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QCD factorization for semi-inclusive deep-inelastic scattering at low transverse momentum

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We argue a factorization formula for semi-inclusive deep-inelastic scattering with hadrons in the current fragmentation region detected at low transverse momentum. To facilitate the factorization, we introduce the transverse-momentum dependent parton distributions and fragmentation functions with gauge links slightly off the light cone, and with soft-gluon radiations subtracted. We verify the factorization to one-loop order in perturbative quantum chromodynamics and argue that it is valid to all orders in perturbation theory.

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I. INTRODUCTION

In recent years, semi-inclusive deep-inelastic (SIDIS) lepton-nucleon scattering has emerged as an important tool to learn various aspects of perturbative and nonperturbative quantum chromodynamics (pQCD), the internal structure of the nucleon, in particular. The European Muon Collaboration experiment at CERN has provided us valuable information about the flavor dependence of quark fragmentation functions [1]. The H1 and ZEUS collaborations at the DESY HERA collider have measured the topology of the hadron final states in great detail and have compared them with the predictions of perturbative QCD [2]. In the area of polarized semi-inclusive DIS, the Spin Muon Collaboration, and recently the HERMES collaboration at DESY, have extracted the sea quark distributions and the polarized gluon distribution with controlled accuracy [3,4]. More recently, the target single-spin asymmetry measured by HERMES in semi-inclusive DIS is a new observable sensitive, for example, to the quark transversity distribution through the transverse-momentum dependence of the produced hadron [5].

In the semi-inclusive production of DIS, both the longitudinal momentum fraction z and the transverse momentum $P_{h\perp}$ of the hadron yield can be measured. When the transverse momentum is integrated over or when it is comparable to the hard photon-mass scale, $P_{h\perp} \sim Q$, the cross sections can be calculated from the standard pQCD formalism similar to inclusive DIS and Feynman parton distributions [6]. In these cases, the theoretical tool has been well tested against experimental data with notable successes. When the transverse momentum is much smaller than Q, but is still hard, $P_{h\perp} \gg \Lambda_{\rm QCD}$, the cross section can be calculated again with integrated parton distributions augmented by small nonperturbative QCD corrections. The hard part contains the large double logarithms of the type $\alpha_s \ln^2 P_{h\perp}/Q^2$. To make reliable predictions, these large logarithms must be summed [7,8]. An adequate formalism was developed by Collins and Soper in the case of e^+e^- annihilation [9], and shortly thereafter applied to the Drell-Yan process by Collins, Soper, and Sterman (CSS) [10]. A first application of the CSS approach to SIDIS was made by Meng, Olness, and Soper [11]. Recently, a quantitative comparison between this theory and data from HERA collider has been made by Nadolsky, Stump, and Yuan [12].

In this paper, we are interested in a special kinematic regime in SIDIS where $P_{h\perp}$ is soft, i.e., on the order of $\Lambda_{\rm QCD}$, and Q^2 is not too large, for example, on the order of tens or hundreds of GeV^2. When Q^2 is large, the soft-gluon radiations become important and can easily generate a large transverse momentum $\gg \Lambda_{\rm QCD}$. Then the cross section for the hadron yield with $P_{h\perp} \sim \Lambda_{\rm QCD}$ is exponentially suppressed. To have a significant fraction of events with $P_{h\perp} \sim \Lambda_{\rm QCD}$, fixed-target experiments with lepton beam energies on the order of tens to hundreds of GeV are preferred. The above kinematic regime is in fact ideal for studying transverse-momentum dependent (TMD) parton distributions in the nucleon and the related quark fragmentation functions. Recent interest in this subject has been stimulated by Collins' observation that semi-inclusive DIS at low-$P_\perp$ provides a tool to measure the quark transversity distribution [13]. The physics potential has been reinforced by the rediscovery of Siver's effect [14] by Brodsky, Hwang, and Schimdt [15].

The main result of this paper is a QCD factorization theorem for the SIDIS cross section in the above kinematics region, accurate up to the power corrections $(P_{h\perp}^2/Q^2)^n$ and to all orders in perturbation theory. This factorization has been conjectured by Collins [13] [Eq. (13)], following the early work of Collins and Soper on e^+e^- annihilation [9]. However, an exact statement of the factorization theorem requires an adequate definition of the TMD parton distributions and fragmentation functions in QCD and a systematic factorization (and subtraction) of soft, collinear, and hard gluon contributions. In light of the recent development in this area [16–19], here we provide a first detailed examination of QCD radiative corrections in SIDIS, following the methodology of Ref. [9].

The factorization theorem we propose for the leading spin-independent structure function is

$$F(x_B, z_h, P_{h\perp}, Q^2) = \sum_{q=u,d,s,\ldots} e_q^2 \int d^2\vec{k}_\perp d^2\vec{p}_\perp d^2\vec{\ell}_\perp q(x_B, k_\perp, \mu^2, x_B\zeta, \rho) \times \hat{q}(z_h, p_\perp, \mu^2, \hat{\zeta}/z_h, \rho) S(\vec{\ell}_\perp, \mu^2, \rho) H(Q^2, \mu^2, \rho) \delta^2(z_h\vec{k}_\perp + \vec{p}_\perp + \vec{\ell}_\perp - \vec{P}_{h\perp}), \tag{1}$$

where μ is a renormalization (and collinear factorization) scale; ρ is a gluon rapidity cutoff parameter; *the μ and ρ dependence cancels among various factors.* In a special system of coordinates in which $x_B\zeta = \hat{\zeta}/z_h$, one has $\zeta^2 x_B^2 = \hat{\zeta}^2/z_h^2 = Q^2\rho$. The physical interpretation of the factors are as follows: q is the TMD quark distribution function depending on, among others, the Bjorken x_B; $\hat{q}$ is the TMD quark fragmentation function depending on, among others, the hadron momentum fraction z_h; H represents the contribution of parton hard scattering and is a perturbation series in α_s; and, finally, the soft factor S comes from soft-gluon radiations and is defined by a matrix element of Wilson lines in a QCD vacuum. The above result shows that the hadron transverse momentum is generated from the combined effects of transverse momentum of the quarks in the nucleon, soft-gluon radiation, and the transverse momentum of the quark fragmentation.

There is no contribution from the TMD gluon distributions and fragmentation functions at the leading twist. For the gluons to contribute, one must introduce the soft quark lines. According to the power counting in Sec. IV A, the process is power-suppressed.

The main steps to establish the above factorization are as follows. In Sec. II, we introduce the TMD parton distribution and fragmentation function, and calculate them to one-loop order in perturbative QCD. The result contains collinear divergence and obeys the Collins and Soper evolution equation in the rapidity cutoff. We study the factorization of the TMD distributions by subtracting away the soft contributions. In Sec. III, one-loop result for semi-inclusive DIS scattering is obtained, and the factorization is shown to be true on the diagram-by-diagram basis. In Sec. IV, we generalize the one-loop result to all orders by identifying the leading regions for an arbitrary Feynman diagram using soft and collinear power counting. We then argue that a systematic factorization of the leading region leads to the general formula in Eq. (1). In Sec. V, the large logarithms in the perturbative expression are summed through solving evolution equations. We conclude the paper in Sec. VI.

The factorization considered here can also be studied in the framework of soft-collinear effective theory developed recently in Refs. [20–23]. We will leave this subject for a future publication.

II. TRANSVERSE-MOMENTUM DEPENDENT PARTON DISTRIBUTION

In the factorization formula [Eq. (1)], there is a factor $q(x, k_\perp, \mu^2, x_B\zeta, \rho)$ representing a TMD parton distribution, which differs from the usual Feynman parton distribution where the parton transverse momentum has already been integrated over. This object was introduced by Collins and Soper in the axial gauge and has a number of interesting properties [9]. In particular, it has a light cone singularity and hence depends on the energy of the parent nucleon (related to ζ) in addition to the parton's longitudinal momentum fraction x and transverse momentum $k_\perp$. The sensitivity to the small-x gluon physics is controlled by a parameter ρ.

In this paper, we follow a definition of TMD distribution in Feynman gauge with explicit gauge links [16]. We avoid the axial gauge because of the potential existence of gauge links at space-time infinity [18]. We calculate the TMDPD at one-loop order and show that it obeys the Collins-Soper evolution equation. The simplest definition of the distribution contains the soft gluon effect which must be subtracted; we show how this can be done at the one-loop level. We also discuss theoretical difficulty to recover the integrated parton distribution from a direct transverse-momentum integration.

A. Definition of TMDPD

Consider a hadron, a nucleon, for example, with four-momentum P. For convenience, we choose $\vec{P}$ along the z-direction, $P^\mu = (P^0, 0, 0, P^3)$. In the limit $P^3 \to \infty$, the P^μ is proportional to the light cone vector $(1, 0, 0, 1)$. From now on, we use the light cone coordinates $k^\pm = (k^0 \pm k^3)/\sqrt{2}$, and write any four-vector k^μ in the form of $(k^-, \vec{k}) = (k^-, k^+, \vec{k}_\perp)$, where $\vec{k}_\perp$ represents two perpendicular components (k^x, k^y). Let $(xP^+, \vec{k}_\perp)$ represent the momentum of a parton (quark or gluon) in the hadron. Let us start with the following definition of the transverse-momentum dependent quark distribution in a class of nonsingular gauges [16,19],

$$\mathcal{Q}\,(x, k_\perp, \mu, x\zeta) = \frac{1}{2}\int \frac{d\xi^-}{2\pi} e^{-ix\xi^- P^+} \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{k}_\perp} \langle P|\bar{\psi}_q(\xi^-, 0, \vec{b}_\perp)\mathcal{L}_v^\dagger(\infty; \xi^-, 0, \vec{b}_\perp)\gamma^+ \mathcal{L}_v(\infty; 0)\psi_q(0)|P\rangle, \quad (2)$$

where the quark color indices are implicit, ψ_q is the quark field, v^μ is a timelike dimensionless ($v^2 > 0$) four-vector with zero transverse components ($v^-, v^+, \vec{0}$), and $\mathcal{L}_v$ is a gauge link along v^μ,

$$\mathcal{L}_v(\infty; \xi) = \exp\left[-ig\int_0^\infty d\lambda v \cdot A(\lambda v + \xi)\right]. \quad (3)$$

The sign convention for the gauge coupling is $D^\mu = \partial^\mu + igA^\mu$. As mentioned before, μ is an ultraviolet (UV) renormalization (or cutoff) scale.

It is convenient to introduce the "gauge-invariant" quark field,

$$\Psi_v(\xi) = \mathcal{L}_v(\infty; \xi)\psi(\xi). \quad (4)$$

The quark distribution becomes simply

$$\mathcal{Q}\,(x, k_\perp, \mu, x\zeta) = \frac{1}{2}\int \frac{d\xi^-}{2\pi} e^{-ix\xi^- P^+} \int \frac{d^2 b_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{k}_\perp} \langle P|\bar{\Psi}_v(\xi^-, 0, \vec{b}_\perp)\gamma^+ \Psi_v(0)|P\rangle. \quad (5)$$

The variable ζ^2 denotes the combination $(2P\cdot v)^2/v^2 = \zeta^2$.

