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Perturbative spectrum of the dressed sliver

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We analyze the fluctuations of the dressed sliver solution found in a previous paper, hep-th/0311198, in the operator formulation of Vacuum String Field Theory. We derive the tachyon wave function and then analyze the higher level fluctuations. We show that the dressing is responsible for implementing the transversality condition on the massless vector. In order to consistently deal with the singular $k = 0$ mode we introduce a string midpoint regulator and we show that it is possible to accommodate all the open string states among the solutions to the linearized equations of motion. We finally show how the dressing can give rise to the correct ratio between the energy density of the dressed sliver and the brane tension computed via the three-tachyons-coupling.

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I. INTRODUCTION

In a companion paper [1], in a search for nonperturbative solutions with the appropriate features to represent D-branes, we found a solution of Vacuum String Field Theory (VSFT), the dressed sliver, endowed with some interesting properties, the most important being the possibility to define, via a suitable regularization, a finite energy density. The purpose of this paper is to analyze the perturbative spectrum around this solution.

Before we plunge into the technicalities of the analysis of the spectrum, it is worth summarizing the motivations for this research. VSFT [2,3] is conjectured to represent String Field Theory (SFT) [4] at the tachyon condensation vacuum, and it is expected to represent a new theory, radically different from the SFT constructed on the initial (unstable) vacuum. It was argued that this new vacuum can only be the critical bosonic closed string vacuum. If so it should harbor nonperturbative solutions representing (unstable) D-branes of any dimensions. As it turned out, such solutions were found in copious variety [5–9], lending evidence to the above conjecture. A particularly simple and appealing one (both from the algebraic and geometric point of view) was the sliver.

However some important problems have remained unsolved. First, the energy density of these solutions, in particular, of the sliver, is evanescent and one is obliged to postulate an infinite multiplicative constant in front of the action to account for a physical solution [10–13]. Second, the attempts at finding the perturbative spectrum around these solutions (i.e., the spectrum of the open strings attached to the D-branes) have been successful in the BCFT approach , but it has proven to be remarkably hard (particularly in regards to the task of reconstructing the Virasoro constraints) in the operator approach [16,17].

These failures of the operator approach should not be underestimated. The issue here is not the practical use one can make of the spectrum of, say, the D25-brane obtained in this way, i.e., of the fluctuations around a nonperturbative solution of VSFT. If one wishes to describe some physics of the D25-brane, one had better use the formalism of the initial SFT about the trivial vacuum. The crucial issue here is rather the consistency of VSFT itself. Let us recall that VSFT is a simplified version of SFT, the simplification being determined by the very simple form taken by the Becchi-Rouet-Stora-Tyutin (BRST) charge $\mathcal{Q}$, which is expressed only in terms of the ghost oscillators c_n (see below). This may at first look as an oversimplification: one can derive in a simple and elegant manner the above-mentioned nonperturbative solutions, but one may have to pay a cost in terms of loss of information, which seems to show up when we come to determining the spectrum. To sum up: can we regard the operator formulation of VSFT as a reliable approach or not? We would like to explore in this paper the possibility that the answer to this question is yes. We believe in fact that the true reason for the above failures is that the sliver solution is too singular and needs to be regularized, but once this is properly done the above-mentioned problems disappear.

We have proven in [1] that, using a new type of solution akin to the sliver, but somehow deformed with respect to it, which we dubbed dressed sliver, the problem of the energy density may find a solution. In this paper we continue the analysis started there by studying the perturbative spectrum around the dressed sliver, i.e., the solutions to the linearized equations of motion (LEOM). We prove that not

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only can the full open string mass spectrum be recovered, but that the right Virasoro constraints come out naturally from the LEOM. The key ingredients that make this possible are, on the one hand, the dressing and, on the other hand, a careful analysis about the string midpoint. These are the new elements we introduce in the analysis with respect to [9], which otherwise we follow rather closely. In this paper we limit ourselves to the D25-brane case, although, as shown in [1], the analysis can be extended to lower dimensional branes.

This result seems to justify on a more substantial basis the claim that VSFT has solutions that can be interpreted as D-branes of any dimensions. The existence of such solutions are expected on the basis of the lore that D-branes are sources of closed strings and on the conjecture that the tachyon condensation vacuum corresponds to the closed string vacuum. In turn this adds support to our philosophy that VSFT is a complete theory. As an aside, we notice that the D-brane solutions being expressed in terms of the (initial) open string creation operators, may be considered a clear manifestation of the open-closed string duality advocated by A. Sen [18].

It is fair now to also mention that we have not been able to find an algorithmic way to construct all the solutions to the linearized equations of motion. As a consequence we have explicitly analyzed only the first few levels, although there seem to be no conceivable obstacles to the extension to higher levels. Our analysis of the cohomological structure, still far from being exhaustive, confirms that for dressed sliver solutions physical excitations are concentrated around the "midpoint" $k = 0$ of the continuous basis, as claimed in [17,19] for previous classical solutions of VSFT. We also find an additional set of solutions: they are constructed by means of polynomials of $\xi a^\dagger$ (ξ is the dressing vector) applied to any physical state. As a consequence, any physical state is accompanied by an infinite tower of descendants which seem to contain the same information as far as observables are concerned. These extra solutions might be remnants of the full gauge symmetry of VSFT.

The paper is organized as follows. Section II is a review about the dressed sliver solution. Section III is a general and preliminary presentation of the LEOM in VSFT: we introduce there all the general objects which are needed in the course of our analysis. In Sec. IV we find solutions for the infinite vector **t** and the number G which play a crucial role in the construction of all the open strings states as solutions to the LEOM.

Next, in Sec. V, we analyze the lowest level excitations: the tachyon state is found at the correct mass -shell. The vector excitation is also rather straightforward: we show not only how to define the corresponding string field solution but, in particular, how transversality is implemented thanks to dressing. In order to continue with the analysis of higher level string modes it turns out that the relevant information is concentrated around the midpoint $k = 0$ of the continuous basis. We therefore need a technique to probe this region. To this end in Sec. VI we introduce a new regulator η by means of which we are able to control the singularity at the k-basis midpoint. In Sec. VII we determine the solutions corresponding to the string modes of level 2 and 3 of the canonical quantization. In Sec. VIII we briefly discuss the cohomology properties of the previous states. In Sec. IX we show that states that represent physical modes are accompanied by an infinite tower of dressing excitations which, as far as we could verify, does not alter any physical observable. In Sec. X, we evaluate the three-tachyon coupling and comment on how the well-known puzzle that links such coupling to the energy density of the classical solution may be resolved in our approach, thanks to the dressing. The copious calculations that underpin the above results are presented or summarized in several appendices.

II. THE DRESSED SLIVER: A REVIEW

In order to render this paper as self-contained as possible, in this section we review the main properties of the dressed sliver solution.

