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Loop corrections to volume moduli and inflation in string theory

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The recent progress in embedding inflation in string theory has made it clear that the problem of moduli stabilization cannot be ignored in this context. In many models a special role is played by the volume modulus, which is modified in the presence of mobile branes. The challenge is to stabilize this modified volume while keeping the inflaton mass small compared to the Hubble parameter. It is then crucial to know not only how the volume modulus is modified, but also to have control over the dependence of the potential on the inflaton field. We address these questions within a simple setting: toroidal $\mathcal{N} = 1$ type IIB orientifolds. We calculate corrections to the superpotential and show how the holomorphic dependence on the properly modified volume modulus arises. The potential then explicitly involves the inflaton, leaving room for lowering the inflaton mass through moderate fine-tuning of flux quantum numbers.

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I. INTRODUCTION

Inflation has become one of the cornerstones of our picture of the early universe and its evolution. The successes of inflation include the explanation of the apparent homogeneity and isotropy of the universe at large scales, and the prediction of a spectrum of density fluctuations in the cosmic microwave background that agrees with observation. Recently there have been several interesting attempts to embed inflation in string theory by combining elements of string theory model building such as background fluxes and D-branes. In this paper, we will focus mainly on the model of Kachru, Kallosh, Linde, Maldacena, McAllister, and Trivedi (KKLMMT) [1,2], but our conclusions are equally relevant for, e.g., D3/D7-brane inflation [3–5].

The central idea of inflation in D-brane models is to realize inflaton fields by open string moduli that parametrize the positions of branes.[1] The motion of the brane then roughly corresponds to the rolling of the inflaton. More precisely, these models typically use the standard framework of single-field slow-roll inflation, where the flatness of the effective potential for the inflaton is measured by the slow-roll parameters

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \ll 1, \qquad \eta = M_{\rm Pl}^2\left|\frac{V''(\varphi)}{V(\varphi)}\right| \ll 1, \tag{1}$$

primes denoting derivatives of $V(\varphi)$ with respect to the canonically normalized inflaton field φ. Early models of brane inflation relied on the assumption that all other scalars, in particular the geometrical moduli for the background, can be ignored or frozen while the brane-position scalars evolve. It may be considered one of the main merits of [1,2] that this issue was addressed in a model that in principle allows stabilization of all geometrical moduli of the compactification space. However, the situation is complicated by the mixing of the geometrical background moduli and the open string moduli in the effective action; it is not obvious that one can fix the former while evolving the latter [1].

In particular, let us consider the volume modulus. Under the assumption that a model with just a single Kähler modulus can be found, it was argued in [2] that the volume of the internal space can be stabilized in type IIB orientifold compactifications with 3-form fluxes [7], if one includes nonperturbative effects: either superpotential contributions from Euclidean D3-brane instantons [8], or gaugino condensation on the world-volume of wrapped D7-branes (we will concentrate on the latter in the following). In either case, a superpotential is generated that depends holomorphically on the volume modulus and on the inflaton field, as dictated by supersymmetry. Now, in the presence of mobile D3-branes, the Kähler modulus includes not only the volume but also the D3-brane scalars, as alluded to in the previous paragraph. It was argued in [1] that it is this combination, as opposed to the actual "geometrical" volume, that is stabilized along the lines described in [2]. This mixing produces a mass term for the D3-brane scalars, a combination of which can naturally serve as a candidate for the inflaton field. This mass gives a contribution of order one to the slow-roll parameter η and thus seems to spoil inflation in this class of models if no additional contribu-

[1]Other more exotic candidates for the inflaton were proposed in [6].

tion to the mass arises, e.g. through quantum corrections.[2]

Of course, the actual mass depends crucially on *what* combination of the volume modulus and the D3-brane scalars is stabilized, and how the inflaton enters the superpotential. The key to understanding both issues is the gauge kinetic function that appears in the effective Lagrangian for the gauge fields on the D7-branes. This is because the gauge kinetic function determines the nonperturbative superpotential, and holomorphy of the superpotential allows one to read off the correct Kähler modulus. The dependence of the superpotential on the inflaton field, which we determine, leads to additional contributions to its mass and confirms the expectation of [1] that it can be fine-tuned to small values.

An important potential problem with the form of the Kähler modulus ρ suggested in [1] is that at first sight it seems to be in conflict with supersymmetry of the effective theory [4,11], because it appears to violate holomorphy of the gauge kinetic function.[3] More concretely, one might compute the Wilsonian coupling of the D7-brane gauge group by reduction of the Dirac-Born-Infeld (DBI) action of the D7-branes on the 4-cycle the branes are wrapped around, as we review in Sec. II. The D7-brane gauge coupling that results from this reduction does not seem to be the real part of a holomorphic function of the corrected Kähler modulus ρ.[4] To summarize, there are three questions we would like to address:

(i) How does the modified modulus ρ depend on the D3-brane scalars?

(ii) How does the gauge kinetic function become a holomorphic function of this modified modulus? (We call these two issues collectively the "rho problem").

(iii) How does the nonperturbative superpotential depend on the open string scalars, in particular, on the inflaton candidate φ?

It is the main purpose of this paper to shed some light on the solution to these problems. We propose that the dependence of the gauge kinetic function on the D3-brane scalars due to *open-string one-loop corrections* leads both to a solution of the rho problem and to additional dependence of the superpotential on the open-string scalars. These corrections arise from the Möbius and annulus diagrams at Euler characteristic zero.[5] By comparison, the tree-level action from dimensional reduction of the bulk supergravity and the D-brane DBI action come with powers $e^{-2\Phi}$ and $e^{-\Phi}$ of the string coupling. Thus, the three questions above can be addressed simultaneously by calculating the gauge kinetic function of the D7-branes at the open-string one-loop level.

The actual KKLMMT setup is reviewed in the next section, but for the computations later in the paper we consider a simplified setting: toroidal $\mathcal{N} = 1$ type IIB orientifolds (in the following we will drop "toroidal" and simply talk about type IIB orientifolds). In doing so, we are certainly not able to capture the details of a hypothetical analogous calculation in the KKLMMT background, but it enables us to understand the basic qualitative picture in a controlled way. Also, we actually perform the calculation in a T-dual picture, where 6 T-dualities are performed to turn the D3- and D7-branes into D9- and D5-branes, respectively. Working in this T-dual picture makes it easier to compare our results to the existing literature on open string loop corrections in orientifolds, where the equivalent D9/D5-brane language is usually preferred. In this language, the D3-brane scalars are mapped to continuous Wilson line moduli. This means that for our purpose of investigating the rho problem, we want to compute the dependence of the D5-brane gauge coupling on the D9-brane Wilson lines. In particular, we want the dependence due to closed string exchange between D-branes (and O-planes), or equivalently, due to open string one-loop diagrams. Such corrections are usually referred to as one-loop *threshold corrections* to the gauge coupling constants (see e.g. [14] for earlier work in this context, including the heterotic string).[6] In fact, without much additional effort, we can ask the slightly more general question of how both the D5-brane and D9-brane gauge couplings depend on both the D5-brane and D9-brane Wilson line moduli. Hence, our proposal for an answer to the three questions above appears as a special case of a more general result.

Our conclusions are that the D3-brane scalars and thus the inflaton field φ indeed appear in the modified ρ modulus in a form that was qualitatively anticipated in [1]. The one-loop correction to the gauge kinetic function is found to be of exactly the right form to reinstate its

[2]This result was confirmed using a completely different method in [9]. Also, in the following we will always use the term "inflaton mass" instead of "η", although the mass of the inflaton is strictly speaking only defined at the minimum of the potential. Finally, note that the parameter ϵ is usually much smaller than η in the KKLMMT model, at least if the inflaton field is much smaller than the Planck mass [1,10]. We, therefore, concentrate on the inflaton mass problem in the following.

[3]See [12] for a nice introduction to holomorphic couplings in string theory.

[4]We would like to stress that this puzzle is not restricted to the present class of cosmological models; it is a general problem of the effective supergravity action that follows from string theory in the presence of D-branes. It seems to us that up to now, the problem has simply been ignored.

[5]The order of string perturbation theory, which is given in terms of the Euler characteristic χ and the dilaton Φ as $e^{-\chi\Phi}$, should not be confused with the open (loop) vs closed (tree) channel interpretations; the annulus diagram always has $\chi = 0$, but it can be computed two different ways. See [13] for a comprehensive introduction to open strings.

[6]The role of threshold corrections for moduli stabilization through nonperturbative superpotentials due to gaugino condensation has recently been discussed also in [15].

holomorphy, when expressed in terms of the modified modulus ρ. Hence the rho problem is solved in this (simplified) setting, and the inflaton mass problem of [1] is manifest. Happily, the inflaton mass problem may be alleviated by certain additional corrections to the gauge kinetic function, and thus to the nonperturbative superpotential, which depend on the inflaton. This can help lowering the inflaton mass. We expect there to be quantitative modification of our orientifold results in the KKLMMT and D3/D7-brane inflationary models, but qualitatively, we expect our conclusions to remain the same.

The paper is organized as follows: In the next section, we review the two models of inflation where we want to apply our results: the KKLMMT model and D3/D7-brane inflation. We then proceed by describing our method of calculation, the background field method in type IIB orientifolds [16–18], in Sec. III A, and we compute the one-loop corrections to the D9- and D5-brane gauge couplings in the $\mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$ orientifold [19–21], and their dependence on the Wilson lines along the $\mathbb{T}^2$, in Sec. III C. This model actually has unbroken $\mathcal{N} = 2$ supersymmetry in four dimensions, but the computation can easily be generalized to $\mathcal{N} = 1$ orientifolds on $\mathbb{T}^6/\mathbb{Z}_N$ with even N (see e.g. [22,23]), or to $\mathbb{T}^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ models [24,25]. We carry out this generalization in Sec. III D, focusing on the examples of $\mathbb{Z}_6'$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Finally, in Sec. IV we interpret our results in the context of string-theoretic models for inflation. Some of the relevant formulas and more technical details are collected in the appendices.

Reading Guide—For the reader who is interested in results and not details, we propose reading the review Sec. II, and then jumping straight to Sec. IV, where the implications for inflation in string theory are discussed. It requires some of the notation introduced in Sec. III, but it does not rely on understanding the calculations of Sec.III in any detail. This reader might also want to have a look at our two "side remarks" about the prepotential and the special coordinates in the $\mathcal{N} = 2$ case, Sec. III C2.

II. THE KKLMMT MODEL FOR D-BRANE INFLATION

This section is a brief review of the basic ingredients that go into the string-theoretic models of inflation that our results can be applied to. The one we shall be concentrating on was introduced in [1] and is often referred to as the KKLMMT model.[7] It is based on a type IIB compactification on a Calabi-Yau manifold that has a discrete symmetry, which is quotiented out together with the world-sheet parity Ω, producing an orientifold (an example with $\mathbb{Z}_2$ symmetry is sketched in Fig. 1). This is analogous to the T-dualized Ω-projection applied in [26] and allows turning on imaginary self-dual 3-form fluxes despite the fact that the fluctuations of the corresponding potentials are projected out of the spectrum.[8]

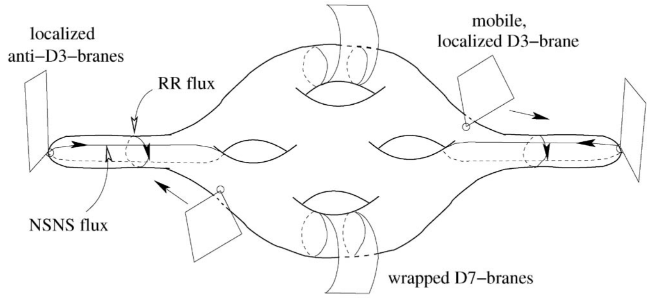

FIG. 1. The KKLMMT model

The Calabi-Yau manifold has deformed conifold singularities, with a deformation parameter that is fixed by the values of the fluxes [7]. These fluxes stabilize not only the deformation parameter, but also *all* the other complex structure moduli and the complexified dilaton.[9] However, the Kähler moduli, such as the overall volume modulus, remain unfixed by this flux stabilization, as is manifest in the no-scale structure of the resulting effective potential. Although the no-scale structure is broken by α'-corrections [34], the known α'-corrections are not sufficient to argue for stabilization of the Kähler moduli. In order to fix also them, Kachru, Kallosh, Linde and Trivedi (KKLT) [2] resorted to nonperturbative effects like Euclidean D3-brane instantons [8] or gaugino condensation on wrapped D7-branes in order to generate an additional contribution to the superpotential that explicitly depends on the Kähler moduli. In this way all closed string moduli are stabilized, albeit in an AdS minimum.

Momentarily we will recall how to lift this AdS minimum to a dS minimum, but let us first elaborate a bit more on the nonperturbative superpotential, focusing on the version using gaugino condensation. In this case it takes the form

$$W_{\text{nonpert}} \sim e^{-\alpha f}, \tag{2}$$

where f is the D7-brane gauge kinetic function. To leading order in string perturbation theory, and ignoring the open

[7]We would like to emphasize that our results are important also for any other effective theory in which closed-string and open-string moduli appear simultaneously. Another example, D3/D7-brane inflation, will be mentioned later on.

[8]The effective action of these orientifold models has for instance been discussed in [27–31]. Further generalizations with non-Abelian gauge groups and chiral matter have also been proposed in [32].

[9]See [33] for explicit examples of this stabilization of complex structure moduli in Calabi-Yau compactifications with fluxes.

string moduli ϕ^i for the moment, we have $f = -i\rho$ and hence

$$W_{\text{nonpert}} \sim Ce^{i\alpha\rho}, \qquad (\phi^i = 0), \tag{3}$$

where the imaginary part of ρ is the volume of the 4-cycle that the D7-branes are wrapped around, measured in the Einstein-frame metric. The constants C and α depend on e.g. the beta function of the D7-brane gauge theory. To derive (3), let us assume that there is only one Kähler modulus ρ. Then the volume of the wrapped 4-cycle is given by the $2/3$ power of the overall six-dimensional volume, and the reduction of the DBI action leads to

$$\mathcal{L}_{\text{DBI}} \sim -\frac{1}{4}\mathcal{V}^{2/3}\text{tr } F_{\mu\nu}F^{\mu\nu}, \tag{4}$$

among other terms. From (4) one can read off the real part of the gauge kinetic function:

$$\text{Re } f = \mathcal{V}^{2/3} = -\frac{i}{2}(\rho - \bar{\rho}), \tag{5}$$

which can be taken as a defining equation for the imaginary part of ρ.[10] As f has to be holomorphic in the field variables, it follows that $f(\rho) = -i\rho$, which leads to (3).

This nonperturbative superpotential stabilizes the volume, but in an AdS minimum. In order to lift the negative cosmological constant to a positive value, several possibilities have been proposed.[11] The original KKLT approach [2] was to add anti-D3-branes at the tip of the deformed conifolds, as in Fig. 1. This breaks supersymmetry explicitly, so the authors of [36] suggested replacing the effect of the anti-D3-branes by a D-term potential due to the introduction of a background for the gauge fields on the world-volume of the D7-branes. In this scenario, supersymmetry is only broken spontaneously, and $\mathcal{N} = 1$ supersymmetric Lagrangians can be used straightforwardly.[12] Here we focus on the original model involving anti-D3-branes, but like in [1,2], we mostly ignore their effects except for their contribution to the vacuum energy.[13]

It was the idea of KKLMMT [1] to study brane-antibrane inflation [39][14] in the previously described KKLT background, by adding mobile D3-branes to the D7- and anti-D3-branes already present in the KKLT model. This approach solves one of the generic problems that brane-antibrane inflation had struggled with; in a flat geometry, the attractive potential between a brane and an antibrane is too steep to allow for slow-roll inflation. Placing the antibranes at the tip of the curved-geometry throat as in Fig. 1 reduces the attractive force between the mobile D3- and the anti-D3-branes by gravitational redshift (due to the warp factor in the metric), and the potential can become flat enough to allow for slow-roll inflation, at least in principle.

In practice, the story is more complicated due to the "rho problem" outlined in the introduction, and this was realized in [1]. To decide whether slow-roll inflation is possible or not, it is not sufficient to consider only the attractive force between the branes and antibranes, but one has to take into account the other contributions to the potential as well. The potential generated by fluxes and gaugino condensation leads to a stabilization of the geometric moduli, but in the presence of mobile D3-branes, the Kähler modulus ρ that is fixed is not the geometric volume, rather it is a combination of the volume and the D3-brane scalars. As the inflaton field φ is supposed to be represented by D3-brane scalars, expanding the potential around the minimum shows that in the KKLMMT model the inflaton has a mass that (after canonically normalizing the field) is of the order of the Hubble parameter:

$$m_\varphi^2 = 2H^2 = \frac{2}{3}V_{\text{dS}}, \tag{6}$$

with V_{dS} the vacuum energy density at the de Sitter minimum. This mass is too large to allow for slow-roll inflation. Still there is hope; this m_φ^2 was derived by neglecting any explicit dependence of the superpotential on the open string scalars, and thus on the inflaton. It was already indicated in [1] that such a dependence could contribute to the inflaton mass, allowing the value (6) to be lowered. It is clear from (2) that the open string scalars can enter the superpotential directly, i.e. $W_{\text{nonpert}} = W_{\text{nonpert}}(\rho, \phi)$, if the gauge kinetic function receives corrections depending on them. In other words, apart from the dependence through $f(\rho(\phi))$ that we already argued for, W could also depend on ϕ through additional explicit dependence $f = f(\rho(\phi), \phi)$. We will see that such corrections typically do arise at the open-string one-loop level.

One of the important lessons here is that the question of the inflaton mass cannot be discussed separately from the issue of volume stabilization, and that the precise form in

[10] Of course, one has to make sure that this definition leads to a viable Kähler coordinate on the moduli space. This is clear from the appendix of [7].

[11] Note that nonperturbative effects release us from the shackles of the no-go theorems [35] that prohibit compactifications with fluxes and/or branes to four dimensions with a positive cosmological constant.

[12] A different approach has recently been put forward in [37], where the potential energy is positive because one expands around a relative dS minimum as opposed to an absolute AdS minimum.

[13] Consequences of soft supersymmetry breaking in effective actions derived from D-brane models (in orientifolds) have recently been discussed in [30,38].

[14] For a recent review of D-brane cosmology and more references see [40].

which the open string scalars enter into the definition of ρ can have a large impact on the physical outcome. This precise form can be determined by direct computation, as we will discuss later, but let us now briefly review general arguments why such a dependence is expected at all, following [1]. It was conjectured in [41] that the Kähler potential for the volume modulus ρ in the presence of D3-brane scalars ϕ^i, $i = 1, 2, 3$, should be modified to

$$K(\rho, \bar{\rho}, \phi, \bar{\phi}) = -3\ln[-i(\rho - \bar{\rho}) + k(\phi, \bar{\phi})], \qquad (7)$$

where $k(\phi, \bar{\phi})$ is the (so far unknown) Kähler potential of the metric on the Calabi-Yau manifold. This leads to a kinetic term for the 3-brane scalars of the form

$$\mathcal{L} \sim \frac{k_{i\bar{j}}\partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}}}{-i(\rho - \bar{\rho}) + k(\phi, \bar{\phi})} + \ldots, \qquad (8)$$

where $k_{i\bar{j}}$ is the derivative of $k(\phi, \bar{\phi})$ with respect to ϕ^i and $\bar{\phi}^{\bar{j}}$, and the dots include further contributions to the kinetic terms of the 3-brane scalars involving single derivatives of $k(\phi, \bar{\phi})$. Comparing this to the kinetic term that one would infer from considering the DBI action of a D3-brane transverse to the Calabi-Yau, i.e.

$$\mathcal{L} \sim \frac{k_{i\bar{j}}\partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}}}{\mathcal{V}^{2/3}} + \ldots, \qquad (9)$$

where $\mathcal{V}$ is the volume of the Calabi-Yau as measured in the Einstein frame, suggests the identification

$$-i(\rho - \bar{\rho}) = \mathcal{V}^{2/3} - k(\phi, \bar{\phi}), \qquad (10)$$

so that the imaginary part of the Kähler modulus is indeed a mixture of the geometric volume and a function of the 3-brane scalars. Thus we see that the form (10) follows from the conjectured form (7).[15]

The identification (10) leads to an intriguing puzzle [4,11]. Comparing with (4) shows that the gauge coupling of the D7-brane gauge group as derived from a reduction of the DBI action is not the real part of a holomorphic function in the corrected ρ. More precisely, in (4) a term of the type $k(\phi, \bar{\phi})\text{tr}(F_{\mu\nu}F^{\mu\nu})$ is missing to complete the imaginary part of the modified ρ. The restoration of the holomorphy of the gauge kinetic functions is the "rho problem" described in the introduction. As we will argue in the following and as already anticipated in the introduction, this missing term should arise at open-string one-loop level. The computation we will present confirms the conjectured form of the corrected ρ, making it compatible with supersymmetry, and thus solves the puzzle (at least in our simplified setting, but as mentioned, we expect this to be true more generally). Moreover, additional terms depending on the open string scalars arise at this order.

The reasons why the corrections must arise at open-string one-loop level are easy to state. First, the missing term does not involve the ten-dimensional dilaton, whereas the term from reduction of the DBI action has a factor $e^{-\Phi_{10}}$, i.e., it is open string tree-level.[16] Therefore, the missing term comes with a power of the dilaton appropriate for string diagrams of Euler characteristic zero. Second, for several coincident D-branes the scalars ϕ^i would carry a representation of the world-volume gauge group, and non-Abelian versions of the missing term, e.g. $\text{tr}[k(\phi, \bar{\phi})]\text{tr}(F_{\mu\nu}F^{\mu\nu})$, would involve at least two traces over gauge indices. This requires an open-string diagram with two boundaries, but there is no such diagram at tree-level.

