Precise analytical description of the Earth matter effect on oscillations of low energy neutrinos

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We present a formalism for the matter effects in the Earth on low energy neutrino fluxes which is both accurate and has all the advantages of a full analytic treatment. The oscillation probabilities are calculated up to the second order term in $\epsilon(x) \equiv 2V(x)E/\Delta m^2$, where $V(x)$ is the neutrino potential at position x. We show the absence of large undamped phases which makes the expansion in ϵ well behaved. An improved expansion is presented in terms of the variation of $V(x)$ around a suitable mean value which allows one to treat energies up to those relevant for supernova neutrinos. We discuss also the case of three-neutrino mixing.

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I. INTRODUCTION

The propagation of low energy neutrinos in the Earth [1–4] is an important aspect of physics of solar [1–14] and supernova (SN) neutrinos [15–22]. It will be useful in determining the oscillation parameters and, in future, to search for effects of 1-3 mixing [23] and for a ''tomography'' of the Earth (see, e.g., [20,21]). It might even be possible to look for small structures of the density profile [20].

In the existing calculations of Earth matter effects (see, e.g., [1–20,23]), the density profile is often approximated by one, two, or several layers (mainly mantle and core) with constant densities or a direct numerical integration of the evolution equation is performed. However, the emergence of the large mixing angle (LMA) Mikheyev-Smirnov-Wolfenstein (MSW) solution to the solar neutrino problem opens a more efficient approach to the oscillation effects in the Earth. Indeed, for the LMA parameters, the oscillations of the solar and (lower energy) supernova neutrinos inside the Earth occur in a ''weak'' regime, where the matter potential V is much smaller than the ''kinetic energy'' of the neutrino system, i.e.,

$$V(x) \ll \frac{\Delta m^2}{2E}. \tag{1}$$

Here $V(x) \equiv \sqrt{2} G_F N_e(x)$, G_F is the Fermi constant, $N_e(x)$ is the number density of the electrons, $\Delta m^2 \equiv m_2^2 - m_1^2$ is the mass squared difference, and E is the neutrino energy.

In this case, one can introduce a small parameter

$$\epsilon(x) \equiv \frac{2EV(x)}{\Delta m^2} \simeq 0.02 \left[\frac{E}{10\ \mathrm{MeV}}\right]\left[\frac{N_e(x)}{N_A}\right]\left[\frac{8\times 10^{-5}\ \mathrm{eV}^2}{\Delta m^2}\right], \tag{2}$$

where N_A is the Avogadro number, and consider an expansion of the oscillation probabilities in $\epsilon(x)$.

In Ref. [24], the ϵ perturbation theory was formulated in the basis of neutrino mass states $\nu_{\mathrm{mass}} \equiv (\nu_1, \nu_2)^T$. The oscillation probabilities and the regeneration factor were calculated to first order in ϵ. The expressions obtained are valid for arbitrary density profiles with sufficiently low density (1). They simplify the numerical calculations substantially and allow one to understand in detail all features of the oscillation effects. The method reproduced immediately the analytic result obtained in Ref. [25] for an approximate but realistic density profile. Similar integral expression for the regeneration factor has been obtained in Ref. [26].

Since $\epsilon(x)$ increases with energy, the lowest approximation in $\epsilon(x)$ may not be enough for larger energies. For instance, if $E \simeq 50$ MeV (possible for SN neutrinos), we find $\epsilon(x) \simeq 0.6$ at the center of the Earth.

The purpose of this paper is to improve on this method and obtain accurate formulas which are valid for higher energies. In Sec. II the oscillation probabilities are calculated in second order in $\epsilon(x)$ and the convergence of the ϵ expansion is commented on. In Sec. III we suggest an improved perturbation theory which allows one to extend the expansion to higher energies. The generalization to three neutrinos is given in Sec. IV and a brief conclusion in Sec. V.

II. SECOND ORDER CORRECTIONS TO THE OSCILLATION PROBABILITIES

In this and the following section, we consider the mixing of two (active) neutrinos $\nu_f = U(\theta)\nu_{\mathrm{mass}}$, where $\nu_f \equiv (\nu_e, \nu_a)^T$ and $\nu_{\mathrm{mass}} \equiv (\nu_1, \nu_2)^T$ are the flavor and mass states, respectively, and ν_a is a linear combination of ν_μ and ν_τ. $U(\theta)$ and θ are the mixing matrix and mixing angle in vacuum, respectively. We define the matrix $U(\alpha)$ as

$$U(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{3}$$

In Ref. [24] the following expression for the S matrix in the mass eigenstates basis was derived [27]:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} - i\int_{x_0}^{x_f} dx \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x\to x_f}} \end{pmatrix} \Upsilon(x) \times \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0\to x}} \end{pmatrix} - \int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x\to x_f}} \end{pmatrix} \Upsilon(x) \times \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{y\to x}} \end{pmatrix} \Upsilon(y) \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0\to y}} \end{pmatrix} + \cdots, \tag{4}$$