To start with we recall some formulas relevant to VSFT. The action is

$$S(\Psi) = -\frac{1}{g_0^2}\left(\frac{1}{2}\langle\Psi|\mathcal{Q}|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right), \quad (2.1)$$

where

$$\mathcal{Q} = c_0 + \sum_{n>0}(-1)^n(c_{2n} + c_{-2n}). \quad (2.2)$$

The equation of motion is

$$\mathcal{Q}\Psi = -\Psi * \Psi, \quad (2.3)$$

and the ansatz for nonperturbative solutions has the factorized form

$$\Psi = \Psi_m \otimes \Psi_g, \quad (2.4)$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then Eq. (2.3) splits into

$$\mathcal{Q}\Psi_g = -\Psi_g *_g \Psi_g, \quad (2.5)$$

$$\Psi_m = \Psi_m *_m \Psi_m, \quad (2.6)$$

where $*_g$ and $*_m$ refer to the star product involving only the ghost and matter part. The action for this type of solution becomes

$$S(\Psi) = -\frac{1}{6g_0^2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\langle\Psi_m|\Psi_m\rangle. \quad (2.7)$$

$\langle\Psi_m|\Psi_m\rangle$ is the ordinary inner product, $\langle\Psi_m|$ being the bpz conjugate of $|\Psi_m\rangle$ (see below).

The $*_m$ product is defined as follows

$$ {}_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 = {}_3\langle\Psi_1 *_m \Psi_2|, \tag{2.8}$$

where the three strings vertex V_3 is

$$|V_3\rangle_{123} = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \times \exp(-E)|0, p\rangle_{123}, \tag{2.9}$$

with

$$E = \sum_{a,b=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\right). \tag{2.10}$$

Summation over the Lorentz indices $\mu, \nu = 0, \ldots, 25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the nonzero modes matter oscillators of the ath string, which satisfy

$$[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}, \qquad m, n \geq 1. \tag{2.11}$$

$p_{(r)}$ is the momentum of the ath string and $|0, p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings with definite c.m. momentum . $|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ ($m \geq 1$) and it is eigenstate of the momentum operator $\hat{p}_{(a)}^{\mu}$ with eigenvalue $p_{(a)}^{\mu}$. The normalization is

$$\langle p_{(a)}|p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p + p'). \tag{2.12}$$

The symbols V_{nm}^{ab} denote the coefficients computed in [20–25]. We will use them in the notation of Appendix A and B of [6].

To complete the definition of the $*_m$ product we must specify the bpz conjugation properties of the oscillators

$$bpz(a_n^{(a)\mu}) = (-1)^{n+1}a_{-n}^{(a)\mu}. \tag{3.13}$$

The sliver solution to Eq. (2.6) is given by

$$|\Xi\rangle = \mathcal{N}e^{-\frac{1}{2}a^\dagger S a^\dagger}|0\rangle, \qquad a^\dagger S a^\dagger = \sum_{n,m=1}^{\infty}a_n^{\mu\dagger}S_{nm}a_m^{\nu\dagger}\eta_{\mu\nu}, \tag{2.14}$$

where the Neumann matrix S is as follows. Let us introduce the twisted matrices $X = CV^{11}, X_+ = CV^{12}$ and $X_- = CV^{21}$, together with $T = CS = SC$, where $C_{nm} = (-1)^n\delta_{nm}$. They are mutually commuting and T is given by

$$T = \frac{1}{2X}[1 + X - \sqrt{(1 + 3X)(1 - X)}] \tag{2.15}$$

The normalization constant $\mathcal{N}$ is

$$\mathcal{N} = [\mathrm{Det}(1 - \Sigma\mathcal{V})]^{D/2}, \tag{2.16}$$

and

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}, \tag{2.17}$$

The norm of the sliver is

$$\langle\Xi|\Xi\rangle = \frac{\mathcal{N}}{2}[\det(1 - S^2)]^{D/2}. \tag{2.18}$$

Both Eqs. (2.16) and (2.18) are ill-defined and need to be regularized, after which they both result to be vanishing [10].

The dressed sliver is a sort of deformation of the sliver, which is obtained as follows. We first introduce the infinite real vector $\xi = \{\xi_n\}$, which is chosen to satisfy the condition

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \tag{2.19}$$

where $\rho_1(\rho_2)$ are the Fock-space projectors into the right (left) half of the string (see Appendix A). Next we set

$$\langle\xi|\frac{1}{1 - T^2}|\xi\rangle = 1, \qquad \langle\xi|\frac{T}{1 - T^2}|\xi\rangle = \kappa. \tag{2.20}$$

The first equation sets the normalization of ξ. The second tells us that κ is a real negative number, see [1].

The *dressed sliver* solution is given by an ansatz similar to (2.14)

$$|\hat{\Xi}\rangle = \hat{\mathcal{N}}e^{-1/2a^\dagger\hat{S}a^\dagger}|0\rangle, \tag{2.21}$$

with S replaced by

$$\hat{S} = S + R, \qquad R_{nm} = \frac{1}{\kappa + 1}[\xi_n(-1)^m\xi_m + \xi_m(-1)^n\xi_n]. \tag{2.22}$$

Alternatively

$$\hat{T} = T + P, \qquad P = \frac{1}{\kappa + 1}(|\xi\rangle\langle\xi| + |C\xi\rangle\langle C\xi|). \tag{2.23}$$

As shown in [1] the matrix $\hat{S}$ satisfies the equation

$$\hat{S} = V^{11} + (V^{12}, V^{21})(1 - \hat{\Sigma}\mathcal{V})^{-1}\hat{\Sigma}\begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix}, \tag{2.24}$$

where

$$\hat{\Sigma} = \begin{pmatrix} \hat{S} & 0 \\ 0 & \hat{S} \end{pmatrix}, \tag{2.25}$$

which is required for $\hat{\Xi}$ to be a projector.

The normalization is given by

$$\hat{\mathcal{N}} = [\mathrm{Det}(1 - \Sigma\mathcal{V})]^{D/2}\left(\frac{1}{\kappa + 1}\right)^{D}. \tag{2.26}$$

This determinant, after regularization, turns out to vanish [10]. The action corresponding to $\hat{\Xi}$ is ill-defined. In [1] we devised a way to regularize the action by introducing a deformation parameter ϵ multiplying R as follows

$$|\hat{\Xi}\rangle \to |\hat{\Xi}_\epsilon\rangle, \qquad \hat{S} \to \hat{S}_\epsilon = S + \epsilon R. \tag{2.27}$$

Therefore $\hat{\Xi}_\epsilon$ coincides with the sliver for $\epsilon = 0$ and with the dressed sliver for $\epsilon = 1$. For $\epsilon \neq 0, 1$, the new state is in general not a projector. However its limit for $\epsilon \to 1$ allows us to define a finite norm $\langle\hat{\Xi}|\hat{\Xi}\rangle$. This goes according to the following recipe. We regularize the determinants involving the sliver matrices X, X_+, X_-, and T by truncating them at a finite level L. In the limit $L \to \infty$ these determinants behave like fractional powers of L. The next ingredient consists in using the nested limits prescription

$$\lim_{\epsilon_1\to 1}(\lim_{\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon 1}|\hat{\Xi}_{\epsilon 2}\rangle), \tag{2.28}$$

which gives

$$\frac{1}{\langle 0|0\rangle}\lim_{\epsilon_1\to 1}(\lim_{\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon 1}|\hat{\Xi}_{\epsilon 2}\rangle) \tag{2.29}$$

$$= \lim_{\epsilon_1\to 1}\left[\frac{1}{(\kappa+1)^2}\right]^{D/2}\left(\frac{L^{-5/36}}{1-\epsilon}\right)^D + \dots$$
$$= \left[\frac{1}{(\kappa+1)^2 s^2}\right]^{D/3}, \tag{2.30}$$

provided we use the last ingredient, i.e., the tuning

$$1 - \epsilon_1 = sL^{-5/36} \tag{2.31}$$

in the limit $\epsilon_1 \to 1, L \to \infty$. This recipe guarantees that the equation of motion is satisfied when evaluated on a very large class of states (including the entire Fock-space and the dressed sliver itself).