We conclude this review section with a few remarks on D3/D7-inflation; the rho problem arises also in that context. There, the idea is to consider a system of D3- and D7-branes that have four noncompact directions in common. In the absence of world-volume fluxes, this system is supersymmetric and the distance between the D3- and D7-branes along the two directions transverse to the D7-branes is a massless modulus. Turning on a non-self-dual magnetic background flux for the gauge fields on the D7-branes breaks supersymmetry, and leads to an attraction between the D3- and D7-branes [3]. In other words, the scalar parametrizing the distance feels a potential, and this potential turns out to be flat enough to allow for slow-roll inflation with the distance scalar as the inflaton field. The rho problem then arises in this string-theoretic model of inflation, just like in the KKLMMT model.

III. ONE-LOOP CORRECTIONS TO THE VOLUME MODULUS

In this section we are going to compute the dependence of D3- and D7-brane gauge kinetic functions on the open string scalars by determining the complete one-loop correction, which also involves terms depending on the background complex structure moduli. As explained in the previous section, this produces an explicit expression for how the nonperturbative superpotential due to gaugino condensation depends on the open string scalars, and addresses the rho problem and the inflaton mass problem.

One could in principle obtain the desired renormalization from string amplitudes with vertex operator insertions, depicted in Fig. 2. The wiggly lines are vector insertions, and the dashed lines are (open string) scalar insertions. For the KKLMMT model, Fig. 2(a) with one end on D5-branes and the other on D9-branes would be sufficient to determine the renormalization of the D5-brane gauge kinetic function and its dependence on the

[15] Another argument comes from the analogy to the heterotic string. In compactifications on either a Calabi-Yau manifold or a torus, the Kähler moduli are corrected in the presence of Wilson line moduli, the heterotic analogs of open-string scalars [42,43].

[16] Note that this $e^{-\Phi_{10}}$ is implicit in (4) and (5) but appears when we express the volume in the string frame.

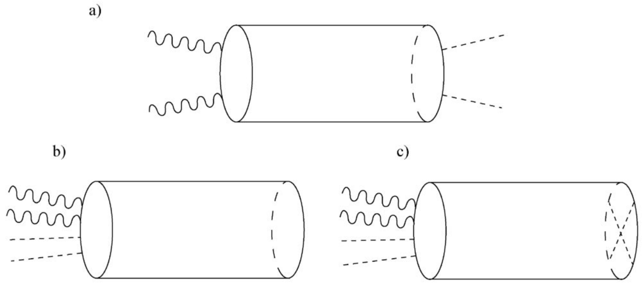

FIG. 2. D-brane gauge kinetic term corrections. a) Nonplanar annulus amplitude, b) Planar annulus amplitude, c) Möbius amplitude.

D9-brane Wilson lines to quadratic order (recall from Sec. I that we work in the D9/D5-picture). For the other dependences we are interested in, computation of Figs. 2(b) and 2(c) would also be necessary.

Here, however, we will use a convenient shortcut to gauge coupling renormalization: the background field method.

A. The background field method

The background field method was introduced in [16] and used there and in [17,18] to calculate threshold corrections to gauge coupling constants in type IIB orientifolds. The basic idea is to study how the one-loop vacuum energy is deformed by the presence of a constant background gauge field strength, where the background field is turned on in a $U(1)$ subgroup of the gauge group factor of interest. The deformed vacuum energy is straightforward to calculate, since the background field only modifies the boundary conditions; the worldsheet conformal field theory is still free. By expanding the deformed vacuum energy for small background fields, one can extract the zero-momentum limits of the string amplitudes in Fig. 2. In models with both D5- and D9-branes, one can consider a background in either the D5- or D9-brane gauge group, or both. Here we will consider the most general case, in which we turn on constant background gauge field strengths in both gauge groups at the same time, denoted by $\mathcal{F}_i$ and $\mathcal{F}_a$, respectively. The indices i and a enumerate gauge group factors in the D9- and D5-brane gauge groups. In components, the gauge field background reads

$$A_\mu = \mathcal{F}_{\mu\nu} x^\nu, \qquad \text{only e.g. } \mathcal{F}_{23} \neq 0. \tag{11}$$

The expressions for the relevant one-loop diagrams are then expanded to quadratic order in the field strengths $\mathcal{F}_i$ or $\mathcal{F}_a$ around $\mathcal{F}_i = \mathcal{F}_a = 0$, which yields the correction to the gauge coupling. In principle one could limit oneself to turn on only one type of background field and invoke T-duality to infer the corresponding terms for the other gauge groups. However, since we want to consider the effect of both types of Wilson lines (denoted by $\vec{a}_i$ and $\vec{a}_a$) on both types of gauge couplings g_i and g_a, we turn on both types of gauge fields concurrently.[17]

Let us concretize these introductory words in three schematic formulas. The one-loop vacuum energy in type I theory receives four contributions, from the torus, Klein bottle, Möbius strip and annulus diagrams,[18]

$$\Lambda_{1-\text{loop}}(\mathcal{F}, \vec{a}) = \mathcal{T} + \mathcal{K} + \mathcal{M}(\mathcal{F}, \vec{a}) + \mathcal{A}(\mathcal{F}, \vec{a}), \tag{12}$$

where we made explicit the fact that only diagrams with boundaries can have insertions of background gauge fields and Wilson lines. If one expands the vacuum energy to second order in the background field, the coefficient of the quadratic term directly gives the one-loop threshold correction to the gauge group for which the background was turned on. Omitting indices (that enumerate the gauge group factors) for backgrounds and Wilson lines, the expansion schematically looks like

$$\Lambda_{1-\text{loop}}(\mathcal{F}, \vec{a}) = \Lambda^{(0)} + \frac{1}{2}\left(\frac{\mathcal{F}}{2\pi}\right)^2 \Lambda^{(2)}(\vec{a}) + \ldots, \tag{13}$$

where $\Lambda^{(0)}$ is the one-loop induced cosmological constant, and the one-loop corrected gauge coupling can be identified as

$$\frac{4\pi^2}{g^2}\bigg|_{1-\text{loop}} = \frac{4\pi^2}{g^2}\bigg|_{\text{tree}} + \frac{1}{\sqrt{-g_4}}\Lambda^{(2)}(\vec{a}). \tag{14}$$

Here "tree" signifies open string tree-level, which in principle means disk diagram, although in practice the gauge kinetic term is of course easier to obtain by dimensional reduction of the DBI action. Unfortunately, the type I literature is littered with pitfalls when it comes to the precise meaning of "tree-level", so this would be an appropriate place to elaborate on this issue.

B. Tree-level effective action of type I on $\mathbb{T}^2 \times$ K3

Before we present the relevant loop calculations in Sec. III C, let us collect some results on the known "tree-level" supergravity action for the model considered in that section: the orientifold $\mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$.

This orientifold is a special case of a type I compactification on $\mathbb{T}^2 \times$ K3, with the K3 at a particular orbifold point. We are interested in computing corrections to the gauge kinetic terms of the Yang-Mills theories supported by D9- and D5-branes. The reader may rightly wonder why we begin by considering an $\mathcal{N} = 2$ orientifold when we are interested in $\mathcal{N} = 1$ models; the detailed reason is explained at the beginning of Sec. III D, but the basic

[17] More precisely, for non-Abelian gauge groups, it is in fact sufficient to turn on only one of the gauge fields and both types of Wilson lines, but for Abelian gauge groups it is not; one would miss cross-term corrections such as Eq. (49).

[18] Note that we have moved the factor of 1/2 that appears explicitly in $\Lambda_{1-\text{loop}}(\mathcal{F}, \vec{a})$ in much of the literature, e.g. in [18], to our definition of the amplitudes, Eq. (A2).

idea is that the *relevant part* of the full result in $\mathcal{N} = 1$ models is very similar to the result in this $\mathcal{N} = 2$ model, and the latter is simpler.

We have already bemoaned the fact that some of the literature on Kaluza-Klein (KK) reduction of type I is not precise in the usage of the term "tree-level". More concretely, it secretly incorporates various terms that really only arise at string one-loop level, and one may wonder whether including some terms but excluding others is consistent from the point of view of string perturbation theory—hence our quotation marks on "tree-level". This potentially confusing situation comes about because part of the relevant literature concerns heterotic-type I duality, and certain terms are loop corrections on the type I side but tree-level on the heterotic side. Here we use type I terminology exclusively.

Some general aspects of compactifications of type I strings on $\mathbb{T}^2 \times$ K3 have been discussed in [44], and we review the relevant results here. However, since the explicit factors will turn out to be important for our conclusions later on, we redo some of their analysis and adapt it to our conventions.

The closed string spectrum contains hypermultiplets and vector multiplets. The hypermultiplets, which will not be of great concern here, consist of the geometric moduli of the K3, moduli from an expansion of the antisymmetric tensors into the harmonic forms of the K3, and the six-dimensional dilaton. More important for our purposes are the vector multiplets. There are $3 + N_9 + N_5$ of them, where N_9 and N_5 denote the number of vector fields from the open string sector of the 9- and 5-branes, respectively. The three additional vector multiplets arise in the closed string sector. There are four KK vectors from the metric and the antisymmetric 2-form due to the presence of the two 1-forms of the torus. One of them, the graviphoton, resides in the supergravity multiplet and the other three are contained in the three closed string vector multiplets. Their scalar components are given as follows. Let

$$G = \frac{\sqrt{G}}{\mathrm{Im}(U)}\begin{pmatrix} 1 & \mathrm{Re}(U) \\ \mathrm{Re}(U) & |U|^2 \end{pmatrix} \tag{15}$$

be the metric of the torus in string frame.[19] Then $U = (G_{45} + i\sqrt{G})/G_{44}$ is its complex structure modulus, which belongs to one of the closed string vector multiplets. In the absence of Wilson line moduli the other two scalars are given by

$$\begin{aligned} S &= \frac{1}{2\pi\sqrt{2}}(b + ie^{-\Phi_{10}}\mathcal{V}_{\mathrm{K3}}^{(\mathrm{str})}\sqrt{G}\alpha'^{-3}), \\ S' &= \frac{1}{2\pi\sqrt{2}}(B_{45} + ie^{-\Phi_{10}}\sqrt{G}\alpha'^{-1}), \end{aligned} \tag{16}$$

where b is the scalar dual to $B_{\mu\nu}$, Φ_{10} is the ten-dimensional dilaton and $\mathcal{V}_{\mathrm{K3}}^{(\mathrm{str})}$ denotes the volume of the K3-manifold measured in the string frame metric. Moreover, we keep α' explicit in this section to have better control over numerical factors. The three scalars U, S, S' span the moduli space $[SU(1,1)/U(1)]^3$ with prepotential $\mathcal{F}^{(0)} = SS'U$.

We included an extra factor $1/(2\pi\sqrt{2})$ in (16) as compared to the definition of [44] because we want the relations $g_{(9)}^{-2} = \mathrm{Im}(S)$ and $g_{(5)}^{-2} = \mathrm{Im}(S')$ to hold. Let us check this explicitly for the case of S', by reducing the 5-brane DBI action on the torus. To this end, we start with the standard expression for the DBI action

$$\begin{aligned} &T_5 \int d^6\xi e^{-\Phi_{10}}\sqrt{\det(-g_6 + 2\pi\alpha' F_{(5)})} \\ &= -\frac{1}{4}T_5(2\pi\alpha')^2 \int d^6\xi\sqrt{-g_6}e^{-\Phi_{10}}\mathrm{tr}F_{(5)}^2 + \cdots, \end{aligned} \tag{17}$$

where $T_5 = (1/\sqrt{2})2\pi(4\pi^2\alpha')^{-3}$. The factor of $1/\sqrt{2}$ in T_5 arises in type I theory and is absent in type IIB, see e.g. [45]. Reducing (17) on a torus of volume $(2\pi)^2\sqrt{G}$ leads to a gauge coupling

$$g_{(5)}^{-2} = \frac{1}{2\pi\sqrt{2}}e^{-\Phi_{10}}\sqrt{G}\alpha'^{-1} = \mathrm{Im}(S'), \tag{18}$$

where we choose the normalization such that $-\frac{1}{4}(g^{-2}) \times \int d^4x\sqrt{-g_4}$ tr F^2 is the kinetic term for vector fields in four dimensions.

Including Wilson line moduli of the open string vector fields, i.e., components of the (higher-dimensional) vectors along the $\mathbb{T}^2$, leads to a modification of the expression (16) for the scalars S and S' cf. [43,44]. Explicitly, we parametrize the Wilson line moduli by a 2-vector $\vec{a} = (a_4, a_5)$ with an index i or a for the stack of D9- or D5-branes, respectively, where a_4 and a_5 are related to the internal components V_4 and V_5 of the corresponding higher-dimensional vectors as $(a_4, a_5) = \sqrt{\alpha'}(V_4, V_5)$. To be precise, the $\vec{a}$ are defined as components with respect to the basis of the dual lattice $(\vec{e}^4, \vec{e}^5)$. The original compactification lattice is then $(\vec{e}_4, \vec{e}_5)$ with conventions such that

$$\begin{aligned} (a_I\vec{e}^I)\cdot(a_J\vec{e}^J) = G^{IJ}a_Ia_J, \qquad \vec{e}^I\cdot\vec{e}_J = \delta_J^I, \\ I, J \in \{4, 5\}, \end{aligned} \tag{19}$$

where G^{IJ} is the inverse of the metric (15). Note that [44] considers the case when the gauge group is broken to the Abelian subgroup, so for notational simplicity we restrict to that case in this section, but the string computations in

[19] We use the same letter G for both the torus metric and its determinant, but the determinant always occurs in the form $\sqrt{G}$, so no confusion should arise.

later sections will be performed also for the non-Abelian case. Adopting to our conventions, the modified scalars are given by[20]

$$S = S|_{A=0} + \frac{1}{8\pi}\sum_a a_4^a A_a, \qquad S' = S'|_{A=0} + \frac{1}{8\pi}\sum_i a_4^i A_i. \tag{20}$$

Here we used the complex Wilson line modulus

$$A = Ua_4 - a_5, \tag{21}$$

which is the complex combination that makes the metric of the four-dimensional scalar manifold manifestly Kähler. The modifications (20) of the scalars S and S' are not supposed to be obvious; one way to see that they must be modified is to consider the reduction of the kinetic term of the 3-form RR field strength in six dimensions. This term includes a Chern-Simons correction in the presence of open string fields.

In fact, we can fix the relative factor $1/8\pi$ between the leading term and the one-loop correction in (20) by inspection of precisely this Chern-Simons-corrected kinetic term. It can be reduced on the torus according to

$$dB - \frac{\kappa_{10}^2}{g_{10}^2}\omega_3 \stackrel{\mathbb{T}^2}{\rightarrow} \partial_\mu B_{45} - \frac{1}{2\sqrt{2}}\sum_i[(\partial_\mu a_4^i)a_5^i - (\partial_\mu a_5^i)a_4^i], \tag{22}$$

where ω_3 is the standard Yang-Mills Chern-Simons 3-form. We used the type I relation $g_{10}^2\kappa_{10}^{-1} = 2(2\pi)^{7/2}\alpha'$ and $\kappa_{10}^2 = \frac{1}{2}(2\pi)^7\alpha'^4$ cf. [46]. This leads to $\kappa_{10}^2/g_{10}^2 = \alpha'/(2\sqrt{2})$ and we absorbed α' in the definition of the fields $\vec{a}_i$ as explained above (19). Thus the relative factor between $\mathrm{Re}(S')$ and $\sum_i a_4^i \overleftrightarrow{\partial} a_5^i$ contains an additional $1/\sqrt{2}$ as compared to (3.6) of [44]. Taking into account the overall factor introduced in (16), we arrive at the $1/8\pi$ factor in the modification of S' given in (20). A similar argument should hold for the modification of S.

The full moduli space of the vector multiplets was identified in [47] to be a space called $L(0, N_5, N_9)$ in [48], which is homogeneous but not symmetric. The corresponding Kähler potential is determined by a holomorphic prepotential that was derived in [44]. An explicit KK-reduction of the ten-dimensional type I action (including the 9-brane vector fields but not those from the 5-branes) leads to $\mathcal{F}^{(0)} = S(S'U - \frac{1}{8\pi}\sum_i A_i^2)$, where we adjusted the formula of [44] to our conventions. To derive the form of the prepotential including the 5-brane vectors, one first has to compactify to six dimensions and add in the kinetic term for the 5-brane vectors (17) and the Chern-Simons term that is needed to cancel anomalies [49]

$$\mathcal{L}_{\rm CS}\, d\mathrm{vol} \sim -B \wedge F_{(5)} \wedge F_{(5)}, \tag{23}$$

where $d$vol is the six-dimensional volume form. Taking these terms into account in a further reduction to four dimensions leads to what is called the "tree-level" prepotential in [44], i.e.[21]

$$\mathcal{F}^{(0)} = SS'U - \frac{1}{8\pi}S\sum_i A_i^2 - \frac{1}{8\pi}S'\sum_a A_a^2. \tag{24}$$

We will come back to this prepotential in Sec. III C2.

The important bottom line of this section is that the geometric closed string moduli are corrected in the presence of open strings and D-branes, cf. (20). As discussed in the previous section, we want to show that the gauge kinetic functions are holomorphic in the *corrected* closed string moduli fields S, S', or more precisely, in their $\mathcal{N} = 1$ analogs. We will see that, in the $\mathcal{N} = 2$ case, the correction terms present in (20) arise as open string one-loop contributions to the tree-level result (18).

C. One-loop threshold corrections in $\mathcal{N} = 2$

Let us now specialize to the orbifold limit of K3 by considering the $\mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$ orientifold. (The parts of the following computation that can be considered standard are collected in appendix A). The orientifold group is generated by $\{\Omega, \Theta\}$, where Ω is world-sheet parity and Θ is a reflection along the $\mathbb{T}^4$. Threshold corrections to gauge couplings in this model were studied in [17], for the case where the background field and Wilson lines are turned on only on D9-branes. It is true that this situation is T-dual to the case with background field and Wilson lines on D5-branes, but as we already pointed out, mixing between D9-brane and D5-brane gauge groups (that may occur for $U(1)$ group factors) cannot be obtained by T-dualizing the results in [17], and the same holds for any dependence of the D9-brane coupling on the D5-brane Wilson lines and vice versa; this dependence is crucial for the application we are interested in.

We first summarize a few important features of the $\mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$ orientifold. Tadpole conditions (cancellation of RR charge) imply the presence of 32 units of D9- and 32 units of D5-brane charge, with maximal gauge group $U(16)_{D9} \times U(16)_{D5}$. The one-loop vacuum energy (12) becomes

[20]In [44] it was shown that there appears a further correction to Im(S), given by $\mathrm{Im}(S) \rightarrow \mathrm{Im}(S) + \sqrt{G}\delta/[2\mathrm{Im}(S')]$, where δ is the correction to the Einstein-Hilbert term arising at open string one-loop level. This redefinition is even higher order in an expansion in $e^{\Phi_{10}}$ and we will ignore it in the following. It is, however, important to establish the duality to the heterotic string [44].

[21]To be more precise, one has to take into account additional counterterms in the derivation, cf. [44,50].

$$\Lambda_{1-\text{loop}} = \mathcal{T} + \mathcal{K} + \mathcal{M}_9 + \mathcal{M}_5 + \mathcal{A}_{99} + \mathcal{A}_{55} + \mathcal{A}_{95} + \mathcal{A}_{59}.$$

We use labels i and a for stacks of $D9_i$- and $D5_a$-branes, and assume that the maximal gauge group is broken to a subgroup

$$G = \bigotimes_i U(N_i) \times \bigotimes_a U(N_a), \qquad \sum_i N_i = \sum_a N_a = 16, \tag{25}$$

through the presence of Wilson lines along $\mathbb{T}^2$, denoted by $\vec{a}_i$ or $\vec{a}_a$. For example, one may want to consider breaking to the Abelian subgroup $N_i = N_a = 1$; we will consider both Abelian and non-Abelian groups. The overall $U(1)$ factors in the two $U(16)$'s are actually anomalous, and will decouple from the low energy theory [49]. Given a configuration of branes, a background gauge field strength $\mathcal{F}_i$ or $\mathcal{F}_a$ can be turned on on any individual stack of branes. Each stack is represented in the CFT by a boundary state

$$|D9_i, IJ\rangle = |D9_i(\mathcal{F}_i, \vec{a}_i), IJ\rangle, \qquad |D5_a, IJ\rangle = |D5_a(\mathcal{F}_a, \vec{a}_a), IJ\rangle, \tag{26}$$

with Chan-Paton (CP) indices I, J. The explicit form of the boundary states is standard (for a review see [51]), but will not be needed here. The elements of the orientifold group act on the boundary states, e.g. by

$$\Omega|D9_i, IJ\rangle = (\gamma_{\Omega i})_{IK}|\Omega \cdot D9_i, LK\rangle(\gamma_{\Omega i}^{-1})_{LJ}, \qquad \Theta|D9_i, IJ\rangle = (\gamma_{\Theta i})_{IK}|\Theta \cdot D9_i, KL\rangle(\gamma_{\Theta i}^{-1})_{LJ}, \tag{27}$$

where $\Omega \cdot D9_i$ and $\Theta \cdot D9_i$ schematically represent the action on the string world-sheet fields. The unitary matrices γ summarize the action on the gauge bundle. In this notation it is evident that a 32×32 matrix γ labeled by i or a only acts on the respective stack, with all other entries vanishing, and

$$\gamma_{\Omega 9} = \sum_i \gamma_{\Omega i}, \qquad \gamma_{\Theta 9} = \sum_i \gamma_{\Theta i}, \tag{28}$$

and so on. The solution of the tadpole constraints fixes the only nonvanishing blocks to the form given in (A9), (A10), and (A12), in the appendix.[22] In this 32×32 matrix formulation, the original 32 D-branes of either type are pairwise related under Ω, breaking $U(32)$ to $SO(32)$ as in type I, and further subjected to the Θ-projection. This breaks the gauge group to $U(16)$, without further rank reduction. Therefore, the $32 + 32$ Wilson lines in the Cartan subalgebra are really $16 + 16$ independent pairs. T-dualizing to D3- and D7-branes localized on the 2-torus, this means the $32 + 32$ branes can be moved pairwise out of the fixed locus of the T-dual Ω-projection $\Omega R(-1)^{F_L}$, R being a reflection of all six internal coordinates, and F_L the left-moving space-time fermion operator. Apart from breaking the gauge group, the Wilson lines have the effect of introducing shifts $\vec{a}_i$ and $\vec{a}_a$ in the spectrum of KK states.