Physically, a parton interpretation of $\mathcal{Q}(x, k_\perp, \mu, \zeta)$ is the most natural if v is chosen along the conjugating light cone direction of P^μ, i.e., $v^\mu \sim (1, 0, 0_\perp)$. However, as has been known in the literature and reemphasized recently by Collins [19], the distribution in this limit has logarithmic divergences (also called light cone singularity) corresponding to contributions of virtual gluons with zero plus momentum ℓ^+, or infinitely negative rapidity, $\ln \ell^+/\ell^-$. [In a physical process, the plus momentum of a parton is limited by the kinematics of scattering.] To avoid the divergence, we must introduce a rapidity cutoff for the gluons. One way to accomplish this is to introduce a non-lightlike v^μ, such as with a $v^+ \neq 0$ [16]. Then the contribution of the virtual gluons with rapidity smaller than $\ln v^+/v^-$ is excluded from the parton distribution. As a consequence, a dimensional scalar $\zeta^2 = (2P\cdot v)^2/v^2$ emerges in the distribution. The limit of lifting the cutoff, $v^+ \to 0$, corresponds to $\zeta \to \infty$. [In the following expressions, we will take this limit whenever we can.] The light cone divergences are now reflected in the large logarithms involving ζ. The ζ-evolution of the TMD distribution can be viewed as either the evolution in the gluon rapidity cutoff through v^μ or that in the energy of the incoming hadron. The evolution is calculable in perturbation theory when $k_\perp$ is hard, i.e., $\gg \Lambda_{\rm QCD}$ [9].

Unless stated otherwise, we work in nonsingular gauges, such as covariant gauges (including the Feynman gauge), for which the gauge potential vanishes at space-time infinity. In light cone gauge, however, it is well known that the gauge potential is finite at infinity. Otherwise, the single-spin asymmetry discussed by Brodsky *et al.* would disappear in such a gauge [15]. Belitsky, Ji, and Yuan have shown that in singular gauges, one generally has to include gauge links at infinity [18]. These extra links can be found by imposing the gauge invariance of the parton densities starting from their definition in nonsingular gauges.

Since the two quark fields in Eq. (2) are separated along the spatial directions, the only ultraviolet divergence in $\mathcal{Q}(x, k_\perp, \mu, \zeta)$ comes from the wave function renormalization of the quark fields and the gauge links. In this paper, we use dimensional regularization (DR) and modified minimal subtraction ($\overline{\rm MS}$) to treat ultraviolet divergences. If we use Eq. (2) naively in the axial gauge $v\cdot A = 0$, then the ultraviolet divergence of $\mathcal{Q}(x, k_\perp)$ is the same as the quark wave function renormalization in that gauge. Then the renormalization group equation becomes simple,

$$\mu\frac{d\mathcal{Q}(x, k_\perp, \mu, x\zeta)}{d\mu} = 2\gamma_F\mathcal{Q}(x, k_\perp, \mu, x\zeta), \quad (6)$$

where γ_F is the anomalous dimension of the quark field in the axial gauge: $\gamma_F = (3\alpha_s/4\pi)C_F + \mathcal{O}(\alpha_s^2)$.

B. One-loop calculation

In this subsection, we present the one-loop result for a quark TMD distribution in an "on shell" quark. The calculation is important for a number of reasons. First, it shows clearly that the TMD distribution contains double logarithms in ζ because of the collinear and light cone divergences. It also serves as an explicit check for the evolution equation in rapidity cutoff. More importantly, the one-loop result allows one to devise QCD factorizations both for the distribution itself and for the one-loop DIS cross section to be presented in the next section.

We use a nonzero gluon mass λ as an infrared regulator since there is no nonlinear gluon coupling at one-loop. One can use dimensional regularization beyond the leading order. The factorization is, of course, independent of the infrared regulator. Collinear singularities are regulated by nonzero quark masses.

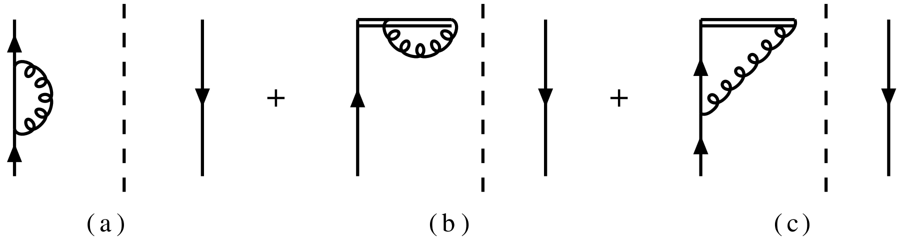

FIG. 1. Virtual gluon contribution to one-loop transverse-momentum dependent quark distribution in an on shell quark. The asymmetric diagrams from left-right reflection are not shown, but are included in the result.

Let us first consider the virtual contribution shown in Fig. 1. For the self-energy diagram on the incoming quark leg, one has a contribution $\mathcal{Q}(x, k_\perp) = \delta(x-1)\delta^2(\vec{k}_\perp) \times (Z_F - 1)$ with

$$Z_F = 1 + \frac{\alpha_s C_F}{4\pi}\left(-\ln\frac{\mu^2}{m^2} + 2\ln\frac{m^2}{\lambda^2} - 4\right), \tag{7}$$

where m and λ are the masses of the quark and gluon, respectively, $C_F = (N_c^2 - 1)/(2N_c)$ with $N_c = 3$; a term linear in $N_\epsilon = 2/\epsilon - \gamma_E + \ln 4\pi$, where $\epsilon = 4 - d$ and γ_E the Euler constant, has been removed according to the $\overline{\rm MS}$ scheme. The on-shell renormalization introduces the soft divergence in Z_F, reflecting in the gluon mass dependence. For the self-energy on the gauge link, one has a similar contribution with Z_F replaced by,

$$Z_W = 1 + \frac{C_F \alpha_s}{4\pi}\left(2\ln\frac{\mu^2}{\lambda^2}\right). \tag{8}$$

Finally, the diagram with the virtual gluon vertex again has a similar contribution with Z_F replaced by

$$Z_V = 1 + \frac{\alpha_s C_F}{4\pi}\left[2\ln\frac{\mu^2}{m^2} + 2\ln\frac{\zeta^2}{m^2} - \ln^2\frac{\zeta^2}{m^2} - 2\ln\frac{m^2}{\lambda^2}\ln\frac{\zeta^2}{m^2} - \frac{2\pi^2}{3} + 4\right], \tag{9}$$

where we have made the approximation that $\zeta^2 = \frac{4(P\cdot v)^2}{v^2}$ is much larger than any other soft QCD scales. When ζ is large, the double logarithms slow down the convergence of the pQCD series and call for a resummation which can be accomplished with the Collins-Soper equation (see the next subsection). In summary, the virtual diagrams give,

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.1} = \delta(x-1)\delta^2(\vec{k}_\perp) \times (Z_F + Z_W + Z_V - 3), \tag{10}$$

where the dependence on soft scales m and λ is implicit on the left.

Now turn to the real gluon emission contributions shown in Fig. 2. The contribution from Fig. 2(a) without the light cone link,

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.2a} = \frac{\alpha_s C_F |1-x|}{2\pi^2}\left[\frac{1}{k_\perp^2 + x\lambda^2 + (1-x)^2 m^2} - \frac{2xm^2}{(k_\perp^2 + x\lambda^2 + (1-x)^2 m^2)^2}\right], \tag{11}$$

where we have taken $\epsilon \to 0$. This must be done if we treat the TMDPD in the factorization formula as a physical observable. This, however, introduces certain problems in integrating out $k_\perp$ in DR, and we will discuss this more thoroughly in Sec. II F. The transverse momentum $k_\perp$ can go to zero, and therefore we cannot set the quark and gluon masses to zero too soon. However, the nonperturbative QCD physics will erase this sensitivity after factorization is formulated.

The contribution from Fig. 2(b), including the Hermitian conjugation term, is,

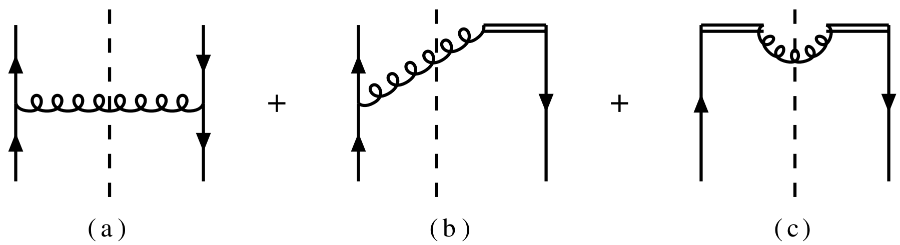

FIG. 2. Same as Fig. 1: real gluon contribution.

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.2b} = \frac{\alpha_s C_F}{\pi^2}\frac{x}{|1-x|}\left[\frac{1}{k_\perp^2 + x\lambda^2 + (1-x)^2 m^2} - \frac{1}{k_\perp^2 + \lambda^2 + \zeta^2(1-x)^2}\right]. \quad (12)$$

The second term regularizes the light cone singularity at $x = 1$. If one uses the usual regularization method of a plus function [24] and takes the limit that ζ is large, the above can be transformed into the following form,

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.2b} = \frac{\alpha_s C_F}{\pi^2}\frac{x}{(1-x)_+}\frac{1}{k_\perp^2 + x\lambda^2 + (1-x)^2 m^2} + \frac{\alpha_s C_F}{2\pi^2}\delta(x-1)\frac{1}{k_\perp^2+\lambda^2}\ln\frac{\zeta^2}{k_\perp^2+\lambda^2}, \quad (13)$$

which has a delta function at $x = 1$.

Finally the contribution from Fig. 2(c) with two gauge links is

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.2c} = \frac{\alpha_s C_F}{2\pi^2}\frac{2|1-x|\zeta^2}{(k_\perp^2 + \lambda^2 + (1-x)^2\zeta^2)^2} \quad (14)$$

Formally, it vanishes when ζ is large, except in the region where $(1-x)^2\zeta^2$ is small. Therefore in the limit $\zeta \to \infty$, the above is the same as a δ function at $x = 1$,

$$\mathcal{Q}(x, k_\perp, \mu, x\zeta)|_{\rm fig.2c} = \delta(x-1)\frac{\alpha_s}{2\pi^2}C_F\frac{1}{k_\perp^2+\lambda^2}. \quad (15)$$

We caution the reader that taking the $\zeta \to \infty$ limit conflicts with the $k_\perp \to \infty$ limit. In fact, in the above example, it turns a $k_\perp$-convergent integral into a divergent one.

C. Collins-Soper evolution in hadron energy or gluon rapidity cutoff

As we have seen from the previous subsection, unlike the Feynman parton distributions which contain just the collinear singularities from the quark masses, the TMD distributions contain in addition the light cone singularities which are regulated by ζ^2. Since $\zeta \to \infty$ corresponds to the high-energy limit, ζ dependence of the parton distribution is calculable in perturbative QCD, just like the renormalization scale dependence in μ. It turns out this is true only for large $k_\perp$.

The ζ-evolution equation for $\mathcal{Q}(x, k_\perp, \mu, x\zeta)$ has been derived by Collins and Soper in the large ζ limit [9]. Normally, we use $\mathcal{Q}(x, k_\perp)$ in the small $k_\perp$ region. Let us extend this dependence to large $k_\perp$ and introduce the Fourier (or impact-parameter) representation,

$$\mathcal{Q}(x, b_\perp, \mu, x\zeta) = \int d^2k_\perp e^{i\vec{b}_\perp\cdot\vec{k}_\perp}\mathcal{Q}(x, k_\perp, \mu, x\zeta). \quad (16)$$

The above integral should be convergent for nonzero $b_\perp$. [There are UV divergences when $\vec{b} = 0$ which we will not consider here.] The Collins-Soper evolution equation is

$$\zeta\frac{\partial}{\partial\zeta}\mathcal{Q}(x, b, \mu, x\zeta) = (K(\mu, b) + G(\mu, x\zeta))\mathcal{Q}(x, b, \mu, x\zeta), \quad (17)$$

where K depends on the UV renormalization scale μ and infrared impact parameter b, and is nonperturbative when b is large; G is perturbative because μ and ζ are hard; and both are free of gluon and quark mass singularity. The sum $K + G$ is independent of UV scale μ and hence,

$$\mu\frac{d}{d\mu}K = -\gamma_K = -\mu\frac{d}{d\mu}G, \quad (18)$$

where γ_K is the cusp anomalous dimension [25] and is a series in α_s free of infrared singularities. The derivation of the above equation in Feynman gauge has been given in Refs. [16,26]. In the above equation, any power correction of $(\Lambda_{\rm QCD}/\zeta)^n$ has been ignored and hence it is true only when $\zeta \gg \Lambda_{\rm QCD}$.

