not the matter is in the phase? We cannot observe the color-magnetic field, which is confined in the matter. But we show that the matter in the phase possesses a large observable magnetic moment, in other words, it produces strong observable magnetic fields outside of the matter. The point is that the difference between the number of positively color-charged quarks and that of negatively charged quarks generates a rotation of the quark matter as a whole. The difference is a result of the realization of the QHS. When the quark matter is not electrically neutral, the rotation generates a magnetic moment. Suppose that the quark matter is composed of up and down quarks, and that the number difference between positively and negatively color-charged quarks is identical in each flavor. We consider the QHS with $\nu = 1/2$. Then, the density difference, $\rho_f^{(-)} - \rho_f^{(+)}$, of each flavor is given by the color-charge density of gluons in the QHS; $\Delta\rho = \rho_f^{(-)} - \rho_f^{(+)} = g\mathcal{B}/8\pi\ell$. A quark with electromagnetic charge $e_{\rm q} = 2/3$ (or $-1/3$) $\times\, e$ generates a magnetic moment ($= \partial E(g\mathcal{B} + e_{\rm q}B)/\partial B|_{B=0}$), which is of the order of $e_{\rm q}/(2\sqrt{g\mathcal{B}})$ in the strong magnetic field $\sqrt{g\mathcal{B}} \gtrsim m_q$. Hence, the magnetization of the quark matter is given by

$$(2/3 - 1/3)e\frac{\Delta\rho}{2\sqrt{g\mathcal{B}}} = \frac{e\sqrt{g\mathcal{B}}}{48\pi\ell} \sim (5\text{ MeV})^2 \frac{\sqrt{g\mathcal{B}}}{200\text{MeV}}\frac{3\text{fm}}{\ell} \simeq 3.5 \times 10^{14}\text{ G}, \tag{24}$$

where we have assumed that the strength of the color-magnetic field is $\sqrt{g\mathcal{B}} = 200$ MeV and the size of the quark matter is $\ell = 3$ fm. We have taken the value of $g\mathcal{B}$ as a reference point based on the discussion in the previous section. The observation of this strong magnetic field can be an evidence of the presence of the ferromagnetic phase in the quark matter.

VII. DISCUSSION

As is well known, the Savvidy vacuum is unstable; when the color-magnetic field is spontaneously generated some gluonic modes become unstable. It has been a long standing problem how the unstable modes form a new stable state, i.e., the physical vacuum, which has not been fully solved yet. In this paper we have shown that in quark matter the Savvidy vacuum is stabilized by the condensation of the unstable gluons, which leads to a ferromagnetic QHS of gluons.

In general, there exist two possibilities in order to stabilize the Savvidy vacuum in quark matter: one is that large masses for unstable gluons are generated by dynamical gauge symmetry breaking or Higgs mechanism and the other is that the gauge symmetry is not broken but the average value of the color-magnetic field vanishes due to large fluctuations of the magnetic field. In the color ferromagnetic phase the gauge symmetry is broken due to the spontaneous generation of the color-magnetic field and the condensation of unstable gluons. Also in the color superconducting phase the gauge symmetry is broken by the effects of quarks. In these two phases the first possibility is realized. On the other hand, in the hadronic phase the gauge symmetry is not broken and the second possibility is expected to be realized.

In this paper we have discussed mainly the case of the SU(2) gauge theory. Similar results hold even in the SU(3) gauge theory although possible structures of QHSs are much richer in SU(3) case than in SU(2) case because of the presence of more unstable modes [4]. A particular point in the SU(3) gauge theory is the presence of a phase with the coexistence of color ferromagnetism and color superconductivity (so-called 2SC) at large chemical potential. This is because the direction of the magnetic field in the color space is normal to the direction of the quark pair condensate. For more details, refer to our paper [4].

Quark confinement (hadron phase) is caused mainly by gluon's dynamics, namely, the SU(3) gauge theory. Ground-state structure of QCD is determined by analyzing nonperturbative dynamics of gluons in the phase. Especially, we need fully dynamical treatment in the gauge theory for revealing the property of the confinement. The quarks play no dominant roles for the confinement. For example, in the large N expansion of the SU(N) gauge theory the confinement is realized at 0th order of the expansion, in which the quark loops do not arise. The contribution of the quarks appears in the higher order of the expansion so that the effects of the quarks can be treated perturbatively in the expansion.

On the other hand, the dynamics of the quarks play an important role for the color superconductivity in the region of large chemical potential of the quark number. Gluons simply give perturbation, an attractive force in an appropriate channel of the quarks; it makes the Fermi gas of free quarks unstable and realizes the superconducting state of the quarks.

The color ferromagnetic phase is realized between the hadron phase and the color superconducting phase when the chemical potential is varied. Thus, it is natural to expect that both gluon and quark dynamics play important roles for the phase. As we have explained, indeed, the gluon dynamics plays the main role in leading to the stable ferromagnetic phase along with quantum Hall state of gluons when the quark matter is present. In such a case, the quark dynamics plays a role of choosing the ferromagnetic phase when the chemical potential is small. On the other hand at large chemical potential the quark dynamics plays a role of leading to and of choosing the superconducting phase. In other words, the gluon dynamics plays a main role in realizing the ferromagnetic state at small chemical potential, while the quark dynamics plays a main role in realizing the color superconducting state at large chemical potential.

In this way, in QCD the player of the main role for determining various phases of quark matter changes from the gluons to the quarks when we increase the number density of the quarks. Probably, recent observations of multiquark hadrons are a step toward revealing the presence of these phases we have discussed.

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