The ghost part of the solution, i.e., a solution to (2.5), can be found much in the same way. It has the form of a squeezed state

$$|\hat{\tilde{\Xi}}_{\tilde{\epsilon}}\rangle = \hat{\tilde{\mathcal{N}}}_{\tilde{\epsilon}} e^{c^\dagger \hat{\tilde{S}}_\epsilon b^\dagger} c_1|0\rangle, \tag{2.32}$$

with

$$\hat{\tilde{S}}_{\tilde{\epsilon}} = \tilde{S} + \tilde{\epsilon}\tilde{R}, \qquad \tilde{R} = \frac{1}{\tilde{\kappa}+1}(|C\delta\rangle\langle\beta| + |\delta\rangle\langle C\beta|)$$

for $\epsilon = 1$. Here we have introduced directly the deformation parameter $\tilde{\epsilon}$, which is necessary in order to regularize the action. The vectors β and δ are such that

$$\tilde{\rho}_1\beta = \tilde{\rho}_1\delta = 0, \qquad \tilde{\rho}_2\beta = \beta, \qquad \tilde{\rho}_2\delta = \delta, \tag{2.33}$$

with

$$\langle\beta|\frac{1}{1-\tilde{T}^2}|\delta\rangle = 1, \qquad \langle\beta|\frac{\tilde{T}}{1-\tilde{T}^2}|\delta\rangle = \tilde{\kappa}. \tag{2.34}$$

The first equation fixes the normalization of β and δ, and the second is the definition of $\tilde{\kappa}$. Once again, the quantity $\langle\hat{\tilde{\Xi}}_1|\hat{\tilde{\Xi}}_1\rangle$ is ill-defined but, by using an ordered limiting prescription, one can show that

$$\lim_{\tilde{\epsilon}_1\to 1}(\lim_{\tilde{\epsilon}_2\to 1}\langle\hat{\tilde{\Xi}}_{\tilde{\epsilon}_1}|\mathcal{Q}|\hat{\tilde{\Xi}}_{\tilde{\epsilon}_2}\rangle) = (\tilde{\kappa}+1)^2\tilde{s}^2, \tag{2.35}$$

provided

$$1 - \tilde{\epsilon}_1 = \tilde{s}L^{-11/36}, \tag{2.36}$$

for some constant $\tilde{s}$. Equation (2.35) together with (2.30) allows us to define a finite value for the action (2.7). This relates the physical coupling constant to the regularization procedure (see below).

From now on, to simplify the notation, we often understand the limit for $\epsilon \to 1$. In the following we will therefore represent the dressed sliver solution as the factorized product $|\hat{\Xi}\rangle \otimes |\hat{\tilde{\Xi}}\rangle$. We will however reintroduce the ϵ-dependence whenever a risk of ambiguity arises.

To end this summary we would like to make a comment on the eigenvalues of the Neumann matrix of the dressed sliver, which hopefully clarifies some of the formulas used below. As we have remarked, this Neumann matrix does not commute with the sliver matrix T, so they cannot share their eigenvectors. However much can be said about the eigenvalues of $\hat{T}$. If the vector ξ is square-summable (as we suppose), P is a compact operator. Perturbing T by a compact operator does not modify its continuous spectrum [26]. Therefore $\hat{T}$ must have the same continuous spectrum as T. In addition, however, it might have isolated eigenvalues of its own. It is easy to show that $\hat{T}$ does develop an extra discrete eigenvalue 1. This fact can be easily guessed from the result of [1]

$$\det(1 - \hat{T}_\epsilon) = (1-\epsilon)^2\det(1-T), \tag{2.37}$$

which suggests that $\hat{T}$ has a doubly degenerate eigenvalue 1. It turns out that the corresponding eigenvectors have definite twist and are given by

$$|\chi_\pm\rangle = \frac{1}{1-T}(1 \pm C)|\xi\rangle, \tag{2.38}$$

as can be easily proved by applying (2.23) to the above expression.

This is in fact the reason why the bpz norm of the dressed sliver can be made finite by appropriately tuning the vanishing behavior induced by the midpoint $k = 0$ and the divergent one induced by this discrete eigenvalue. We will see, in the study of the spectrum, that these new eigenvectors are responsible for creating an infinite tower of "descendants" of every physical state, with same mass and same polarization conditions as the initial state.

III. THE LINEARIZED EQUATION OF MOTION

Let us call for simplicity $\Phi_0 = |\hat{\Xi}\rangle \otimes |\hat{\hat{\Xi}}\rangle$ the overall (matter + ghost) solution we have just reviewed. If we write $\Psi = \Phi_0 + \phi$, the action becomes

$$S(\Psi) = S(\Phi_0) - \frac{1}{g_0^2}\left(\frac{1}{2}\langle\phi|\mathcal{Q}_0|\phi\rangle + \frac{1}{3}\langle\phi|\phi * \phi\rangle\right), \tag{3.1}$$

where

$$\mathcal{Q}_0\phi = \mathcal{Q}\phi + \Phi_0 * \phi + \phi * \Phi_0. \tag{3.2}$$

The equation of motion for small fluctuations around the solution Φ_0 is therefore

$$\mathcal{Q}\,\phi + \Phi_0 * \phi + \phi * \Phi_0 = 0. \tag{3.3}$$

The solutions to this LEOM are expected to encompass all the modes of the open strings with endpoints on the D25-brane represented by Φ_0 as well as all the states which are $\mathcal{Q}_0$-exact.

To find the solutions to (3.3) we follow [9], but we introduce some significant changes: the dressing and the midpoint regularization. The ansatz for a general solution of momentum p is as follows

$$\begin{aligned}|\hat{\phi}_e(P, \mathbf{t}, p)\rangle &= \mathcal{N}_e\mathcal{P}(a^\dagger)\exp[-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu]|\hat{\Xi}_e\rangle \otimes |\hat{\hat{\Xi}}\rangle e^{ipx}\\ &\equiv |\varphi_e(\mathcal{P}, \mathbf{t}, p)\rangle \otimes |\hat{\hat{\Xi}}\rangle,\end{aligned} \tag{3.4}$$

where $\mathbf{t} = \{t_n\}$, $\mathcal{P}(a^\dagger)$ is some polynomial of expressions of the type $\sum_n \zeta_n a_n^\dagger$, and

$$\hat{p}e^{ipx} = pe^{ipx}, \qquad bpz(\hat{p}) = -\hat{p}.$$

We will often drop the labels $\mathbf{t}$, $\mathcal{P}$ and p when no ambiguities are possible. The factorized form of (3.4) allows us to split the linearized equation of motion into ghost and matter part

$$\mathcal{Q}\,|\hat{\hat{\Xi}}\rangle + |\hat{\hat{\Xi}}\rangle *_g |\hat{\hat{\Xi}}\rangle = 0, \tag{3.5}$$

$$|\hat{\varphi}_e\rangle = |\hat{\Xi}\rangle *_m |\hat{\varphi}_e\rangle + |\hat{\varphi}_e\rangle *_m |\hat{\Xi}\rangle. \tag{3.6}$$

The ghost part will remain the same throughout the paper, and from now on we simply forget it and concentrate on the matter part.