1. Couplings of non-Abelian gauge groups

We will first determine threshold corrections to non-Abelian gauge group factors, i.e., to $SU(N)$ groups, postponing the discussion of $U(1)$ factors to the next subsection.

To describe the embedding of the Wilson line in the gauge group we introduce charge matrices of the form

$$W_i = \text{diag}(\underbrace{0, \ldots, 0}_{p_i \text{ entries}}, \mathbf{1}_{N_i}, -\mathbf{1}_{N_i}, 0, \ldots, 0) \tag{29}$$

with nonvanishing entries in the block of the i-th factor of the gauge group, and similarly for W_a. The two factors of the gauge group (25) are just by definition the subgroups of the two $U(16)$ that commute with all W_i or W_a. For example, the W_i belong to $U(1)_i$, the overall factor in $U(N_i) = U(1)_i \times SU(N_i)$. The Wilson lines take their values in these $U(1)$ factors.

To specify the background gauge fields in some direction of the $SU(N_i)$ subgroup, we can choose the matrices

$$Q_i = \frac{1}{2}\text{diag}(\underbrace{0, \ldots, 0}_{p_i}, \underbrace{1, -1, 0, \ldots, 0}_{N_i}, \underbrace{-1, 1, 0, \ldots, 0}_{N_i}, \underbrace{0, \ldots, 0}_{32-p_i-2N_i}) \tag{30}$$

with just four nonvanishing entries. The first $1, -1$ pair occurs at the position of the first nonvanishing block $\mathbf{1}_{N_i}$ in W_i, the second pair at the position of the block $-\mathbf{1}_{N_i}$ in W_i. Together, these matrices specify the background in 32×32 matrix notation, and the matrix valued gauge field strengths and Wilson lines are $Q_i\mathcal{F}_i$ and $W_i\vec{a}_i$, etc. To keep the expressions for the annulus diagrams compact, we introduce the following notation for the background fields:

$$\mathbf{F}_{ij} = (Q_i\mathcal{F}_i \otimes \mathbf{1}_{32}) \oplus [\mathbf{1}_{32} \otimes (-Q_j\mathcal{F}_j)], \qquad \vec{\mathbf{A}}_{ij} = (W_i\vec{a}_i \otimes \mathbf{1}_{32}) \oplus [\mathbf{1}_{32} \otimes (-W_j\vec{a}_j)], \tag{31}$$

and similarly for $(\mathbf{F}_{ab}, \vec{\mathbf{A}}_{ab})$, $(\mathbf{F}_{ai}, \vec{\mathbf{A}}_{ai})$. The background is now tensor-valued, with one factor for each end of the

[22] We actually use the solution presented e.g. in [18], not that of the original literature [20]. The former has the advantage that $\gamma_{\Theta i}$ is diagonal.

open string in question. The matrices γ are then also tensor-valued: $\boldsymbol{\gamma}_i = \gamma_i \otimes \mathbf{1}_{32}$ or $\boldsymbol{\gamma}_j = \mathbf{1}_{32} \otimes \gamma_j$ etc., depending on whether the matrix $\boldsymbol{\gamma}$ acts on the left or the right end of the string. The trace on CP indices is defined as the product of the traces on both ends, e.g.

$$\begin{aligned}\mathrm{tr}(\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta j}\mathbf{F}_{ij}^2) &= \mathrm{tr}[(\gamma_{\Theta i}\otimes \mathbf{1}_{32})(\mathbf{1}_{32}\otimes\gamma_{\Theta j})\{[(Q_i\mathcal{F}_i)^2\otimes\mathbf{1}_{32}]\oplus 2[(Q_i\mathcal{F}_i)\otimes(-Q_j\mathcal{F}_j)]\oplus[\mathbf{1}_{32}\otimes(-Q_j\mathcal{F}_j)^2]\}] \\ &= \mathcal{F}_i^2\mathrm{tr}(\gamma_{\Theta i}Q_i^2)\mathrm{tr}(\gamma_{\Theta j}) - 2\mathcal{F}_i\mathcal{F}_j\mathrm{tr}(\gamma_{\Theta i}Q_i)\mathrm{tr}(\gamma_{\Theta j}Q_j) + \mathcal{F}_j^2\mathrm{tr}(\gamma_{\Theta i})\mathrm{tr}(\gamma_{\Theta j}Q_j^2) \\ &= 0, \end{aligned} \tag{32}$$

where for the last equality we used (30) and (A12). More generally, one has

$$\begin{gathered}\mathrm{tr}(\gamma_{\Theta i}Q_i^{2n-1}W_i^n) = \mathrm{tr}(\gamma_{\Theta i}Q_i^{2n}W_i^{2n}) = \mathrm{tr}(\gamma_i Q_i^{2n-1}W_i^n) = \mathrm{tr}(\gamma_i Q_i^{2n}W_i^{2n-1}) = 0, \\ \mathrm{tr}(\gamma_{\Theta i}Q_i^{2n}W_i^{2n-1}) = \frac{4i}{2^{2n}}, \qquad \mathrm{tr}(\gamma_i Q_i^{2n}W_i^{2n}) = \frac{4}{2^{2n}}, \qquad \mathrm{tr}(\gamma_{\Theta i}W_i^{2n-1}) = 2iN_i, \qquad \mathrm{tr}(\gamma_i W_i^{2n}) = 2N_i. \end{gathered} \tag{33}$$

For the Möbius strip there is only one boundary, so we write

$$\mathbf{F}_i = Q_i\mathcal{F}_i, \qquad \vec{\mathbf{A}}_i = W_i\vec{a}_i \tag{34}$$

without tensor products. To exclude the two overall $U(1)$ factors in the two $U(16)$ gauge groups, as mentioned above, we impose the additional conditions

$$\sum_i N_i\vec{a}_i = \sum_a N_a\vec{a}_a = 0 \tag{35}$$

on the Wilson line moduli. Expanding the amplitudes to leading quadratic order in the background fields, Eq. (A14) in the appendix gives the total one-loop correction

$$\begin{gathered}\tilde{\mathcal{M}}_i = -\pi^{-2}\sqrt{G}\,\mathrm{tr}\Big(\gamma_{\Omega\Theta i}^{-1}\gamma_{\Omega\Theta i}^{\mathrm{T}}\mathbf{F}_i^2\vartheta[^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_i, 8ilG)\Big), \qquad \tilde{\mathcal{M}}_a = -\pi^{-2}\sqrt{G}\,\mathrm{tr}\Big(\gamma_{\Omega a}^{-1}\gamma_{\Omega a}^{\mathrm{T}}\mathbf{F}_a^2\vartheta[^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_a, 8ilG)\Big), \\ \tilde{\mathcal{A}}_{ij} = (16\pi^2)^{-1}\sqrt{G}\,\mathrm{tr}\Big(\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta j}^{-1}\mathbf{F}_{ij}^2\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ij}, 2ilG)\Big), \qquad \tilde{\mathcal{A}}_{ab} = (16\pi^2)^{-1}\sqrt{G}\,\mathrm{tr}\Big(\boldsymbol{\gamma}_{\Theta a}\boldsymbol{\gamma}_{\Theta b}^{-1}, \mathbf{F}_{ab}^2\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ab}2ilG)\Big), \\ \tilde{\mathcal{A}}_{ia} + \tilde{\mathcal{A}}_{ai} = (32\pi^2)^{-1}\sqrt{G}\,\mathrm{tr}\Big((\boldsymbol{\gamma}_i\boldsymbol{\gamma}_a^{-1} + \boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta a}^{-1})\mathbf{F}_{ia}^2\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}2ilG)\Big), \end{gathered} \tag{36}$$

where we set $\alpha' = 1/2$. Here, G is the metric on the torus (15). The sums over string oscillators have collapsed to numbers, due to $\mathcal{N} = 2$ supersymmetry [17] and only KK states originating in the torus reduction from $D = 6$ to $D = 4$ contribute. These contributions appear in the form of Wilson-line-shifted KK momentum sums, that we have written as (genus-two) Jacobi theta functions. Some useful properties of theta functions are collected in appendix C.

One can proceed to directly evaluate the traces in the amplitudes (36) with the help of (33) as in the example (32). This evaluation is straightforward but fairly tedious, and we will not repeat it here. Instead, we will condense the trace evaluation to a simple prescription that is hopefully more transparent. Let us first note that all traces in (36) are effectively over $2N \times 2N$ matrices for some $N \in \{N_i, N_j, N_a, N_b\}$. Moreover, these $2N \times 2N$ matrices are diagonal, and the first N elements on the diagonal are either the same as or the negative of the next N elements. Thus one can express each trace in terms of traces over $N \times N$ matrices that consist of the first N elements of the corresponding $2N \times 2N$ matrix. In the following we denote such a trace with tr_N, and use the same letter for the matrix. With this prescription, we arrive at the following form of the one-loop amplitudes:

$$\begin{aligned}
\tilde{\mathcal{M}}_i &= -\pi^{-2}\sqrt{G}\mathcal{F}_i^2 \mathrm{tr}_{N_i}(\gamma_{\Omega\Theta i}^{-1}\gamma_{\Omega\Theta i}^{\mathrm{T}} Q_i^2)[\vartheta(2\vec{a}_i) + \vartheta(-2\vec{a}_i)] \\
&= -(2\pi^2)^{-1}\sqrt{G}\mathcal{F}_i^2[\vartheta(2\vec{a}_i) + \vartheta(-2\vec{a}_i)], \\
\tilde{\mathcal{M}}_a &= -\pi^{-2}\sqrt{G}\mathcal{F}_a^2 \mathrm{tr}_{N_a}(\gamma_{\Omega a}^{-1}\gamma_{\Omega a}^{\mathrm{T}} Q_a^2)[\vartheta(2\vec{a}_a) + \vartheta(-2\vec{a}_a)] \\
&= -(2\pi^2)^{-1}\sqrt{G}\mathcal{F}_a^2[\vartheta(2\vec{a}_a) + \vartheta(-2\vec{a}_a)], \\
\tilde{\mathcal{A}}_{ij} &= (16\pi^2)^{-1}\sqrt{G}[\mathcal{F}_i^2 \mathrm{tr}_{N_i}(\gamma_{\Theta i} Q_i^2)\mathrm{tr}_{N_j}(\gamma_{\Theta j}^{-1}) + \mathcal{F}_j^2 \mathrm{tr}_{N_i}(\gamma_{\Theta i})\mathrm{tr}_{N_j}(\gamma_{\Theta j}^{-1} Q_j^2)] \\
&\quad \times[\vartheta(\vec{a}_i - \vec{a}_j) + \vartheta(-\vec{a}_i + \vec{a}_j) - \vartheta(\vec{a}_i + \vec{a}_j) - \vartheta(-\vec{a}_i - \vec{a}_j)] \\
&= (32\pi^2)^{-1}\sqrt{G}(\mathcal{F}_i^2 N_j + \mathcal{F}_j^2 N_i)[\vartheta(\vec{a}_i - \vec{a}_j) + \vartheta(-\vec{a}_i + \vec{a}_j) - \vartheta(\vec{a}_i + \vec{a}_j) - \vartheta(-\vec{a}_i - \vec{a}_j)], \qquad (37) \\
\tilde{\mathcal{A}}_{ab} &= (16\pi^2)^{-1}\sqrt{G}[\mathcal{F}_a^2 \mathrm{tr}_{N_a}(\gamma_{\Theta a} Q_a^2)\mathrm{tr}_{N_b}(\gamma_{\Theta b}^{-1}) + \mathcal{F}_b^2 \mathrm{tr}_{N_a}(\gamma_{\Theta a})\mathrm{tr}_{N_b}(\gamma_{\Theta b}^{-1} Q_b^2)][\vartheta(\vec{a}_a - \vec{a}_b) \\
&\quad + \vartheta(-\vec{a}_a + \vec{a}_b) - \vartheta(\vec{a}_a + \vec{a}_b) - \vartheta(-\vec{a}_a - \vec{a}_b)] \\
&= (32\pi^2)^{-1}\sqrt{G}(\mathcal{F}_a^2 N_b + \mathcal{F}_b^2 N_a)[\vartheta(\vec{a}_a - \vec{a}_b) \\
&\quad + \vartheta(-\vec{a}_a + \vec{a}_b) - \vartheta(\vec{a}_a + \vec{a}_b) - \vartheta(-\vec{a}_a - \vec{a}_b)], \\
\tilde{\mathcal{A}}_{ia} + \tilde{\mathcal{A}}_{ai} &= (32\pi^2)^{-1}\sqrt{G}[\mathcal{F}_i^2 \mathrm{tr}_{N_i}(\gamma_i Q_i^2)\mathrm{tr}_{N_a}(\gamma_a^{-1}) + \mathcal{F}_a^2 \mathrm{tr}_{N_i}(\gamma_i)\mathrm{tr}_{N_a}(\gamma_a^{-1} Q_a^2)][\vartheta(\vec{a}_i - \vec{a}_a) + \vartheta(-\vec{a}_i + \vec{a}_a) \\
&\quad + \vartheta(\vec{a}_i + \vec{a}_a) + \vartheta(-\vec{a}_i - \vec{a}_a)] + (32\pi^2)^{-1}\sqrt{G}[\mathcal{F}_i^2 \mathrm{tr}_{N_i}(\gamma_{\Theta i} Q_i^2)\mathrm{tr}_{N_a}(\gamma_{\Theta a}^{-1}) + \mathcal{F}_a^2 \mathrm{tr}_{N_i}(\gamma_{\Theta i})\mathrm{tr}_{N_a}(\gamma_{\Theta a}^{-1} Q_a^2)] \\
&\quad \times[\vartheta(\vec{a}_i - \vec{a}_a) + \vartheta(-\vec{a}_i + \vec{a}_a) - \vartheta(\vec{a}_i + \vec{a}_a) - \vartheta(-\vec{a}_i - \vec{a}_a)] \\
&= (32\pi^2)^{-1}\sqrt{G}(\mathcal{F}_i^2 N_a + \mathcal{F}_a^2 N_i)[\vartheta(\vec{a}_i - \vec{a}_a) + \vartheta(-\vec{a}_i + \vec{a}_a)].
\end{aligned}$$

For notational simplicity, we abbreviated $\vartheta[^{\vec{0}}_{\vec{0}}]$ as ϑ and left the second argument of the theta functions implicit. Moreover, we directly omitted terms that vanish in the non-Abelian case due to the appearance of a single factor of Q_i or Q_a in the trace.

To finish the computation, we want to integrate these expressions over the world-sheet modulus l (see Eq. (A6)). One has to take extra care of massless fields propagating in the loop due to coincident D-branes. In particular, for each of the theta functions in (37) we must distinguish between zero and nonzero argument. When an argument is zero, massless modes appear, and we have to introduce an explicit IR cutoff[23] to regulate the integral over l for the massless states (i.e. the zero modes $\vec{n}^{\mathrm{T}} = (0, 0)$ in the theta function (C1)). If an argument is nonzero, the Wilson lines act as an effective IR cutoff for that integral and all states are massive; then the only contributions from massless states come from the $\mathcal{A}_{ii}$ and $\mathcal{A}_{aa}$ amplitudes. As far as UV divergences are concerned, we know that they all cancel in the end, but it is still useful to introduce a UV cutoff in addition to the IR cutoff, to check that this indeed happens.

Now for the explicit integration, beginning with the case of vanishing first argument of the theta function. Using [18] we have for annulus amplitudes that

$$\begin{aligned}
\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{0}) &= \int_0^{2\Lambda^2} dl\,\vartheta[^{\vec{0}}_{\vec{0}}](\vec{0}, 2ilG)e^{-\pi\chi/l} \\
&= \frac{1}{2\sqrt{G}}\int_{1/\Lambda^2}^{\infty}\frac{dt}{t}\vartheta[^{\vec{0}}_{\vec{0}}](0, itG^{-1})e^{-2\pi\chi t} \\
&= \frac{1}{2\sqrt{G}}(\Lambda^2\sqrt{G} - \ln[8\pi^3\chi\sqrt{G}U_2|\eta(U)|^4]),
\end{aligned} \qquad (38)$$

where $8\pi^3\chi$ corresponds to μ^2 in [18]. This integral is truly divergent for $\chi \rightarrow 0$; this is the usual field-theory IR divergence due to massless modes.

For the nonvanishing first argument, we argued above that there is no need to introduce the IR cutoff χ; the Wilson lines act as IR regulators. Let us check that this works, by keeping χ for now. Using [52] we have

$$\begin{aligned}
\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{a}) &= \int_0^{2\Lambda^2} dl\,\vartheta[^{\vec{0}}_{\vec{0}}](\vec{a}, 2ilG)e^{-\pi\chi/l} \\
&= \frac{1}{2\sqrt{G}}\int_{1/\Lambda^2}^{\infty}\frac{dt}{t}\vartheta[^{\vec{a}}_{\vec{0}}](0, itG^{-1})e^{-2\pi\chi t} \\
&= \frac{1}{2\sqrt{G}}\bigg[-\ln\Big(2\pi\chi + \frac{|A|^2}{\sqrt{G}U_2}\Big) + \Lambda^2\sqrt{G} \\
&\qquad - \ln\frac{\sqrt{G}U_2}{|A|^2} - \ln\bigg|\frac{\vartheta_1(A, U)}{\eta(U)}\bigg|^2 + 2\pi U_2 a_4^2\bigg],
\end{aligned} \qquad (39)$$

[23]Here we mean IR in the open string channel, i.e., cutting off large values of t, cf. (A2), or equivalently small values of $l \sim 1/t$.

where we used the complex Wilson line A introduced in (21), and $U_2 = \mathrm{Im}(U)$. The first logarithm in (39) is the contribution of states with $\vec{n}^{\mathrm{T}} = (0, 0)$, which was the source of the IR divergence in the previous case. It is clear that as long as $|A| \neq 0$, we can set $\chi = 0$ in this first term, and then it cancels against the third term. Thus, provided $|A| \neq 0$, we can remove the IR regulator $\chi = 0$ as promised and find the answer for nonvanishing first argument of the theta function:

$$\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{a}) = \frac{1}{2\sqrt{G}}\bigg(\Lambda^2\sqrt{G} - \ln\left|\frac{\vartheta_1(A, U)}{\eta(U)}\right|^2 + 2\pi U_2 a_4^2\bigg). \tag{40}$$

Before continuing, let us make a quick consistency check. In the form (39), that includes the IR cutoff χ, we could have taken $|A| \to 0$ which should yield (38). To see this, it suffices to know that the other terms in (39) are actually finite in the limit $|A| \to 0$ as can be seen from the expansion (cf. (C6) together with Eq. (C7))

$$\ln\left|\frac{\vartheta_1(A, U)}{\eta(U)}\right|^2 = \ln(2\pi)^2 + \ln|\eta(U)|^4 + \ln|A|^2 - \frac{\pi^2}{3}\mathrm{Re}[E_2(U)A^2] + \mathcal{O}(A^4), \tag{41}$$

where $E_2(U)$ is the holomorphic second Eisenstein series (C8). The logarithmic term $\ln|A|^2$ cancels the third term in (39), the contribution of massive modes is manifestly IR finite, and taking $|A| \to 0$ we recover (38).

