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Teleparallel limit of Poincaré gauge theory

M. Leclerc
Section of Astrophysics and Astronomy, Department of Physics, University of Athens, Greece
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We will address the question of the consistency of teleparallel theories in presence of spinning matter which has been a controversial subject of discussion over the last 20 years. We argue that the origin of the problem is not simply the symmetry or asymmetry of the stress-energy tensor of the matter fields, which has been recently analyzed by several authors, but arises at a more fundamental level, namely, from the invariance of the field equations under a frame change, a problem that has been discussed long time ago by Kopczynski in the framework of the teleparallel equivalent of general relativity. More importantly, we show that the problem is not only confined to the purely teleparallel theory but arises actually in every Poincaré gauge theory that admits a teleparallel geometry in the absence of spinning sources, i.e., in its classical limit.

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I. INTRODUCTION

Recently [1,2] there has been a revival of the discussion on whether or not the Dirac field can be consistently coupled to gravity in the framework of the teleparallel equivalent of general relativity (TEGR). The authors of [2] came to the conclusion that the theory, with the usual minimal coupling prescription (which we consider exclusively in this paper), is not consistent. The reason for this is simply the fact that the theory leads to a symmetric Einstein equation and thus requires the right-hand side of this equation, namely, the stress-energy tensor of the Dirac particle, to be symmetric too. Clearly, the stress-energy tensor of the Dirac particle, as well as of any other particle with intrinsic spin (when minimally coupled), is not symmetric by itself. In other words, requiring its symmetry is a constraint on the fermion field. Especially, the spin tensor would have to be conserved (covariantly), a condition that is not even satisfied in the absence of gravitational fields.

On the other hand, the inconsistency of TEGR has already been claimed 20 years ago in [3] (see also [4–7]), using a different argumentation. It has been noted that TEGR Lagrangian possesses a symmetry that is not inherit of the matter Lagrangian of a spinning particle. Namely, the Lagrangian and the field equations (in the absence of spinning matter) are invariant under what is called a frame transformation, i.e., a Lorentz transformation of the tetrad field with the connection held fixed [see Eq. (7) below]. As a consequence of this symmetry, the torsion tensor is not entirely determined by the field equations. Since spinning matter fields do not present the same invariance (in other words, they couple directly to the torsion), their behavior, when treated as test fields, cannot be predicted by the theory. Actually, the authors of [3–7] do not confine their analysis to TEGR. Rather, they consider the so-called one-parameter teleparallel Lagrangian, which leads to the most general teleparallel geometry that is consistent with the experimental situation. In this article, we confine ourselves to those Lagrangians that present a classical limit that is completely equivalent to general relativity. The discussion is easily generalized to the more general case (see remark at the end of Sec. III).

The scope of this article is to show that the problem described in [1,2] is actually directly related to the frame invariance of the teleparallel Lagrangian analyzed in [3] and that it is not confined to the teleparallel equivalent of general relativity, but is present in any Poincaré gauge theory that leads to a teleparallel geometry (with equations equivalent to those of general relativity) in its classical limit, i.e., in the absence of spinning matter fields.

In order to fix our notations and conventions, we briefly review the basic concepts of Riemann-Cartan geometry which is the basis of Poincaré gauge theory. For a detailed introduction, consult the standard reference [8]. Latin letters from the beginning of the alphabet $(a, b, c \dots)$ run from zero to three and are (flat) tangent space indices. Especially, η_{ab} is the Minkowski metric diag$(1, -1, -1, -1)$ in tangent space. Latin letters from the middle of the alphabet $(i, j, k \dots)$ are indices in a curved spacetime with metric g_{ik}. We introduce the independent gauge fields, the tetrad e^a_m and the connection $\Gamma^{ab}{}_m$ (antisymmetric in ab) and the correspondent field strengths, the curvature and torsion tensors

$$R^{ab}{}_{lm} = \Gamma^{ab}{}_{m,l} - \Gamma^{ab}{}_{l,m} + \Gamma^a{}_{cl}\Gamma^{cb}{}_m - \Gamma^a{}_{cm}\Gamma^{cb}{}_l, \quad (1)$$

$$T^a{}_{lm} = e^a_{m,l} - e^a_{l,m} + e^b_m\Gamma^a{}_{bl} - e^b_l\Gamma^a{}_{bm}. \quad (2)$$

The spacetime connection Γ^i_{lm} and the spacetime metric g_{ik} can now be defined through

$$e^a_{m,l} + \Gamma^a{}_{bl}e^b_m = e^a_i\Gamma^i_{ml} \quad \text{and} \quad e^a_i e^b_k \eta_{ab} = g_{ik}. \quad (3)$$