where

$$\phi^m_{x_1\to x_2} \equiv \int_{x_1}^{x_2} dx \Delta^m(x) \tag{5}$$

is the adiabatic phase difference acquired by the neutrino eigenstates in matter on their trajectory between two points x_1 and x_2. $\Delta^m(x)$ is defined as

$$\Delta^m(x) \equiv \frac{\Delta m^2}{2E}\sqrt{1 - 2\epsilon(x)\cos 2\theta + \epsilon(x)^2}; \tag{6}$$

in vacuum we obviously have

$$\Delta^m \to \Delta \equiv \frac{\Delta m^2}{2E}. \tag{7}$$

The S matrix in (4) is written as a perturbative expansion in $\Upsilon(x)$ where

$$\Upsilon(x) = \frac{\sin 2\theta}{2} V(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\Delta^m(x) \sin^2\theta' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

θ' is the mixing angle of the mass eigenstates in matter,

$$\sin 2\theta' = \frac{\epsilon \sin 2\theta}{\sqrt{(\cos 2\theta - \epsilon)^2 + \sin^2 2\theta}} = \epsilon \sin 2\theta^m, \tag{9}$$

and $\theta^m = \theta + \theta'$ is the corresponding mixing angle of the flavor states.

The S matrix in Eq. (4) refers to a straight path through the Earth from the entry point x_0 to an exit point x_f and the coordinate x is measured along the path. For notational convenience, we do not put labels x_0, x_f, etc., on S.

Using Eq. (8), we obtain the S matrix in terms of the potential V:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} - i\frac{\sin 2\theta}{2}\int_{x_0}^{x_f} dx V(x) \times \begin{pmatrix} 0 & e^{-i\phi^m_{x_0\to x}} \\ e^{-i\phi^m_{x\to x_f}} & 0 \end{pmatrix} - i\frac{\sin^2 2\theta}{4\Delta} \begin{pmatrix} 1 & 0 \\ 0 & -e^{-i\phi^m_{x_0\to x_f}} \end{pmatrix} \int_{x_0}^{x_f} dx V(x)^2 - \frac{\sin^2 2\theta}{4}\int_{x_0}^{x_f} dx \int_{x_0}^{x} dy V(x)V(y) \times \begin{pmatrix} e^{-i\phi^m_{y\to x}} & 0 \\ 0 & e^{-i\phi^m_{x_0\to x_f} + i\phi^m_{y\to x}} \end{pmatrix}. \tag{10}$$

The two last terms (proportional to ϵ^2) come from the first order in Υ [term proportional to $\sin^2\theta'$ in Eq. (8)] and the second order in Υ [see Eq. (4)], correspondingly.

Using the evolution matrix in the mass state basis (10), we can calculate the amplitudes and probabilities of various transitions. The evolution matrix from the mass states to the flavor states relevant for the solar and SN neutrinos equals US, where U is the vacuum mixing matrix (3). Consequently, the amplitude of the mass-to-flavor transition is given by

$$A_{\nu_i\to\nu_\alpha} = U_{\alpha j}(\theta) S_{ji}. \tag{11}$$

The probability of the $\nu_2 \to \nu_e$ transition $P_{\nu_2\to\nu_e} = |A_{\nu_2\to\nu_e}|^2 = |U_{ej}(\theta)S_{j2}|^2$ is then found to be

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx V(x) \sin\phi^m_{x\to x_f} + \frac{1}{4}\sin^2 2\theta \cos 2\theta \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy V(x)V(y) \times \cos\phi^m_{y\to x}, \tag{12}$$

where the last term is the ϵ^2 correction. The integrations over x and y can be disentangled. Indeed, writing $\phi^m_{y\to x} = \phi^m_{y\to z} + \phi^m_{z\to x}$, where z is an arbitrary point of the trajectory, we find

$$\int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy V(x)V(y)\cos\phi^m_{y\to x} = \left[\int_{x_0}^{x_f} dx V(x)\cos\phi^m_{z\to x}\right]^2 + \left[\int_{x_0}^{x_f} dx V(x)\sin\phi^m_{z\to x}\right]^2. \tag{13}$$

This shows that the second order correction is positive for all V which do not vanish.