According to the result in the previous section, G gets a contribution from Fig. 1(c) only, whereas K gets a contribution from Fig. 2(b). The sum is

$$K(b, \mu) + G(x\zeta, \mu) = -\frac{\alpha_s C_F}{\pi}\ln\frac{x^2\zeta^2 b^2 e^{2\gamma_E - 1}}{4}, \quad (19)$$

which is valid when b^2 is small and where γ_E is the Euler constant. The one-loop anomalous dimension is then,

$$\gamma_K = \frac{\alpha_s}{\pi}2C_F, \quad (20)$$

which is well known. Using the above renormalization group Eq. (18), one can sum over large logarithms $\ln\zeta^2 b^2$ in $K + G$ when b is small (otherwise K is nonperturbative). Substituting the result into Eq. (17), one finds a resummed double-leading logarithm in ζb (see Sec. V).

D. Factorization of soft gluons in the TMD parton distribution

From the viewpoint of QCD factorization, parton distributions are introduced to absorb collinear divergences when the quark masses are zero. From the one-loop result, it is seen that the TMD distribution contains both collinear and soft contributions. In this subsection, we will attempt to isolate and subtract the soft contribution from the above definition of TMDPD.

Let us first consider the self-energy diagram in Fig. 1. Figure 1(a) contains the soft gluon contribution which is obtained by making the soft approximation to the quark propagator and quark gluon coupling. Briefly, the soft approximation corresponds to neglecting the soft-gluon momentum in the numerator and the gluon momentum squared in the denominator; more discussion on the soft approximation is provided in Sec. IV. In the soft region where all of the components of ℓ^μ are small, the self-energy becomes

$$\int \frac{d^4\ell}{(4\pi)^2} \frac{1}{(\ell \cdot p + i\epsilon)^2} \frac{1}{(\ell^2 - \lambda^2 + i\epsilon)}. \tag{21}$$

We can factorize the above contribution out of the parton distribution by subtracting it from the one-loop result of Z_F. To make the subtraction mathematically convenient, we extend the soft approximation to the whole integration region of ℓ and use the DR and $\overline{\text{MS}}$ scheme to get rid of the UV contribution. [There is no contribution from the collinear region because the integral is convergent in the massless quark limit.] The result is a contribution similar to the self-energy of an eikonal line.

We can do the similar subtraction for Fig. 1(c), by forming a soft approximation for the gluon interacting with the quark line

$$-ig^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{2p \cdot v}{(v \cdot \ell + i\epsilon)(p \cdot \ell + i\epsilon)(\ell^2 - \lambda^2 + i\epsilon)}, \tag{22}$$

with ℓ^μ restricted to the soft region. This time, however, the situation is more complicated. If one extends the integration ℓ^μ to all regions, there is also a collinear contribution coming from virtual gluons with momentum parallel to p^μ, as signified by the divergence of the zero quark mass. In other words, the simplified approach of subtracting away the whole integral will also take away a part of the collinear contribution.

One may get around this by excluding the collinear gluon contribution with small ℓ^-. This can be achieved by introducing a four-vector $\tilde{v}^\mu$ which has a large $\tilde{v}^+$ but relatively small $\tilde{v}^-$, and approximating the above soft contribution with the following integral,

$$-ig^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{2\tilde{v} \cdot v}{(v \cdot \ell + i\epsilon)(\tilde{v} \cdot \ell + i\epsilon)(\ell^2 - \lambda^2 + i\epsilon)}, \tag{23}$$

where we have replaced p by $\tilde{v}$. The above soft contribution includes soft gluons with ℓ^+/ℓ^- limited by v^+/v^- and ℓ^-/ℓ^+ by $\tilde{v}^-/\tilde{v}^+$.

Subtracting the above contribution from the Fig. 1(c), the remainder has a soft divergence in the gluon mass. This indicates that the soft and collinear divergences cannot be completely separated, as there are regions of loop momentum where collinear and soft divergences overlap. Therefore, one could in principle *define* Eq. (22) with unrestricted ℓ-integration as the "soft contribution." With this approach, the subtracted Fig. 1(c) has no soft divergence. However, as we have mentioned before, the soft contribution is then not entirely soft. Two different approaches may be considered as two different subtraction schemes. Here, we use the first one.

If we follow the above procedure, one finds the complete soft contribution in terms of the matrix element of Wilson lines,

$$S(\vec{b}_\perp, \mu^2, \rho) = \frac{1}{N_c} \langle 0| \mathcal{L}^\dagger_{\tilde{v}il}(\vec{b}_\perp, -\infty) \mathcal{L}^\dagger_{vlj}(\infty; \vec{b}_\perp) \mathcal{L}_{vjk}(\infty; 0) \mathcal{L}_{\tilde{v}ki}(0; -\infty)|0\rangle, \tag{24}$$

where $\rho = \sqrt{v^- \tilde{v}^+ / v^+ \tilde{v}^-}$. We have made the color indices explicit ($i, j, k, l = 1, 2, 3$). The subtracted parton distribution can be defined as

$$q(x, k_\perp, \mu, x\zeta, \rho) = \frac{1}{2} \int \frac{d\xi^-}{2\pi} e^{-ix\xi^- P^+} \int \frac{d^2 b_\perp}{(2\pi)^2} e^{i\vec{b}_\perp \cdot \vec{k}_\perp} \frac{\langle P|\overline{\Psi}_v(\xi^-, 0, \vec{b}_\perp)\gamma^+ \Psi_v(0)|P\rangle}{S(\vec{b}_\perp, \mu^2, \rho)}. \tag{25}$$

This definition differs from that of Collins [19] in that the soft contribution here has no dependence in ξ^-. Moreover, from our one-loop calculation, it is not clear that the above distribution has a well-defined limit when $\zeta \to \infty$ or v^μ becomes lightlike, $v^2 = 0$, as claimed in [19].

Let us calculate the one-loop soft subtraction, shown in Fig. 3. First, the diagrams with self-energy on all four of the Wilson lines,

$$\Delta_{\text{soft}} q(x, k_\perp)|_{\text{diag.3a}} = -\delta(x-1)\delta^2(\vec{k}_\perp) 2(Z_W - 1), \tag{26}$$

half of which cancels the self-energy of the gauge-link in Eq. (10).

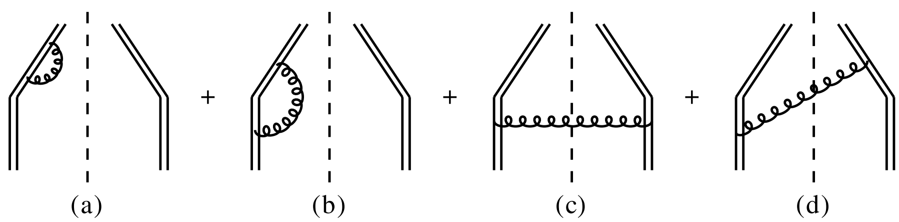

FIG. 3. Soft gluon contribution to the TMD parton distribution at one-loop order. The double lines represent eikonal line.

The vertex correction of the gauge links, for which there is a factor of 2 to account for two virtual vertices, is

$$\begin{aligned}\Delta_{\rm soft} q(x, k_\perp)|_{\rm diag.3b} &= -\delta(x-1)\delta^2(\vec{k}_\perp)2\int \frac{1}{v\cdot l}\frac{1}{\tilde{v}\cdot l}\frac{1}{l^2-\lambda^2}\\ &= \delta(x-1)\delta^2(\vec{k}_\perp)\frac{\alpha_s}{2\pi}C_F \ln\frac{4(v\cdot\tilde{v})^2}{v^2\tilde{v}^2}\cdot\ln\frac{\mu^2}{\lambda^2},\end{aligned} \tag{27}$$

where the coefficient is just the cusp anomalous dimension [25].

Now we consider the soft contribution from the real emission. Again there are two types of diagrams. The first type is the one with two gluons emitted from v or $\tilde{v}$. The contribution to the parton distribution is

$$\Delta_{\rm soft} q(x, k_\perp)|_{\rm diag.3c} = \delta(x-1)\frac{\alpha_s}{\pi^2}C_F\frac{1}{k_\perp^2+\lambda^2}, \tag{28}$$

which, when integrated over $k_\perp$, cancels out the self-energy contribution. The second type is the interference from the real gluon emission of the v and $\tilde{v}$ lines. Plugging in the soft factor,

$$\Delta_{\rm soft} q(x, k_\perp)|_{\rm diag.3d} = -\delta(x-1)\frac{\alpha_s}{2\pi^2}C_F\ln\frac{4(v\cdot\tilde{v})^2}{v^2\tilde{v}^2}\frac{1}{k_\perp^2+\lambda^2}. \tag{29}$$

When integrated over $k_\perp$, it cancels out the vertex corrections.

E. Final result for the one-loop quark distribution

Adding all diagrams, we get the final result for the quark distribution at one-loop level, following the definition in Eq. (25),

$$\begin{aligned} q(x, k_\perp, \mu, x\zeta, \rho) &= \delta(x-1)\delta^2(k_\perp)\left[Z_F + Z_V + Z_W - 3 + \frac{\alpha_s C_F}{2\pi}\ln\frac{\mu^2}{\lambda^2}(\ln\rho^2 - 2)\right]\\ &\quad + \delta(x-1)\frac{\alpha_s C_F}{2\pi^2}\frac{1}{k_\perp^2+\lambda^2}\left[\ln\frac{\zeta^2}{k_\perp^2+\lambda^2} - \ln\rho^2 + 1\right] + \frac{\alpha_s C_F}{2\pi}\left[\frac{1-x}{k_\perp^2 + x\lambda^2 + (1-x)^2 m^2}\right.\\ &\quad \left. - \frac{2x(1-x)m^2}{[k_\perp^2 + x\lambda^2 + (1-x)^2m^2]^2} + \frac{2x}{(1-x)_+}\frac{1}{k_\perp^2 + x\lambda^2 + (1-x)^2m^2}\right],\end{aligned} \tag{30}$$

where the constants Z_F, Z_W, and Z_V are defined in Eqs. (7)–(9). The soft divergence (λ^2 dependence) in the right-hand side cancels out among the terms, leaving the total result free of divergence. This fact can be more easily seen from the impact parameter b-space expressions. Similar to Eq. (16), we can define the impact parameter space TMD quark distribution as Fourier transformation of that in the momentum space,

$$q(x, b_\perp, \mu, x\zeta, \rho) = \int d^2k_\perp e^{i\vec{b}_\perp\cdot\vec{k}_\perp} q(x, k_\perp, \mu, x\zeta, \rho). \tag{31}$$

After a tedious but straightforward calculation, we can get the one-loop result for the TMD quark distribution in the impact parameter space,

$$\begin{aligned} q(x, b, \mu, x\zeta, \rho) &= \frac{\alpha_s C_F}{2\pi}\left\{\left(\frac{1+x^2}{1-x}\right)_+ \ln\frac{4}{b^2m^2}e^{-2\gamma_E} - \left(\frac{2x}{1-x}\right)_+ - \left(\frac{1+x^2}{1-x}\ln\frac{1}{(1-x)^2}\right)_+\right.\\ &\quad \left. + \delta(x-1)\left[\left(\frac{1}{2} - \ln\rho^2\right)\ln\frac{4}{b^2\mu^2}e^{-2\gamma_E} - \frac{1}{2}\ln^2\left(\frac{\zeta^2b^2}{4}e^{2\gamma_E - 1}\right) - \frac{2+\pi^2}{2}\right]\right\}.\end{aligned} \tag{32}$$

It is obvious that the above result is free of soft divergence, where the explicit dependence on λ^2 disappears.