It is understood that there exists an inverse to the tetrad, such that $e^a_i e^i_b = \delta^a_b$. It can now be shown that the connection splits in two parts,

$$\Gamma^{ab}{}_m = \hat{\Gamma}^{ab}{}_m + K^{ab}{}_m, \quad (4)$$

such that $\hat{\Gamma}^{ab}{}_m$ is torsion free and the contortion $K^a{}_{bm}$ is related to the torsion through $T^a{}_{ik} = K^a{}_{bi}e^b_k - K^a{}_{bk}e^b_i$. Especially, the spacetime connection $\hat{\Gamma}^i_{lm}$ constructed from $e^a_{m,l} + \hat{\Gamma}^a{}_{bl} e^b_m = e^a_i \hat{\Gamma}^i{}_{ml}$ is just the Christoffel connection of general relativity, a function of the metric only.

All quantities constructed with the torsion free connection $\hat{\Gamma}^{ab}{}_m$ or $\hat{\Gamma}^i_{lm}$ will be denoted with a hat. Thus, for instance, $\hat{R}^i{}_{lkm}$ is the usual Riemann curvature tensor.

The gauge fields e^a_m and $\Gamma^{ab}{}_m$ are vector fields with respect to the spacetime index m. Under a local gauge transformation in tangent space, $\Lambda^a{}_b(x^m)$, they transform as

$$e^a_m \to \Lambda^a{}_b e^b_m, \qquad \Gamma^a{}_{bm} \to \Lambda^a{}_c \Lambda^d{}_b \Gamma^c{}_{dm} - \Lambda^a{}_{c,m}\Lambda_b{}^c. \quad (5)$$

The transformation (5) is the basis of Poincaré gauge theories. Under this transformation, the torsion and the curvature transform homogeneously. We will refer to it as Poincaré gauge transformation, although it is actually only the Lorentz part of a Poincaré transformation after having fixed the translational part to the so-called physical gauge. This conception of the Poincaré transformation is described in [9]. (For a fundamental treatment in a more general framework, see [10].) Every Lagrangian, gravitational or not, should be invariant under (5).

In addition, one can consider the pure Lorentz gauge transformations

$$e^a_m \to e^a_m, \qquad \Gamma^a{}_{bm} \to \Lambda^a{}_c \Lambda^d{}_b \Gamma^c{}_{dm} - \Lambda^a{}_{c,m}\Lambda_b{}^c, \quad (6)$$

as well as the frame transformations

$$e^a_m \to \Lambda^a{}_b e^b_m, \qquad \Gamma^a{}_{bm} \to \Gamma^a{}_{bm}. \quad (7)$$

Clearly, neither (6) nor (7) are symmetries of the Dirac Lagrangian (always speaking of the minimally coupled Lagrangian) nor of the Einstein-Cartan Lagrangian for instance. Note also that the transformation (5)–(7) are not independent. Clearly, a Lorentz transformation (6) followed by a frame transformation (7) (with the same parameters) is equivalent to a Poincaré transformation (5).

In the next section, we will investigate under which conditions the stress-energy tensor of the matter fields is symmetric. Then, in Sec. III, we construct the family of Lagrangians that present a teleparallel limit in the spinless case and discuss the problem of the inconsistency of such theories in the presence of spinning particles in relation with their invariance under a frame change (7).

II. FRAME INVARIANCE AND SYMMETRY OF THE STRESS-ENERGY TENSOR

We now deduce the conservation laws that follow from the symmetries (5)–(7) of a general matter Lagrangian density $\mathcal{L}_m$, which may depend on e^a_m, Γ^{ab}_m (as well as on their derivatives) and on matter fields that we summarize under the notation ψ.

As usual, we use the canonical definitions of the stress-energy tensor and of the spin density under the form

$$T^a{}_m = \frac{1}{2e}\frac{\delta \mathcal{L}_m}{\delta e^m_a}, \qquad \sigma_{ab}{}^m = \frac{1}{e}\frac{\delta \mathcal{L}_m}{\delta \Gamma^{ab}{}_m}.$$

We consider infinitesimal transformations $\Lambda^a{}_b = \delta^a_b + \epsilon^a{}_b$ with $\epsilon^{ab} = -\epsilon^{ba}$. (As tangent space indices, $a, b\ldots$ are and lowered with η_{ab}.)