To use the same formula for the Möbius strip amplitude, one simply needs to keep track of the different modular transformation, $t = 1/(8l)$ instead of $t = 1/(2l)$ that we use for the annulus, and one finds

$$\begin{aligned}\mathcal{M}_{\mathrm{KK}}(\Lambda^2, \vec{a}) &= \int_0^{2\Lambda^2} dl\,\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](2\vec{a}, 8ilG) \\ &= \frac{1}{8\sqrt{G}}\int_{1/(4\Lambda^2)}^{\infty}\frac{dt}{t}\,\vartheta\big[{}^{2\vec{a}}_{\vec{0}}\big](0, itG^{-1}) \\ &= \frac{1}{4}\mathcal{A}_{\mathrm{KK}}(4\Lambda^2, 2\vec{a}).\end{aligned} \tag{42}$$

These expressions can now be used when integrating (37). We only give the result for the correction to the D5-brane gauge couplings; the correction for D9-branes can be obtained from this via the replacement $(a, b, i) \to (i, j, a)$. It is

$$\begin{aligned}\delta\Big(\frac{4\pi^2}{g_a^2}\Big) = &-2\sqrt{G}\mathcal{A}_{\mathrm{KK}}(4\Lambda^2, 2\vec{a}_a) + \sqrt{G}\sum_{b\neq a}N_b \\ &\times[\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{a}_a - \vec{a}_b) - \mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{a}_a + \vec{a}_b)] \\ &+\sqrt{G}N_a[\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{0}) - \mathcal{A}_{\mathrm{KK}}(\Lambda^2, 2\vec{a}_a)] \\ &+\frac{1}{2}\sqrt{G}\sum_i N_i\mathcal{A}_{\mathrm{KK}}(\Lambda^2, \vec{a}_a - \vec{a}_i).\end{aligned} \tag{43}$$

One important check of this result is that it is UV finite, due to tadpole cancellation. It is easy to carry over the actual check from [17], using (38) and (39). In these two equations it is obvious that the Λ-dependent terms are independent of the Wilson lines, so they drop out of the contribution to (43) from the 55-annulus amplitudes $\mathcal{A}_{ab}$ (given in the second and third row). Moreover, using $\sum_i N_i = 16$, it is also evident that the UV-divergent terms cancel between the Möbius and 95-annulus amplitudes, leaving a UV-finite result in which the cutoff Λ can be taken to infinity. Using (38) and (39) and assuming for concreteness that all $\vec{a}_a$ and $\vec{a}_i$ are distinct and nonzero, the result (43) can finally be expressed as

$$\begin{aligned}\delta\Big(\frac{4\pi^2}{g_a^2}\Big) = &-\frac{1}{2}N_a\ln(8\pi^3\chi\sqrt{G}U_2) + (6 - 3N_a)\ln|\eta(U)| \\ &+\frac{1}{2}\pi U_2\sum_i N_i(a_i)_4^2 + (2 + N_a)\ln|\vartheta_1(2A_a, U)| \\ &-\frac{1}{2}\sum_i N_i\ln|\vartheta_1(A_a - A_i, U)| \\ &+\sum_{b\neq a}N_b\ln\left|\frac{\vartheta_1(A_a + A_b, U)}{\vartheta_1(A_a - A_b, U)}\right|.\end{aligned} \tag{44}$$

This formula is one of the main results of this paper. Note that we have used the extra condition (35) that excludes Wilson lines in the anomalous $U(1)$ factors. The first term is the contribution of massless fields. All the other terms—except the third one—are the real part of a holomorphic function in the variables A and U. We will come back to the interpretation of the third term in Sec. IV.

2. Couplings of Abelian gauge groups

Before we go on to generalize the result to the $\mathcal{N} = 1$ case and draw our conclusions for inflationary models in string theory, let us first, for completeness, also discuss the corrections to the gauge couplings of (nonanomalous) $U(1)$ group factors, although it is not the case relevant for the discussion of the KKLMMT model. Readers who are more interested in the application of our result to that model can therefore skip this subsection.

To deal with the $U(1)$ case, we choose the generators specifying the background fields to lie in the $U(1)$ factors. For each of the $U(N)$ factors in (25) we have one $U(1)$, whose background can be characterized by replacing the

Q_i of (30) with matrices equal to the W_i of (29), i.e. take

$$Q_i = W_i, \qquad Q_a = W_a, \tag{45}$$

and the W_i and W_a defined as before. As the overall two $U(1)$ inside $U(16)_{D9} \times U(16)_{D5}$ are anomalous [49], we also have to impose

$$\sum_i N_i \mathcal{F}_i = 0, \qquad \sum_a N_a \mathcal{F}_a = 0 \tag{46}$$

for the field strengths in this case. Repeating the steps of the non-Abelian case leading to (37), we end up with

$$\begin{aligned}
\tilde{\mathcal{M}}_i &= -\pi^{-2}\sqrt{G}N_i\mathcal{F}_i^2[\vartheta(2\vec{a}_i) + \vartheta(-2\vec{a}_i)], \qquad \tilde{\mathcal{M}}_a = -\pi^{-2}\sqrt{G}N_a\mathcal{F}_a^2[\vartheta(2\vec{a}_a) + \vartheta(-2\vec{a}_a)],\\
\tilde{\mathcal{A}}_{ij} &= (16\pi^2)^{-1}\sqrt{G}N_iN_j(\mathcal{F}_i^2 + \mathcal{F}_j^2)[\vartheta(\vec{a}_i - \vec{a}_j) + \vartheta(-\vec{a}_i + \vec{a}_j) - \vartheta(\vec{a}_i + \vec{a}_j) - \vartheta(-\vec{a}_i - \vec{a}_j)]\\
&\quad + (16\pi^2)^{-1}\sqrt{G}N_iN_j\mathcal{F}_i\mathcal{F}_j[\vartheta(\vec{a}_i - \vec{a}_j) + \vartheta(-\vec{a}_i + \vec{a}_j) + \vartheta(\vec{a}_i + \vec{a}_j) + \vartheta(-\vec{a}_i - \vec{a}_j)],\\
\tilde{\mathcal{A}}_{ab} &= (16\pi^2)^{-1}\sqrt{G}N_aN_b(\mathcal{F}_a^2 + \mathcal{F}_b^2)[\vartheta(\vec{a}_a - \vec{a}_b) + \vartheta(-\vec{a}_a + \vec{a}_b) - \vartheta(\vec{a}_a + \vec{a}_b) - \vartheta(-\vec{a}_a - \vec{a}_b)]\\
&\quad + (16\pi^2)^{-1}\sqrt{G}N_aN_b\mathcal{F}_a\mathcal{F}_b[\vartheta(\vec{a}_a - \vec{a}_b) + \vartheta(-\vec{a}_a + \vec{a}_b) + \vartheta(\vec{a}_a + \vec{a}_b) + \vartheta(-\vec{a}_a - \vec{a}_b)],\\
\tilde{\mathcal{A}}_{ia} + \tilde{\mathcal{A}}_{ai} &= (16\pi^2)^{-1}\sqrt{G}N_iN_a(\mathcal{F}_i^2 + \mathcal{F}_i\mathcal{F}_a + \mathcal{F}_a^2)[\vartheta(\vec{a}_i - \vec{a}_a) + \vartheta(-\vec{a}_i + \vec{a}_a)].
\end{aligned} \tag{47}$$

The main difference to the non-Abelian case appears in the annulus diagrams. Now there are also off-diagonal terms present, mixing different gauge groups, and in the given basis the gauge kinetic terms will no longer be a simple sum of terms for the stacks labeled by i and a, but of the bilinear form

$$-\frac{1}{4}\sqrt{-g_4}\Big(\sum_{ij} g_{ij}^{-2}\mathcal{F}_i\mathcal{F}_j + \sum_{ia} g_{ia}^{-2}\mathcal{F}_i\mathcal{F}_a + \sum_{ab} g_{ab}^{-2}\mathcal{F}_a\mathcal{F}_b\Big). \tag{48}$$

In the non-Abelian case the cross-terms are not allowed by gauge invariance, which is encoded in the fact that $\mathrm{tr}_{N_i}(Q_i) = 0$ for the non-Abelian Q_i of (30), whereas $\mathrm{tr}_{N_i}(Q_i) = N_i$ for the Abelian ones cf. (29), which we use presently. To remove the extra factors of N_i from the gauge couplings, we could redefine the gauge fields by $\sqrt{N_i}$, but here we leave the expressions as they are.

For the terms proportional to $\mathcal{F}_i^2$ and $\mathcal{F}_a^2$ the cancellation of the UV-divergence proceeds in the same way as in the non-Abelian case. On the other hand, for the cancellation in the mixed terms, it is important that we decoupled the anomalous $U(1)$ by imposing (46). The same condition also implies that the mixed terms would be absent for vanishing Wilson lines.

We then derive, for Abelian gauge groups,

$$\begin{aligned}
\delta\Big(\frac{4\pi^2}{g_{ab}^2}\Big) &= \delta_{ab}\Big[-\frac{3}{2}N_a^2\ln(8\pi^3\chi\sqrt{G}U_2) + (12 - 9N_a)N_a\ln|\eta(U)| + \pi U_2 N_a\sum_i N_i(a_i)_4^2 + (4 + N_a)N_a\ln|\vartheta_1(2A_a, U)|\\
&\quad - N_a\sum_i N_i\ln|\vartheta_1(A_a - A_i, U)|\\
&\quad + 2N_a\sum_{c\neq a} N_c\ln\Big|\frac{\vartheta_1(A_a + A_c, U)}{\vartheta_1(A_a - A_c, U)}\Big|\Big] - (1 - \delta_{ab})N_aN_b\ln|\vartheta_1(A_a - A_b, U)\vartheta_1(A_a + A_b, U)|,\\
\delta\Big(\frac{4\pi^2}{g_{ai}^2}\Big) &= -N_aN_i[2\pi U_2(a_a)_4(a_i)_4 + \ln|\vartheta_1(A_a - A_i, U)|].
\end{aligned} \tag{49}$$

There are no summations over repeated indices here; rather, all summations have been written explicitly. Moreover, we have used (35) and (46) in the derivation. Again, the correction to g_{ij}^{-2} can be recovered from δg_{ab}^{-2} by replacing $(a, b, c, i) \to (i, j, k, a)$ and we assume also here that all $\vec{a}_i$ and $\vec{a}_a$ are nonzero and distinct. As before, there is a contribution from massless modes, one term depending on the open string scalars that is not the real part of a holomorphic function in U and A and various others that are.

Let us close this section with two side remarks. We stress again that the result for the couplings of the 9-brane gauge group and the 5-brane group are exactly the same, related just by exchanging indices $a \leftrightarrow i$. This means that, in the case where the gauge group has been broken to the Abelian subgroup, string theory seems to choose a different symplectic section than the one used in the supergravity literature, see e.g. [47,53], where the gauge groups are treated asymmetrically.[24] As we only derived the gauge couplings for the open string vectors and not for the closed string KK vectors, we cannot say precisely what symplectic section is used by string theory. Let us discuss this point in a little more detail. Consider the "tree-level" prepotential of [44] given in (24), when absorbing the $1/(4\pi)$ in the definition of the Wilson line moduli. If one calculates the gauge couplings according to[25]

$$g^{-2}_{\Lambda\Sigma} = \frac{i}{2}(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}),$$
$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\frac{\mathrm{Im}(F_{\Lambda\Xi})\mathrm{Im}(F_{\Sigma\Upsilon})X^{\Xi}X^{\Upsilon}}{\mathrm{Im}(F_{\Xi\Upsilon})X^{\Xi}X^{\Upsilon}}, \tag{50}$$

and performs the symplectic transformation à la [47] one derives the following coupling constants for the open string gauge groups (for notational simplicity we consider only one 9-brane vector A and one 5-brane vector A' here):

$$g^{-2}_{99} = S_2,$$
$$g^{-2}_{55} = S'_2 + 2\frac{S'_2(2U_2S_2 - A'^2_2)(A'^2_1 + A'^2_2) + (A'^2_2 - A'^2_1)S_2A^2_2}{(2U_2S_2 - A'^2_2)^2},$$
$$g^{-2}_{95} = 2\frac{A_2A'_2S_2}{2U_2S_2 - A'^2_2}. \tag{51}$$

Obviously, the 55 and 99 gauge couplings are rather different and therefore correspond to a different symplectic section as chosen by the string in our background field calculation. In the absence of charged fields, the two choices of symplectic section lead to an equivalent set of equations of motion and Bianchi identities, see e.g. [54,55]. In the closed string sector such charged fields would arise e.g. through gauging the theory by turning on background fluxes [47,56].

Finally, let us make a remark about the "tree-level" approximation using the prepotential (24). The term "tree-level" is used in analogy to the perturbative heterotic string, where the last term of (24) is absent. In the heterotic theory S' is usually called T and is a Kähler modulus that is independent of the ten-dimensional dilaton, i.e., the string coupling constant g_s. Thus both terms in the heterotic analog of (24) have the same dependence on the dilaton and their sum does, in fact, correspond to a consistent tree-level truncation cf. [54,55]. Here, however, both S_2 and S'_2 depend on the dilaton, as is obvious from (16), and therefore the first and the second two terms of (24) have a different dilaton dependence. This fact, that there are two independent gauge couplings S_2 and S'_2 which should both be large in perturbation theory, raises the question in which sense it is possible in open string perturbation theory to truncate the prepotential to the "tree-level" terms of (24). To see the problem more explicitly, consider, for example, the 55 gauge coupling in (51). Expanding the second term in the couplings S_2 and S'_2 gives to leading order

$$\frac{S'_2(A'^2_1 + A'^2_2)}{S_2U_2}. \tag{52}$$

This term is of order $\mathcal{O}(g^0_s)$. A term of the same order would, however, be generated, e.g., by a "one-loop"-correction to the prepotential of the form[26]

$$\delta\mathcal{F} \sim UA'^2. \tag{53}$$

We call this "one-loop" because it is neither multiplied by S nor S'. Taking such a term into account and performing the same symplectic transformation as in [47] leads to an additional contribution to the 55 gauge coupling of order $\mathcal{O}(g^0_s)$ and proportional to U_2. Thus it is doubtful whether a truncation to the prepotential (24) and the form (51) of the gauge couplings would be consistent in string perturbation theory, since it does not appear to correspond to a systematic expansion of the effective Lagrangian (in particular, of the gauge couplings) in powers of the string coupling. For example, only some terms of the order $\mathcal{O}(g^0_s)$ would be included, others left out.

We do not claim that previous literature on the subject is wrong; we merely wish to emphasize that "tree-level" should not be taken too literally.

D. Generalization to $\mathcal{N} = 1$

In this section we want to generalize our results to the case of interest with only $\mathcal{N} = 1$ supersymmetry. In terms of toroidal orientifold models, we will use a background $\mathbb{T}^6/\mathbb{Z}_N$ or $\mathbb{T}^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$. In order to be able to employ the results of [18], we first concentrate on the $\mathbb{Z}'_6$ orientifold. Another reason for choosing this model is that the discussion of Wilson lines is rather similar to the one in the $\mathbb{Z}_6$ orientifold given in [57]. It turns out that the one-loop corrections to the gauge kinetic function (and thus to the nonperturbative superpotential) do not contain any terms quadratic in the Wilson line moduli and thus cannot help to reduce the inflaton mass in a KKLMMT-like scenario. To show that

[24] Note that the coordinates used in [47] are related to ours in the following way: $s \leftrightarrow S'$, $t \leftrightarrow U$, $u \leftrightarrow S$, $x^k \leftrightarrow A_a$ (resp. A' in (51)), $y^r \leftrightarrow A_i$ (resp. A in (51)).

[25] We refer again to [12] for notations and conventions on $\mathcal{N} = 2$ gauge couplings. Note, though, that we include an additional factor of 2 in the definition of the couplings as compared to [12], in order to get the same normalization in the relations $g^{-2}_{99} = \mathrm{Im}(S)$, etc., as in Sec. III B cf. (51) below.

[26] The argument does not depend on the particular form chosen here. Any $\delta\mathcal{F} \sim f_1(U)f_2(A)A'^n$, with $n \geq 2$ and f_1, f_2 some arbitrary functions, would lead to the same conclusion. Comparing this with (49), it is obvious that such terms indeed do appear at one-loop level.

this is not a generic problem, we also consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model of [24]. We do not go into the details as much as in the $\mathbb{Z}_6'$ case but our results show that the one-loop corrections in this model are capable to lower the inflaton mass by fine-tuning.

1. The $\mathbb{Z}_6'$ model

This orientifold is defined in terms of the eigenvalues $\exp(2\pi i v)$, $v = (1, -3, 2)/6$, of the generator Θ, acting on the three complex coordinates of a $\mathbb{T}^6 = \mathbb{T}_1^2 \times \mathbb{T}_2^2 \times \mathbb{T}_3^2$. The first and third of the three 2-tori are assumed to allow a crystallographic $\mathbb{Z}_6$ and $\mathbb{Z}_3$ operation, respectively. Since Θ^3 is just identical to the geometric operation of the generator of the $\mathbb{Z}_2$ we considered in the previous section on $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{T}^2$, the $\mathbb{Z}_6'$ model includes 32 D5-branes extended along the third 2-torus $\mathbb{T}_3^2$, in addition to the 32 space-time filling D9-branes. The moduli space of the untwisted moduli is given by three copies of $SU(1, 1)/U(1)$ for each of the three generic Kähler moduli of the three 2-tori, and one extra copy each for the dilaton and the only complex structure modulus of the model, the complex structure of the second 2-torus [58]. The complex structure of the third 2-torus $\mathbb{T}_3^2$, that we denote by U, will actually appear in the one-loop correction to the gauge couplings discussed below, but this is not a modulus, since it is fixed to a rational value by compatibility with the orbifold action.

For application to the KKLMMT model, we are mainly interested in the dependence of the 5-brane gauge couplings on the Wilson line moduli of the 9-branes along their common world volume directions, which is the third torus. This dependence is completely contained in the 95 annulus. Moreover, the only sectors that can depend on Wilson lines along the third torus are those in which this torus is left invariant, i.e., those with insertions of the identity or Θ^3. The relevant amplitudes are given in (B2) of appendix B, where we also give all the other amplitudes for completeness. The important point that allows to reduce much of the calculation to the $\mathcal{N} = 2$ case of the previous section is the fact that the amplitudes in the sectors with insertions Θ^k, $k = 0, 3$, are formally identical to those arising in the case of $\mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$. Because of the fact that the element Θ^3 of the orbifold group is exactly the same as the $\mathbb{Z}_2$ generator in the $\mathcal{N} = 2$ case discussed above, the result formally exactly carries over to the case at hand, up to an overall factor that we will determine.

T-duality along all six internal directions again maps 9- and 5-branes to 3- and 7-branes, localized on the third torus. It is clear that these now have to be moved in sets of six at least: the orbifold generator Θ identifies three of them and the T-dual world sheet parity $\Omega R(-1)^{F_L}$ acts geometrically as a reflection on the 2-torus, and thus identifies these three with another set of three images. When analyzing the allowed Wilson lines in the next subsection, we will use this geometric intuition of moving sets of six branes.

Thus, the most important difference to the $\mathcal{N} = 2$ case is that the gauge group and the allowed Wilson lines are different. The latter have to be compatible with the operation of the orbifold generator on the third torus, while in the previous section, the Wilson lines were turned on in a 2-torus that was invariant under the orbifold action. Thus, here in the $\mathcal{N} = 1$ case we have to go into some detail to solve this compatibility condition. As usual, the action of Θ^k on the CP labels is encoded in a 32×32 matrix γ_{Θ^k}. Without Wilson lines, the tadpole cancellation conditions for twisted tadpoles are [23]

$$\mathrm{tr}(\gamma_{\Theta^k 9}) = \mathrm{tr}(\gamma_{\Theta^k 5}) = 0, \qquad k = 1, 3, 5,$$
$$\mathrm{tr}(\gamma_{\Theta^2 9}) = \mathrm{tr}(\gamma_{\Theta^2 5}) = -8, \qquad \mathrm{tr}(\gamma_{\Theta^4 9}) = \mathrm{tr}(\gamma_{\Theta^4 5}) = 8 \tag{54}$$

and the solution with the maximal gauge group is given by

$$\gamma_{\Theta 9} = \gamma_{\Theta 5} = \mathrm{diag}(\beta \mathbf{1}_4, \beta^5 \mathbf{1}_4, \beta^9 \mathbf{1}_8, \bar{\beta} \mathbf{1}_4, \bar{\beta}^5 \mathbf{1}_4, \bar{\beta}^9 \mathbf{1}_8) \tag{55}$$

and $\gamma_{\Theta^k 9} = \gamma_{\Theta 9}^k$, $\gamma_{\Theta^k 5} = \gamma_{\Theta 5}^k$, where we used $\beta = e^{i\pi/6}$. This choice of matrices γ implies the gauge group

$$[U(4)^2 \times U(8)]_{\mathrm{D9}} \times [U(4)^2 \times U(8)]_{\mathrm{D5}}. \tag{56}$$

We will see in the next subsection how this gauge group can be broken by turning on continuous Wilson lines.