Furthermore, for a symmetric density profile (with respect to the middle point of the trajectory) the second term in (13) vanishes. This can be seen immediately by choosing $z = \bar{x} \equiv (x_f + x_0)/2$ in the center of the trajectory. So, finally we obtain for a symmetric profile

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx V(x) \sin\phi^m_{x\to x_f} + \frac{1}{4}\sin^2 2\theta \cos 2\theta \left[\int_{x_0}^{x_f} dx V(x) \cos\phi^m_{\bar{x}\to x}\right]^2 \quad (14)$$

or (again using the symmetry of V)

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \frac{1}{2}\sin^2 2\theta \sin\phi^m_{\bar{x}\to x_f} \int_{x_0}^{x_f} dx V(x) \cos\phi^m_{\bar{x}\to x} + \frac{1}{4}\sin^2 2\theta \cos 2\theta \left[\int_{x_0}^{x_f} dx V(x) \cos\phi^m_{\bar{x}\to x}\right]^2. \quad (15)$$

The phase $\phi^m_{\bar{x}\to x_f}$ should be calculated according to (5).

The two last terms in (15) determine the regeneration parameter defined as $f_{\rm reg} \equiv P_{\nu_2\to\nu_e} - \sin^2\theta$ (see, e.g., [11]). The probability of the $\nu_1 \to \nu_e$ oscillations can be obtained immediately from the unitarity condition $P_{\nu_1\to\nu_e} = 1 - P_{\nu_2\to\nu_e}$.

We see from Eq. (15) that the effective expansion parameter of the series is

$$I \equiv \int_{\bar{x}}^{x_f} dx V(x) \cos\phi^m_{\bar{x}\to x}, \quad (16)$$

so that

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \sin^2 2\theta [\sin\phi^m_{\bar{x}\to x_f} I + \cos 2\theta I^2 + \ldots]. \quad (17)$$

Notice that here the adiabatic phase should be calculated from the center of trajectory to a given point x, which corresponds to the explicit analytic expression obtained in Ref. [25].

Equation (17) tells us that the first order correction is absent for trajectories with $\phi^m_{\bar{x}\to x_f} = \pi k$, ($k$ = integer) and the second order correction would be zero for maximal vacuum mixing.

Taking $\Delta_m \approx \Delta$, we obtain the useful bound

$$I \sim \frac{2E}{\Delta m^2} \int_{y(\bar{x})}^{y(x_f)} dy V(y) \cos y \le \frac{2EV_{\max}}{\Delta m^2} = \epsilon_{\max}. \quad (18)$$

$V_{\max}$ is the maximum value of the potential on the trajectory and $y(x) = (\Delta m^2/2E)x$.

In Eq. (10) we note the presence of a possibly large phase $\phi^m_{x_0\to x_f}$ and an undamped integral in the term $\sim V(x)^2$ (see the 1-1 element of the matrix). It originates from the ϵ^2 term in Υ. (The undamped terms are absent in the linear term in ϵ [28].) This could be a problem, because the potential (squared) is integrated over a large distance without an oscillatory damping, and this might give rise to a large second order term in the expansion. However, by a simple partial integration of the last, $\sim V(x)V(y)$, term in (10) one can see that the undamped integral cancels. We have verified that this also happens in order V^3 for constant potentials. Therefore, the ϵ expansion appears to be well behaved (see also [26]).

In order to estimate the accuracy of (17), we compare the exact values $P^{(\rm exact)}_{2e}$ of the probabilities, found numerically, with the values $P^{(\rm appr)}_{2e}$, calculated using Eqs. (16) and (17). In Fig. 1 are shown the differences

$$\delta \equiv \frac{1}{\bar{f}}(f^{(\rm appr)} - f^{(\rm exact)}) = \frac{1}{\bar{f}}(P^{(\rm appr)}_{2e} - P^{(\rm exact)}_{2e}) \quad (19)$$

as functions of neutrino energy. Here

$$\bar{f} = \frac{1}{2}\epsilon_f \sin^2 2\theta \quad (20)$$

is the averaged regeneration factor in the layer with the surface density (or in the adiabatic case). Essentially, δ is the relative errors of the approximate formulas.

For illustration we use the simplified Earth density profile which consists of five spherical shells with constant densities. The external radiuses of the shells (in units of the solar radius $R_\odot$) and the electron number densities in units of the Avogadro number $(R_i/R_\odot, n_i/N_A)$ equal, correspondingly, (0.192, 6.05), (0.546, 5.24), (0.895, 2.47), (0.937, 1.92), and (1, 1.67).

In Fig. 1 we show the differences (19) for P_{2e} calculated in the first order, $\delta^{(1)}$ [the term $\sim I$ in (17) only], and in the second order $\delta^{(2)}$ (17). The lines correspond to the trajectory which crosses the center of the Earth. Apparently, the second order gives much better approximation. Notice that δ is the oscillatory function of energy. Its period increases with energy as $\sim El_\nu/D \propto E^2$, where D is the diameter of the Earth and l_ν is the vacuum oscillation length.