The Poincaré transformation (5) now takes the form

$$\delta\Gamma^{ab}{}_m = -\epsilon^{ab}{}_{,m} + \epsilon^a{}_c \Gamma^{cb}{}_m + \epsilon^b{}_c \Gamma^{ac}{}_m, \; \delta e^a_m = \epsilon^a{}_c e^c_m. \quad (8)$$

The inverse of the tetrad transforms with the inverse transformation, i.e., $\delta e^m_a = \epsilon_a{}^c e^m_c$. The matter action $S_m = \int \mathcal{L}_m \mathrm{d}^4x$ therefore undergoes the following change (up to a boundary term):

$$\delta S_m = \int\left(\frac{\delta \mathcal{L}_m}{\delta e^m_a}\delta e^m_a + \frac{\delta \mathcal{L}_m}{\delta \Gamma^{ab}{}_m}\delta\Gamma^{ab}{}_m\right)\mathrm{d}^4x$$
$$= \int e(2T^{[ab]} + D_m\sigma^{abm})\epsilon_{ab}\mathrm{d}^4x,$$

where D_m is the covariant derivative that acts with $\Gamma^{ab}{}_m$ on the tangent space indices and with $\hat{\Gamma}^i_{kl}$ (torsion less) on the spacetime indices. We conclude that, if the matter Lagrangian possesses the symmetry (5), we have the following (well known) conservation law

$$D_m\sigma^{abm} + 2T^{[ab]} = 0. \quad (9)$$

If the matter fields ψ too are subject to a gauge transformation (for instance $\delta\psi = i\epsilon^{ab}\sigma_{ab}\psi$ in the Dirac case, with the Lorentz generators σ_{ab}), the action undergoes an additional change $\frac{\delta\mathcal{L}_m}{\delta\psi}\delta\psi$, but this does not contribute, due to the field equations of the matter fields, which are derived from $\frac{\delta\mathcal{L}_m}{\delta\psi} = 0$.

Clearly, the same argument if applied to the transformation (6) instead of (5) leads to $D_m\sigma^{abm} = 0$ and if applied to the frame change (7) to $T^{[ab]} = 0$. Since we consider only Lagrangians that possess the Poincaré symmetry, the symmetry (7) will imply the symmetry (6) and vice versa. Therefore, we can state that if the Lagrangian is frame invariant, then we have the conservation laws

$$D_m\sigma^{abm} = 0 \quad \text{and} \quad T^{[ab]} = 0. \quad (10)$$

Until now, we have considered only the matter part of the Lagrangian. Similar arguments can be applied to the gravitational Lagrangian $\mathcal{L}_0$ itself, which depends only on e^a_m, Γ^{ab}_m and their first derivatives. If we define $C_{ab}{}^m = -e^{-1}\delta\mathcal{L}_0/\delta\Gamma^{ab}{}_m$ and $E^a_m = -(2e)^{-1}\delta\mathcal{L}_0/\delta e^m_a$, the gravitational field equations arising from $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m$ have the form

$$E^a{}_m = T^a{}_m, \qquad C_{ab}{}^m = \sigma_{ab}{}^m, \quad (11)$$

where as usual we refer to the first equation as Einstein equation and to the second one as Cartan equation.

Using the same argumentation as before, we can show that every Poincaré invariant Lagrangian $\mathcal{L}_o$ will satisfy the Bianchi identity

$$D_mC^{abm} + 2E^{[ab]} = 0. \quad (12)$$

If $\mathcal{L}_0$ is in addition frame invariant, we have the relations

$$D_mC^{abm} = 0 \quad \text{and} \quad E^{[ab]} = 0. \quad (13)$$

III. POINCARÉ GAUGE THEORY WITH TELEPARALLEL LIMIT

A major problem in Poincaré gauge theory consists in reducing the 11 parameter Lagrangian (see [11] for instance) to those Lagrangians which are compatible with the classical experimental situation. Since our experiments until today are confined to the metrical structure of space-time, we can be sure to be in agreement with the experiments if the metric obeys the classical Einstein equations $\hat{G}_{ik} = T_{ik}$. Therefore, we will look for Lagrangians whose Einstein equation $E^a{}_m = T^a{}_m$, in the case of a vanishing spin density of the matter fields, reduces to $\hat{G}_{ik} = T_{ik}$. We know at least two such theories, namely, general relativity (GR) itself (which can be seen as the classical limit of Einstein-Cartan (EC) theory, the zero spin condition leading to zero torsion) and the teleparallel equivalent of GR (TEGR) where $R^{ab}{}_{lm} = 0$.