2. Wilson lines in the $\mathbb{Z}_6'$ orientifold

The classification of Wilson lines in the $\mathbb{Z}_6'$ orientifold has not been considered in the literature so far. However, our discussion will be very similar to that for the $\mathbb{Z}_6$ orientifold [57], and we will be able to make use of the results for $\mathbb{Z}_3$ [59] as well. To introduce Wilson lines it is convenient to reorder the blocks in $\gamma_{\Theta 9}$ in the following way (by abuse of notation we still use $\gamma_{\Theta 9}$ after the reordering):

$$\gamma_{\Theta 9} = \mathrm{diag}(\beta \mathbf{1}_{4-n_9}, \beta^5 \mathbf{1}_{4-n_9}, \beta^9 \mathbf{1}_{8-n_9}, \bar{\beta} \mathbf{1}_{4-n_9}, \bar{\beta}^5 \mathbf{1}_{4-n_9}, \bar{\beta}^9 \mathbf{1}_{8-n_9}, \gamma_{\Theta 9}^{[6n_9]}) \tag{57}$$

with

$$\gamma_{\Theta 9}^{[6n_9]} = \mathrm{diag}(\beta, \beta^5, \beta^9, \bar{\beta}, \bar{\beta}^5, \bar{\beta}^9) \otimes \mathbf{1}_{n_9} = \gamma_{\Theta 9}^{[6]} \otimes \mathbf{1}_{n_9} \tag{58}$$

and similarly for the 5-branes, using n_5 and $\gamma_{\Theta 5}^{[6]}$. (We use bracketed superscripts to denote the size of the matrix). Obviously, $n_5, n_9 \leq 4$ has to hold. The most general ansatz for the Wilson lines that leaves (at least) the gauge group

$$[U(4-n_9)^2 \times U(8-n_9) \times U(n_9)]_{\rm D9} \times [U(4-n_5)^2 \times U(8-n_5) \times U(n_5)]_{\rm D5} \tag{59}$$

intact is

$$\begin{aligned} \gamma_{W9} &= \mathrm{diag}(\mathbf{1}_{32-6n_9}, \gamma_{W9}^{[6]} \otimes \mathbf{1}_{n_9}), \\ \gamma_{W5} &= \mathrm{diag}(\mathbf{1}_{32-6n_5}, \gamma_{W5}^{[6]} \otimes \mathbf{1}_{n_5}). \end{aligned} \tag{60}$$

In order for the matrices γ_W to describe Wilson lines along the third 2-torus, they have to satisfy three conditions. Tadpole cancellation has to remain fulfilled, they need to be compatible with the orientifold operation on the third 2-torus, and finally they have to be unitary. The tadpole constraints are satisfied if [57,59]

$$\begin{aligned} \mathrm{tr}[(\gamma_{\Theta 9}^{[6]})^k (\gamma_{W9}^{[6]})^p] = \mathrm{tr}[(\gamma_{\Theta 5}^{[6]})^k (\gamma_{W5}^{[6]})^p] = 0, \\ k = 1, 2, 4, 5, \qquad p = 0, 1, 2. \end{aligned} \tag{61}$$

Note that there is no condition for $k = 3$ because Θ^3 acts as the identity on the third torus. Further, consistency conditions for the Wilson line to be compatible with the action of Θ on the third 2-torus have to be satisfied,

$$\begin{aligned} (\gamma_{\Theta 9}^{[6]} \gamma_{W9}^{[6]})^6 &= -\mathbf{1}_6, \qquad (\gamma_{\Theta 5}^{[6]} \gamma_{W5}^{[6]})^6 = -\mathbf{1}_6, \\ [(\gamma_{\Theta 9}^{[6]})^2 \gamma_{W9}^{[6]}]^3 &= -\mathbf{1}_6, \qquad [(\gamma_{\Theta 5}^{[6]})^2 \gamma_{W5}^{[6]}]^3 = -\mathbf{1}_6. \end{aligned} \tag{62}$$

The T-dual geometrical interpretation goes as follows: Originally, with the maximal gauge group given above, all 32 + 32 D7- and D3-branes are located at the origin. One can then move six in a $\mathbb{Z}_3 \times \mathbb{Z}_2$ invariant fashion, two sets of three being identified under the T-dual world sheet parity, and the elements of each set of three are identified under Θ, leaving just a single independent brane. Moving n_9 coinciding sets of 6 D3-branes then leaves a $U(n_9)$ on the mobile stack, while reducing the rank of the total gauge group by $3n_9 - n_9 = 2n_9$. The tadpole consistency requires that one takes one brane each from the three sets that made up $[U(4) \times U(4) \times U(8)]_{\rm D9}$, which explains the breaking pattern. Guided by the geometrical intuition that the Wilson line that corresponds to the T-dual separation of D3-branes from the origin should reflect the fact that there are two triplets of branes, which are separately identified under Θ, but not mixed, we now make an ansatz, where the Wilson line is block diagonal in 3×3 blocks, i.e., we choose

$$\gamma_{W9}^{[6]} = \mathrm{diag}(\gamma_{W9}^{[3]}, \bar{\gamma}_{W9}^{[3]}). \tag{63}$$

Here $\bar{\gamma}_{W9}^{[3]}$ is the complex conjugate of $\gamma_{W9}^{[3]}$. For the blocks, we adopt the form of the most general Wilson line consistent with a $\mathbb{Z}_3$ twist [59][27]

$$\gamma_{W9}^{[3]} = b_1 \mathbf{1}_3 + b_2 \zeta + b_3 \zeta^2, \tag{64}$$

having defined the permutation matrices of three elements via

$$\zeta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \zeta^2 = \zeta^{\rm T}. \tag{65}$$

The three coefficients b_i are *a priori* free complex parameters. This choice automatically satisfies the tadpole constraints (61). Evaluating the consistency conditions (62), one finds

$$\begin{aligned} (\gamma_{\Theta 9}^{[6]} \gamma_{W9}^{[6]})^6 &= -\mathbf{1}_6 (b_1^3 + b_2^3 + b_3^3 - 3b_1 b_2 b_3)^2, \\ [(\gamma_{\Theta 9}^{[6]})^2 \gamma_{W9}^{[6]}]^3 &= -\mathbf{1}_6 (b_1^3 + b_2^3 + b_3^3 - 3b_1 b_2 b_3). \end{aligned} \tag{66}$$

Upon diagonalizing the matrix ζ via the unitary transformation

$$P_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}, \tag{67}$$

$\alpha = e^{2\pi i/3}$, one has

$$P_3 \gamma_{W9}^{[3]} P_3^\dagger = \mathrm{diag}(b_1 + b_2 + b_3, b_1 + b_2\alpha^2 + b_3\alpha, b_1 + b_2\alpha + b_3\alpha^2). \tag{68}$$

In the diagonal form, unitarity is most easily imposed, and implies that the three diagonal elements are just phases. The extra condition $b_1^3 + b_2^3 + b_3^3 - 3b_1 b_2 b_3 = 1$ means that the determinant has to be one as well, so that we can finally write

$$P_3 \gamma_{W9}^{[3]} P_3^\dagger = \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{-i(\varphi_1 + \varphi_2)}). \tag{69}$$

This provides an explicit parametrization in terms of two periodic variables, which are related to the T-dual positions of the branes on the 2-torus. In order to implement the Wilson line in the open string KK spectrum as shifts of momenta, we define $\vec{a}$ through

[27]Despite such claims in the literature, this does not seem to imply that the Wilson lines in the $\mathbb{Z}_6'$ model on the third torus, where Θ is of order 3, are fully classified by the solution for $\mathbb{Z}_3$.

$$\varphi_1 = \vec{e}\,\vec{a}, \qquad \varphi_2 = \vec{e}\vec{a}^{\Theta}, \qquad -\varphi_1 - \varphi_2 = \vec{e}\vec{a}^{\Theta^2}, \tag{70}$$

where $\vec{a}$ is the Wilson line on the first D9-brane, i.e., the T-dual of the D3-brane position, and $\vec{a}^{\Theta}$ and $\vec{a}^{\Theta^2}$ are its images under the orbifold generator, acting on the third 2-torus. Explicitly, the action is $(a_4, a_5)^{\Theta} = (-a_5, a_5 - a_4)$. Moreover, $\vec{e}$ can be chosen to be one of the basic lattice vectors $(\vec{e}_4, \vec{e}_5)$. Thus the complete Wilson-line on the 9-branes is given by

$$P_{32}\gamma_{W9}P_{32}^{\dagger} = \mathrm{diag}(\mathbf{1}_{32-6n_9}, \gamma_{W9}^{[6n_9]}),$$
$$\gamma_{W9}^{[6n_9]} = \mathrm{diag}(e^{i\vec{e}\,\vec{a}}, e^{i\vec{e}\vec{a}^{\Theta}}, e^{i\vec{e}\vec{a}^{\Theta^2}}, e^{-i\vec{e}\,\vec{a}}, e^{-i\vec{e}\vec{a}^{\Theta}}, e^{-i\vec{e}\vec{a}^{\Theta^2}}) \otimes \mathbf{1}_{n_9} \tag{71}$$

with $P_{32} = \mathbf{1}_{32-6n_9} \oplus (P_3 \otimes \mathbf{1}_{2n_9})$. In the T-dual picture this describes $6n_9$ mobile D3-branes at positions given through $\pm\vec{a}^{\Theta^n}$, $n = 0, 1, 2$, and supporting a mobile $\mathrm{U}(n_9)$ gauge group.[28] This is depicted in Fig. 3. The points labeled by $\vec{a}$ have coordinates $a_I\vec{e}^I$, the $\vec{e}^I$ being the basis of the dual lattice, as in (19).

In this basis, where the Wilson line is diagonal, the operation of the orbifold no longer is, except for

$$P_6(\gamma_{\Theta 9}^{[6]})^3 P_6^{\dagger} = (\gamma_{\Theta 9}^{[6]})^3 = P_6(\gamma_{\Theta 5}^{[6]})^3 P_6^{\dagger} = (\gamma_{\Theta 5}^{[6]})^3 = \mathrm{diag}(i\mathbf{1}_3, -i\mathbf{1}_3), \tag{72}$$

where we have defined $P_6 = P_3 \otimes \mathbf{1}_2$, acting block-diagonally. This is, however, all we need to evaluate the amplitudes in the $k = 0, 3$ sectors explicitly. This matrix is identical to the matrix representation (A9) of the $\mathbb{Z}_2$ generator in the $\mathcal{N} = 2$ model of the previous chapter. In the basis where the Wilson line is diagonal, we expect the orbifold generator to act in a way on the CP labels that matches with our geometrical intuition. Indeed, one finds that

$$P_6\gamma_{\Theta 9}^{[6]}P_6^{\dagger} = \mathrm{diag}(\beta\zeta, \bar{\beta}\zeta^2). \tag{73}$$

Thus, $\gamma_{\Theta 9}$ really just permutes the three CP labels of each of the two sets separately, as expected.

To determine the matrix representation $\lambda = \mathrm{diag}(\mathbf{0}_{32-6n_9}, \lambda^{[6n_9]})$ of the surviving, say, 9-brane gauge fields in the mobile $U(n_9)$, one has to regard the projections

$$\lambda^{[6n_9]} = \gamma_{W9}^{[6n_9]}\lambda^{[6n_9]}(\gamma_{W9}^{[6n_9]})^{-1},$$
$$\lambda^{[6n_9]} = \gamma_{\Theta 9}^{[6n_9]}\lambda^{[6n_9]}(\gamma_{\Theta 9}^{[6n_9]})^{-1}. \tag{74}$$

This leads to gauge fields represented by CP matrices

[28]Note that the lattice and the dual lattice are exchanged via T-duality.

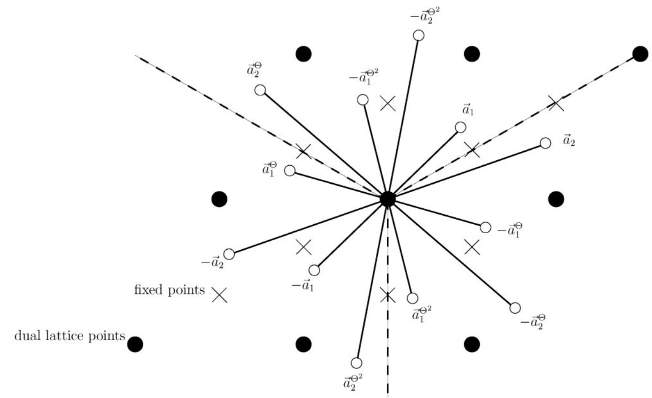

FIG. 3. $\mathbb{Z}_3$-symmetric Wilson lines in $\mathbb{Z}_6'$

$$\lambda^{[6n_9]} = \mathrm{diag}[\mathbf{1}_3 \otimes Q_{n_9}, \mathbf{1}_3 \otimes (-Q_{n_9})], \tag{75}$$

where Q_{n_9} is an arbitrary $n_9 \times n_9$ matrix in the adjoint of $U(n_9)$. A field strength in the Cartan subalgebra would now be given by e.g.

$$Q_{n_9} = \frac{1}{2\sqrt{3}}\mathrm{diag}(1, -1, 0, \ldots, 0), \tag{76}$$

where we included the factor of $1/\sqrt{3}$ in order to normalize $\mathrm{tr}(\lambda^{[6n_9]})^2 = 1$. All the above works analogously for the D5-branes, respectively, their T-dual D7-branes.

3. Results for $\mathbb{Z}_6'$

In this subsection we would like to use the above insights into the breaking of the gauge group to determine the dependence of the 5-brane gauge couplings on the 9-brane scalars, i.e., we consider the case with vanishing Wilson lines on the 5-branes so that their gauge group is the unbroken $[U(4) \times U(4) \times U(8)]_{\mathrm{D5}}$. Moreover, being interested in the 5-brane gauge couplings, we only consider a background for the 5-brane gauge fields, specified by a matrix similar to (30), where the position of the nonvanishing entries depends on the gauge group factor according to $Q_{U(4)_1} = \frac{1}{2}\mathrm{diag}(1, -1, 0^{14}, -1, 1, 0^{14})$, $Q_{U(4)_2} = \frac{1}{2}\mathrm{diag}(0^4, 1, -1, 0^{14}, -1, 1, 0^{10})$ and $Q_{U(8)} = \frac{1}{2}\mathrm{diag}(0^8, 1, -1, 0^{14}, -1, 1, 0^6)$, where the ordering is chosen to be consistent with (55). Furthermore, we only give the dependence on the 9-brane scalars. The full gauge couplings could be extracted from the formulas given in [18] without much more difficulty. An important point to mention is that, as opposed to the $\mathcal{N} = 2$ case, it is no longer possible to go to an Abelian limit (the Coulomb branch) by just turning on Wilson lines of the specified type, since a remnant non-Abelian $U(4)_{\mathrm{D9}} \times U(4)_{\mathrm{D5}}$ cannot be broken this way. In the T-dual version this implies that one cannot move all the branes away from the origin.

Recall that the two amplitudes of interest for us are formally given by the 95 annulus of the $\mathcal{N} = 2$ case discussed above, up to an overall factor of 1/3. The main difference of the two cases is the fact that the Wilson lines in the $\mathcal{N}=1$ model have to be consistent with the orbifold operation, which amounts to moving the T-dual D3-branes in groups of multiples of six. In other words, for any Wilson line $\vec{a}$ one has to turn on $\vec{a}^{\Theta}$ and $\vec{a}^{\Theta^2}$ at the same time. In addition, as in the $\mathcal{N}=2$ case, also the negative of these values appear due to the world sheet parity projection.

Thus we can just copy the result from the last line of (37) including the different normalization factor to get

$$\tilde{\mathcal{A}}^{(0)}_{ia} + \tilde{\mathcal{A}}^{(0)}_{ai} + \tilde{\mathcal{A}}^{(3)}_{ia} + \tilde{\mathcal{A}}^{(3)}_{ai} = \frac{1}{3}(32\pi^2)^{-1}\sqrt{G}\mathcal{F}^2_a n_i \sum_{m=0}^{2}[\vartheta(-\vec{a}_i^{\Theta^m}) + \vartheta(\vec{a}_i^{\Theta^m})], \tag{77}$$

where we allowed for various mobile stacks now, using n_i instead of n_9 as before. As we already mentioned, the total sum of all n_i is limited to four now. Moreover, in (77) we only wrote down the amplitudes for 9-branes with a nonvanishing Wilson line. Other D9-branes will only contribute universal terms, independent of the open string scalars. From (77) we can read off the dependence of the 5-brane gauge couplings on the 9-brane scalars according to

$$\delta\left(\frac{4\pi^2}{g_a^2}\right) = \frac{1}{6}\pi U_2 \sum_i n_i \sum_{m=0}^{2}(a_i^{\Theta^m})_4^2 - \frac{1}{6}\sum_i n_i \sum_{m=0}^{2}\ln|\vartheta_1(A_i^{\Theta^m}, U)| + \ldots, \tag{78}$$

where the dots stand for correction terms that are independent of the Wilson line moduli. We now note that the Wilson line moduli on the images under Θ are related by multiplication with a phase, i.e. $A^{\Theta} = e^{2\pi i/3}A$, which can be verified using the action of Θ on $\vec{a}$ given above Eq. (71).[29] This implies that $A^{2m} + (A^{\Theta})^{2m} + (A^{2\Theta})^{2m} = 0$ for all integers m that are not multiples of 3. Thus, using the fact that ϑ_1 is an odd function in A, we see that for small $|A|$

$$\delta\left(\frac{4\pi^2}{g_a^2}\right) = \frac{1}{3}\pi U_2 \sum_i n_i[(a_i)_4^2 + (a_i)_5^2 - (a_i)_4(a_i)_5] - \frac{1}{2}\sum_i n_i \ln|A_i| + \mathcal{O}(A^6) + \ldots. \tag{79}$$

The terms quadratic in the A_i have canceled out. Since the A_i are the candidate fields for the inflaton in the T-dual setting with D3-branes, this implies that the above gauge kinetic function would produce no extra contribution to the inflaton mass. This is not generic in $\mathcal{N}=1$ orientifolds, but an accidental consequence of the global $\mathbb{Z}_3$ symmetry of the Wilson lines, as we shall demonstrate in the following section.

[29]This mapping under Θ ensures that in the $\mathcal{N}=1$ analog of the Abelian gauge coupling g_{ai}^{-2} (49), the first term on the right-hand side drops out when summing over orbits of Θ. This is necessary to guarantee that the gauge coupling is the real part of a holomorphic function.

4. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ model

We would now like to discuss another Calabi-Yau orientifold model with $\mathcal{N}=1$ supersymmetry, the type IIB $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$ orientifold [24]. We have not worked out all the details, but we intend to stress its qualitative features here. In particular, we shall point out why the one-loop correction in the $\mathbb{Z}_2\times\mathbb{Z}_2$ model relevant for the correction of the inflaton mass does not cancel out, as it did in the previous section for $\mathbb{Z}_6'$.[30]

This orientifold is defined by the action of the three orbifold group elements Θ_p, $p = 1, 2, 3$, on $\mathbb{T}^6 = \mathbb{T}_1^2 \times \mathbb{T}_2^2 \times \mathbb{T}_3^2$, where each Θ_p is a reflection along two 2-tori, leaving the torus $\mathbb{T}_p^2$ invariant. The orbifold symmetry does not impose any further requirements on the background tori, such that all three complex structure moduli remain in the spectrum, together with the generic three complexified Kähler parameters. Thus, the untwisted moduli space consists of a total of six copies of $SU(1,1)/U(1)$ plus one for the dilaton [58]. When the 3-form fluxes are turned on, the complex structure and the string coupling are assumed to get fixed. But compared to the previous $\mathbb{Z}_6'$ example, the modulus U that appears in the relevant one-loop corrections to the gauge coupling is not fixed universally by the orbifold symmetry, and can, without fluxes, take any value. In analogy with the $\mathbb{Z}_2$ K3-orientifold there are four types of untwisted tadpole divergences in the Klein bottle, canceled by 32 D9-branes plus three sets of 32 D5_p-branes each, wrapped around $\mathbb{T}_p^2$ respectively. Together they support the maximal gauge symmetry $Sp(8)_{D9}\times Sp(8)_{D5_1}\times Sp(8)_{D5_2}\times Sp(8)_{D5_3}$.[31]

Again, we are now only interested in that part of the one-loop amplitude that depends on the Wilson lines on the D9-branes and on the gauge field background on one of the three stacks of D5_p-brane. The latter is T-dual to the stack of D7-branes that undergoes gaugino condensation, while the other D5-branes are ignored for the moment. It is evident that again, the only relevant amplitudes are

[30]We hope to give a more complete analysis of this example in a forthcoming publication [60].

[31]This phenomenologically less interesting gauge group was actually one of the main reasons to concentrate on the $\mathbb{Z}_6'$ orientifold in the first place. It has, however, also been argued that the gauge group may be changed to a group of unitary factors in the presence of discrete torsion [23]. The $\mathbb{Z}_2\times\mathbb{Z}_2$ orientifold was also the starting point of constructing supersymmetric intersecting brane models in [61], which do possess unitary gauge symmetries plus chiral matter. Hence, it may turn out that this model allows for better phenomenology than the standard solution with symplectic gauge groups suggests.

$$\tilde{\mathcal{A}}^{(0)}_{95_p} + \tilde{\mathcal{A}}^{(0)}_{5_p9} + \tilde{\mathcal{A}}^{(\Theta_p)}_{95_p} + \tilde{\mathcal{A}}^{(\Theta_p)}_{5_p9}, \tag{80}$$

where the upper index (Θ_p) stands for the insertion of Θ_p in the trace, and (0) for the identity as before. Formally, i.e., up to the concrete charge matrices to be used in evaluating the traces, this amplitude is again identical to the last equation of (37) up to a different overall normalization, which this time is $1/4$ compared to $1/2$ (or $1/6$ in the $\mathbb{Z}_6'$ case). Now using the solution of [24] for the operation of the orientifold group elements on the CP indices, one can pick matrices $\gamma_{\Theta_p 9}$ and $\gamma_{\Theta_p 5_p}$ with eigenvalues $\pm i$, that after diagonalizing become again identical to $\gamma_{\Theta 9}$ and $\gamma_{\Theta 5}$ of the $\mathbb{Z}_2$ K3-orientifold as given in (A9) with $N = 16$.[32] Given this, we now would have to determine the consistent forms of Wilson lines on the p-th 2-torus, defined by matrices γ_{W9p}, along the same lines as for $\mathbb{Z}_6'$, and find the patterns of gauge symmetry breaking.[33] However, for the time being, we will not go through the procedure explicitly, leaving it to future work [60], and simply follow geometric intuition. Thus, we just use the analog of (77), but now summing over the images of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $\mathbb{Z}_6'$.

The two orbifold elements Θ_q, $q \neq p$, only act by reflection on $\mathbb{T}^2_p$, and so on $\vec{a}_i$. Thus the final form of the relevant one-loop correction reads

$$\tilde{\mathcal{A}}^{(0)}_{95_p} + \tilde{\mathcal{A}}^{(0)}_{5_p9} + \tilde{\mathcal{A}}^{(p)}_{95_p} + \tilde{\mathcal{A}}^{(p)}_{5_p9} = \frac{1}{2}(32\pi^2)^{-1}\sqrt{G}\mathcal{F}_a^2 n_i \sum_{q\neq p}\sum_{m=0}^{1}[\vartheta(-\vec{a}_i^{\Theta_q^m}) + \vartheta(\vec{a}_i^{\Theta_q^m})], \tag{81}$$

where now $\vec{a}_i^{\Theta_q} = -\vec{a}_i$. Since the theta function is even, all contributions add up. In particular, the outcome is identical to our result (37) for the $\mathcal{N} = 2$ model, up to an overall numerical factor. The expression for the correction to the gauge coupling on the stack of D5-branes labeled by a then reads

$$\delta\left(\frac{4\pi^2}{g_a^2}\right) = \pi U_2 \sum_i n_i (a_i)_4^2 - \sum_i n_i \ln|\vartheta_1(A_i, \mathrm{U})| + \dots \tag{82}$$

Unlike in the $\mathbb{Z}_6'$ model, the terms quadratic in A_i do not cancel out. This leads to a new contribution to the inflaton mass, as we will discuss in the next section.