F. Integrating over transverse momentum in TMDPD

One would expect that after integrating over transverse momentum, a TMD parton distribution reduces to the usual Feynman parton distribution. This, in fact, is not straightforward. This can be seen from the one-loop result presented in the last subsection. Integrating over transverse momentum, one cannot get the integrated quark distribution at one-loop order.

First of all, we have chosen DR and $\overline{\rm MS}$ to regulate ultraviolet divergences. In the TMDPD, the $\epsilon \to 0$ limit and $\overline{\rm MS}$ subtraction have already been performed as it represents an observable in 4-dimension. On the other hand, a Feynman parton distribution is obtained first by integrating the transverse momentum and then performing UV subtraction. Since the integral is a divergent one, the procedure of integration and subtraction is not interchangeable. To get around this, one may use a momentum cutoff to regularize the UV divergence [27]; but this is hard to implement beyond one-loop without destroying the gauge symmetry. [A consistent regularization might be a discrete space-time lattice, however, in practice this is hard to implement in Minkowski space.]

Even when there is a consistent cutoff regulator, one may still have a problem with the light cone singularities. While the gauge link in Feynman distributions is along the light cone, we have chosen an off-light cone gauge link to regulate these divergences. Therefore, one cannot expect that after integrating over a TMDPD with a non-lightlike gauge the Feynman distribution recovers.

What happens if one takes the light cone limit of the non-lightlike gauge link after integrating over the transverse momentum in a TMDPD? The standard integrated parton distribution still does not emerge if dimensional regularization is used. If one chooses the non-lightlike gauge from the beginning, the gluon propagator has a term proportional to v^2 (if v is the direction of the gauge link). This term contributes to the TMDPD in loop calculations. When the light cone limit, $v^2 \to 0$, is taken, these contributions would vanish if there were no axial-gauge singularities at $k \cdot v = 0$. In practice, however, the singularities are present and the limit $v^2 \to 0$ does not reproduce the result obtained with $v^2 = 0$ set in the beginning in dimensional regularization.

To summarize, it is nontrivial to recover a Feynman parton distribution by integrating over the transverse momentum in a TMDPD. One could cutoff the integral by hand, but a UV regularization scheme must be used which implements the same cutoff in the loop integrals. This is difficult to construct beyond one-loop. For the same reason, the light cone limit of the vector v^μ is not analytical.

G. Connection with the integrated parton distribution

As explained above, integrating over the transverse momentum of the TMD parton distribution does not yield the integrated parton distribution. However, at large transverse momentum, $k_\perp \gg \Lambda_{\rm QCD}$, the TMD parton distribution can be calculated from the integrated parton distributions. This is because large transverse momentum is generated from hard gluon radiations, which can be calculated in perturbative QCD. We can predict the power behavior ($k_\perp$) for the TMD parton distribution, using the standard power counting rule [28,29]. Indeed, from power counting rule we find that the TMD quark distribution scales as $1/k_\perp^2$ at large transverse momentum, which is consistent with our result in Eq. (30).

In a form of QCD factorization, we can write down the large $k_\perp$ TMD quark distribution in terms of integrated quark distribution,

$$q(x, k_\perp, \mu^2, x\zeta^2, \rho) = \frac{1}{k_\perp^2}\int_x^1 \frac{dy}{y}\tilde{C}_k\left(\frac{x}{y}, \mu^2, \zeta^2/k_\perp^2, \rho\right) \times q(y, \mu^2), \tag{33}$$

where $q(x, \mu^2)$ is the ordinary Feynman parton distribution. The $1/k_\perp^2$ behavior comes from the power counting, and $\tilde{C}_k$ is the coefficient function. At one-loop level, from Eq. (30) one can easily find,

$$\tilde{C}_k(x, \mu^2, \zeta^2/k_\perp^2, \rho) = \frac{\alpha_s}{2\pi^2}C_F\left[\left(\frac{1+x^2}{1-x}\right)_+ + \delta(x-1)\left(\ln\frac{\zeta^2}{k_\perp^2} - \ln\rho - \frac{1}{2}\right)\right]. \tag{34}$$

This result can be used to analyze the large transverse momentum behavior for the SIDIS processes.

Similarly, in the impact parameter b-space, when b is small, the TMD parton distribution can be calculated from the integrated parton distribution by using another factorization theorem [9], e.g.,

$$q(x, b, \mu^2, x\zeta, \rho) = \int_x^1 \frac{dy}{y}\tilde{C}\left(\frac{x}{y}, b^2, \mu^2, \bar{\mu}^2, x^2\zeta^2, \rho\right) \times q(y, \bar{\mu}^2), \tag{35}$$

where $q(x, \bar{\mu}^2)$ is again the integrated quark distribution. $\tilde{C}$ is the coefficient function depending on the scales μ, $\bar{\mu}$, and other variables. At one-loop order, from Eq. (32) one finds,

$$\tilde{C}(x, b^2, \mu^2, \bar{\mu}^2, x^2\zeta^2, \rho) = \frac{\alpha_s}{2\pi}C_F\left\{(1-x) + \left(\frac{1+x^2}{1-x}\right)_+ \ln\frac{4}{b^2\bar{\mu}^2}e^{-2\gamma_E} + \delta(x-1)\left[\left(\frac{1}{2} - \ln\rho^2\right)\ln\frac{4}{b^2\mu^2}e^{-2\gamma_E} - \frac{1}{2}\ln^2\left(\frac{\zeta^2 b^2}{4}e^{2\gamma_E - 1}\right) - \frac{3+\pi^2}{2}\right]\right\}. \tag{36}$$

This result will be useful to relate to the CSS resummation [10].

Note that there is no simple connection between the two factorization expressions above, because to get the parton distribution at small-b one needs the distribution at large $k_\perp$ as well.

H. TMD fragmentation function

The transverse-momentum-dependent fragmentation function (TMDFF) has a similar definition as the TMD parton distribution. Many of the results discussed in the previous subsections can be immediately translated into those for TMDFF. Here, we sketch the main results briefly.

Use $\hat{q}_h(z, k_\perp, \mu, \zeta)$ as a notation for a subtracted quark fragmentation function into the hadron h,

$$\hat{q}_h(z, P_{h\perp}, \mu, \hat{\zeta}/z, \rho) = \frac{1}{2z}\int \frac{d\xi^-}{2\pi}\frac{d^2\vec{b}}{(2\pi)^2}e^{-i(k^+\xi^- - \vec{k}_\perp\cdot\vec{b}_\perp)}\sum_X \frac{1}{3}\sum_a \langle 0|\mathcal{L}_{\tilde{v}}(-\infty;0)\psi_{\beta a}(0)|P_h X\rangle\gamma^+_{\alpha\beta}\langle P_h X|\overline{\psi}_{\alpha a}(\xi^-, \vec{b}) \times \mathcal{L}^\dagger_{\tilde{v}}(\xi^-, \vec{b}; -\infty)|0\rangle / S(b_\perp, \mu, \rho), \quad (37)$$

where $\tilde{v}$ is mainly along the light cone direction conjugating to P_h; $k^+ = P_h^+/z$ and $k_\perp = -\vec{P}_{h\perp}/z$; and a is a color index. The variable $\hat{\zeta}$ is defined as

$$\hat{\zeta}^2 = 4(P_h\cdot\tilde{v})^2/\tilde{v}^2. \quad (38)$$

For a quark fragmenting into a quark, the leading order result is normalized to $\hat{q}(z, P_\perp) = \delta(z-1)\delta^2(P_\perp)$.

It is not difficult to see that the quark fragmentation function in a quark can be obtained by a simple substitution of the corresponding quark distribution in a quark,

$$\hat{q}_h(z, P_\perp, \mu^2, \hat{\zeta}/z, \rho) = \frac{1}{z}q\left(\frac{1}{z}, \frac{P_\perp}{z}, \mu^2, \hat{\zeta}/z, \rho\right) = q(z, P_\perp, \mu^2, z\hat{\zeta}, \rho), \quad (39)$$

where the second equality holds at one-loop order. Moreover, $\hat{q}_h$ satisfies the same Collins-Soper equation in $\hat{\zeta}$ evolution as that of q in ζ evolution.

I. Soft contribution

According to the definition of the soft contribution in Eq. (24), the one-loop result for the soft factor can be constructed from the above soft subtraction contributions,

$$S(k_\perp, \mu, \rho) = \delta^2(k_\perp) - \frac{\sum\Delta_{\rm soft}q(x, k_\perp)}{\delta(x-1)} = \delta^2(k_\perp) + \frac{\alpha_s}{2\pi^2}C_F\left[\ln\frac{4(v\cdot\tilde{v})^2}{v^2\tilde{v}^2} - 2\right]\cdot\left[\frac{1}{k_\perp^2+\lambda^2} - \pi\delta^2(k_\perp)\ln\frac{\mu^2}{\lambda^2}\right]. \quad (40)$$

When Fourier transformed to b-space, it becomes,

$$S(b, \mu^2, \rho) = 1 + \frac{\alpha_s C_F}{2\pi}(2 - \ln\rho^2)\ln\left(\frac{\mu^2 b^2}{4}e^{2\gamma_E}\right). \quad (41)$$

This, however, cannot be used for $b = 0$, for which the integration over $k_\perp$ must be regularized so that $S(\vec{b} = 0) = 1$.

III. ONE-LOOP FACTORIZATION

In this section, we show that the factorization formula, Eq. (1), is valid at one-loop order. To accomplish this, semi-inclusive DIS on a single-quark target is studied. The result can be easily translated into that for a nonperturbative hadronic target.

In the first subsection, we establish notation and normalization for the tree-level result. In the second subsection, we state and explain the content of the factorization theorem. In the following two subsections, we will verify its correctness for a single-quark target on the diagram-by-diagram basis: first for the virtual corrections, and then for the real corrections.

A. Notation and tree normalization

We choose a coordinate system for semi-inclusive DIS in which the nucleon is traveling along the z-direction. Introduce the light cone vectors $(p^0, p^x, p^y, p^z) = \Lambda(1, 0, 0, 1)$, $(n^0, n^x, n^y, n^z) = (1, 0, 0, -1)/2\Lambda$, and $p\cdot n = 1$, where Λ is an arbitrary parameter. The initial nucleon momentum P can be written as,

$$P^\mu = p^\mu + (M^2/2)n^\mu, \quad (42)$$

where M is the nucleon mass. The photon momentum is $q = \ell - \ell'$, where ℓ and ℓ' are the initial and final lepton momenta, respectively. We choose the photon momentum in the negative-z direction,

$$q^\mu = -\xi p^\mu + \frac{Q^2}{2\xi}n^\mu, \quad (43)$$

where $\xi \sim x_B = Q^2/2(P\cdot q)$ when M^2/Q^2 is neglected and $Q^2 = -q^2$. The so-called *hadron frame* is obtained by making a particular choice of Λ [6].

One has the option of either fixing the lepton plane as the xz plane or the hadron plane as the xz plane. In either case, there is an azimuthal angle between the two planes, and for the simplicity of our discussion, we integrate out this angle. The detected hadron has a momentum P_h mainly along the n^μ (negative z) direction with z_h fraction of the photon momentum component in the same direction, and with transverse momentum $P_{h\perp}$ which is invariant under the boost along the z-direction. As indicated earlier, $P_{h\perp}$ is considered to be soft (on the order of $\Lambda_{\rm QCD}$). If $P_{h\perp} \gg \Lambda_{\rm QCD}$, a different factorization formula exists in which only the integrated parton distributions and fragmentation functions enter.

The semi-inclusive DIS cross section under the one-photon exchange is

$$\frac{d\sigma}{dx_B dy dz_h d^2P_{h\perp}} = \frac{2\pi\alpha_{\rm em}^2}{Q^4} y\ell_{\mu\nu}W^{\mu\nu}(P, q, P_h), \tag{44}$$

where the unpolarized lepton tensor is

$$\ell^{\mu\nu} = 2(\ell^\mu\ell'^\nu + \ell^\mu\ell'^\nu - g^{\mu\nu}Q^2/2) = (Q^2/y^2)(1 - y + y^2/2)(-2g_\perp^{\mu\nu}) + \cdots, \tag{45}$$

where y is the fraction of the lepton energy loss, $1 - E'/E$. Since we are going to integrate over the azimuthal angle ϕ, only the structure $g_\perp^{\mu\nu} = g^{\mu\nu} - p^\mu n^\nu - p^\nu n^\mu$ will survive.