One goal of Poincaré gauge theory is to generalize the above theories to allow for both dynamical torsion and curvature. This means that we have to include at least one term quadratic in the curvature into the Lagrangian. If we seek for a classical limit with zero torsion, this term will certainly contribute to the Einstein equation even in the classical limit, except if it is of a very special (and unnatural) form like $R^{i[klm]}R_{i[klm]}$ or $R^{[lm]}R_{[lm]}$ (here, $[ikl]$ means total antisymmetrization of the three indices). Such terms actually depend only on torsion derivatives and vanish in the zero torsion limit via the Bianchi identities in Riemannian space.

On the other hand, if we are looking for a teleparallel limit in the zero spin case, we can add all kinds of terms quadratic in the curvature, $R_{ik}R^{ik}$, $R^2 \ldots$, without changing the classical limit of the theory. Such terms will lead only to contributions that vanish in the zero curvature limit. These are the Lagrangians we investigate in this paper.

Apart from the quadratic curvature terms, we have to modify the TEGR Lagrangian such that it is suitable for a first order variation without the use of Lagrange multipliers (see [12]). The suitable Lagrangian can be found (in a more general framework) in [13]. It consists of the sum of the teleparallel and the EC Lagrangian ($eL_0 = \mathcal{L}_0$),

$$L_0 = R - \tfrac{1}{4}T^{ikl}T_{ikl} - \tfrac{1}{2}T^{ikl}T_{lki} + \tfrac{1}{2}T^k{}_{ik}T^{mi}{}_m. \qquad (14)$$

This Lagrangian, apart from a divergence term, is essentially the Einstein-Hilbert Lagrangian (expressed in terms of the tetrad) (see [13] or [14]). It leads, in the absence of spinning matter, to the GR equation $\hat{G}_{ik} = T_{ik}$ and the Cartan equation is identically fulfilled. (In other words, $\delta\mathcal{L}_0/\delta\Gamma^{ab}{}_m = 0$.) This means that $\Gamma^{ab}{}_m$ remains completely undetermined.

Note that (14) is frame invariant and consistently, the Einstein tensor is symmetric. Let us now look at the Lagrangian

$$L = L_0 + aR^{ab}{}_{lm}R_{ab}{}^{lm} + L_m, \qquad (15)$$

with L_0 from (14) and L_m some matter Lagrangian. The field equations now read

$$\hat{G}_{ik} = \tau_{ik} + T_{ik}, \qquad (16)$$

$$D_m R^{ablm} = \sigma^{abl}, \qquad (17)$$

with $\tau_{ik} = -2a[R^{ab}{}_{li}R_{ab}{}^l{}_k - (1/4)R^{ab}{}_{lm}R_{ab}{}^{lm}]$. We chose (15) as an illustrative example because of its simple structure. Its field equations are exactly those of an Einstein-Yang-Mills system. Instead of $R^{ab}{}_{ik}R_{ab}{}^{ik}$ we can take any combination of quadratic curvature terms, because in the following, we are interested mainly in the classical, teleparallel limit.

Clearly, if the source is spinless, we get $R^{ab}{}_{lm} = 0$ as ground state solution. With this solution, we have $\tau_{ik} = 0$, and (16) reduces to the Einstein equation of GR.

We now come to the discussion of references [1,2]. The main statement in [2] is the fact that TEGR is not consistent when coupled to the Dirac particle because its Einstein equation has a symmetric left-hand side but the stress-energy tensor of the Dirac particle is asymmetric. We agree completely with this view, but we will show that the roots of the problem can be traced back to the frame invariance not only of the field equations, but of their classical limit (i.e., even in the absence of the Dirac particle as source). Therefore, the discussion should not be confined to the symmetry properties of T_{ik}.

Indeed, the Lagrangian (15) is again frame invariant, and thus Eq. (16) has the same symmetry problem as the corresponding one considered in [1,2]. However, this problem can be cured very easily: We simply add a term bR^2 (with the curvature scalar $R = e^i_a e^k_b R^{ab}{}_{ik}$) to (15). This term is clearly not frame invariant (although Poincaré invariant) and thus breaks the unwanted symmetry. (Any other quadratic curvature term that is not frame invariant does the same job. Again, the term R^2 serves as illustrative example.) Especially, we will get an additional asymmetric contribution $\sim R(4R_{ik} - g_{ik}R)$ to (16), allowing therefore for an asymmetric T_{ik}. Further, we get a contribution to the Cartan Eq. (17) of the form $\sim \mathrm{D}_i(e^i_{[a}e^k_{b]}R)$. Therefore, from the point of view of the discussion in [1,2], which focuses on the symmetry properties of the Einstein equation, the problem has been solved.