The accuracy worsens with energy. In the first approximation, as expected, $|\delta^{(1)}| \propto \epsilon \propto E$, since $|P_{2e} - P^{(1)}_{2e}| = O(\epsilon^2)$. The second order $\delta^{(2)}$ is very small at small energies; however, it increases with energy faster: $|P_{2e} - P^{(2)}_{2e}| \sim \epsilon^3$, so that

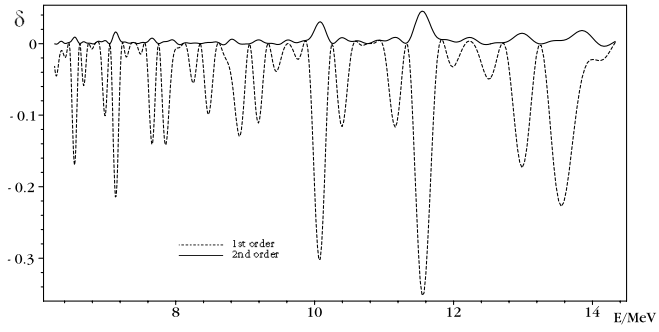

FIG. 1. The relative errors δ of the first approximation (dashed curve) and second approximation (solid curve) in ϵ as functions of the neutrino energy. The lines correspond to the neutrino trajectory which crosses the center of the Earth and to the oscillation parameters $\Delta m^2 = 8 \times 10^{-5}$ eV^2 and $\tan^2\theta = 0.4$.

$$|\bar{\delta}^{(2)}| \approx \frac{2\epsilon^2}{\sin^2 2\theta_{12}}. \tag{21}$$

As an example, for $E = 40$ MeV we find $|\bar{\delta}^{(2)}| \approx 0.04$ and maxima of the peaks $|\bar{\delta}^{(2)}| \sim 0.08$. In the first order approximation the corresponding numbers equal $|\bar{\delta}^{(1)}| \approx 0.3$ and 0.6.

These features can be explicitly seen in Fig. 2 in which we show the reduced values of errors $\delta^{(1)}/E$ and $\delta^{(2)}/E^2$ as functions of $1/E$. In average the reduced deviations do not depend on energy and the frequency of the peaks changes weakly in $1/E$ scale.

III. IMPROVED PERTURBATION THEORY

As mentioned before, the accuracy of our expressions decreases for higher densities and energies. However, the expansion parameter can be reduced and therefore the expansion can be improved. This can be achieved by considering a perturbation around some average potential V_0 rather than around the vacuum value $V_0 = 0$ [29]. In this case, we expect the expansion parameter to be

$$\epsilon = \frac{2E\Delta V}{\Delta m^2} = \frac{2E(V - V_0)}{\Delta m^2}. \tag{22}$$

The corresponding results can be immediately obtained from the original perturbation theory. Indeed, the transition to an average potential V_0 is equivalent to considering the problem in the basis $\nu_m^0 = (\nu_1^0, \nu_2^0)$, where ν_i^0 are the eigenstates of the Hamiltonian in matter with a constant potential V_0. These states are analogous to mass eigenstates in the $V_0 = 0$ theory. Therefore, the S matrix S^0 for (ν_1^0, ν_2^0) follows from the S matrix for mass eigenstates (10) by the substitution

$$V \to \Delta V \equiv V - V_0, \qquad \theta \to \theta_0^m, \tag{23}$$

where θ_0^m is the flavor mixing angle in matter with the potential V_0:

$$\sin 2\theta_0^m = \frac{\sin 2\theta}{\sqrt{1 - 2\epsilon_0 \cos 2\theta + \epsilon_0^2}} \tag{24}$$

and

$$\epsilon_0 \equiv \frac{2EV_0}{\Delta m^2}. \tag{25}$$

The adiabatic phase differences generated for the eigenstates traveling in matter with true V are invariant under a shift of the average potential, so that the phases $\phi^m_{x_i \to x_j}$ are unchanged. Therefore,

$$S^0 = S(\Delta V, \theta_0^m). \tag{26}$$

We introduce the mixing matrix

$$U_0' \equiv U(\theta_0'), \tag{27}$$

which relates the eigenstates of neutrinos in the potential V_0 to the mass eigenstates in vacuum: $\nu_{\text{mass}} = U_0' \nu_m^0$. The angle θ_0' is given by

$$\sin 2\theta_0' = \epsilon_0 \sin 2\theta_0^m \tag{28}$$

and it is easy to check that $\theta = \theta_0^m - \theta_0'$.