The hadron tensor has the following expression in QCD,

$$W^{\mu\nu}(P, q, P_h) = \frac{1}{4z_h}\sum_X \int \frac{d^4\xi}{(2\pi)^4} e^{iq\cdot\xi}\langle P|J_\mu(\xi)|XP_h\rangle \times \langle XP_h|J_\nu(0)|P\rangle, \tag{46}$$

where J^μ is the electromagnetic current of the quarks, X represents all other final-state hadrons other than the observed particle h. The variable z_h can be defined as $P\cdot P_h/P\cdot q$ or P_h^-/q^-.

A simple calculation on the single-quark target yields that,

$$W^{\mu\nu} = -\frac{1}{2}g_\perp^{\mu\nu}\delta(x_B - 1)\delta(z_h - 1)\delta^2(\vec{P}_{h\perp}) + \cdots. \tag{47}$$

In this case, it is known

$$q^{(0)}(x_B, k_\perp) = \delta(x_B - 1)\delta^2(\vec{k}_\perp), \hat{q}^{(0)}(z_h, p_\perp) = \delta(z_h - 1)\delta^2(\vec{p}_\perp). \tag{48}$$

It is easy to translate the above into a result for a physical hadron,

$$W^{\mu\nu} = -\frac{1}{2}g_\perp^{\mu\nu}\int d^2\vec{k}_\perp q(x_B, k_\perp) \times \int d^2\vec{p}_\perp \hat{q}(z_h, p_\perp)\delta^2(z_h\vec{k}_\perp + \vec{p}_\perp - \vec{P}_{h\perp}). \tag{49}$$

Therefore the cross section is,

$$\frac{d\sigma}{dx_B dy dz_h d^2P_{h\perp}} = \frac{4\pi\alpha_{\rm em}^2 s}{Q^4}(1 - y + y^2/2)x_B\sum_q e_q^2\int d^2\vec{k}_\perp q(x_B, k_\perp)\int d^2\vec{p}_\perp \hat{q}_h(z_h, p_\perp)\delta^2(z_h\vec{k}_\perp + \vec{p}_\perp - \vec{P}_{h\perp}), \tag{50}$$

where $s = (P + \ell)^2$, and we have kept only the ϕ-independent term. This result is known in the literature [30].

B. General form of factorization

In the following discussion, we are interested in the leading structure $F(x_B, z_h, P_{h\perp}, Q^2)$ only,

$$W^{\mu\nu} = -\frac{1}{2}g_\perp^{\mu\nu}F(x_B, z_h, P_{h\perp}, Q^2) + \cdots. \tag{51}$$

The other structures factorize in a similar way. The form of the factorization theorem we want to show is

$$F(x_B, z_h, P_{h\perp}, Q^2) = \sum_{q=u,d,s,\ldots} e_q^2\int d^2\vec{k}_\perp d^2\vec{p}_\perp d^2\vec{\ell}_\perp q(x_B, k_\perp, \mu^2, x_B\zeta, \rho)\hat{q}_h(z_h, p_\perp, \mu^2, \hat{\zeta}/z_h, \rho)S(\vec{\ell}_\perp, \mu^2, \rho) \times H(Q^2, \mu^2, \rho)\delta^2(z_h\vec{k}_\perp + \vec{p}_\perp + \vec{\ell}_\perp - \vec{P}_{h\perp}), \tag{52}$$

where in a special system of coordinates: $\zeta^2 = (Q^2/x_B^2)\rho$ and $\hat{\zeta}^2 = (Q^2z_h^2)\rho$ and $\rho = \sqrt{v^-\tilde{v}^+/v^+\tilde{v}^-}$ is a gluon rapidity cutoff parameter. The above result is accurate up to powers in $(P_{h\perp}^2/Q^2)^n$ for soft $P_{h\perp} \sim \Lambda_{\rm QCD}$. There is no direct contribution from the gluon distribution in this kinematic region. There is no convolution involving the longitudinal momentum fractions, x_B and z_h, typical in other hard processes. The transverse-momentum integrals show that the hadron transverse momentum can be generated from the initial state parton, final-state fragmentation, and the soft-gluon radiation.

The renormalization and collinear factorization scale μ cancels among the four factors, as the structure function should be μ-independent. As in the inclusive case, one can choose $\mu^2 = Q^2$ to eliminate the large logarithms in the hard factor. The soft-factorization parameter ρ depends on the directions of the Wilson lines and must also be canceled among all factors. The Collins-Soper equation allows studying the double logarithmic dependence of Q^2 in the distribution and fragmentation functions.

In principle, for $P_{h\perp} \sim Q$, the above factorization formula breaks down because of the power corrections, and because the transverse momentum $P_{h\perp}$ is now mainly generated from multijets production. However, it is convenient to extrapolate the above factorization to all $P_{h\perp}$, and introduce the impact-parameter space representation,

$$F(x_B, z_h, b, Q^2) = \sum_{q=u,d,s,\ldots} e_q^2 q(x_B, z_h b, \mu^2, x_B\zeta, \rho) \times \hat{q}(z_h, b, \mu^2, \hat{\zeta}/z_h, \rho) S(b, \mu^2, \rho) \times H(Q^2, \mu^2, \rho). \tag{53}$$

The convolution in the transverse momentum becomes a product of Fourier factors.

At tree level, $S^{(0)}(b, \mu^2, \rho) = 1$, and $H^{(0)} = 1$. Let us show that at one-loop, the above factorization is still valid, and calculate the correction to the hard part at this order.

C. Virtual corrections

We first consider factorization of one-loop virtual corrections to the tree process. The factorization actually holds diagram-by-diagram, and therefore we will study the momentum flow in individual diagrams and extract the corresponding hard factor.

Three diagrams shown in Fig. 4 correspond to the initial and final-state wave function renormalization and vertex corrections.

The self-energy correction is straightforward,

$$F = \delta(x_B - 1)\delta(z_h - 1)\delta^2(\vec{P}_{h\perp})(1 + 2(Z_F - 1)) + \ldots \tag{54}$$

where Z_F is given in Eq. (7). Z_F contains both soft and collinear contribution in the on shell scheme. If the self-energy is associated with the initial state quark, the collinear part of Z_F is absorbed by the self-energy correction on the parton distribution, corresponding to Fig. 1(a) subtracted by Fig. 3(a). The remaining soft contribution is attributed to the soft factor shown in Fig. 3(a). There is no contribution from the fragmentation function, nor is there a contribution from the hard part. A similar argument shows the self-energy correction to the final-state quark can be absorbed by the fragmentation function and the soft factor, yielding no contribution to the hard part.

The vertex correction produces exactly the same expression for F as the above, except Z_F is replaced by,

$$\hat{Z}_V = 1 - \frac{\alpha_s}{4\pi} C_F\left(\ln\frac{Q^2}{\mu^2} + \ln^2\frac{Q^2}{m^2} + 2\ln\frac{m^2}{\lambda^2}\ln\frac{Q^2}{m^2} - 4\ln\frac{Q^2}{m^2} - \frac{\pi^2}{3}\right). \tag{55}$$

The UV divergence in the above expression cancels that in Z_F because the sum of one-loop virtual corrections has no UV divergence. The above result contains a quark distribution part shown in Fig. 1(c) subtracted by Fig. 3(b), and a fragmentation function with a similar structure, and a soft contribution in Fig. 3(b). Subtracting all of the above from $\hat{Z}_V - 1$, we find a leftover hard contribution,

$$H^{(1)}(Q^2, \mu^2, \rho) = \frac{\alpha_s}{2\pi} C_F\left[(1 + \ln\rho^2)\ln\frac{Q^2}{\mu^2} - \ln\rho^2 + \frac{1}{4}\ln^2\rho^2 + \pi^2 - 4\right], \tag{56}$$

where we have chosen a coordinate system in which $x_B\zeta = \hat{\zeta}/z_h$ and therefore the dependence on the quasi-lightlike vectors v and $\tilde{v}$ is simply through a combination, $\rho = \sqrt{v^-\tilde{v}^+/v^+\tilde{v}^-}$. The dependence on Q^2 is of the form of single logarithms and can be controlled by a renormalization group equation because it contains no additional scale other than μ.

D. Real corrections

The one-loop real corrections are shown in Fig. 5. Since we are interested in the topology of the final-state in which the struck quark carries the dominant part of the energy-momentum of the current region, the emitted gluons are considered to be either soft or collinear. Therefore, there is no contribution to the hard scattering kernel from any of these diagrams. Our job is to show that these diagrams can be properly taken into account by the known one-loop parton distribution, fragmentation function, and the soft factor.

Let us start with the ladder diagram shown in Fig. 5(a). The soft-gluon radiation generates a transverse momentum for the struck quark. There is no contribution from the fragmentation function because the contribution from the final state with a gluon in the n^μ direction and a soft quark is power suppressed. Therefore, the diagram must

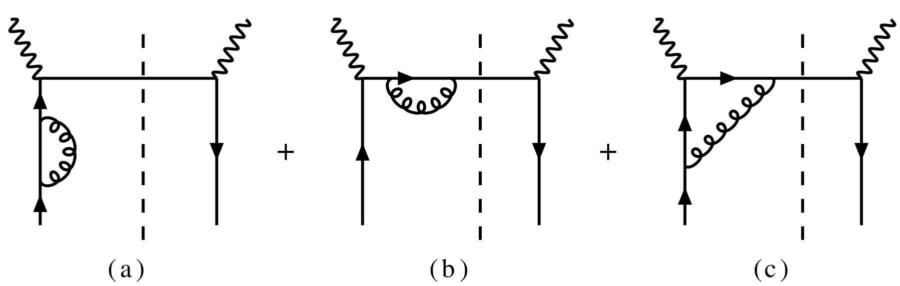

FIG. 4. One-loop virtual correction to semi-inclusive DIS.

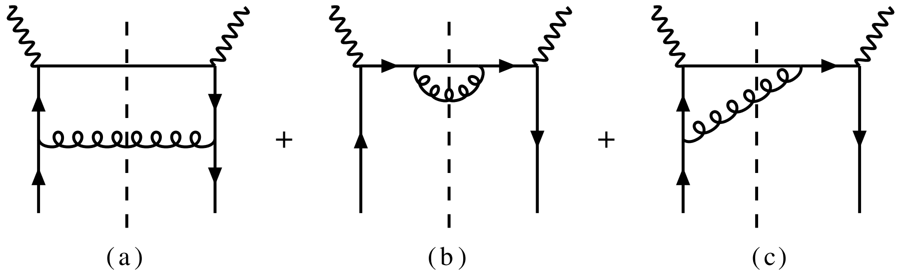

FIG. 5. One-loop real correction to semi-inclusive DIS.

be factorizable into the parton distribution in Fig. 2(a) subtracted off Fig. 3(c), and the soft factor in Fig. 3(c). A simple calculation of the diagram yields,

$$F = \frac{\alpha_s}{2\pi^2} C_F \delta(z_h - 1)(1 - x_B)\left[\frac{1}{P_{h\perp}^2 + x_B\lambda^2 + (1 - x_B)^2 m^2} - \frac{2xm^2}{[P_{h\perp}^2 + x_B\lambda^2 + (1 - x_B)^2 m^2]^2}\right]. \tag{57}$$

Indeed, the above expressions can easily be reproduced by the factorization formula with a one-loop result for $q^{(1)}$ and $S^{(1)}$ and tree-level $\hat{q}^{(0)}$ and $H^{(0)}$.