In the above equation $|\hat{\Xi}_e\rangle$ formally coincides with $|\hat{\Xi}_\epsilon\rangle$, with ϵ replaced by e. The reason for this seemingly bizarre change of notation is because the parameter e plays a different role from ϵ. While ϵ is a deformation parameter and we are only interested in the limit $\epsilon \to 1$ [recall that for $\epsilon \neq 0, 1$ $\hat{\Xi}_\epsilon$ is not a solution to (2.6)], we will find that the linearized equation of motion can be solved for any value of e. The reason of this lies in a result we found in [1], see Eq. (4.15) there,

$$|\hat{\Xi}_\epsilon\rangle * |\hat{\Xi}_e\rangle = |\hat{\Xi}_{\epsilon\star e}\rangle \tag{3.7}$$

where $|\hat{\Xi}_\epsilon\rangle$ is the same as in (2.27) with

$$\hat{\mathcal{N}}_\epsilon = \mathcal{N}\left(\frac{1 + (1-\epsilon)\kappa}{\kappa + 1}\right)^D,$$
$$\hat{\mathcal{N}}_e = \mathcal{N}\left(\frac{1 + (1-e)\kappa}{\kappa + 1}\right)^D,$$

and

$$\epsilon \star e = \frac{\epsilon e}{1 + (1-\epsilon)(1-e)\kappa}. \tag{3.8}$$

The $\star$-multiplication is isomorphic to ordinary multiplication between real numbers: using the reparametrization

$$f_\epsilon = \frac{1 + (1-\epsilon)\kappa}{\epsilon} = 1 + (\kappa + 1)\frac{1-\epsilon}{\epsilon}, \tag{3.9}$$

it is easy to check that $f_{\epsilon\star e} = f_\epsilon f_e$.

It is evident from (3.7) that

$$|\hat{\Xi}_e\rangle * |\hat{\Xi}\rangle = |\hat{\Xi}\rangle * |\hat{\Xi}_e\rangle = |\hat{\Xi}_e\rangle, \tag{3.10}$$

for any value of the parameter e. This basic equality will allow us to construct solutions to the LEOM that contain the free parameter e. We anticipate that eventually, in order to guarantee finiteness of the three-tachyons coupling, e will have to be set to 1.

Let us see all this in more detail, i.e., let us find the general conditions for solving the LEOM (3.3). To this end we introduce the general state

$$|\hat{\varphi}_{e,\beta}\rangle = \mathcal{N}_e\exp\left[-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu - \sum_{n\geq 1}\beta_n^\mu a_n^{\nu\dagger}\eta_{\mu\nu}\right]|\hat{\Xi}_e\rangle e^{ipx}, \tag{3.11}$$

where, with respect to (3.4), we have inserted the parameters β_n^μ. By differentiating with respect to it the appropriate number of times and setting afterwards $\beta_n^\mu = 0$, we will be able to generate any polynomial in $a_n^\dagger$ and therefore reproduce any state of the form (3.4).

Now we need

$$\begin{aligned}{}_1\langle\hat{\Xi}_\epsilon|_2\langle\hat{\varphi}_{e,\beta}|V_3\rangle = \frac{\hat{\mathcal{N}}_e\hat{\mathcal{N}}_e}{(\det\hat{K}_\epsilon e)^{D/2}}&\exp\Big[-\chi^T\hat{K}_{\epsilon e}^{-1}\lambda - \frac{1}{2}\chi^T\hat{K}_{\epsilon e}^{-1}\chi - \frac{1}{2}\lambda^T\mathcal{V}\hat{\mathcal{K}}_{\epsilon\mathcal{E}}^{-1}\lambda\Big]\\ &\times\exp\Big[-\frac{1}{2}\sum_{n,m\geq 1}a_n^{(3)\dagger}V_{n,m}^{33}a_m^{(3)\dagger} - a_n^{(3)\dagger}(\mathbf{v}_{0n} - \mathbf{v}_{+n})p\Big]|0\rangle_3 e^{-pV_{00}p}e^{ipx},\end{aligned} \tag{3.12}$$

where we introduced

$$\hat{\mathcal{K}}_{\epsilon e} = 1 - \hat{S}_{\epsilon e}\mathcal{V}, \qquad \hat{S}_{\epsilon e} = \begin{pmatrix} \hat{S}_\epsilon & 0 \\ 0 & \hat{S}_e \end{pmatrix},$$

together with

$$\chi = \begin{pmatrix} V^{21}a^{(3)\dagger} + p(\mathbf{v}_+ - \mathbf{v}_-) \\ V^{12}a^{(3)\dagger} + p(\mathbf{v}_- - \mathbf{v}_0) \end{pmatrix}, \qquad \lambda = C\begin{pmatrix} 0 \\ \beta - p\mathbf{t} \end{pmatrix}. \tag{3.13}$$

In all these formulas we have introduced infinite vectors β^μ, $\mathbf{t}$, $\mathbf{v}_0$, $\mathbf{v}_+$, $\mathbf{v}_-$ with components

$$\beta^\mu_n, \qquad t_n, \qquad \mathbf{v}_{0n} = V^{11}_{0n} = V^{22}_{0n}, \qquad \mathbf{v}_{+n} = V^{12}_{0n}, \qquad \mathbf{v}_{-n} = V^{21}_{0n}, \tag{3.14}$$

respectively. We are interested in the above formula in the limit $\epsilon \to 1$, while keeping e fixed.

Let us recall from [1], Appendix B.2, that

$$\hat{\mathcal{N}}_\epsilon = [\mathrm{Det}(1 - \Sigma\mathcal{V})]^{D/2}\left(\frac{f_\epsilon}{\kappa + f_\epsilon}\right)^D,$$

$$\mathrm{Det}(1 - \hat{\Sigma}_{\epsilon e}\mathcal{V}) = \left[\frac{\kappa + f_\epsilon f_e}{(\kappa + f_\epsilon)(\kappa + f_e)}\right]^2 \mathrm{Det}(1 - \Sigma\mathcal{V}),$$

from which we get the important relation

$$\lim_{\epsilon\to 1}\frac{\hat{\mathcal{N}}_\epsilon}{(\sqrt{\det}\hat{\mathcal{K}}_{\epsilon e})^D} = \lim_{f_\epsilon \to 1}\left[\frac{f_\epsilon(\kappa + f_e)}{\kappa + f_\epsilon f_e}\right]^D = 1. \tag{3.15}$$