Now the amplitude of the mass-to-flavor transition, $\nu_i \to \nu_\alpha$, equals

$$A_{\nu_i \to \nu_\alpha} = U_{\alpha j}(\theta_0^m)(S^0)_{jk} U^\dagger_{ki}(\theta_0'). \tag{29}$$

A straightforward calculation leads to the $\nu_2 \to \nu_e$ oscillation probability $P_{\nu_2 \to \nu_e} = |A_{\nu_2 \to \nu_e}|^2$,

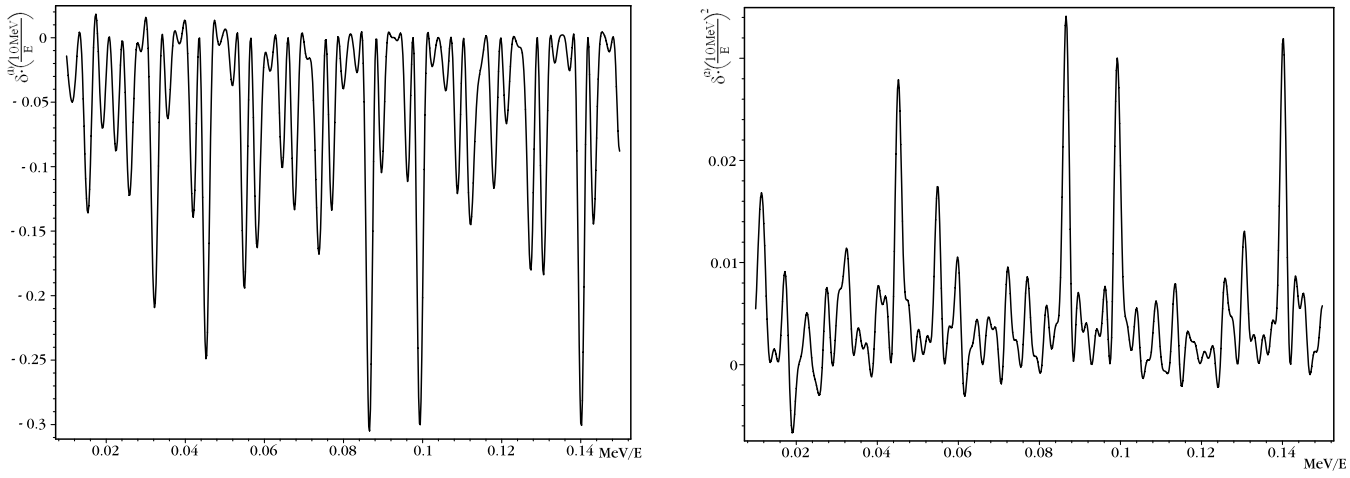

FIG. 2. The reduced errors $\delta^{(1)}/E$ (left) and $\delta^{(2)}/E^2$ (right) as functions of $1/E$. The lines correspond to the neutrino trajectory which crosses the center of the Earth and to the oscillation parameters $\Delta m^2 = 8 \times 10^{-5}$ eV^2 and $\tan^2\theta = 0.4$.

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \epsilon_0 \sin^2 2\theta_0^m \sin^2 \frac{\phi^m_{x_0\to x_f}}{2} + \frac{1}{2}\sin^2 2\theta_0^m \cos 2\theta'_0 \int_{x_0}^{x_f} dx \Delta V(x) \sin\phi^m_{x\to x_f}$$
$$+ \frac{\epsilon_0}{2}\sin^2 2\theta_0^m \cos 2\theta_0^m \int_{x_0}^{x_f} dx \Delta V(x) \sin\phi^m_{x_0\to x} - \frac{\epsilon_0}{8}\sin^4 2\theta_0^m \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \Delta V(x)\Delta V(y) \cos(\phi^m_{x_0\to x} - \phi^m_{y\to x_f})$$
$$+ \frac{1}{8}\sin^2 2\theta_0^m (\cos 2\theta_0^m + \cos 2\theta'_0 - 2\sin^2\theta - \epsilon_0 \sin^2 2\theta_0^m) \int_{x_0}^{x_f} dx \int_{x_0}^{x_f} dy \Delta V(x)\Delta V(y) \cos\phi^m_{y\to x}. \quad (30)$$

We note that there are two first order (in ΔV) terms, one containing $\phi^m_{x_0\to x}$, the other $\phi^m_{x\to x_f}$, in contrast to the original theory which contains the phase $\phi^m_{x\to x_f}$ only.

For $V_0 = 0$, Eq. (30) coincides with the previous result (12).

For a symmetric density profile we obtain

$$P_{\nu_2\to\nu_e} = \sin^2\theta + \epsilon_0 \sin^2 2\theta_0^m \sin^2\phi^m_{\bar{x}\to x_f} + \frac{1}{2}\sin^2 2\theta_0^m (\cos 2\theta'_0 + \epsilon_0 \cos 2\theta_0^m) \sin\phi^m_{\bar{x}\to x_f} \int_{x_0}^{x_f} dx \Delta V(x) \cos\phi^m_{\bar{x}\to x}$$
$$+ \frac{1}{8}\sin^2 2\theta_0^m (\cos 2\theta_0^m + \cos 2\theta'_0 - 2\sin^2\theta - 2\epsilon_0 \sin^2 2\theta_0^m) \left[\int_{x_0}^{x_f} dx \Delta V(x) \cos\phi^m_{\bar{x}\to x}\right]^2. \quad (31)$$

In the limit $V \to 0$ the second, the third, and the fourth terms in (31) cancel each other (up to ϵ_0^3), and the probability reduces to $\sin^2\theta$.