Similarly for Fig. 5(b), we have

$$F = \frac{\alpha_s}{2\pi^2} C_F \delta(x_B - 1)(1 - z)\left[\frac{1}{P_{h\perp}^2 + z_h\lambda^2 + (1 - z_h)^2 m^2} - \frac{2z_h m^2}{[P_{h\perp}^2 + z_h\lambda^2 + (1 - z_h)^2 m^2]^2}\right], \tag{58}$$

which again can be reproduced by the factorization formula with the one-loop fragmentation function and the soft factor S, and the tree-level parton distribution and the hard part.

Finally, let us consider the diagram Fig. 5(c) and its Hermitian conjugate. In the region where $P_{h\perp}$ is small, we find three distinct contributions:

$$\begin{aligned} F = {} & \frac{\alpha_s C_F}{2\pi^2} \delta(z_h - 1) \frac{2x_B}{(1 - x_B)_+}\left[\frac{1}{P_{h\perp}^2 + x_B\lambda^2 + (1 - x_B)^2 m^2}\right] \\ & + \frac{\alpha_s C_F}{2\pi^2} \delta(x_h - 1) \frac{2z_h}{(1 - z_h)_+}\left[\frac{1}{P_{h\perp}^2 + z_h\lambda^2 + (1 - z_h)^2 m^2}\right] + \frac{\alpha_s C_F}{2\pi^2} 2\delta(x_B - 1)\delta(z_h - 1)\frac{1}{P_{h\perp}^2 + \lambda^2} \ln\frac{Q^2}{P_{h\perp}^2 + \lambda^2}, \end{aligned} \tag{59}$$

where the first term corresponds to a gluon collinear to the initial quark, the second term a gluon collinear to the final-state quark, and the third term a soft gluon. All these terms are reproduced by the factorization formula with one-loop parton distribution, fragmentation function, and the soft factor.

Therefore we conclude that at the one-loop level, the general factorization formula holds.

IV. FACTORIZATION TO ALL ORDERS IN PERTURBATION THEORY

In this section, we argue that the factorization formula we stated in the previous section holds to all orders in perturbative QCD. To make such arguments, we follow the steps outlined in an excellent review article by Collins, Sterman, and Soper [31]. One must consider a general Feynman diagram and study its leading contributions to the SIDIS cross section. The contributions from different regions of the internal momentum integrations are characterized by the reduced diagrams which correspond to pinched surfaces in the space of integration variables. The leading reduced diagrams can be determined by infrared power counting. The remaining steps involve decoupling the Lorentz and color indices, and using soft approximation to disentangle momentum integrals in the different parts of the reduced diagrams. To simplify the derivation, one must use the (generalized) Ward identities extensively. After decoupling and replacing the various factors by the parton distribution, fragmentation function, and the soft function, one finally arrives at the general form of factorization.

Using the fact that the physical observables are independent of renormalization and soft-collinear factorization scales, large double and single logarithms involved in the factorization formula can be summed. The final expression is useful to describe experimental data when combined with result from perturbative calculations of a fixed order.

A. Reduced diagrams, power counting, and leading regions

The contribution of an arbitrary (cut) Feynman diagram to the SIDIS cross section can be classified in terms of pinched surfaces corresponding to the solutions of Landau equations [32–34]. Coleman and Norton observed that these pinched surfaces can be pictured in terms of physical space-time processes (or reduced diagrams) [35]. According to the kinematic constraints of SIDIS, it is not difficult to see that the most general reduced diagrams have the structure shown in Fig. 6, in which the initial nucleon evolves into a target fragmentation jet J_t plus a set of collinear quarks and gluons (solid lines) entering the hard-interaction vertex with the highly virtual photon γ^*. A new set of collinear quarks and gluons (solid lines) emerges from the hard vertex in a new direction (opposite direction in the collinear frames), and fragments into the observed hadron and the unobserved jet J_c. Figure 6 is actually a cut diagram including the complex-conjugated amplitudes, corresponding to the measured cross section. Therefore, we will use additional indices L and R to label jets on the left and right sides of the cut (indicated by the vertical dashed line), respectively. For example, J_{cR} labels the current jet on the right-hand side of the cut. In addition, there is a soft subdiagram S with soft quark and gluon lines (shown by dashed lines) connecting the jets and hard parts.

Let us count the degree of infrared divergence $\omega(G)$ of each reduced diagrams G. It can be constructed from the sum of the degrees of divergences for the jets and the soft part,

$$\omega(G) = \omega_{J_{tL}} + \omega_{J_{tR}} + \omega_{J_{cL}} + \omega_{J_{cR}} + \omega_S. \tag{60}$$

The power counting for the soft function is straightforward: If we use E^b and E^f to denote the number of soft boson (gluon) and fermion external lines, then it is well known that

$$\omega_S = E^b + \frac{3}{2}E^f \tag{61}$$

from a simple dimensional analysis in coordinate space. Note that ω_S includes the propagators of the external lines and the associated integration measure.

Let us use p_J to denote the number of collinear quark or gluon-with-physical-polarization lines entering the hard part from jet J; use l_J to represent the number of collinear gluons of longitudinal polarization through a similar attachment; use $E_J^{b,f}$ to denote the number of soft boson or fermion lines connecting the soft part to the jets; use $E_{HL,R}^{b,f}$ to label the number of soft bosons or fermions connecting to the left or right hard part; and finally use $v_J^{(3)}$ to label the number of three-point vertices in the jet, and s_J the number of soft gluons with scalar polarization attaching to the jet. Then it is easy to see that

$$\begin{aligned} E^b &= E^b_{J_{tL}} + E^b_{J_{tR}} + E^b_{J_{cL}} + E^b_{J_{cR}} + E^b_{HL} + E^b_{HR}, \\ E^f &= E^f_{J_{tL}} + E^f_{J_{tR}} + E^f_{J_{cL}} + E^f_{J_{cR}} + E^f_{HL} + E^f_{HR}. \end{aligned} \tag{62}$$

The soft power associated with each collinear jet is

$$\omega_J = 2L_J - N_J + t_J, \tag{63}$$

where L_J is the number of loops in the jet (each contributing two powers), and N_J is the number of internal lines (each contributing one power), and t_J is the numerator suppression factor which in Feynman gauge is equal to $\max[v_J^{(3)} - l_J - s_J, 0]/2$ [33]. The number of loops can be calculated using

$$N_J - v_J^{(3)} - v_J^{(4)} = L_J, \tag{64}$$

where $v_J^{(3,4)}$ are the number of three- and four-point vertices, respectively. The relation between the number of vertices and lines is,

$$3v_J^{(3)} + 4v_J^{(4)} + p_J + l_J = 2N_J + E_J^b + E_J^f + I_J, \tag{65}$$

where I_J is the number of (initial and final) external lines in the jet. From the above, it is easy to see that

$$\omega_J = \frac{1}{2}(p_J - s_J - E_J^b - E_J^f - I_J) + \frac{1}{2}(s_J + l_J - v_J^{(3)})\theta(s_J + l_J - v_J^{(3)}). \tag{66}$$

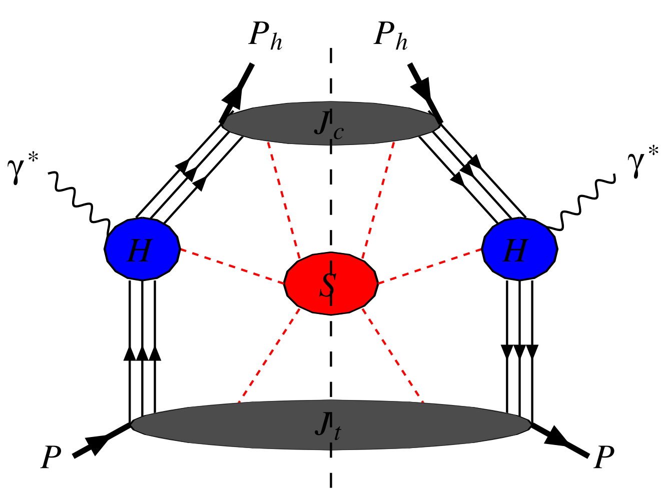

FIG. 6 (color online). A general reduced diagram for semi-inclusive DIS.

Combining the results from four jets, one finds,

$$\omega(G) \geq \frac{1}{2}(p_{J_{tL}} + p_{J_{tR}} + p_{J_{cL}} + p_{J_{cR}}) - \frac{1}{2}(I_t + I_c) + \frac{1}{2}(E^b_{J_{tL}} + E^b_{J_{tR}} + E^b_{J_{cL}} + E^b_{J_{cR}}) - \frac{1}{2}(s_{J_{tL}} + s_{J_{tR}} + s_{J_{cL}} + s_{J_{cR}}) + E^f_{J_{tL}} + E^f_{J_{tR}} + E^f_{J_{cL}} + E^f_{J_{cR}} + E^b_{HL} + E^b_{HR} + \frac{3}{2}(E^f_{HL} + E^f_{HR}) + \frac{1}{2}(s_{J_{tL}} + l_{J_{tL}} - v^{(3)}_{J_{tL}})\theta(s_{J_{tL}} + l_{J_{tL}} - v^{(3)}_{J_{tL}}) + \frac{1}{2}(s_{J_{tR}} + l_{J_{tR}} - v^{(3)}_{J_{tR}})\theta(s_{J_{tR}} + l_{J_{tR}} - v^{(3)}_{J_{tR}}) + \frac{1}{2}(s_{J_{cL}} + l_{J_{cL}} - v^{(3)}_{J_{cL}})\theta(s_{J_{cL}} + l_{J_{cL}} - v^{(3)}_{J_{cL}}) + \frac{1}{2}(s_{J_{cR}} + l_{J_{cR}} - v^{(3)}_{J_{cR}})\theta(s_{J_{cR}} + l_{J_{cR}} - v^{(3)}_{J_{cR}}). \quad (67)$$

From the above, the largest possible degree of infrared divergence is 0 if the initial and final-state hadrons are replaced by a perturbative parton ($I_t = I_c = 1$).

According to the above result for $\omega(G)$, leading reduced diagrams (leading region) must satisfy the following conditions:

(i) No soft fermion lines,
(ii) No soft-gluon lines attached to the hard parts,
(iii) Soft-gluon lines attached to jets must be longitudinally polarized,
(iv) In each jet, one quark line plus an arbitrary number of longitudinally-polarized gluons attached to the corresponding hard part,
(v) The number of three-point vertices in a jet must be larger or equal to the number of soft and longitudinally-polarized gluon attachments.

In Fig. 7, we show the leading reduced diagrams satisfying the above conditions. As indicated already, the collinear gluons are longitudinally polarized.

B. Factorization of collinear gluons

Let us first factorize the longitudinally-polarized collinear gluons from the hard parts. This can be done using the approach discussed in [31]. For definiteness, let us consider the collinear gluons from the initial state nucleon. Because the gluons are longitudinally polarized, the gluon gauge potential can be replaced by

$$A^\mu = A \cdot n p^\mu, \quad (68)$$

where p^μ is the light cone momentum to which the initial nucleon momentum is proportional. The effects of these gluons on the hard part can be factorized through the Ward identity,

$$\langle f|T\partial_{\mu_1}A^{\mu_1}(\xi_1)\partial_{\mu_2}A^{\mu_2}(\xi_2)\ldots\partial_{\mu_n}A^{\mu_n}(\xi_n)|i\rangle = 0, \quad (69)$$

where $|i\rangle$ and $|f\rangle$ are physical states. Applying this identity repeatedly leads to the conclusion that the collinear gluons can be viewed as attaching to an eikonal line in the conjugating light cone direction n^μ. This result can be understood in an intuitive way: The longitudinally-polarized gluons cannot resolve the internal dynamics of the hard scattering. It can, however, be sensitive to the overall flow of the color-charge. The hard-interaction is a light cone dominated process along the n^μ direction in the coordinate space. This is also the direction along which the final state jet is formed. Thus the collinear gluons mainly scatter with the color-charge flow in this direction.