[32] See equation (4.7) and the table above (4.15) in [24]. Pick $p = 1$ which implies $\gamma_{\Theta_p 9} = \gamma_{\Theta_p 5_p} = -M_1$ with the claimed property $M_1^2 = -\mathbf{1}_{32}$.

[33] In [24] it was already argued that moving D5-branes out of the fixed points of the orbifold group would still lead to symplectic gauge groups on the various stacks.

IV. INTERPRETATION

In this section we would like to interpret our results of Sec. III and draw some conclusions for inflationary models in string theory. For readers who decided to skip Sec. III, a few key facts from that section will be repeated here. We state our results for the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ cases in turn. For $\mathcal{N} = 2$ we find a clean solution to the rho problem and for $\mathcal{N} = 1$ we describe the implications of our results for the inflaton mass problem. Let us also remind the reader that, as already stated in the introduction, the results have to be understood as giving a qualitative picture. Our toy models are not close enough to the actual KKLMMT model to allow for reliable quantitative predictions (e.g. our calculation of the one-loop corrections to the gauge kinetic function of the 7-branes in Sec. III neglected the warp factor and the fluxes; it would be very nice, but with present techniques very difficult, to perform our calculation in a more realistic setting). We will come back to this issue at the very end of this section.

A. $\mathcal{N} = 2$

Let us start by reviewing the $\mathcal{N} = 2$ model that we discussed in Secs. III B and III C. Readers who have gone through these sections can skip the following paragraph.

The model under consideration is the type IIB $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{T}^2$ orientifold [19–21], i.e., we consider an orbifold limit of K3 $\times\, \mathbb{T}^2$. It contains 32 D9-branes and 32 D5-branes, wrapped around the torus $\mathbb{T}^2$. This leads to a gauge group $SU(16)_{\mathrm{D9}} \times SU(16)_{\mathrm{D5}}$ if all 5-branes are at the origin of the $\mathbb{T}^4$ and there are no Wilson lines on the 9-branes. The closed string spectrum contains hypermultiplets and vector multiplets but for our purposes we can restrict to vector multiplets only. In addition to those from the open string sector, there are three vector multiplets from the closed string sector, whose complex scalars are given by the complex structure modulus of the torus, U, and the two scalars S and S', given in (16) for the case of vanishing Wilson line moduli and in (20) for the case with corrections due to nonvanishing Wilson line moduli [43,44]. The scalars in the vector multiplets of the open string sector, on the other hand, are given by the Wilson line moduli on the 5- and 9-branes along the torus and are defined according to (21). Turning on Wilson lines breaks the gauge group to a product of unitary groups (25), where the overall $U(1)$ factors are anomalous for both the 5- and the 9-branes and therefore become massive [49]. In the T-dual picture (with 6 T-dualities along all compact directions), the breaking of the gauge group can be understood in terms of D7- and D3-branes that are moved away from the origin of the torus $\mathbb{T}^2$. The main results are formulas (43) and (44), which give the one-loop correction to the couplings of non-Abelian gauge groups on the 5-branes, in particular displaying the dependence on the open string scalars of both 5- and 9-branes.

Having repeated the relevant aspects of this model, let us now draw some conclusions for the rho problem described in the introduction. To do so, we have to make use of the relation between the variables common in the inflationary literature (e.g. in the KKLMMT model) introduced in Sec. II and those common in the orientifold literature used in Sec. III.[34] The volume modulus ρ corresponds to the field S' and the combination of the D3-brane scalars ϕ denoting the inflaton field corresponds to the Wilson line modulus A on the D9-brane which is T-dual to the mobile D3-brane, i.e.

$$\rho \quad \longleftrightarrow \quad S', \qquad \phi \quad \longleftrightarrow \quad A. \tag{83}$$

Other fields present in the $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{T}^2$ orientifold are the modulus U, which corresponds to one of the complex structure moduli that are supposed to be fixed in the inflationary models by fluxes, the modulus S, which, in the T-dual picture, gives the gauge coupling on the D3-branes, i.e., the dilaton, which is also supposed to be fixed, and finally the Wilson line moduli other than those corresponding to the inflaton. These do not have any direct counterpart in the original KKLMMT model, which only considered a single mobile D3-brane. In Sec. III we denoted all Wilson line moduli (including the one corresponding to the inflaton) by A_i, where i enumerated the different stacks of branes. In the following, we will use the notation introduced in Sec. II, because we want to interpret our results in the context of inflationary models in string theory. In order to translate the formulas from Sec. III, we have to use the dictionary just outlined, in particular (83). However, we continue to use the formulas derived in the T-dual D9/D5-picture, and it is understood that the D9-branes (resp. D5-branes) are mapped to D3-branes (resp. D7-branes) after six T-dualities. Our formulas equally hold in the T-dual (D3/D7) picture if one maps the fields in the usual way (see e.g. [63]). At most instances we give our formulas including the other Wilson line moduli (corresponding to positions of further D3-branes in the T-dual picture that are present for consistency in our toy models but not in the KKLMMT model). We denote all Wilson line moduli collectively as

$$\phi_i \quad \longleftrightarrow \quad A_i, \tag{84}$$

where as in (21) the ϕ_i are related to the D3-brane positions $(a_i)_4$ and $(a_i)_5$ on the third torus according to $\phi_i = U(a_i)_4 - (a_i)_5$. Note that the index i on ϕ_i has a completely different meaning now than the index in (8), where the i denoted the internal direction. Here, it enumerates the stacks of branes; all ϕ_i correspond to locations of the branes along the third torus.

The general form of one-loop physical gauge couplings in string theory is given by[35]

$$g^{-2}(\mu) = \mathrm{Re}(f) + \frac{b}{16\pi^2}\ln\left(\frac{M_{\rm str}^2}{\mu^2}\right) + \ldots, \tag{85}$$

where b is the one-loop beta function coefficient and the dots stand for moduli-dependent terms that are not real parts of holomorphic functions, as opposed to the $\mathrm{Re}(f)$ term. This first term is the Wilsonian coupling from integrating out heavy fields, whereas the second term and the nonholomorphic contributions are due to light fields, with masses below the scale μ^2 at which the coupling is probed. Formula (85) is valid both for all gauge couplings in $\mathcal{N}=1$ (where f is the gauge kinetic function) and for non-Abelian couplings in $\mathcal{N}=2$ supersymmetric theories (in which case f is related to the prepotential). Abelian gauge couplings in $\mathcal{N}=2$ theories are a bit more complicated due to a possible mixing of the Abelian gauge fields with the graviphoton.

For simplicity, let us now consider the case with vanishing Wilson lines on the 5-branes, such that the 5-brane gauge group is the unbroken $SU(16)$, i.e., we take all $\vec{a}_a = \vec{0}$. Moreover, in order to make contact to the formulas of Sec. III B, we assume that the 9-brane gauge group is completely broken to its Abelian subgroup so that all $N_i = 1$ and all $\vec{a}_i$ are distinct and nonzero. In this case we can read off the one-loop corrected gauge coupling of the 5-brane gauge group from the sum of (18) (with $\alpha' = 1/2$) and (43), using $\chi \sim \mu^2$, and find

$$\begin{aligned} g_{(5)}^{-2} &= \frac{1}{\pi\sqrt{2}}e^{-\Phi_{10}}\sqrt{G} + \frac{1}{8\pi}U_2\sum_i (a_i)_4^2 \\ &\quad - \frac{1}{16\pi^2}\sum_i \ln\left|\frac{\vartheta_1(\phi_i, U)}{\sqrt{\eta(U)}}\right|^2 \\ &\quad + \frac{1}{4\pi^2}\ln(8\pi^3\mu^2\sqrt{G}U_2), \end{aligned} \tag{86}$$

where $U_2 = \mathrm{Im}(U)$. Comparing the first two terms on the right-hand side with (16) (again for $\alpha' = 1/2$) and (20) we see that they combine to $\mathrm{Re}(-i\rho)$ with the modified field ρ. The third term on the right-hand side of (86) is an additional one-loop contribution to the gauge coupling which is the real part of a holomorphic function in the variables U and ϕ_i, and the last term corresponds to contributions from massless modes that are not given by the real part of a holomorphic function. From (86) we read off

$$f_{(5)} = -i\rho - \frac{1}{8\pi^2}\sum_i \ln\vartheta_1(\phi_i, U) + \frac{1}{\pi^2}\ln\eta(U), \tag{87}$$

[34] Strictly speaking, the KKLMMT model has $\mathcal{N}=1$ before supersymmetry breaking through antibranes and in this subsection we are considering our $\mathcal{N}=2$ orientifold example. However, the notational dictionary works the same way as in the $\mathcal{N}=1$ examples of the next subsection. Also, inflation based directly on $\mathrm{K3}\times\mathbb{T}^2$ compactifications was studied in [3,53,62].

[35] Cf. [12] and references therein.

involving the modified Kähler modulus but also including an additional dependence on the 9-brane scalars. This solves the rho problem described in the introduction. Of course, we could have done the same analysis with 5- and 9-branes exchanged, in which case we would have found that the 9-brane gauge kinetic coupling depends holomorphically on the modified field S of (20).

B. $\mathcal{N} = 1$

There are two different $\mathcal{N} = 1$ models that we considered in Sec. III D, the $\mathbb{T}^6/\mathbb{Z}'_6$ model [18,23] and the $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ model [24]. Again, readers already familiar with that section can skip the following two paragraphs.[36]

The $\mathbb{T}^6/\mathbb{Z}'_6$ model is defined in terms of the eigenvalues $\exp(2\pi i v)$, $v = (1, -3, 2)/6$, of its generator Θ. The open string sector is similar to the $\mathcal{N} = 2$ model just discussed, i.e., it has 32 D9-branes and 32 D5-branes wrapped around the third torus. This leads to a gauge group $[U(4)^2 \times U(8)]_{\rm D9} \times [U(4)^2 \times U(8)]_{\rm D5}$. The moduli space of the untwisted moduli is given by three copies of $SU(1,1)/U(1)$ for each of the three generic Kähler moduli of the three 2-tori, and one extra copy each for the dilaton and the complex structure modulus of the second torus. The complex structures of the first and the third torus, around which the 5-branes are wrapped, are no moduli. Rather, they are fixed to some rational values.[37] In this case the dependence of the one-loop correction to the 5-brane gauge couplings on the 9-brane Wilson line moduli is given in (78) and (79).

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ model is defined by the action of the three orbifold group elements Θ_p, $p = 1, 2, 3$, on $\mathbb{T}^6 = \mathbb{T}^2_1 \times \mathbb{T}^2_2 \times \mathbb{T}^2_3$, where each Θ_p is a reflection along two 2-tori, leaving the torus $\mathbb{T}^2_p$ invariant. Now the open string sector is more complicated. In addition to 32 D9-branes there are three sets of 32 D5-branes, wrapped around the three tori that make up the $\mathbb{T}^6$. The resulting gauge group is symplectic, specifically $Sp(8)_{\rm D9} \times Sp(8)_{{\rm D5}_1} \times Sp(8)_{{\rm D5}_2} \times Sp(8)_{{\rm D5}_3}$.[38] Another difference to the $\mathbb{Z}'_6$ case is that the complex structure moduli of all three tori are moduli and therefore the untwisted moduli space consists of six copies of $SU(1,1)/U(1)$ for the geometric moduli and one copy for the dilaton. The dependence of the one-loop correction to one of the 5-brane gauge couplings on the 9-brane Wilson line moduli is given in (82).

We start the discussion of our results with the $\mathbb{Z}'_6$ model. Adding (78) to (18) we derive

$$g_{(5)}^{-2} = \frac{1}{3\pi\sqrt{2}} e^{-\Phi_{10}}\sqrt{G} + \frac{1}{12\pi} U_2 \sum_i n_i [(a_i)_4^2 + (a_i)_5^2 - (a_i)_4(a_i)_5] - \frac{1}{48\pi^2} \sum_i n_i \sum_{m=0}^{2} \ln|\vartheta_1(\phi_i^{\Theta^m}, U)|^2 + \ldots, \tag{88}$$

where compared to the $\mathcal{N} = 2$ case there is an additional factor of 1/3 in the tree-level contribution due to the smaller volume of the orbifolded torus and n_i denotes the number of 9-branes in the i-th stack. This formula is valid for all three factors of the gauge group $U(4)^2 \times U(8)$ on the 5-branes. The dots stand for further one-loop corrections that do not depend on the Wilson line moduli. These could in principle be extracted from [18] and contain terms that would depend on the only complex structure modulus U' of the model, in the form $\ln|\eta(U')|$, whereas the U appearing in (88) is the complex structure of the third torus, which is not a modulus, as we already mentioned.

In the $\mathcal{N} = 1$ case we do not know of a derivation of the proper Kähler coordinates in the presence of open string scalars from a KK reduction. The analogy to the $\mathcal{N} = 2$ case suggests that the sum of the first two terms in (88) should form the imaginary part of the Kähler modulus (of the third torus) in the $\mathbb{Z}'_6$ model, i.e.

$$\mathrm{Im}(\rho) = \frac{1}{3\pi\sqrt{2}} e^{-\Phi_{10}}\sqrt{G} + \frac{1}{12\pi} U_2 \sum_i n_i [(a_i)_4^2 + (a_i)_5^2 - (a_i)_4(a_i)_5]. \tag{89}$$

Geometrically, this just means we propose to define the $\mathcal{N} = 1$ version of the corrected coordinate by summing over the three Θ-images of the correction that appeared in $\mathcal{N} = 2$, and properly normalized. This is supported by the fact that the one-loop correction included in (88) originates from open strings stretched between 5-branes and 9-branes that, for large enough Wilson line moduli, do not have any light fields in their spectrum. Thus they only contribute to the Wilsonian gauge coupling, i.e., their contribution to the gauge coupling should be the real part of a holomorphic function in the proper Kähler coordinates. In addition, (88) contains further dependence on the open string scalars, one of which is meant to be interpreted as the inflaton. Assuming that the Wilson line moduli are still given in the form (21), we read off the gauge kinetic function for the D5-brane gauge groups

$$f_{(5)} = -i\rho - \frac{1}{24\pi^2} \sum_i n_i \sum_{m=0}^{2} \ln\vartheta_1(\phi_i^{\Theta^m}, U) + \ldots = -i\rho - \frac{1}{8\pi^2} \sum_i n_i \ln(\phi_i) + \mathcal{O}(\phi^6) + \ldots, \tag{90}$$

where, again, we neglected one-loop corrections that are independent of the Wilson line moduli and the second equality comes from the fact that the Wilson line moduli

[36] As for the $\mathcal{N} = 2$ case also here we use the notation of Sec. II. The dictionary to the variables of Sec. III is basically the same as the one given in the last subsection.

[37] For the moduli spaces of the untwisted moduli in $\mathcal{N} = 1$ orientifolds see e.g. [58].

[38] Note footnote 31, however.

on the images under Θ are related by multiplication with a phase, i.e. $\phi^\Theta = e^{2\pi i/3}\phi$. This implies that $\phi^{2m} + (\phi^\Theta)^{2m} + (\phi^{2\Theta})^{2m} = 0$ for all integers m that are not multiples of 3. A few comments are in order here.

(i) First, we see that (90) contains the modified Kähler modulus (in fact, it was defined so that this would be the case). This is the solution to the rho problem that we propose in the $\mathcal{N} = 1$ case. Note that there is a slight difference to the model of KKLMMT. In their case only one Kähler modulus is present, whereas in our case there are three untwisted Kähler moduli. It is the one measuring the volume of the third torus that enters $f_{(5)}$ at tree-level, so this is the relevant Kähler modulus for the rho problem in our case. This modulus is mapped via T-duality to the volume of the four-cycle (transverse to the third torus) around which the 7-branes (the T-duals of the 5-branes) are wrapped.[39]

(ii) Moreover, the gauge kinetic function (90) blows up for $\phi_i \to 0$. The beta function coefficient of the $SU(8)$-factor of the 5-brane gauge group without Wilson line moduli is $b(SU(8)) = -6$ [18] and it can only become more negative if bifundamental matter turns massive for nonvanishing Wilson lines. Therefore the gauge group is asymptotically free, and at low energies a nonperturbative superpotential due to gaugino condensation is generated. From (2) we read off

$$W_{\mathrm{nonpert}} \sim \exp\left\{\alpha\left[i\rho + \frac{1}{8\pi^2}\sum_i n_i \ln(\phi_i) + \mathcal{O}(\phi^6) + \ldots\right]\right\}. \tag{91}$$

As α is positive for a negative beta function coefficient b, we see that the nonperturbative superpotential vanishes in the limit when, in the T-dual language, the 3-branes hit the cycle on which the 7-branes are wrapped, in accord with the results found in [64].[40]

(iii) Furthermore, the superpotential develops an explicit dependence on the open string scalars at one-loop level, hence the shift symmetry discussed in [4,53,65] is violated by the one-loop corrections.

(iv) Finally, there is no quadratic term in the open string fields in (90). Thus the one-loop corrections to the superpotential in the $\mathbb{Z}_6'$ model turn out to be incapable of reducing the inflaton mass. However, this is just an accident occurring in this model due to the $\mathbb{Z}_3$ symmetry of the Wilson line.[41]

[39]Note that for this mapping of volumes, it is important that there is a factor of $e^{-\Phi_{10}}$ in the definition of ρ, cf. (16).

[40]Strictly speaking, it is no longer valid to integrate out the modes from 59 strings when determining the gauge kinetic function in this limit; one has to introduce an IR cutoff as in (39).

[41]We do expect a mass term to appear also in this model if one deforms away from the orbifold limit, but we will not do so here.

This last problem is absent in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model to which we turn now. In this case the one-loop corrected gauge coupling of one of the 5-brane gauge groups is

$$g_{(5)}^{-2} = \frac{1}{4\pi\sqrt{2}} e^{-\Phi_{10}}\sqrt{G} + \frac{1}{4\pi} U_2 \sum_i n_i (a_i)_4^2 - \frac{1}{4\pi^2}\sum_i n_i \ln|\vartheta_1(\phi_i, U)| + \ldots, \tag{92}$$

where, as in the $\mathbb{Z}_6'$ model, the tree-level term contains an additional factor (1/4, this time) due to the orbifolding of the torus and the dots stand for all the one-loop corrections that do not depend on the Wilson line moduli. In analogy to [14,18] we expect them to again include terms of the form $\ln|\eta(U^{(p)})|$ for all three complex structure moduli $U^{(p)}$, $p = 1, 2, 3$. Formula (92) holds true for each of the three different types of 5-branes, and $\sqrt{G}$ and $U \in \{U^{(p)}\}$ denote the volume and complex structure of the corresponding torus around which they are wrapped. To keep the notation simple we just focus on one of them, without explicitly indexing the coupling or the volume and complex structure of the torus. Moreover, (92) also holds if the gauge group on the stack of 5-branes is broken to some smaller Sp group by moving some of them out of the origin.

As above, we suggest that the Kähler modulus for the torus, around which the 5-branes are wrapped, is modified at one-loop level to

$$\mathrm{Im}(\rho) = \frac{1}{4\pi\sqrt{2}} e^{-\Phi_{10}}\sqrt{G} + \frac{1}{4\pi} U_2 \sum_i n_i (a_i)_4^2 \tag{93}$$

and the gauge kinetic function is given by

$$f_{(5)} = -i\rho - \frac{1}{4\pi^2}\sum_i n_i \ln\vartheta_1(\phi_i, U) + \ldots. \tag{94}$$

To calculate the beta function coefficient of the orientifold model we need the charged spectrum derived in [24]. At a generic point in the moduli space of the $\mathcal{N} = 1$ supersymmetric theory considered there, all bifundamental matter is massive due to Wilson lines, and the only massless charged matter resides in the vector multiplet and three chiral multiplets transforming in the antisymmetric representation of the unbroken Sp group of the 5-branes under consideration. Since the KKLMMT model furthermore involves (at least spontaneous) supersymmetry breaking, one would expect that mass terms will be generated even for these matter fields, since only chiral fermions should generically remain massless.[42] In any case, if the rank of the symplectic gauge group has been broken to a small enough value, the beta function is negative even with the antisymmetric matter remaining

[42]For possible forms of soft breaking terms in (orientifold) models with D-branes, see [30,38].

massless, and gaugino condensation occurs at low energies. Substituting (94) into the formula for the resulting superpotential (2) now gives

$$W_{\rm nonpert} \sim \exp\{\alpha[i\rho + \frac{1}{4\pi^2}\sum_i n_i \ln\vartheta_1(\phi_i, U) + \ldots]\}. \tag{95}$$

When expanded around generic values for the open string scalars, the potential that follows from this superpotential in general possesses both linear and quadratic terms in the ϕ_i, whose coefficients depend on all the complex structure moduli. Thus the inflaton mass correction depends on the values at which the complex structure moduli are fixed by the background fluxes. It is then plausible that it is possible to fine-tune the fluxes to achieve a correction that leads to a value for the mass that is small enough to allow for slow roll inflation, a possibility that was anticipated in [1]. However, to obtain a conclusive answer, the one-loop corrections to the Kähler potential have to be known as well. Since they are not known, we hope to come back to them in a future publication [60].[43]

Our lack of knowledge of the corrections to the Kähler potential notwithstanding, and ignoring the fact that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold is at best a toy model for the actual KKLMMT setup, let us conclude by combining our result (95) with the analysis of [1],[44] in order to get a rough picture of fine-tuning the inflaton mass. To do so, we focus on a single dynamical D3-brane (i.e. $n = 1$ in (95)) whose scalar ϕ we interpret as the inflaton field. We then want to expand the superpotential to quadratic order in ϕ. In the KKLMMT model one considers a D3-brane that is well separated from both the D7-branes and the anti D3-branes at the tip of the throat. In principle, any such value would be a valid expansion point. In practice, however, it is most convenient to perform the explicit expansion of $\vartheta_1(\phi, U)$ either around $\phi = 0$ or $\phi = 1/2$. As we do not want to consider the special point $\phi = 0$, at which the gauge symmetry gets enhanced and new massless states appear, we choose to expand (95) around $\phi = 1/2$ for definiteness. Shifting $\phi \to \phi + 1/2$, so that ϕ now denotes the fluctuations around 1/2, we can use the relation $\vartheta_1(\phi + 1/2, U) = \vartheta_2(\phi, U)$ and (C6) and (C9), to expand

$$\vartheta_1(\phi + 1/2, U) = \vartheta_2(0, U)\Big(1 - \frac{\pi^2}{6}[E_2(U) + \vartheta_3^4(0, U) + \vartheta_4^4(0, U)]\phi^2 + \ldots\Big). \tag{96}$$

Substituting this into the formula for the superpotential (95) we obtain

$$W \sim W_0(U^{(p)}) + C(U^{(p)}, \phi_i)e^{\alpha i\rho}\Big(1 - \frac{\alpha}{24}[E_2(U) + \vartheta_3^4(0, U) + \vartheta_4^4(0, U)]\phi^2 + \ldots\Big), \tag{97}$$

where we reinstated the contribution W_0 to the superpotential coming from the 3-form fluxes, and the function C depends on all three complex structure moduli $U^{(p)}$ and the Wilson line moduli ϕ_i other than the inflaton field ϕ. In principle the additional dependence of the nonperturbative superpotential (and possibly also of the Kähler potential) on the complex structure moduli would require reminimizing the potential with respect to them. It is conventional, however, to assume a separation of scales, such that the complex structure moduli receive a flux-induced mass term that is much bigger than the scale of nonperturbative physics leading to gaugino condensation. In this case one can assume that the additional $U^{(p)}$-dependence in (97) does not alter their values at the minimum very much, so that they can be considered constant. This is also the philosophy that we follow here.