Defining

$$I_\Delta \equiv \int_{\bar{x}}^{x_f} dx \Delta V(x) \cos\phi^m_{\bar{x}\to x}, \quad (32)$$

we can rewrite the probability (31) as

$$P_{2e} = \sin^2\theta + \epsilon_0 \sin^2 2\theta_0^m \sin^2\phi^m_{\bar{x}\to x_f} + B_1(\epsilon_0) \sin\phi^m_{\bar{x}\to x_f} I_\Delta + B_2(\epsilon_0) I_\Delta^2, \quad (33)$$

where the prefactors in front of powers of I_Δ equal

$$B_1 \equiv \sin^2 2\theta_0^m (\cos 2\theta'_0 + \epsilon_0 \cos 2\theta_0^m), \quad (34)$$

$$B_2 \equiv \frac{\sin^2 2\theta_0^m}{2}(\cos 2\theta'_0 - 2\sin^2\theta + \cos 2\theta_0^m - 2\sin^2 2\theta_0^m). \quad (35)$$

The prefactors as functions of ϵ_0 are shown in Fig. 3.

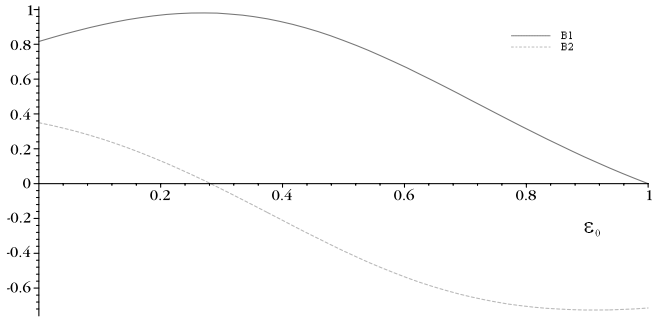

FIG. 3. The dependence of the prefactors in the improved perturbation formula for P_{2e} on the shift parameter ϵ_0. We use $\tan^2\theta = 0.4$.

Notice that $|B_i| < 1$. For $\epsilon_0 \to 0$ ($V_0 \to 0$) we have, as expected, $B_1 \to \sin^2 2\theta$ and $B_2 \to \sin^2 2\theta \cos 2\theta$. With increase of ϵ_0 the prefactor B_1 first slightly increases, reaches maximum at $\epsilon_0 = 0.28$, and then decreases. In contrast, B_2 decreases and crosses zero at $\epsilon_0 = 0.28$. For energies $E <$ (50–60) MeV, relevant for the solar and SN neutrinos, we have $\epsilon_0 < 0.2$.

Figure 4 illustrates an accuracy of the improved perturbation theory. Shown are the reduced errors, δ/E^2, calculated to second order of perturbation theory for different values of the average potential, parametrized as $V_0 = \sqrt{2} G_F N_0^e$. As follows from the figure, the best approximation is achieved (for not too high energies) when $V_0 \sim V_f$, where V_f is the potential at the surface of the Earth. With further increase of V_0, the error increases. In particular, for an average potential $V_0 \sim 2V_f$ along the neutrino trajectory, the errors are large.

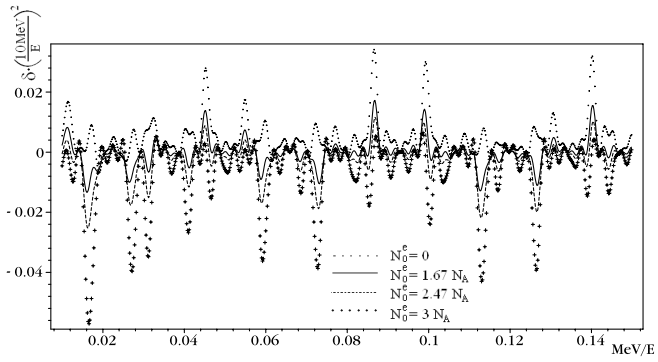

FIG. 4. The reduced errors δ/E^2 of the improved perturbation expansion (to second order) as functions of the inverse neutrino energy for different values of the shift N_0^e. The surface density is $1.67 N_A$. The second order of perturbation is taken. Other characteristics are as in Fig. 2.