To start with, let us consider the simplest example, i.e., $\beta = 0$, which means $\mathcal{P}(a^\dagger) = 1$ in (3.4) and define the candidate for the tachyon wave function. We will denote $\hat{\varphi}_e(1, \mathbf{t}, p)$ by $\hat{\varphi}_e(\mathbf{t}, p)$ or simply by $\hat{\varphi}_e$. We find that (3.12) takes the following form

$$_1\langle\hat{\Xi}|_2\langle\hat{\varphi}_e|V_3\rangle = \lim_{\epsilon\to 1} {}_1\langle\hat{\Xi}_\epsilon|_2\langle\hat{\varphi}_e|V_3\rangle = \exp\left[-\mathbf{t}a^\dagger p - \frac{1}{2}G_1 p^2\right]|\hat{\Xi}_e\rangle e^{ipx}, \tag{3.16}$$

where $\mathbf{t}$ is a solution to

$$\mathbf{t} = \mathbf{v}_0 - \mathbf{v}_+ + (V^{12}, V^{21})\hat{\mathcal{K}}^{-1}_{1e}\hat{S}_{1e}\begin{pmatrix} \mathbf{v}_+ - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_0 \end{pmatrix} + (V^{12}, V^{21})\hat{\mathcal{K}}^{-1}_{1e}C\begin{pmatrix} 0 \\ \mathbf{t} \end{pmatrix} \tag{3.17}$$

and

$$G_1 = 2V_{00} + (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}^{-1}_{e1}\hat{S}_{e1}\begin{pmatrix} \mathbf{v}_+ - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_0 \end{pmatrix} + 2(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}^{-1}_{e1}C\begin{pmatrix} 0 \\ \mathbf{t} \end{pmatrix} + (0, \mathbf{t})C\mathcal{V}\hat{\mathcal{K}}^{-1}_{e1}C\begin{pmatrix} 0 \\ \mathbf{t} \end{pmatrix}, \tag{3.18}$$

where $\hat{\mathcal{K}}_{1e}$ and $\hat{S}_{1e}$ equal $\hat{\mathcal{K}}_{\epsilon e}$ and $\hat{S}_{\epsilon e}$ when $\epsilon = 1$, respectively.

If we repeat the same derivation for the other star product, we find

$$_1\langle\hat{\varphi}_e|_2\langle\hat{\Xi}|V_3\rangle = \lim_{\epsilon\to 1} {}_1\langle\hat{\varphi}_e|_2\langle\hat{\Xi}_\epsilon|V_3\rangle = \exp\left[-\mathbf{t}a^\dagger p - \frac{1}{2}G_2 p^2\right]|\hat{\Xi}_e\rangle e^{ipx}, \tag{3.19}$$

where, this time, $\mathbf{t}$ is a solution to

$$\mathbf{t} = \mathbf{v}_0 - \mathbf{v}_- - (V^{12}, V^{21})\hat{\mathcal{K}}^{-1}_{e1}\hat{S}_{e1}\begin{pmatrix} \mathbf{v}_0 - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_- \end{pmatrix} + (V^{12}, V^{21})\hat{\mathcal{K}}^{-1}_{e1}C\begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix} \tag{3.20}$$

and

$$G_2 = 2V_{00} + (\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\hat{\mathcal{K}}^{-1}_{e1}\hat{S}_{e1}\begin{pmatrix} \mathbf{v}_0 - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_- \end{pmatrix} - 2(\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\hat{\mathcal{K}}^{-1}_{e1}C\begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix} + (\mathbf{t}, 0)C\mathcal{V}\hat{\mathcal{K}}^{-1}_{\mathcal{E}1}C\begin{pmatrix} \mathbf{t} \\ 0 \end{pmatrix}, \tag{3.21}$$

where $\hat{\mathcal{K}}_{e1}$ and $\hat{S}_{e1}$ equal $\hat{\mathcal{K}}_{e\epsilon}$ and $\hat{S}_{e\epsilon}$ when $\epsilon = 1$, respectively.

The two couples of expressions (3.17) and (3.20) and (3.18) and (3.21) are formally different. Of course they must give rise to the same result. If we require twist invariance for $\mathbf{t}$, i.e., $C\mathbf{t} = \mathbf{t}$, it is easy to see that the two couples of equations collapse to a single one. However, for reasons that will become clear later on, we will not require twist invariance for $\mathbf{t}$ (see Sec. V C for more comments on this point). This is why we wrote the two couples of equations explicitly. In general, therefore, $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$. Hermiticity of the string field requires that $C\mathbf{t} = \mathbf{t}^*$, i.e., $\mathbf{t}^*_+ = \mathbf{t}_+$ and $\mathbf{t}^*_- = -\mathbf{t}_-$.

We remark now that, if the above equations have a nontrivial solution for $\mathbf{t}$ and

$$e^{-Gp^2/2} = \frac{1}{2}, \tag{3.22}$$

where $G = G_1 = G_2$, then $|\hat{\varphi}_e\rangle$ is a solution to the LEOM (3.6).

We also notice, for future use, that for a state of the general form (3.4) to satisfy the LEOM, the equation for $\mathbf{t}$ and G remain the same. The presence of a polynomial $\mathcal{P}(a^\dagger)$ does not affect the exponents, but only implies new conditions for the parameters in $\mathcal{P}(a^\dagger)$ (see below).

IV. SOLUTION FOR t AND G

In this section we study the solutions to Eqs. (3.17) and (3.20) and evaluate G. Since, due to the structure of these equations, *a priori* one cannot exclude the possibility of a singularity in $1-\epsilon$, we insert ϵ at the right places and take the limit $\epsilon \longrightarrow 1$ on the solution.

A. The solutions for t

Let us see first the relation between these two equations. We write $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$, where $C\mathbf{t}_\pm = \pm\mathbf{t}_\pm$ and apply C to (3.17). Keeping track of the ϵ dependence, we obtain

$$\mathbf{t}_+ - \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ - \mathbf{t}_- \end{pmatrix}. \tag{4.1}$$

Doing the same with (3.20) we get

$$\mathbf{t}_+ - \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_+ - (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_0 - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_+ \end{pmatrix} + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} \mathbf{t}_+ - \mathbf{t}_- \\ 0 \end{pmatrix}. \tag{4.2}$$

Next we introduce the operator σC, where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$(\sigma C)^2 = 1, \qquad (\sigma C)\hat{\mathcal{K}}_{e\epsilon}^{-1}(\sigma C) = \hat{\mathcal{K}}_{\epsilon e}^{-1}, \qquad (\sigma C)\hat{\mathcal{T}}_{e\epsilon}(\sigma C) = \hat{\mathcal{T}}_{\epsilon e}. \tag{4.3}$$

Therefore, by suitably inserting $(\sigma C)^2$ in (4.2), applying the above transformations and applying C to the resulting equation we find

$$\mathbf{t}_+ + \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ + \mathbf{t}_- \end{pmatrix}. \tag{4.4}$$

Taking the sum and the difference of (4.1) and (4.4) we find separate equations for $\mathbf{t}_+$ and $\mathbf{t}_-$:

$$\mathbf{t}_+ = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix}, \tag{4.5}$$

$$\mathbf{t}_- = (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix}. \tag{4.6}$$

Now we have to solve these two equations. The rather lengthy calculations are left for Appendix B. From the results therein one can see that, for $\epsilon = 1$ and setting $\mathbf{t}_+ = \mathbf{t}_0 + \mathbf{t}\alpha$, the first equation reduces to

$$\mathbf{t}_0 = 3\frac{T^2 - T + 1}{T+1}\mathbf{v}_0, \tag{4.7}$$

$$\left[1 - \frac{1}{\kappa + f_e}(|\xi\rangle + |C\xi\rangle)\langle\xi|\frac{f_e + T}{1 - T^2}\right]|\mathbf{t}\alpha\rangle = 0. \tag{4.8}$$

where $\mathbf{t}_0$ is the result obtained in [9] (multiplied by $\sqrt{2}$). It is easy to see that (4.8) has the general solution

$$\mathbf{t}_\alpha = \alpha\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle(1 + C)\xi, \tag{4.9}$$

for any number α. The factor $\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle$ has been introduced for later convenience.