However, in the absence of spinning sources, we get as before the ground state solution $R^{ab}_{ik} = 0$, and therefore the (teleparallel) Einstein equation $\hat{G}_{ik} = T_{ik}$. Note that T_{ik} is now supposed to be symmetric, since the source is classical. These equations are once again frame invariant.

What does that mean? Well, let us fix the Poincaré gauge by imposing $\Gamma^{ab}{}_m = 0$. Then, from the Einstein equation, we can determine the metric g_{ik}. But the tetrad field will be determined only up to a Lorentz transformation $e^a_m \to \Lambda^a{}_b e^b_m$. This is the problem that has been discussed in [3] twenty years ago in the framework of the teleparallel equivalent of general relativity. For classical matter, this is not a problem, because it couples to the metric alone. Especially, the geodesics of a classical test particle will not depend on the gauge choice. However, spinning particles couple directly to the tetrad [or to the torsion, which is not a tensor under (7)] and the (semiclassical) trajectory of a test particle entering our fields, as well as its spin precession equation, will depend on the specific frame we choose. We can therefore not take the point of view that all the solutions that differ only by a frame change are equivalent.

We can even reduce the whole discussion to the complete groundstate of the field equations. The groundstate solution of the Einstein equation is $g_{ik} = \eta_{ik}$ and that of the Cartan equation is $R^{ab}{}_{lm} = 0$. Without physical consequences, we can fix the Poincaré gauge by the requirement $\Gamma^{ab}{}_{m} = 0$. Obviously, this state is invariant under (7). We can therefore determine neither the tetrad, nor the torsion [which is not a tensor under (7)]. These fields however are measurable since they couple to spinning particles. Clearly, this problem arises in any theory whose field equations reduce in the classical limit to $R^{ab}{}_{lm} = 0$ and $\hat{G}_{ik} = T_{ik}$.

Finally, there is the possibility of adding the term $\lambda T^{[ikl]}T_{[ikl]}$ (the square of the totally antisymmetric torsion part) to the Lagrangian. This changes the classical limit slightly but in a way consistent with the experimental situation, for an arbitrary constant λ. (The so-called one-parameter teleparallel theory, see [15]). This breaks the frame invariance of the classical limit (and even of the groundstate), but it has been shown in [3] that there is a remaining invariance of the form $e^a_m \to \Lambda^a{}_b e^b_m$ with $\Lambda^a{}_b$ a special Lorentz transformation that leaves the axial torsion part unaffected. Therefore, taking into account this new term would solve the problem for the Dirac test particle, which couples only to the axial torsion, but if we consider higher spin fields or macroscopic spin polarized bodies, the problem reappears, since the latter couple also to the other torsion parts (vector and tensor) which remain undetermined (see [16] for semiclassical equations of momentum propagation and of precession for general spinning test bodies). The complete discussion, in the framework of the purely teleparallel theory, can be found in [3]. The results of [3] have been confirmed and analyzed in greater detail in the follow-up articles [4–7].

In order to solve the problem completely, the torsion has to be fixed (determined) completely even in the classical limit (and especially in the ground state of the theory). Therefore, if we want a Poincaré theory to have a general relativity limit in the spinless case, this limit cannot correspond to a teleparallel geometry, but should be described by a fixed torsion, most probably $T^a{}_{ik} = 0$, i.e., a Riemannian geometry.

IV. CONCLUSION

As a result, we conclude that the teleparallel equivalent of general relativity is not consistent in presence of minimally coupled spinning matter. We showed that the argument given in [2], i.e., that the Einstein equation has a symmetric left-hand side whereas the stress-energy tensor of spinning matter field is not symmetric, actually has its roots in the frame invariance of the teleparallel Lagrangian discussed in [3].

Furthermore, we could show that every Poincaré gauge theory that leads, in the absence of spinning matter, to a teleparallel geometry with an Einstein equation equivalent to GR suffers from the same inconsistency. Even if the Lagrangian itself is not frame invariant, the field equations in their classical limit will be frame invariant again. A spinning test particle entering these fields however will couple directly to the torsion (which is not a tensor under the frame change), and its behavior (spin precession, trajectory...) will depend on the arbitrary choice of a specific frame.

The problem with such theories has also been analyzed in [17], based on a completely different argumentation (3 + 1 decomposition). The conclusions are similar, however, our argumentation is much simpler and shows clearly which class of theories suffers from the inconsistency and why there is a relation to the symmetry of the Einstein equation.

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