Comparing (97) with formula (F.1) of [1][45], i.e. $W = W_0 + Ce^{\alpha i\rho}(1 + \delta\phi^2)$ in our notation, we read off

$$\delta = -\frac{\alpha}{24}[E_2(U) + \vartheta_3^4(0, U) + \vartheta_4^4(0, U)]. \tag{98}$$

This quantity determines whether the one-loop correction to the superpotential can help to lower the inflaton mass or not, as can be inferred from (F.8) of [1],

$$\frac{m_\varphi^2}{H^2} = 2 - \frac{2|V_{\rm AdS}|}{V_{\rm dS}}\Delta, \tag{99}$$

where φ is the canonically normalized inflaton field,

$$\Delta = \beta - 2\beta^2 \quad \text{with}$$
$$\beta = -\frac{\delta}{\alpha} = \frac{1}{24}[E_2(U) + \vartheta_3^4(0, U) + \vartheta_4^4(0, U)] \tag{100}$$

and $V_{\rm AdS}$ and $V_{\rm dS}$ are explained in [1]. (However, the mass formula (99) does not include any contributions from one-loop corrections to the Kähler potential.) Obviously, if Δ is positive for some value of U, the inflaton mass is lowered. Note that neither the value of the beta function coefficient nor the function $C(U^{(p)}, \phi_i)$ enter into Δ and so it is insensitive to the uncertainties with which these quantities are afflicted.

[43]One-loop corrections to the Kähler metric in the $\mathbb{Z}_6'$ model without Wilson lines were calculated in [66].

[44]See their appendix F, in particular.

[45]Note that our α corresponds to their a, our $C(U^{(p)}, \phi_i)$ to their A and there is a relative factor of i in our definition of the Kähler coordinates as compared to theirs.

In Fig. 4 we plot the dependence of Δ on the value of U, which, for ease of presentation, we assume to be imaginary at its minimum, i.e., we assume that $\mathrm{Re}(U) = 0$ is consistent with minimization of the flux-induced potential. It is clear in the plot that for a large range of values for $\mathrm{Im}(U)$, the function $\Delta(U)$ is positive, so if the complex structure is stabilized in this range, the inflaton mass is lowered by the one-loop corrections to the superpotential. The explicit value also depends on the values of the other complex structure and open string moduli, as well as on the beta function coefficient and the one-loop corrections to the Kähler potential. Moreover, our analysis is not able to take into account any possible effects of the warp factor present in the actual KKLMMT model. Because of these uncertainties, we refrain from giving a numerical correction to the inflaton mass, which would require computing the value of $|V_{\mathrm{AdS}}|/V_{\mathrm{dS}}$ in (99) by minimizing the full scalar potential.

Nevertheless, our conclusion is that the open string one-loop corrections to the superpotential should, in general, provide the added flexibility needed to fine-tune the inflaton mass to small values. In the philosophy of [1,2,7] this fine-tuning is achieved by choosing appropriate values for the 3-form fluxes, because their values determine the warp factor (and thus the effective tension of the anti-D3-branes at the tip of the throat) and the values at which the complex structure moduli are fixed. (As stressed on p. 37 of appendix F in [1], this fine-tuning is only numerical at the 1% level.) The possibility to lower the inflaton mass in the KKLMMT model via moderate

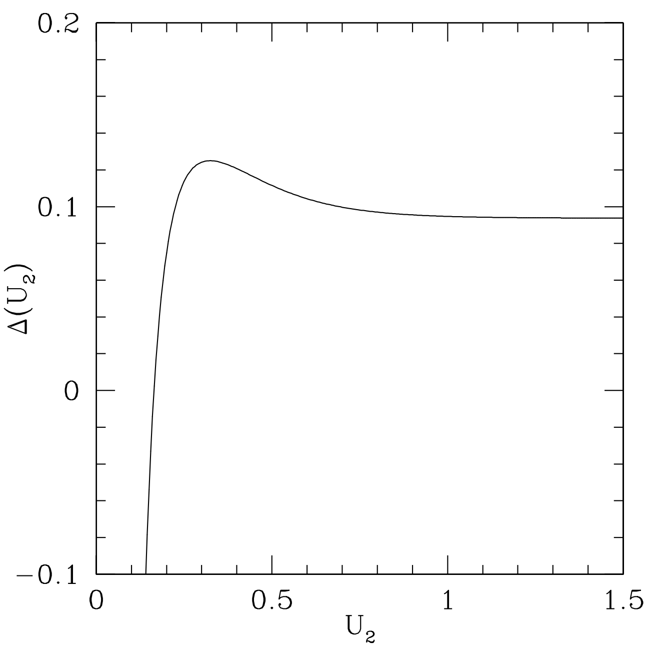

FIG. 4. The function Δ of Eq. (72); for positive values the inflaton mass is lowered by the one-loop corrections to the superpotential.

fine-tuning was already anticipated in [1], assuming the superpotential may contain terms quadratic in the inflaton field with moduli-dependent coefficients. Even though our calculation is not realistic enough to make this completely quantitative, the merit of our result is to show that terms quadratic in the inflaton field with moduli-dependent coefficients do indeed appear in the superpotential in our explicit string theory model. They are induced by open string one-loop (annulus) corrections to the gauge kinetic function on the D7-branes, and this gauge kinetic function appears in the superpotential after gaugino condensation. We believe this qualitative result to be generic in models with D3/D7-branes, not an artifact of the simplifying assumptions in our explicit calculation, and that it will survive in more realistic cases.

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APPENDIX A: ONE-LOOP AMPLITUDE FOR $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{T}^2$

In this appendix we summarize the technical steps to compute the relevant one-loop amplitudes. The conventional method to incorporate the background gauge fields into the amplitude is to introduce them in the loop channel by replacing the momentum integration in the directions with a magnetic background field by the degeneracy per unit area of the Landau levels, and further shift the modings of the string world sheet fields by

$$\boldsymbol{\epsilon}_i = \frac{1}{\pi}\arctan(2\pi\alpha'\mathbf{F}_{\mathrm{i}}), \qquad \boldsymbol{\epsilon}_a = \frac{1}{\pi}\arctan(2\pi\alpha'\mathbf{F}_{\mathrm{a}}) \tag{A1}$$

according to the sigma model boundary conditions [16,17]. We actually prefer to compute the diagrams without the gauge field in the loop channel, and then later directly implement its effects on the tree channel result.

The open string amplitudes in (12) are defined in the loop channel

$$\mathcal{M} = \frac{\sqrt{-g_4}}{(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{(2t)^3}\mathrm{Tr}_{\mathrm{op}}^{\mathrm{NS-R}}\left(\frac{\Omega}{2}\frac{1+(-1)^F}{2}\frac{1+\Theta}{2}e^{-2\pi t\mathcal{H}_{\mathrm{op}}}\right) = \sqrt{-g_4}\int_0^\infty \frac{dt}{t}\sum_{k=0,1}\left[\sum_i \mathcal{M}_i^{(k)}(-q) + \sum_a \mathcal{M}_a^{(k)}(-q)\right],$$

$$\begin{aligned}\mathcal{A} &= \frac{\sqrt{-g_4}}{(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{(2t)^3}\mathrm{Tr}_{\mathrm{op}}^{\mathrm{NS-R}}\left(\frac{1}{2}\frac{1+(-1)^F}{2}\frac{1+\Theta}{2}e^{-2\pi t\mathcal{H}_{\mathrm{op}}}\right) \\ &= \sqrt{-g_4}\int_0^\infty \frac{dt}{t}\sum_{k=0,1}\left[\sum_{i,j}\mathcal{A}_{ij}^{(k)}(q) + \sum_{a,b}\mathcal{A}_{ab}^{(k)}(q) + \sum_{i,a}[\mathcal{A}_{ia}^{(k)}(q) + \mathcal{A}_{ai}^{(k)}(q)]\right],\end{aligned} \tag{A2}$$

where $k = 0, 1$ stands for the power of Θ inserted in the trace. The contributions with $k = 1$ originate from the twisted components of the relevant boundary states, and are thus called twisted contributions. The arguments of theta functions are abbreviated $q = e^{-2\pi t}$ and $\tilde{q} = e^{-4\pi l}$ (used below), i.e., we leave out the second argument of $\vartheta[{}^{\alpha}_{\beta}](\nu, it)$ or $\vartheta[{}^{\alpha}_{\beta}](\nu, 2il)$. In the loop channel, the presence of the background gauge field amounts to a shift of the first argument ν of the theta functions by ϵ_i, etc., This implies that we can ignore the $\mathcal{N} = 4$ "subsectors" of the amplitude, since

$$\mathcal{M}_i^{(0)}, \mathcal{M}_a^{(1)}, \mathcal{A}_{ij}^{(0)}, \mathcal{A}_{ab}^{(0)} = \mathcal{O}(1) + \mathcal{O}(\epsilon_i^4, \epsilon_a^4) \tag{A3}$$

by (C12). The constant tadpole $\mathcal{O}(1)$ cancels after adding up all contributions to the one-loop amplitude. So we are interested in the quadratic term in the expansion of

$$\sum_i \mathcal{M}_i^{(1)} + \sum_a \mathcal{M}_a^{(0)} + \sum_{i,j}\mathcal{A}_{ij}^{(1)} + \sum_{a,b}\mathcal{A}_{ab}^{(1)} + \sum_{k=0,1}\sum_{i,a}(\mathcal{A}_{ia}^{(k)} + \mathcal{A}_{ai}^{(k)}). \tag{A4}$$

To implement the Wilson lines, we use the notation introduced in Sec. III C, and write bold $\vec{\mathbf{A}}$ for the tensor-valued quantities. As well, we extend the usual matrices γ to this notation and formally write a single trace (cf. the discussion below (31)). The results are

$$\begin{aligned}
\mathcal{M}_i^{(1)} + \mathcal{M}_a^{(0)} &= -\frac{1}{16(8\pi^2\alpha')^2 t^2}\mathrm{tr}\left[\gamma_{\Omega\Theta i}^{-1}\gamma_{\Omega\Theta i}^{\mathrm{T}}\vartheta[{}^{2\vec{\mathbf{A}}_i}_{\vec{0}}](0, 2itG^{-1}\alpha') + \gamma_{\Omega a}^{-1}\gamma_{\Omega a}^{\mathrm{T}}\vartheta[{}^{2\vec{\mathbf{A}}_a}_{\vec{0}}](0, 2itG^{-1}\alpha')\right] \\
&\quad\times\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]^2\vartheta[{}^{\alpha}_{\beta+1/2}]^2}{\eta^6\frac{1}{4}\vartheta[{}^{1/2}_{0}]^2}, \\
\mathcal{A}_{ij}^{(1)} + \mathcal{A}_{ab}^{(1)} &= \frac{1}{16(8\pi^2\alpha')^2 t^2}\mathrm{tr}\left[\boldsymbol{\gamma}_{\Theta \mathrm{i}}\boldsymbol{\gamma}_{\Theta j}^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ij}}_{\vec{0}}](0, 2itG^{-1}\alpha') + \boldsymbol{\gamma}_{\Theta a}\boldsymbol{\gamma}_{\Theta b}^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ab}}_{\vec{0}}](0, 2itG^{-1}\alpha')\right]\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]^2\vartheta[{}^{\alpha}_{\beta+1/2}]^2}{\eta^6\frac{1}{4}\vartheta[{}^{1/2}_{0}]^2}, \\
\mathcal{A}_{ia}^{(0)} + \mathcal{A}_{ai}^{(0)} &= \frac{1}{16(8\pi^2\alpha')^2 t^2}\mathrm{tr}\left[\boldsymbol{\gamma}_i\boldsymbol{\gamma}_a^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ia}}_{\vec{0}}](0, 2itG^{-1}\alpha') + \boldsymbol{\gamma}_a\boldsymbol{\gamma}_i^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ai}}_{\vec{0}}](0, 2itG^{-1}\alpha')\right]\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]^2\vartheta[{}^{\alpha+1/2}_{\beta}]^2}{\eta^6\vartheta[{}^{0}_{1/2}]^2}, \\
\mathcal{A}_{ia}^{(1)} + \mathcal{A}_{ai}^{(1)} &= \frac{1}{16(8\pi^2\alpha')^2 t^2}\mathrm{tr}\left[\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta a}^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ia}}_{\vec{0}}](0, 2itG^{-1}\alpha') + \boldsymbol{\gamma}_{\Theta a}\boldsymbol{\gamma}_{\Theta i}^{-1}\vartheta[{}^{\vec{\mathbf{A}}_{ai}}_{\vec{0}}](0, 2itG^{-1}\alpha')\right]\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]^2\vartheta[{}^{\alpha+1/2}_{\beta+1/2}]^2}{\eta^6\vartheta[{}^{0}_{0}]^2}.
\end{aligned} \tag{A5}$$

Transforming to the tree-channel by $t = 1/(2l)$ for the annuli and $t = 1/(8l)$ for the Möbius strip, one finds the amplitude in a form

$$\begin{aligned}\tilde{\mathcal{M}} &= \sqrt{-g_4}\int_0^\infty dl\sum_{k=0,1}\left[\sum_i \tilde{\mathcal{M}}_i^{(k)}(-\tilde{q}) + \sum_a \tilde{\mathcal{M}}_a^{(k)}(-\tilde{q})\right], \\ \tilde{\mathcal{A}} &= \sqrt{-g_4}\int_0^\infty dl\sum_{k=0,1}\left[\sum_{i,j}\tilde{\mathcal{A}}_{ij}^{(k)}(\tilde{q}) + \sum_{a,b}\tilde{\mathcal{A}}_{ab}^{(k)}(\tilde{q}) + \sum_{i,a}[\tilde{\mathcal{A}}_{ia}^{(k)}(\tilde{q}) + \tilde{\mathcal{A}}_{ai}^{(k)}(\tilde{q})]\right]\end{aligned} \tag{A6}$$

with (again focusing on the $\mathcal{N} = 2$ sectors that contribute to the gauge coupling corrections)

$$\tilde{\mathcal{M}}_i^{(1)} + \tilde{\mathcal{M}}_a^{(0)} = -\frac{4}{(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\gamma_{\Omega\Theta i}^{-1}\gamma_{\Omega\Theta i}^{\mathrm{T}}\vartheta[{}^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_i, 4ilG\alpha'^{-1}) + \gamma_{\Omega a}^{-1}\gamma_{\Omega a}^{\mathrm{T}}\vartheta[{}^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_a, 4ilG\alpha'^{-1})\Big]$$
$$\times\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{\alpha}_{\beta}]^2\vartheta[{}^{\alpha}_{\beta+1/2}]^2}{\eta^6\vartheta[{}^{1/2}_{0}]^2},$$
$$\tilde{\mathcal{A}}_{ij}^{(1)} + \tilde{\mathcal{A}}_{ab}^{(1)} = \frac{1}{4(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta j}^{-1}\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ij}, ilG\alpha'^{-1}) + \boldsymbol{\gamma}_{\Theta a}\boldsymbol{\gamma}_{\Theta b}^{-1}\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ab}, ilG\alpha'^{-1})\Big]$$
$$\times\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{-\beta}_{\alpha}]^2\vartheta[{}^{-1/2-\beta}_{\alpha}]^2}{\eta^6\vartheta[{}^{0}_{1/2}]^2},$$
$$\tilde{\mathcal{A}}_{ia}^{(0)} + \tilde{\mathcal{A}}_{ai}^{(0)} = \frac{1}{8(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_i\boldsymbol{\gamma}_a^{-1}\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})\Big]\times\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{-\beta}_{\alpha}]^2\vartheta[{}^{-\beta}_{\alpha+1/2}]^2}{\eta^6\vartheta[{}^{1/2}_{0}]^2},$$
$$\tilde{\mathcal{A}}_{ia}^{(1)} + \tilde{\mathcal{A}}_{ai}^{(1)} = \frac{1}{8(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta a}^{-1}\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})\Big]\times\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta[{}^{-\beta}_{\alpha}]^2\vartheta[{}^{-1/2-\beta}_{1/2+\alpha}]^2}{\eta^6\vartheta[{}^{0}_{0}]^2}. \tag{A7}$$

The Klein bottle is normalized such that the untwisted tadpole cancellation (which involves also contributions from the $\mathcal{N} = 4$ sectors that we did not write out in (A7)) is achieved with

$$\sum_i \mathrm{tr}(\gamma_i) = \sum_a \mathrm{tr}(\gamma_a) = \sum_i \mathrm{tr}(\gamma_{\Omega i}^{-1}\gamma_{\Omega i}^{\mathrm{T}}) = \sum_a \mathrm{tr}(\gamma_{\Omega\Theta a}^{-1}\gamma_{\Omega\Theta a}^{\mathrm{T}}) = 32, \tag{A8}$$

which is solved by

$$\gamma_i = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_i\ \text{entries}}, \mathbf{1}_{2N_i}, 0,\ldots,0),\qquad \gamma_a = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_a\ \text{entries}}, \mathbf{1}_{2N_a}, 0,\ldots,0), \tag{A9}$$

and

$$\gamma_{\Omega i}^{\mathrm{T}} = \gamma_{\Omega i}^{-1},\quad \gamma_{\Omega i}^{-1}\gamma_{\Omega i}^{\mathrm{T}} = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_i\ \text{entries}}, \mathbf{1}_{2N_i}, 0,\ldots,0),$$
$$\gamma_{\Omega\Theta a}^{\mathrm{T}} = \gamma_{\Omega\Theta a}^{-1},\quad \gamma_{\Omega\Theta a}^{-1}\gamma_{\Omega\Theta a}^{\mathrm{T}} = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_a\ \text{entries}}, \mathbf{1}_{2N_a}, 0,\ldots,0), \tag{A10}$$

with the same p_i and p_a as in (29) and (30). The twisted contribution vanishes for

$$\sum_i \mathrm{tr}(\gamma_{\Theta i}) = \sum_a \mathrm{tr}(\gamma_{\Theta a}) = 0. \tag{A11}$$

The solution to the latter condition can be achieved by [18]

$$\gamma_{\Theta i} = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_i\ \text{entries}}, i\mathbf{1}_{N_i}, -i\mathbf{1}_{N_i}, 0,\ldots,0),\qquad \gamma_{\Theta a} = \mathrm{diag}(\underbrace{0,\ldots,0}_{p_a\ \text{entries}}, i\mathbf{1}_{N_a}, -i\mathbf{1}_{N_a}, 0,\ldots,0). \tag{A12}$$

The notation assumes that all 32×32 CP matrices are subdivided into the $(2N_i) \times (2N_i)$ blocks (and similarly for a) referring to the factors of the gauge group (25), such that the matrices in (A9) and (A12) exactly act on the blocks i and a. We are using the conventions of [23], such that $\gamma_{\Theta i}^2 = \gamma_{\Theta a}^2 = -\mathbf{1}_{32}$, and later similarly for the $\mathbb{Z}_6'$ model $\gamma_{\Theta 9}^6 = \gamma_{\Theta 5}^6 = -\mathbf{1}_{32}$. In this basis, the operation of Ω on the CP labels is off-diagonal, given by [23]

$$\gamma_{\Omega 9} = \oplus_i\begin{pmatrix}\mathbf{0}_{N_i} & \mathbf{1}_{N_i}\\ \mathbf{1}_{N_i} & \mathbf{0}_{N_i}\end{pmatrix},\qquad \gamma_{\Omega 5} = \oplus_a\begin{pmatrix}\mathbf{0}_{N_a} & i\mathbf{1}_{N_a}\\ -i\mathbf{1}_{N_a} & \mathbf{0}_{N_a}\end{pmatrix}. \tag{A13}$$

To incorporate the background gauge fields, denoted $\mathbf{F}$ as in Sec. III C, we make the replacement (cf. the discussion in [67])

$$\mathrm{tr}\Big[\gamma\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}, ilG\alpha'^{-1})\Big]\frac{\vartheta[{}^{\alpha}_{\beta}](0)}{\eta^3} \longrightarrow \mathrm{tr}\Big[\gamma[-2\sin(\pi\epsilon)]\vartheta[{}^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}, ilG\alpha'^{-1})\frac{\vartheta[{}^{\alpha}_{\beta}](\epsilon)}{\vartheta[{}^{1/2}_{1/2}](\epsilon)}\Big].$$

Effectively, we have added phase factors for the world sheet oscillators along the space-time directions, where the magnetic field is pointing, and the phase prefactor in the numerator cancels against that of the denominator. Expanding the prefactor $-2\sin(\pi\epsilon)$ in $\mathcal{F}$ to first order gives back the semiclassical result $-4\pi\alpha'\mathbf{F}$, in accord with point particles in a background magnetic field. Using the identity (C15), one can expand the oscillator sums in ϵ and up to $\mathcal{O}(\mathbf{F}^4)$ we find

$$\begin{aligned}
\tilde{\mathcal{M}}_i^{(1)} + \tilde{\mathcal{M}}_a^{(0)} &= -\frac{8}{(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\gamma_{\Omega\Theta i}^{-1}\gamma_{\Omega\Theta i}^{\mathrm{T}}\vartheta[^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_i, 4ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_i)^2 \\
&\quad + \gamma_{\Omega a}^{-1}\gamma_{\Omega a}^{\mathrm{T}}\vartheta[^{\vec{0}}_{\vec{0}}](2\vec{\mathbf{A}}_a, 4ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_a)^2\Big], \\
\tilde{\mathcal{A}}_{ij}^{(1)} + \tilde{\mathcal{A}}_{ab}^{(1)} &= \frac{1}{2(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta i}^{-1}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ij}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ij})^2 + \boldsymbol{\gamma}_{\Theta a}\boldsymbol{\gamma}_{\Theta b}^{-1}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ab}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ab})^2\Big], \\
\tilde{\mathcal{A}}_{ia}^{(0)} + \tilde{\mathcal{A}}_{ai}^{(0)} &= \frac{1}{4(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_i\boldsymbol{\gamma}_a^{-1}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ia})^2\Big], \\
\tilde{\mathcal{A}}_{ia}^{(1)} + \tilde{\mathcal{A}}_{ai}^{(1)} &= \frac{1}{4(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big[\boldsymbol{\gamma}_{\Theta i}\boldsymbol{\gamma}_{\Theta a}^{-1}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ia})^2\Big].
\end{aligned} \tag{A14}$$

For $\alpha' = 1/2$ and vanishing Wilson lines we recover the formulas given in (3.18) of [18] if in the final result we take into account the different factors which were factored out in (2.1) and (2.7) of [18], and we correct their formula (3.11) by adding an additional factor 1/2 in the exponent, which leads to an additional factor of 2 on the right-hand side of their (3.18).