As for Eq. (4.6) for $\epsilon = 1$ it has a nontrivial solution

$$\mathbf{t}_- = \beta(1 - C)\xi, \tag{4.10}$$

with arbitrary β. This solution turns out to have an important role (see below). In conclusion we can say that at $\epsilon = 1$ the solution for $\mathbf{t}$ can be written as

$$\mathbf{t} = \mathbf{t}_0 + \alpha\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle(1 + C)\xi + \beta(1 - C)\xi, \tag{4.11}$$

for arbitrary constants α and β.

B. Calculation of *G*

Once again, in order to compute G, we reintroduce the deformation parameter ϵ as in the previous section (see Appendix B). We rewrite Eqs. (3.18) and (3.21) as follows

$$G_1 = 2V_{00} + (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + 2(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ - \mathbf{t}_- \end{pmatrix} + (0, \mathbf{t}_+ + \mathbf{t}_-)\mathcal{M}\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ - \mathbf{t}_- \end{pmatrix} \tag{4.12}$$

and

$$G_2 = 2V_{00} + (\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\hat{\mathcal{K}}_{e\epsilon}^{-1}\hat{\mathcal{T}}_{e\epsilon}\begin{pmatrix} \mathbf{v}_0 - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_+ \end{pmatrix} - 2(\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\hat{\mathcal{K}}_{e\epsilon}^{-1}\begin{pmatrix} \mathbf{t}_+ - \mathbf{t}_- \\ 0 \end{pmatrix} + (\mathbf{t}_+ + \mathbf{t}_-, 0)\mathcal{M}\hat{\mathcal{K}}_{e\epsilon}^{-1}\begin{pmatrix} \mathbf{t}_+ - \mathbf{t}_- \\ 0 \end{pmatrix}. \tag{4.13}$$

Using (4.11) we obtain

$$G_1 = G_0 - 2(f_\epsilon - 1)\frac{\kappa + f_e}{\kappa + f_\epsilon f_e}\left[\alpha(1 - \kappa\alpha) \times \left(\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)^2 + \beta\left(\kappa\beta + \langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)\right] \tag{4.14}$$

and

$$G_2 = G_0 - 2(f_\epsilon - 1)\frac{\kappa + f_e}{\kappa + f_\epsilon f_e}\bigg[\alpha(1-\kappa\alpha) \times \bigg(\langle \mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\bigg)^2 + \beta\bigg(\kappa\beta - \langle \mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\bigg)\bigg]. \tag{4.15}$$

Therefore, for $\epsilon = 1$ we obtain $G_1 = G_2 = G_0$. Naive manipulations of the relevant formulas lead to the result $G_0 = 0$. However G_0 contains two divergent terms, which need to be regularized. As shown by Hata *et al.* [9,27,28], using level truncation one obtains[1] $G_0 = 2\ln 2$.

V. THE TACHYON AND VECTOR EXCITATIONS

After a long preparation we are now ready to start the analysis of the fluctuations around the dressed sliver.

A. The tachyon excitation

From the results of the previous section it follows that string fields of the form

$$|\hat{\varphi}_e(\mathbf{t}, p)\rangle = \mathcal{N}_e \exp\bigg(-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu\bigg)|\hat{\Xi}_e\rangle e^{ipx} \tag{5.1}$$

with $\mathbf{t}$ as in (4.11), satisfy the LEOM when the momentum fulfills the mass-shell condition $m^2 = -p^2 = -1$. This solution depends on three arbitrary parameters e, α, and β. Eventually we shall see that in fact we have to set $e = 1$. As we will see, the other two parameters never enter the evaluation of physical quantities. There is one more question. We expect the tachyon to be represented by a twist-even state, and we already noticed that (5.1) does not have definite twist parity. We will see at the end of this section how to settle this problem.

B. The vector excitation

Fluctuations other than the tachyon can be obtained by considering nontrivial polynomials in Eq. (3.4). The polynomial will consist of sum of monomials of the type

$$d^{\mu_1\cdots\mu_p}\langle\zeta_1 a_{\mu_1}^\dagger\rangle\ldots\langle\zeta_p a_{\mu_p}^\dagger\rangle, \tag{5.2}$$

where $\langle\zeta_i a^{\mu_i\dagger}\rangle = \sum_{n>0}\zeta_{in}a_n^{\mu_i\dagger}$. As it turns out the ϵ-dependence is trivial as far as higher fluctuations are concerned, therefore we drop it throughout.

Let us find the level one state, corresponding to the massless vector. We start with the following ansatz for the matter part

$$|\hat{\varphi}_{e,v}(d^\mu, \mathbf{t}, p)\rangle = \mathcal{N}_v\mathcal{N}_e d^\mu\langle(1-C)\zeta a_\mu^\dagger\rangle e^{-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu},$$
$$|\hat{\Xi}_e\rangle e^{ipx} = \mathcal{N}_v d^\mu\langle(1-C)\zeta a_\mu^\dagger\rangle|\hat{\varphi}_e(\mathbf{t}, p)\rangle, \tag{5.3}$$

with $\rho_2\zeta = \zeta$ and $\rho_1\zeta = 0$.

Using the results of Appendices D and E we obtain

$$|\hat{\varphi}_{e,v}\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}_{e,v}\rangle = e^{-Gp^2/2}\bigg[d^\mu\langle(1-C)\zeta a_\mu^\dagger\rangle + \frac{1}{\kappa + f_e}\langle\xi|\frac{f_e + T}{1-T^2}|\zeta\rangle d^\mu\langle(1-C)\xi a_\mu^\dagger\rangle + 2\beta(p\cdot d)\langle\xi|\frac{\kappa - T}{1-T^2}|\zeta\rangle\bigg]\mathcal{N}_v|\hat{\varphi}_e(\mathbf{t}, p)\rangle. \tag{5.4}$$

From this result we see that in order to satisfy the LEOM we have to assume that $p^2 = 0$ and to impose the transversality condition

$$p\cdot d = 0. \tag{5.5}$$

Therefore we recover the massless vector state with the correct transversality condition. This result is independent of the value of e. In order to satisfy the LEOM we also have to impose

$$\langle\xi|\frac{f_e + T}{1-T^2}|\zeta\rangle = 0. \tag{5.6}$$

This is to be understood as a condition on the vector ζ and as such it is easy to comply with it. For reasons that will become clear later, eventually we will set $e = 1$. In this case (5.6) becomes simply

$$\langle\xi|\frac{1}{1-T}|\zeta\rangle = 0,$$

which is the condition of orthogonality to the extra eigenvector(s) of the dressed sliver (2.38). To conclude we remark that dressing is essential in order to obtain the transversality condition.