This can be understood as follows. A shift of the potential affects both the prefactors and the integral I_Δ. Since the prefactors do not change by much, we consider only the integral I_Δ. Integrating by parts, it is easy to show that the integral I_Δ can be presented as

$$I_\Delta \approx \frac{2E(V_f - V_0)}{\Delta m^2} \sin\phi_{\bar{x}\to x_f} - \frac{2E}{\Delta m^2} \int_{\bar{x}}^{x_f} dx \frac{dV}{dx} \sin\phi_{\bar{x}\to x_f}, \quad (36)$$

where $V_f = V(x_f)$ and we take in the first approximation $\Delta^m \approx \Delta$. Apparently, the integral in (36) does not depend on the shift of the potential and the dependence appears in the first term which is due to the boundary condition. This explains the dependence of the errors on the shift in Fig. 4. Selecting $V_0 = V_f$, one can eliminate the first term completely. For $V_0 > V_f$ this term changes the sign and, therefore, can in some energy ranges partially compensate the contribution of the integral. However, substantial deviation of V_0 from V_f makes the first term and, therefore, the expansion parameter larger than in the original theory. As follows from Fig. 4, the improved expansion allows one to reduce the relative errors by a factor of 2 (in average), though in some particular energy range the reduction can be stronger.

Apart from specific values of energy, the improved theory gives a better description for $E < (15\text{–}30)$ MeV. Even better approximation for low energies can be obtained for $V_0 < V_f$. This is also related to the attenuation effect: contribution from deep structures is suppressed and the oscillation effect is determined by the outer layers where the subtraction by a suitable potential $V_0 \approx V_f$ can be rather precise. For high energies the effect of the core becomes ''visible.'' It seems that for the average probability over all relevant energies the best approximation corresponds to the surface potential.

IV. CORRECTIONS DUE TO THREE-NEUTRINO MIXING

In the standard parametrization, the lepton mixing matrix is

$$U = O_{23}\mathrm{diag}(1, 1, e^{i\delta_{cp}}) O_{13} \mathrm{diag}(1, 1, e^{-i\delta_{cp}}) O_{12} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{23} & s_{13}e^{-i\delta_{cp}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{cp}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{cp}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{cp}} & c_{13}c_{23} \end{pmatrix}.$$

By a redefinition of the mixing matrix

$$U \to U\mathrm{diag}(1, 1, e^{i\delta_{cp}}), \quad (37)$$

the Hamiltonian becomes real, i.e.,

$$\mathcal{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_s & 0 \\ 0 & 0 & \Delta_a \end{pmatrix} + U^\dagger \begin{pmatrix} V & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \quad (38)$$

$$= \begin{pmatrix} Vc_{13}^2c_{12}^2 & Vc_{13}^2s_{12}c_{12} & Vc_{12}c_{13}s_{13} \\ Vc_{13}^2s_{12}c_{12} & \Delta_s + Vc_{13}^2s_{12}^2 & Vs_{12}c_{13}s_{13} \\ Vc_{12}c_{13}s_{13} & Vs_{12}c_{13}s_{13} & \Delta_a + Vs_{13}^2 \end{pmatrix}, \quad (39)$$

where $\Delta_s \equiv \Delta m^2_\odot/2E$ and $\Delta_a \equiv \Delta m^2_{\mathrm{atm}}/2E - \Delta_s$.

Thus, we see that both the CP phase δ_{cp} and θ_{23} do not influence the propagation in matter (determined by the Hamiltonian). Also, since in (37) the first line does not contain δ_{cp} and θ_{23}, these parameters disappear in the oscillations from ν_e to ν_e or from ν_e to mass eigenstates and vice versa. They manifest themselves only when one considers the flavor states ν_μ or ν_τ.

These arguments are general and are valid in arbitrary matter density.

We now write the Hamiltonian in the form

$$\mathcal{H} = \mathcal{H}^0_{(3\nu)} + \Upsilon_{(3\nu)}, \quad (40)$$

where

$$\mathcal{H}^0_{(3\nu)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_s^m & 0 \\ 0 & 0 & \Delta_a^m \end{pmatrix} \quad (41)$$

and

$$\Upsilon_{(3\nu)} = \mathcal{H} - \mathcal{H}^0_{3\nu} + \mathrm{diag}(0, \Delta_s, \Delta_a) - \frac{V + \Delta_s + \Delta_a - \Delta_s^m - \Delta_a^m}{3} I = Vc_{13}^2 \begin{pmatrix} 0 & \sin2\theta_{12}/2 & c_{12}s_{13}/c_{13} \\ \sin2\theta_{12}/2 & 0 & s_{12}s_{13}/c_{13} \\ c_{12}s_{13}/c_{13} & s_{12}s_{13}/c_{13} & 0 \end{pmatrix} + O(V^2). \quad (42)$$

Δ_s^m and Δ_a^m are the eigenvalues of the Hamiltonian in matter [30]. In Eq. (42) we have subtracted a term proportional to the unit matrix in order to make it traceless and thus convenient for a power expansion.