The Feynman momentum x of a collinear gluon has a lower limit in a physical process. For example, the smallest x that a gluon may have is on the order of M/Q. Only in the limit $Q \to \infty$, can there be near zero-momentum gluons participating in the scattering. When a collinear gluon has a small x, its light cone energy is large, and its contribution to the cross section can be calculated perturbatively. Therefore, one can introduce a parameter that separates contributions of the gluons with different rapidities. The collinear gluons with x larger than a certain cutoff are included in the parton distributions; others are included in the hard factor. Of course, the physical cross section is independent of this parameter. In the inclusive case, the singular contribution from small-x gluons cancels between the real and virtual diagrams.

To define a parton distribution with virtual gluons of limited rapidity, one can introduce a rapidity cutoff. The most straightforward approach is to implement a lower cutoff in x. A more convenient approach, as we discussed in the one-loop case, is to introduce a quasi-light-cone

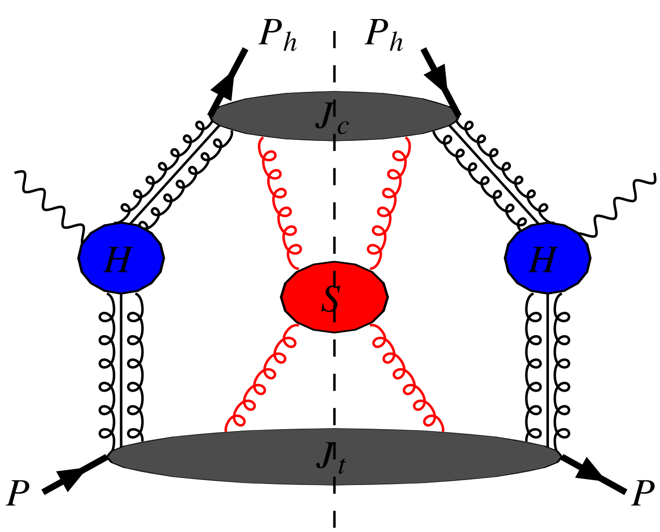

FIG. 7 (color online). The leading region for semi-inclusive DIS.

vector, v^μ, which is close but not exactly in the n^μ direction, and to assume that all collinear gluons couple to a colored jet moving in this direction. It can be checked that in this approach only the gluons with $k^+/k^- > v^+/v^-$ are included in the parton distribution.

The collinear gluons from J_t can be factorized in a similar way. Here a quasi-light-cone vector $\tilde{v}$ must be introduced to limit the small-x gluon contribution to the TMD fragmentation function: only collinear gluons with $k^-/k^+ > \tilde{v}^-/\tilde{v}^+$ are included in the nonperturbative function. The left and right parts of the cut diagrams can be treated in a symmetric way.

C. Soft approximation and soft factor

The soft gluons are attached to the target and current jets, and can be factorized using the Grammer-Yennie approximation [36] (or soft approximation). The approximation consists of two steps. The first step is to neglect any soft momentum in the numerators of the jet factors. One of the consequences is that the gluon polarization is effectively along the conjugating light cone direction of the jet (longitudinally polarized). The second step is to neglect k^2 compared to $k \cdot nk \cdot p$ in the jet denominator. This approximation is not uniformly true in the soft region. In fact, in the so-called Glauber region, where $k_\perp^2 \gg k \cdot nk \cdot p$, the approximation fails [37]. If, however, the momentum k^+ or k^- is not trapped, one can deform the contour integration to a region where $k \cdot nk \cdot p \gg k_\perp^2$, so that the approximation can still be applied. It is known that for semi-inclusive hadron production in e^+e^- annihilation, the deformation can be easily performed [31]. In inclusive Drell-Yan, this happens only after summing the final state interaction diagrams [38]. In the present case, the soft gluons interact with the current and target jets; all of these interactions are final state interactions. Hence, all physical poles appear in the upper-half plane. As such, the contour deformation can be done straightforwardly.

After the soft approximation, one can again use the Ward identity to factorize all the soft gluons from the jets. The physical effect of a jet can be replaced by a Wilson line along the jet direction. Again to avoid the light cone singularity, the Wilson line can be chosen to be off the light cone along the v or $\tilde{v}$ direction. After factorizing the gluons from the jets, and summing over all soft contributions, a soft factor emerges:

$$S(\vec{b}_\perp, \mu^2, \rho) = \frac{1}{N_c} \mathrm{Tr}\langle 0 | \mathcal{L}_{\tilde{v}}^\dagger(\vec{b}_\perp, -\infty) \mathcal{L}_v^\dagger(\infty; \vec{b}_\perp) \mathcal{L}_v(\infty; 0) \times \mathcal{L}_{\tilde{v}}(0; -\infty) | 0 \rangle, \quad (70)$$

which appears as a factor in the factorization theorem. Now the leading region has the form shown in Fig. 8.

The soft factor is renormalization-scale dependent. The renormalization group equation is

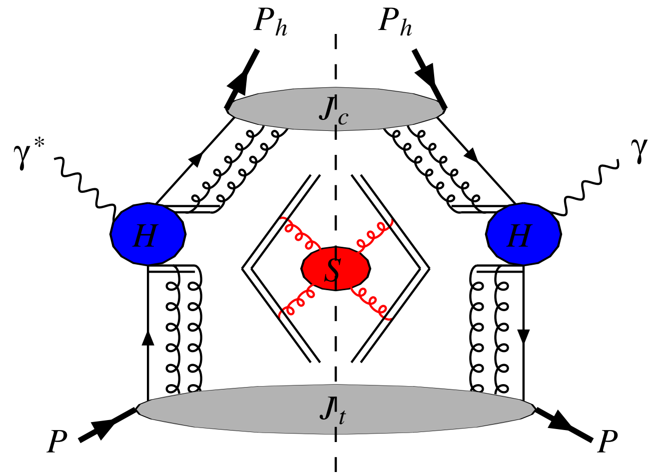

FIG. 8 (color online). The leading region for SIDIS after soft and collinear factorizations.

$$\mu \frac{\partial S(\vec{b}_\perp, \mu^2, \rho)}{\partial \mu} = \gamma_S(\rho) S(\vec{b}_\perp, \mu^2, \rho), \quad (71)$$

where $\gamma_S(\rho)$ is the anomalous dimension of the Wilson lines in the definition. At one-loop order, one has

$$\gamma_S = \frac{\alpha_s}{\pi} C_F [2 - \ln \rho^2] + \cdots, \quad (72)$$

which is ρ dependent. The anomalous dimension at higher-order has been studied in Refs. [25,39].

D. Subtracted and unsubtracted parton distributions and fragmentation functions

Let us consider the target jet factor which has been factored from the hard part, with the soft factor factorized out as well. It is shown on the left-hand side in Fig. 9. The internal loop momenta of the gluons are restricted. Their k^+ components have a lower limit because of the gauge-link direction v. The k^- components also have a lower limit: when the soft gluons are factored, the gluons with k^-/k^+ smaller than $\tilde{v}^-/\tilde{v}^+$ have been factored out of the jet.

Therefore the jet factor is not the same as the parton distribution $\mathcal{Q}(x, b, \mu^2, x\zeta)$ defined in Sec. II. Rather, it is the same as the soft-subtracted parton distribution $q(x, b, \mu^2, x\zeta, \rho)$. The relationship of the two is shown in Fig. 9.

The renormalization group equations for $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ are known. After subtracting the soft factor, the equation has to be modified by including the anomalous dimension for the soft factor:

$$\mu \frac{dq(x, b, \mu^2, x\zeta, \rho)}{d\mu} = (2\gamma_F - \gamma_S(\rho)) q(x, b, \mu^2, x\zeta, \rho). \quad (73)$$

A similar equation holds for the TMD fragmentation func-

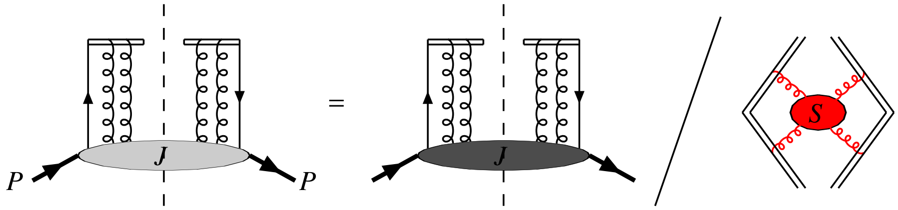

FIG. 9 (color online). Relation between soft-subtracted (left-hand side) and unsubtracted parton distributions. The subtracted distribution has a light-gray blob and the unsubtracted one has a dark blob. The denominator on the right-hand side is a soft factor.

tion. The Collins and Soper equation for q is the same as for Q:

$$\zeta \frac{\partial}{\partial \zeta} q(x, b, \mu^2, x\zeta, \rho) = (K(\mu, b) + G(\mu, x\zeta)) \times q(x, b, \mu^2, x\zeta, \rho). \tag{74}$$

We will solve this equation to sum over large logarithms in Sec. V.

E. Subtraction method

For a given Feynman diagram, there are multiple leading regions. To ensure the factorization works, one, in principle, has to supply a subtraction method which allows separating contributions from different leading regions. In particular, the subtraction method must provide a systematic way of handling the overlapping contribution of different leading regions.

The easiest way to develop a subtraction method in gauge theory is to choose the axial gauge. For example, the factorization for inclusive DIS in the axial gauge can be developed using the Bethe-Salpeter formalism and has been used to calculate the anomalous dimension of parton distributions at two-loop order [40]. For factorization involving collinear and soft divergences, a subtraction method in the axial gauge has been developed by Collins and Soper [9].

In covariant gauge, a systematic subtraction is complicated, and has not yet been fully developed in the literature. At the one-loop level, an example has been provided by Collins and Hautmann [17]. We have checked that the new subtraction method corresponds to a particular choice of ρ in this paper. It would be interesting to pursue this subtraction to higher order. In particular, higher-order calculations help to clarify the roles of different nonperturbative matrix elements which have the same one-loop result.

The best approach to treating overlapping infrared divergences in a multiloop case might be the soft-collinear effective theory mentioned in the introduction. Here we assume this can be done in principle and leave a more careful discussion for future publication.

F. Factorization and ρ-independence

Collecting all factors in Fig. 8, one finally has the following factorization formula:

$$F(x_B, z_h, b, Q^2) = \sum_{q=u,d,s,\ldots} e_q^2 q(x_B, z_h b, \mu^2, x_B\zeta, \rho) \times \hat{q}(z_h, b, \mu^2, \hat{\zeta}/z_h, \rho) S(b, \mu^2, \rho) \times H(Q^2, \mu^2, \rho), \tag{75}$$

advertised earlier.

It can be shown that the above expression is independent of ρ. At one-loop order, this is easy to see,

$$\rho \frac{\partial S(b, \rho)}{\partial \rho} = -\frac{\alpha_s C_F}{\pi} \ln\left(\frac{\mu^2 b^2}{4} e^{2\gamma_E}\right) S(b, \rho), \tag{76}$$

On the other hand,

$$\rho \frac{\partial S(b, \rho) q(x, b, \rho)}{\partial \rho} = -\frac{\alpha_s C_F}{2\pi} \ln\left(\frac{\rho Q^2 b^2}{4} e^{2\gamma_E - 1}\right) S(b, \rho) \times q(x, b, \rho). \tag{77}$$

If the structure function is independent of ρ, the above requires $H(\rho)$ to evolve in ρ,

$$\rho \frac{\partial H(\rho)}{\partial \rho} = \frac{\alpha_s C_F}{\pi} \left[\ln\left(\frac{\rho Q^2}{\mu^2}\right) - 1 \right] H(\rho). \tag{78}$$

It is easy to check that our one-loop hard part satisfies the above equation.