C. Twist parity

Let us see how to implement the requirement that the solutions to the LEOM have definite twist parity.

The twist operator Ω acts on Fock-space mode operators as

$$\Omega(a, b, c)_n\Omega = (-1)^n(a, b, c)_n = (Ca, Cb, Cc)_n, \tag{5.7}$$

and satisfies $\Omega|p\rangle = |p\rangle$. Acting on the state (5.1) one obtains

$$\Omega|\hat{\varphi}_e(\mathbf{t}, p)\rangle = |\hat{\varphi}_e(C\mathbf{t}, p)\rangle. \tag{5.8}$$

It follows that the states we have constructed (tachyon and vector) in general are not eigenstates of Ω. This is in contrast to the expectation that the tachyon should be twist-even, the vector twist-odd, etc. Now, from the properties of the VSFT action it follows that if some string field

[1] Our definitions for $\mathbf{t}$ and G differ from those in [9] by factors of $\sqrt{2}$ and 2, respectively, see Appendix A.

$|\Psi\rangle$ is a solution to the LEOM, then $\Omega|\Psi\rangle$ is also a solution. Using this we can define twist parity eigenstates in the following way:

Tachyon ($\Omega = 1$)
$$|\hat{\varphi}_t(\mathbf{t}, p)\rangle = \frac{1}{2}(1+\Omega)|\hat{\varphi}_e(\mathbf{t}, p)\rangle \tag{5.9}$$

Vector ($\Omega = -1$)
$$|\hat{\varphi}_v(d^\mu, \mathbf{t}, p)\rangle = \frac{1}{2}(1-\Omega)|\hat{\varphi}_{e,v}(d^\mu, \mathbf{t}, p)\rangle, \tag{5.10}$$
where $|\hat{\varphi}_{e,v}(d^\mu, \mathbf{t}, p)\rangle$ is given in (5.3). This naturally generalizes to higher states.

At the end of Sec. III we noted that the string field $|\hat{\varphi}_e(\mathbf{t}, p)\rangle$ will satisfy the reality condition if $C\mathbf{t} = \mathbf{t}^*$, which when applied to (4.11) implies that α is real and β is pure imaginary. Consequently, the same applies to parity eigenstates (5.9) and (5.10). But, now it is easy to see that there is another possibility: (5.9) satisfies the reality condition also when β is real, and the same is true for (5.10) when multiplied by i. In short, we take α to be real while β can be either real or pure imaginary. Having seen how to implement the correct twist parity on the states, it should be mentioned that, in computing physical observables like masses and amplitudes, one could use the twist asymmetric states (5.8) since the twist violating part (which is controlled by the parameter β) never enters in such observables, see also Sec. X.

VI. PROBING THE $k \sim 0$ REGION

Level truncation is a natural regularization in the SFT context. It permits many numerical computations, but it is very unwieldy if one wants to derive analytical results, the lack of analytical control being related to the impossibility of using the analytical machinery of the continuous basis. This is true in particular for the region around $k = 0$, i.e., the string midpoint region, which turns out to be crucial for higher level excitations. In this section we therefore introduce an analytic surrogate of level truncation, at least as far as the $k \sim 0$ region is concerned. It consists of a regulator which mimics the level truncation by regulating the singularities arising when the $k \sim 0$ region is probed, but has the good feature of being defined on the continuous basis (hence permitting analytical control).

To this end the crucial issue is the eigenvalues distribution at $k \sim 0$. As proved in [29] this distribution is divergent, but can be regularized in large-L level truncation
$$\rho(k) = \frac{\ln L}{2\pi} + \rho_{\rm fin}(k), \tag{6.1}$$
the quantity $\rho_{\rm fin}(k)$ is responsible for finite contributions which are relevant for large k, see [30], but it will play no role in the sequel. The eigenvectors of the k-basis have infinite norm due to the continuous orthonormality condition
$$\langle k|k'\rangle = \delta(k-k'), \tag{6.2}$$
Large-L level regularization suggests that their norm is given by[2]
$$\langle k|k\rangle = \delta(0) = \frac{\ln L}{2\pi}. \tag{6.3}$$
Consider now the following half (right) string vector in the k-basis
$$|\eta\rangle = \frac{1}{\eta}\int_{\eta/2}^{3\eta/2} dk|k\rangle, \qquad \eta > 0. \tag{6.4}$$
The norm of this vector is easily computed to be
$$\langle\eta|\eta\rangle = \frac{1}{\eta}. \tag{6.5}$$
From this we define a twist-even and a twist-odd vector as follows
$$|\eta_+\rangle = \frac{1}{\sqrt{2}}(|\eta\rangle + C|\eta\rangle), \qquad |\eta_-\rangle = \frac{1}{\sqrt{2}}(|\eta\rangle - C|\eta\rangle). \tag{6.6}$$
Their norm is given by
$$\langle\eta_-|\eta_-\rangle = \langle\eta_+|\eta_+\rangle = \frac{1}{\eta}. \tag{6.7}$$
These two vectors are the basis of our regularization. In the limit $\eta \to 0^+$ they collapse to the midpoint $k = 0$, and keeping track of the powers of η will allow us to give an unambiguous meaning to the objects we are interested in.

Our first aim is to show that this procedure is inspired by and very close to the level truncation. To this end let us expand these two vectors in the oscillator basis $|n\rangle$. Using
$$\langle n|k\rangle = \sqrt{\frac{nk}{2\sinh\frac{\pi k}{2}}}\oint\frac{dz}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{k}[1-\exp(-k\tan^{-1}z)],$$
a term by term integration yields
$$\langle n|\eta_-\rangle = \sqrt{\frac{2}{\pi}}\left(1, 0, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{5}}, 0, \ldots\right) + O(\eta^2),$$
$$\langle n|\eta_+\rangle = -\frac{\eta}{\sqrt{2\pi}}\left(0, \sqrt{2}, 0, -\frac{4}{3}, 0, \frac{23}{15}\sqrt{\frac{2}{3}}, 0, \ldots\right) + O(\eta^2). \tag{6.8}$$
The first vector is therefore the usual $|k=0\rangle$ twist-odd vector, while every component of the second vanishes in the limit $\eta \to 0$. The latter is $(-\eta\sqrt{2/\pi})$ times the $K^2 = 0$ twist-even vector Rastelli, Sen, and Zwiebach found in [29], that is

[2] Again finite corrections are neglected, as they are not important for our purposes.

$$|0_-\rangle = \lim_{\eta\to 0^+} |\eta_-\rangle = \sqrt{\frac{2}{\pi}}|v^-_{\rm RSZ}\rangle, \tag{6.9}$$

$$|0_+\rangle = \lim_{\eta\to 0^+} |\eta_+\rangle = -\eta\sqrt{\frac{2}{\pi}}|v^+_{\rm RSZ}\rangle. \tag{6.10}$$

It is important to note that although the twist-even vector $|0_+\rangle$ is vanishing, due to (6.7), it has the same infinite norm as $|0_-\rangle$. Like all the vectors which form the continuous basis, this vector does not belong either to the Fock-space, but, unlike all other $|k\rangle$'s, it has vanishing overlap with all oscillators

$$\langle n|0_+\rangle = \lim_{\eta\to 0}\langle n|\eta_+\rangle = 0. \tag{6.11}$$

Nevertheless, as we will see in the sequel, it is crucial for the consistency of the $*$-algebra and, moreover, for accommodating the complete open string D-brane spectrum in the VSFT approach.