A straightforward calculation leads to the transition probabilities of the mass eigenstates to ν_e:

$$P_{\nu_1\to\nu_e} = c_{13}^2c_{12}^2 - \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx V \sin\phi_{x\to x_f} - 2c_{12}^2c_{13}^2s_{13}^2 \int_{x_0}^{x_f} dx V \sin\psi_{x\to x_f}, \quad (43)$$

$$P_{\nu_2\to\nu_e} = c_{13}^2 s_{12}^2 + \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx V \sin\phi_{x\to x_f} - 2 s_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx V \sin(\psi_{x\to x_f} - \phi_{x\to x_f}), \quad (44)$$

$$P_{\nu_3\to\nu_e} = s_{13}^2 + 2 c_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx V \sin\psi_{x\to x_f} + 2 s_{12}^2 c_{13}^2 s_{13}^2 \int_{x_0}^{x_f} dx V \sin(\psi_{x\to x_f} - \phi_{x\to x_f}), \quad (45)$$

where

$$\phi_{a\to b} = \int_a^b \Delta_s^m(x) dx, \qquad \psi_{a\to b} = \int_a^b \Delta_a^m(x) dx. \quad (46)$$

The function $\sin\psi_{x\to x_f}$ oscillates $\Delta_a^m/\Delta_s^m \simeq \Delta m^2_{\rm atm}/\Delta m^2_\odot$ times faster than $\sin\phi_{x\to x_f}$. Thus, the corresponding integral is roughly $\Delta m^2_{\rm atm}/\Delta m^2_\odot$ times smaller than the one which contains the phase ϕ; furthermore, it has a prefactor s_{13}^2. Therefore, we get to a good approximation

$$P_{\nu_1\to\nu_e} = c_{13}^2 c_{12}^2 - \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx V \sin\phi_{x\to x_f}, \quad (47)$$

$$P_{\nu_2\to\nu_e} = c_{13}^2 s_{12}^2 + \frac{\sin^2 2\theta_{12}}{2} c_{13}^4 \int_{x_0}^{x_f} dx V \sin\phi_{x\to x_f}, \quad (48)$$

$$P_{\nu_3\to\nu_e} \simeq s_{13}^2. \quad (49)$$

These results may be also obtained from Eq. (39) [26] (see [23] for some earlier discussion). If $\Delta_a \gg \Delta_s \gg V$ and $s_{13} \ll 1$, the third neutrino decouples and one arrives at the two-neutrino propagation problem in matter with potential $V \to V c_{13}^2$ and mixing angle θ_{12}. Following the procedure of Sec. II and using the full mixing matrix $U\,\mathrm{diag}(1, 1, e^{i\delta_{cp}})$, we easily recover Eqs. (47)–(49).

V. CONCLUSION

Motivated by the large mixing MSW solution to the solar neutrino, we have developed a simple formulation of the Earth matter effects on low energy neutrino beams. Following [24], we derive an expansion for the neutrino transitions in terms of the parameter $\epsilon(x)$ to second order. By choosing a certain constant average value for the neutrino potential as a starting point, the precision can be further improved and it is possible to reach an accuracy of a few percent even for energies near 50–70 MeV. The effective expansion parameter is a simple integral in Eq. (16) [or Eq. (32)] together with Eq. (5) which can be done numerically. The expansion allows for a convenient quantitative discussion of various physical effects such as the attenuation effect to the remote structures of the density profile or the effect of energy resolution of detectors. We also consider the case of three-neutrino mixing.

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[27] This result may be obtained via ordinary perturbation theory with the Hamiltonian $\mathcal{H}(x) = \mathrm{diag}(0, \Delta) + U^{\dagger}\mathrm{diag}[V(x), 0]U = \mathcal{H}^0 + \Upsilon$, where $\mathcal{H}^0$ is the diagonalized Hamiltonian (MSW solution) at point (x). We stress that only that separation of the Hamiltonian into a nonperturbative ($\mathcal{H}^0$) and a perturbative (Υ) part leads to results where terms proportional to the full distance traveled by neutrinos in matter are absent; this is, in fact, guaranteed by the existence of the MSW solution.
[28] It is important to recall that ϵ enters both through V and through the adiabatic phase ϕ^m.
[29] Even more general would be an expansion around a suitable potential for which there is a closed analytic form.
[30] When $V \ll \Delta_s \ll \Delta_a$, then $\Delta_a^m \simeq \Delta_a + O(V)$ and $\Delta_s^m \simeq \Delta_s\sqrt{[\cos 2\theta - (Vc_{13}^2/\Delta_s)]^2 + \sin^2 2\theta} + O[s_{13}^2(V^2/\Delta_a)]$.