At higher orders, the ρ independence is guaranteed because one can view the factorization formula as a definition for the hard part.

V. SUMMING OVER LARGE LOGARITHMS

From the factorization formula and the evolution equations, we can get an expression for the structure function in which the large logarithms involving momentum Q are summed over. Here we consider two cases: In the first case, $P_{h\perp}$ is on the order of $\Lambda_{\rm QCD}$ as we have discussed throughout the paper. In the second case, $\Lambda_{\rm QCD} \ll P_{h\perp} \ll Q$, where one can make additional factorization of the TMD parton distributions and fragmentation function. The result is a summarization formula which has been used in Refs. [11,12].

A. Summation when $P_{h\perp} \sim \Lambda_{\rm QCD}$

First of all, there are large logarithms in $K + G$ (which is independent of the renormalization scale). To sum it, we solve the renormalization group equation to get

$$K(b, \mu) + G(x\zeta, \mu) = K(b, \mu_L) + G(x\zeta, \mu_H) - \int_{\mu_L}^{\mu_H} \frac{d\tilde{\mu}}{\tilde{\mu}} \gamma_K(\alpha(\tilde{\mu})). \quad (79)$$

To isolate the large logarithms, one has to choose μ_L to be on the order of $\Lambda_{\rm QCD}$ and μ_H to be on the order of ζ. Therefore, we let

$$\mu_L = C_1 M_N; \qquad \mu_H = C_2 x\zeta = C_2 Q\sqrt{\rho}, \quad (80)$$

where M_N is the mass of the nucleon.

Substituting the above into the Collins-Soper equation for $q(x, b, \mu^2, x\zeta, \rho)$, the large logarithms in ζ can be factorized,

$$q(x, b, \mu, x\zeta, \rho) = \exp\left\{-\int_{\mu_L}^{C_2 x\zeta} \frac{d\mu}{\mu}\left[\ln\left(\frac{C_2 x\zeta}{\mu}\right)\gamma_K(\alpha(\mu)) - K(b, \mu_L) - G(\mu/C_2, \mu)\right]\right\} \times q(x, b, \mu, x\zeta_0 = \mu_L/C_2, \rho), \quad (81)$$

where the exponential factor contains the entire dependence on ζ, in particular, the large Sudakov double logarithms. However, the above expression contains much more than just the leading double logarithms; it contains all the subleading logs as well.

Similarly, one can find the solution for the fragmentation function,

$$\hat{q}(z, b, \mu, \hat{\zeta}/z, \rho) = \exp\left\{-\int_{\mu_L}^{C_2\hat{\zeta}/z} \frac{d\mu}{\mu}\left[\ln\left(\frac{C_2\hat{\zeta}}{z\mu}\right)\gamma_K(\alpha(\mu)) - K(b, \mu_L) - G(\mu/C_2, \mu)\right]\right\} \times \hat{q}(z, b, \mu, \hat{\zeta}_0/z = \mu_L/C_2, \rho). \quad (82)$$

If we choose a frame in which $x_B\zeta = \hat{\zeta}/z_h$, then the exponential factors in q and $\hat{q}$ become the same, and moreover $\zeta^2 x_B^2 = \hat{\zeta}^2/z_h = Q^2\rho$.

Let us study the renormalization group equation for the hard part. The physical cross section is, of course, independent of the renormalization scale μ. Since we know the renormalization group equation for q, $\hat{q}$, and the soft factor, we can easily derive the renormalization group equation for the hard part,

$$\mu\frac{dH(Q^2/\mu^2, \rho)}{d\mu} = -(4\gamma_F - \gamma_S(\rho))H(Q^2/\mu^2, \rho). \quad (83)$$

The solution is

$$H(Q^2/\mu^2, \rho) = \exp\left\{-\int_{\mu'}^{\mu} \frac{d\mu}{\mu}[4\gamma_F - \gamma_S(\rho)]\right\} \times H(Q^2/\mu'^2, \rho). \quad (84)$$

To factor out the large renormalization logarithms, one can choose μ to be at low scale such as μ_L, and μ' at high scale such as μ_H. Therefore, we write,

$$H(Q^2/\mu_L^2, \rho) = \exp\left\{-\int_{C_2 x\zeta}^{\mu_L} \frac{d\mu}{\mu}[4\gamma_F - \gamma_S]\right\} \times H(Q^2/C_2 x\zeta, \rho), \quad (85)$$

where $H(Q^2/C_2 x\zeta, \rho)$ contains no large logarithms.

Collecting the above results, one has both the renormalization and soft-collinear logarithms summed in the following expression:

$$F(x_B, z_h, b, Q^2) = q(x_B, z_h b, \mu_L^2, \mu_L/C_2, \rho)\hat{q}(z_h, b, \mu_L^2, \mu_L/C_2, \rho)S(b, \mu_L^2, \rho)H(1/C_2^2\rho, \rho) \times \exp\left\{-2\int_{\mu_L}^{C_2 Q\sqrt{\rho}} \frac{d\mu}{\mu}\left[\ln\left(\frac{C_2 Q\sqrt{\rho}}{\mu}\right)\gamma_K(\alpha(\mu)) - K(b, \mu_L) - G(\mu/C_2, \mu) - 2\gamma_F + \frac{1}{2}\gamma_S(\rho)\right]\right\}, \quad (86)$$

where all large logarithms have been factorized in the exponential factor. For the physics discussed in this paper, we do not want significant large logarithms, because otherwise the transverse momentum of the hadron yield is generated mostly by soft-gluon radiations. To avoid them, Q^2 can only be moderately large compared to $P_{h\perp}$. On the other hand, in this kinematic regime, the contributions from power-suppressed terms might not be entirely negligible.

Finally, the choice of ρ. According to its definition, we must have $\rho \gg 1$, although the physics is independent of ρ. However, if ρ is too large, one has large logarithms in the hard part and the convergence of the perturbation series might be spoiled. Therefore in practice one might choose a ρ, for example, somewhere in between 3 and 10.

B. Summation when $\Lambda_{\rm QCD} \ll P_{h\perp} \ll Q$

When $P_{h\perp} \gg \Lambda_{\rm QCD}$, the transverse-momentum dependence in the parton distributions and fragmentation functions can be calculated in terms of the integrated ones, as we have shown in Sec. II G for the quark distribution. From factorization formula and the result in Eq. (33), and taking into account the contributions from the fragmentation function and the soft factor, one finds a structure function the same as the ordinary pQCD prediction, calculated in the limit $\Lambda_{\rm QCD} \ll P_{h\perp} \ll Q$ [11,12]. When carrying this out to higher order in α_s, one has double logarithms $\alpha_s \ln^2 Q^2/k_\perp^2$. To make reliable predictions one has to sum over these double logs.

The best way to make the double-log summation is again in the impact parameter space. For example, in the b space, the structure function reads,

$$F(x_B, z_h, b, Q^2) = q(x_B, b, \mu, \rho Q^2, \rho)\hat{q}(z_h, b, \mu, \rho Q^2, \rho) \times S(b, \mu, \rho)H(Q^2, \mu, \rho), \tag{87}$$

where the sum over quark flavor weighted with charge square is understood at the right-hand side of the equation. From this equation, we can derive the evolution equation depending on Q^2 [10],

$$Q^2 \frac{\partial}{\partial Q^2} F(x_B, z_h, b, Q^2) = [K(b\mu, g(\mu)) + G'(Q/\mu, g(\mu))] \times F(x_B, z_h, b, Q^2), \tag{88}$$

where K is the same as before, and G' contains an additional contribution from the hard part. The ρ dependence in hard part and parton distribution and fragmentation has been canceled out in G'. To one-loop level, we have

$$K + G' = -\frac{\alpha_s C_F}{\pi} \ln \frac{Q^2 b^2 e^{2\gamma_E - 3/2}}{4}. \tag{89}$$

The solution to the differential equation Eq. (87) has the following form,

$$F(x_B, z_h, b, Q^2) = F(x_B, z_h, b, \mu_L^2/C_2^2) e^{-S(Q^2, \mu_L^2, b, C_2)}, \tag{90}$$

where the Sudakov form factor reads,

$$S(Q^2, \mu_L^2, b, C_2) = \int_{\mu_L}^{C_2 Q} \frac{d\bar{\mu}}{\bar{\mu}} \left[\ln\left(\frac{C_2 Q^2}{\bar{\mu}^2}\right) A(b\mu_L, \bar{\mu}) + B(C_2, b\mu_L, \bar{\mu}) \right]. \tag{91}$$

Here C_2 is a parameter in the order of 1, and μ_L is a lower scale as before. The A and B functions are defined as

$$A(b\mu_L, \bar{\mu}) = \gamma_K(\bar{\mu}) + \beta \frac{\partial}{\partial g} K(b\mu_L, g(\bar{\mu})),$$
$$B(C_2, b\mu_L, \bar{\mu}) = -2K(b\mu_L, g(\bar{\mu})) - 2G'(1/C_2, g(\bar{\mu})). \tag{92}$$

If we choose $\mu_L = C_1/b$ and C_1 is a parameter in order 1, the above formulism will reproduce the CSS resummation [10]. This is because when b is small, the TMD parton distribution and fragmentation can be calculated from the integrated parton distribution and fragmentation by using the factorization theorem [9]. We have shown this for the TMD quark distribution in Sec. II G. Substituting Eq. (35) into Eq. (90), we can get the structure function at large $1/b$ calculated in terms of integrated parton distribution and fragmentation function [10],

$$F(x_B, z_h, b, Q) = e^{-S(Q^2, b, C_1, C_2)} \int_{x_B}^1 \frac{dy}{y} C\left(\frac{x_B}{y}, b, Q_0, \bar{\mu}\right) \times q(y, \bar{\mu}) \int_{z_h}^1 \frac{dy'}{y'} C\left(\frac{z_h}{y'}, b, Q_0, \bar{\mu}\right) \times \hat{q}(y', \bar{\mu}), \tag{93}$$

where $Q_0 = C_1/b$ and the coefficient function C are defined as

$$C(x, b, Q_0, \bar{\mu}) = \tilde{C}(x, b^2, \mu^2, \bar{\mu}^2, \rho Q_0^2, \rho) \times \sqrt{S(b, \mu, \rho)H(Q_0, \mu, \rho)}, \tag{94}$$

where the μ and ρ dependence of the various factors on the right-hand side of the equation cancel out. Using the one-loop results for $\tilde{C}$ in Eq. (36) and for the soft and hard factors, we can reproduce the one-loop results for the quark sector C functions used in the literature [10–12].

VI. CONCLUSION

In this paper, we argued that a factorization theorem exists for semi-inclusive deep-inelastic scattering with detected hadron momentum $P_{\perp h} \ll Q$. $P_{h\perp}$ can either be soft, i.e., on the order of $\Lambda_{\rm QCD}$, or in the perturbative domain $\gg \Lambda_{\rm QCD}$. We have mainly focused on the former case although the result is valid also for the latter. For $P_{\perp h} \gg \Lambda_{\rm QCD}$, the theorem can be simplified by an additional factorization of the TMD parton distributions and fragmentation functions [9].

We argued the theorem by first considering the example at the one-loop level. In this case, the calculation of the parton distribution, fragmentation function, soft factor, and the SIDIS cross section was straightforward. The example demonstrated that the factorization indeed works.

At higher order in perturbation theory, one can use the formalism developed by Collins, Sterman, and Soper and others. Starting from the most general reduced diagrams, we showed the factorization of collinear gluons from the hard part and the soft gluons from the collinear part, matching the jet factors with the distribution and fragmentation functions. The factorization scale μ and the ρ independence of the physical cross section allows one to sum over large logarithms involving scales Q and $P_{h\perp}$.

The present result can be easily extended to the situations where the target is polarized or the polarization of the final state hadron is measured. It can also be extended to the case where the transverse momentum of the hadrons are integrated with a weighting factor. Finally, all results here can be obtained also in the framework of the soft-collinear effective theory.

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