APPENDIX B: ONE-LOOP AMPLITUDE FOR $\mathbb{T}^6/\mathbb{Z}_6'$

Here we collect some formulas which are relevant for our discussion of the $\mathbb{Z}_6'$ orientifold in Sec. III D. The one-loop amplitude without Wilson lines can be directly copied from [18], taking into account the remarks at the end of the previous section, and adapting to our notation. We include them here for completeness, and to make our statement concrete that the part important for the discussion of the rho problem is just identical to the $\mathbb{Z}_2$ result, up to an overall numerical factor 1/3.

We use an analogous normalization as for the $\mathbb{Z}_2$ model and absorb all relative factors into the integrands of the amplitudes, writing

$$\begin{aligned}
\mathcal{M} &= \frac{\sqrt{-g_4}}{(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{(2t)^3}\mathrm{Tr}_{\mathrm{op}}^{\mathrm{NS-R}}\Big(\frac{\Omega}{2}\frac{1+(-1)^F}{2}\frac{1+\Theta+\cdots+\Theta^5}{6}e^{-2\pi t\mathcal{H}_{\mathrm{op}}}\Big) \\
&= \sqrt{-g_4}\int_0^\infty \frac{dt}{t}\sum_{k=0}^{5}\Big[\sum_i \mathcal{M}_i^{(k)}(-q) + \sum_a \mathcal{M}_a^{(k)}(-q)\Big] \\
\mathcal{A} &= \frac{\sqrt{-g_4}}{(4\pi^2\alpha')^2}\int_0^\infty \frac{dt}{(2t)^3}\mathrm{Tr}_{\mathrm{op}}^{\mathrm{NS-R}}\Big(\frac{1}{2}\frac{1+(-1)^F}{2}\frac{1+\Theta+\cdots+\Theta^5}{6}e^{-2\pi t\mathcal{H}_{\mathrm{op}}}\Big) \\
&= \sqrt{-g_4}\int_0^\infty \frac{dt}{t}\sum_{k=0}^{5}\Big[\sum_{i,j}\mathcal{A}_{ij}^{(k)}(q) + \sum_{a,b}\mathcal{A}_{ab}^{(k)}(q) + \sum_{i,a}[\mathcal{A}_{ia}^{(k)}(q) + \mathcal{A}_{ai}^{(k)}(q)]\Big].
\end{aligned} \tag{B1}$$

The only amplitudes that depend on the D9-brane Wilson lines on the third 2-torus and on the background field strength on the D5-branes are given (to order $\mathcal{O}(\mathbf{F}^2)$) by

$$\begin{aligned}
\tilde{\mathcal{A}}_{ia}^{(0)} + \tilde{\mathcal{A}}_{ai}^{(0)} &= \frac{1}{12(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\ \mathrm{tr}\Big[\boldsymbol{\gamma}_i\boldsymbol{\gamma}_a^{-1}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ia})^2\Big], \\
\tilde{\mathcal{A}}_{ia}^{(3)} + \tilde{\mathcal{A}}_{ai}^{(3)} &= \frac{1}{12(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\ \mathrm{tr}\Big[\boldsymbol{\gamma}_{\Theta i}^{3}\boldsymbol{\gamma}_{\Theta a}^{-3}\vartheta[^{\vec{0}}_{\vec{0}}](\vec{\mathbf{A}}_{ia}, ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ia})^2\Big].
\end{aligned} \tag{B2}$$

In the $\mathbb{Z}_6'$ model there will also emerge a dependence of the correction to the gauge couplings on the moduli of the second 2-torus, whose metric we denote G_2. This is evident from the classification of all amplitudes in Table I.

Explicitly, the other contributions are given by (suppressing the $\mathcal{N} = 4$ sector again, and leaving out terms that can be restored trivially by using the symmetry between 9- and 5-branes)

$$
\begin{aligned}
\tilde{\mathcal{M}}_i^{(3)} &= -\frac{8}{3(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big(\gamma_{\Omega\Theta i}^{-3}(\gamma_{\Omega\Theta i}^{\mathrm{T}})^3\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](\vec{0},4ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_i)^2\Big),\\
\tilde{\mathcal{A}}_{ij}^{(3)} &= \frac{1}{6(8\pi^2\alpha')^2}\sqrt{G}\alpha'^{-1}\mathrm{tr}\Big(\boldsymbol{\gamma}_{\Theta i}^3\boldsymbol{\gamma}_{\Theta j}^{-3}\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](\vec{0},ilG\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ij})^2\Big),\\
\tilde{\mathcal{M}}_i^{(k=1,5)} &= \frac{2}{3\pi(8\pi\alpha')^2}\mathrm{tr}((\gamma_{\Omega\Theta i}^{\mathrm{T}})^k\gamma_{\Omega\Theta i}^{-k}(2\pi\alpha'\mathbf{F}_i)^2)\prod_{i=1}^3\sin(\pi k v_i)\sum_{i=1}^3\frac{\vartheta'\big[{}^{1/2}_{1/2+kv_i}\big](0)}{\vartheta\big[{}^{1/2}_{1/2+kv_i}\big](0)},\\
\tilde{\mathcal{M}}_i^{(k=2,4)} &= \frac{8}{3(8\pi\alpha')^2}\sqrt{G_2}\alpha'^{-1}\mathrm{tr}\Big((\gamma_{\Omega\Theta i}^{\mathrm{T}})^k\gamma_{\Omega\Theta i}^{-k}\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](\vec{0},4ilG_2\alpha'^{-1})(2\pi\alpha'\mathbf{F}_i)^2\Big)\times\sin(\pi k v_1)\sin(\pi k v_3),\\
\tilde{\mathcal{A}}_{ij}^{(k=1,5)} &= -\frac{1}{3\pi(8\pi\alpha')^2}\mathrm{tr}(\boldsymbol{\gamma}_{\Theta i}^k\boldsymbol{\gamma}_{\Theta j}^{-k}(2\pi\alpha'\mathbf{F}_{ij})^2)\prod_{i=1}^3\sin(\pi k v_i)\sum_{i=1}^3\frac{\vartheta'\big[{}^{1/2}_{1/2+kv_i}\big](0)}{\vartheta\big[{}^{1/2}_{1/2+kv_i}\big](0)},\\
\tilde{\mathcal{A}}_{ij}^{(k=2,4)} &= -\frac{1}{6(8\pi\alpha')^2}\sqrt{G_2}\alpha'^{-1}\mathrm{tr}\Big(\boldsymbol{\gamma}_{\Theta i}^k\boldsymbol{\gamma}_{\Theta j}^{-k}\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](\vec{0},ilG_2\alpha'^{-1})(2\pi\alpha'\mathbf{F}_{ij})^2\Big)\times\sin(\pi k v_1)\sin(\pi k v_3),\\
\tilde{\mathcal{A}}_{ia}^{(k=1,2,4,5)} &= -\frac{2}{3\pi(8\pi\alpha')^2}\mathrm{tr}(\boldsymbol{\gamma}_{\Theta i}^k\boldsymbol{\gamma}_{\Theta a}^{-k}(2\pi\alpha'\mathbf{F}_{ia})^2)\sin(\pi k v_3)\times\Big(\frac{\vartheta'\big[{}^{1/2}_{1/2-kv_3}\big](0)}{\vartheta\big[{}^{1/2}_{1/2-kv_3}\big](0)}+\sum_{i=1}^2\frac{\vartheta'\big[{}^{0}_{1/2-kv_i}\big](0)}{\vartheta\big[{}^{0}_{1/2-kv_i}\big](0)}\Big).
\end{aligned}
\tag{B3}
$$

We have given these amplitudes only for vanishing Wilson lines. Turning on Wilson lines would lead to nonvanishing first arguments $2\vec{\mathbf{A}}_i$ and $\vec{\mathbf{A}}_{ij}$ in the theta functions appearing in $\tilde{\mathcal{M}}_i^{(3)}$ and $\tilde{\mathcal{A}}_{ij}^{(3)}$, respectively. In the other amplitudes (twisted by Θ^k with $k\notin\{0,3\}$), turning on Wilson lines would require summing over fixed points of Θ^k on the third torus and inserting powers of the matrices γ_{W9} (71) (and the analogs γ_{W5} for D5-branes), appropriate to the fixed point, into the traces cf. [23,57]. Obviously, the combined set of $k=0,3$ amplitudes is formally identical to the results (A14) for the $\mathcal{N}=2$ model, up to the overall normalization factor $1/6$ instead of $1/2$, and the necessary replacements of the appropriate matrices γ. The rest will contribute terms independent of the Wilson line moduli, for example, a term $\sim\ln|\eta(U')|$, where U' is the only complex structure modulus of the model as explained in Sec. IV. These contributions could in principle be extracted from [18].

APPENDIX C: FORMULAS

Here we collect some formulas about elliptic functions that are all available in various corners of the literature, but usually in different notation. We make an effort to consistently follow the conventions of the textbook by Polchinski [46].

TABLE I. Amplitude summary

SUSY	Untwisted $\mathbb{T}^2$	Amplitudes		
$\mathcal{N}=1$	–	$\mathcal{A}_{ij}^{(k=1,5)}$,	$\mathcal{A}_{ia}^{(k=1,2,4,5)}$,	$\mathcal{M}_i^{(k=1,5)}$
$\mathcal{N}=2$	2nd	$\mathcal{A}_{ij}^{(k=2,4)}$,	$\mathcal{M}_i^{(k=2,4)}$	
$\mathcal{N}=2$	3rd	$\mathcal{A}_{ij}^{(k=3)}$,	$\mathcal{A}_{ia}^{(k=0,3)}$,	$\mathcal{M}_i^{(k=3)}$

The eta and theta functions we use are

$$
\begin{aligned}
\eta(\tau) &= q^{1/24}\prod_{n=1}^\infty(1-q^n),\\
\vartheta\big[{}^{\vec{\alpha}}_{\vec{\beta}}\big](\vec{\nu},G) &= \sum_{\vec{n}\in\mathbb{Z}^N}e^{i\pi(\vec{n}+\vec{\alpha})^{\mathrm{T}}G(\vec{n}+\vec{\alpha})}e^{2\pi i(\vec{\nu}+\vec{\beta})^{\mathrm{T}}(\vec{n}+\vec{\alpha})},
\end{aligned}
\tag{C1}
$$

where G is an $N\times N$ matrix with $\mathrm{Im}(G)>0$, and $q=e^{2\pi i\tau}$. The case $N=1$ is the usual set of genus one theta functions. For $N=1$ and half-integer characteristics we use the notation

$$
\begin{aligned}
\vartheta\big[{}^0_0\big](\nu,\tau) &= \vartheta_3(\nu,\tau), & \vartheta\big[{}^{1/2}_{0}\big](\nu,\tau) &= \vartheta_2(\nu,\tau),\\
\vartheta\big[{}^{0}_{1/2}\big](\nu,\tau) &= \vartheta_4(\nu,\tau), & \vartheta\big[{}^{1/2}_{1/2}\big](\nu,\tau) &= -\vartheta_1(\nu,\tau).
\end{aligned}
\tag{C2}
$$

Comparing to another good source for theta identities, the lecture notes by Kiritsis [68], we have $\vartheta\big[{}^{\alpha}_{\beta}\big]=\vartheta^{\mathrm{K}}\big[{}^{-2\alpha}_{-2\beta}\big]$, where ϑ^{K} is that of Kiritsis. A word of warning:

$$
\begin{aligned}
\vartheta_1^{\mathrm{K}}(\nu,\tau) &\equiv \vartheta^{\mathrm{K}}\big[{}^1_1\big](\nu,\tau) = \vartheta\big[{}^{-1/2}_{-1/2}\big](\nu,\tau)\\
&= -\vartheta\big[{}^{1/2}_{1/2}\big](\nu,\tau)\equiv\vartheta_1(\nu,\tau).
\end{aligned}
$$

The modular transformation property of (C1) that will be relevant to us is

$$
\vartheta\big[{}^{\vec{\alpha}}_{\vec{0}}\big](0,itG^{-1}) = \sqrt{G}t^{-N/2}\vartheta\big[{}^{\vec{0}}_{\vec{0}}\big](\vec{\alpha},it^{-1}G),
\tag{C3}
$$

where the G under the square root denotes the determinant of the matrix G in the argument. The modular transformations for $N=1$ read

$$\vartheta[^{\alpha}_{\beta}](\nu,\tau) = e^{\pi i\alpha(\alpha+1)}\vartheta[^{\alpha}_{\beta-\alpha-1/2}](\nu,\tau+1),$$
$$\vartheta[^{\alpha}_{\beta}](\nu,\tau) = (-i\tau)^{-1/2}e^{2\pi i\alpha\beta-\pi i\nu^2/\tau}\vartheta[^{-\beta}_{\alpha}](\nu/\tau,-1/\tau). \tag{C4}$$

For the Möbius strip the following sequence of modular transformations is useful:

$$\tau\to-\frac{1}{\tau}\to-\frac{1}{\tau}+2\to-\frac{1}{-\frac{1}{\tau}+2},$$

giving

$$\vartheta[^{\alpha}_{\beta}](\nu,\tau) = (1-2\tau)^{-1/2}e^{-2\pi i\beta}e^{-\pi i\nu^2/(\tau-1/2)}\vartheta[^{\alpha+2\beta}_{\beta}]\left(\frac{\nu}{1-2\tau},\frac{\tau}{1-2\tau}\right). \tag{C5}$$

For ν-derivatives we use the notation $\vartheta'[^{\alpha}_{\beta}](0,\tau) = \partial_\nu\vartheta[^{\alpha}_{\beta}](\nu,\tau)|_{\nu=0}$. For the four special theta functions (C2), we have

$$\vartheta_2'(0,\tau)=\vartheta_3'(0,\tau)=\vartheta_4'(0,\tau)=0,\quad \vartheta_1'(0,\tau)=2\pi\eta(\tau)^3. \tag{C6}$$

In Sec. III C, we also make use of the third ν-derivative (cf. (F.14) in [68])

$$\vartheta_1'''(0,\tau) = -\pi^2\vartheta_1'(0,\tau)E_2(\tau), \tag{C7}$$

where $E_2(\tau)$ is the holomorphic second Eisenstein series,

$$E_2(\tau) = 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{C8}$$

Moreover, in Sec. IV, we use the second ν-derivative (cf. (A.25) in [68])

$$\vartheta_2''(0,\tau) = -\frac{\pi^2}{3}\vartheta_2(0,\tau)[E_2(\tau)+\vartheta_3^4(0,\tau)+\vartheta_4^4(0,\tau)]. \tag{C9}$$

From the basic quartic Riemann identity

$$\frac{1}{2}\sum_{\alpha,\beta}\eta_{\alpha\beta}\prod_{i=1}^4\vartheta[^{\alpha}_{\beta}](g_i,\tau) = -\prod_{i=1}^4\vartheta[^{1/2}_{1/2}](g_i',\tau) \tag{C10}$$

a number of useful identities follow. Here

$$g_1' = \frac{1}{2}(g_1+g_2+g_3+g_4),$$
$$g_2' = \frac{1}{2}(g_1+g_2-g_3-g_4),$$
$$g_3' = \frac{1}{2}(g_1-g_2+g_3-g_4),$$
$$g_4' = \frac{1}{2}(g_1-g_2-g_3+g_4) \tag{C11}$$

are those of [46]. (The identity holds for other sign combinations as well.) For instance, setting $g_1=g_2=g_3=0$, $g_4=\nu$ and expanding in ν, one has

$$\sum_{\alpha,\beta}\eta_{\alpha\beta}\vartheta''[^{\alpha}_{\beta}](0)\vartheta[^{\alpha}_{\beta}]^3(0) = 0. \tag{C12}$$

It will be useful to have the following slightly more general theta identity, which allows for shifts not only in the ν argument (or equivalently in the β characteristic) but also in the α characteristic. It can be proven from the standard one using periodicity properties (see e.g. [68]). For $\sum_i g_i=0$, $\sum_i h_i=0$, the most useful form is to include an additional spin-structure independent denominator:

$$\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta''[^{\alpha}_{\beta}](0)}{\vartheta_1'(0)}\frac{\prod_{i=1}^3\vartheta[^{\alpha+h_i}_{\beta+g_i}](0)}{\prod_{i=1}^3\vartheta[^{1/2+h_i}_{1/2+g_i}](0)} = -\sum_{i=1}^3\frac{\vartheta'[^{1/2+h_i}_{1/2+g_i}](0)}{\vartheta[^{1/2+h_i}_{1/2+g_i}](0)}, \tag{C13}$$

where we set the g_1 of (C11) to zero and relabeled the other $g_i\to g_{i-1}$, $i=2,3,4$. We now turn to two useful special cases.

Special case 1: $h_1=0$, $h_2=1/2$, $h_3=-1/2$:

$$\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta''[^{\alpha}_{\beta}](0)}{\vartheta_1'(0)}\frac{\vartheta[^{\alpha}_{\beta+g_1}](0)\vartheta[^{\alpha+1/2}_{\beta+g_2}](0)\vartheta[^{\alpha-1/2}_{\beta+g_3}](0)}{\vartheta[^{1/2}_{1/2+g_1}](0)\vartheta[^{0}_{1/2+g_2}](0)\vartheta[^{0}_{1/2+g_3}](0)} = -\left(\frac{\vartheta'[^{1/2}_{1/2+g_1}](0)}{\vartheta[^{1/2}_{1/2+g_1}](0)}+\sum_{i=2}^3\frac{\vartheta'[^{0}_{1/2+g_i}](0)}{\vartheta[^{0}_{1/2+g_i}](0)}\right). \tag{C14}$$

In applying this identity, it is useful to note that all theta functions have periodicity one in the upper characteristic.

Special case 2: Let us in addition to the previous assumptions assume $g_1=0$ (untwisted first two-torus). Then the last two terms on the right-hand side cancel, as they must since $\vartheta_1(0)$ must cancel out of the denominators for the expression to remain regular. When $h_1=g_1=0$, we can denote $h_2=-h_3=:h$, and similarly $g_2=-g_3=:g$. This is the familiar case that the sum collapses to a number:

$$\sum_{\alpha,\beta}\eta_{\alpha\beta}\frac{\vartheta''[^{\alpha}_{\beta}](0)\vartheta[^{\alpha}_{\beta}](0)}{\eta^6}\frac{\vartheta[^{\alpha+h}_{\beta+g}](0)\vartheta[^{\alpha-h}_{\beta-g}](0)}{\vartheta[^{1/2+h}_{1/2+g}](0)\vartheta[^{1/2-h}_{1/2-g}](0)} = -4\pi^2. \tag{C15}$$

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