At this stage it should be clear that the η parameter plays the role of an effective large-L truncation of the continuous basis, and that $|\eta_-\rangle$ represents the eigenvector relative to the smallest eigenvalue of T at level $L(\eta)$, which is always twist-odd. From [28] we expect the first eigenvector to be located at $k = \frac{\pi}{\log} L$. This suggests that one should make the identification

$$\eta = \frac{\pi}{\log} L. \tag{6.12}$$

We can verify this assertion by checking that

$$\langle 0_-|0_-\rangle = \langle 0_+|0_+\rangle.$$

Using (6.9), this gives

$$\eta = \sqrt{\frac{\langle v^-_{\rm RSZ}|v^-_{\rm RSZ}\rangle}{\langle v^+_{\rm RSZ}|v^+_{\rm RSZ}\rangle}}. \tag{6.13}$$

Computing the difference between the right-hand side (RHS) of (6.12) and the RHS of (6.13) in level truncation we find that it becomes smaller and smaller as $L \to \infty$. For example at $L = 1000$ we have $\frac{\pi}{\log} L \sim 0.45\,479$ (not very near 0!) and such a difference is $-0.03\,082$, while at $L = 10\,000$ we have $0.34\,109$ and $-0.01\,040$, respectively, which is a 3% agreement. Proceeding further with the level it is easy to verify that the agreement improves.[3]

We have therefore succeeded in relating our regularization parameter η to the cutoff L. With some abuse of language we will call the previous empirical set of rules η-regularization. Now we are going to show that some ambiguities that used to plague the string midpoint analysis are naturally resolved within this regularization scheme. We are interested, in particular, in the action of the half string projectors $\rho_{1,2}$ on the midpoint modes $|0_\pm\rangle$. By using the η-regularization (6.6) we simply get

$$\rho_1|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle + \frac{1}{2}|0_\mp\rangle, \qquad \rho_2|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle - \frac{1}{2}|0_\mp\rangle,$$
$$(\rho_1 - \rho_2)|0_\pm\rangle = |0_\mp\rangle. \tag{6.14}$$

If we contract this result with any Fock-space vector $\langle n|$, we recover the result of [17] that the ρ projectors have $\frac{1}{2}$ eigenvalue at $k = 0$. The latter assertion is however, by itself, not free from ambiguities and/or associativity inconsistencies if we do not want to give up the properties (A9). For example, a naive manipulation leads to

$$0 = (\rho_1\rho_2)|0_-\rangle \neq \rho_1(\rho_2|0_-\rangle) = \frac{1}{4}|0_-\rangle. \tag{6.15}$$

On the contrary, with our regularization it is very easy to check that

$$0 = (\rho_1\rho_2)|0_\pm\rangle = \rho_1(\rho_2|0_\pm\rangle) = 0, \tag{6.16}$$

which is definitely nonambiguous. Other remarkable inconsistencies which would arise using the same kind of naive manipulations would be

$$\frac{1}{2}|0_-\rangle = (\rho_{1,2}\rho_{1,2})|0_-\rangle \neq \rho_{1,2}(\rho_{1,2}|0_-\rangle) = \frac{1}{4}|0_-\rangle,$$
$$|0_-\rangle = (\rho_1 - \rho_2)^2|0_-\rangle \neq (\rho_1 - \rho_2)[(\rho_1 - \rho_2)|0_-\rangle] = 0. \tag{6.17}$$

It is easy to check that, with our regularization, this anomaly disappears and all the properties (A9) are preserved even at $k = 0$. The crucial move was to introduce an extra twist-even midpoint vector which vanishes in the Fock-space, but has nevertheless infinite norm. We will see in the sequel how this vanishing vector is important for the construction of open string states on the dressed sliver. For the time being we only point out that the vector $|0_+\rangle$ cannot create string excitations when contracted with oscillators since, see (6.11),

$$\langle 0_+|a^\dagger\rangle|\text{state}\rangle = \lim_{\eta\to 0}\sum_n a_n^\dagger\langle n|\eta_+\rangle|\text{state}\rangle = 0 \tag{6.18}$$

vanishes. However we can excite Fock-space states if, in η-regularization, we consider the vector

$$\lim_{\eta\to 0^+}\frac{1}{\eta}|\eta_+\rangle \sim |v^+_{\rm RSZ}\rangle. \tag{6.19}$$

From (6.8) it is clear that this vector has finite overlap with any Fock-space vector. We will see that this vector plays a fundamental role in the construction of cohomologically nontrivial open string states. The vector $|0_+\rangle$ can also contribute to matrix elements involving vectors that are finite at the midpoint (hence out of the Fock-space) like the "bare tachyon" $\langle t_0|$. For example the following relations hold in η-regularization

[3] This simple example should warn the reader on how level truncation is slow in probing the midpoint $k = 0$.

$$\langle t_0|0_+\rangle = \sqrt{2}t_0(0) + O(\eta), \tag{6.20}$$

$$\langle t_0|\frac{1}{1+T}|0_+\rangle = \ln 3\frac{2\sqrt{2}}{\pi}t_0(0)\frac{1}{\eta} + O(1). \tag{6.21}$$

In the sequel we will see that, using η-regularization, all the divergent brackets that appear in computing solutions to the LEOM can be explicitly evaluated in terms of some (regularization dependent) function of η. We will comment *a posteriori* on the regularization independence of our final and physical results.

VII. HIGHER LEVEL SOLUTIONS TO LEOM

In the canonical quantization of string theory the tower of massive states is constructed by applying monomials of creation operators on the Fock vacuum. In order for the state to have a definite mass one selects all the monomials of the same level and takes a linear combination thereof, with tensorial coefficients which are generically referred to as polarizations. The latter are not completely free, but must satisfy some constraints, the Virasoro constraints. The construction of analogous states in VSFT proceeds differently. Although we will keep talking about level n solutions in order to relate our results with the familiar ones, the level is not the right issue here, because in VSFT we do not have any explicit realization of the L_0 Virasoro generator. The most general level n state we will consider will take the form

$$|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle \equiv |\hat{\varphi}(\theta_1, \ldots, \theta_n, \mathbf{t}, p)\rangle = \sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a^\dagger_{\mu_1}\zeta_1^{(i)}\rangle\ldots\langle a^\dagger_{\mu_i}\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle \tag{7.1}$$

in analogy with the canonical quantization construction, but without imposing any level restriction. As we shall see below, the request that the state (7.1) satisfy the LEOM will be sufficient to select a definite mass and impose the appropriate Virasoro constraints on the polarizations θ_i.

A. Level 2

The level 2 state in canonical quantization is

$$(h_\mu\alpha_2^{\mu\dagger} + \lambda_{\mu\nu}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger})|0\rangle e^{ipx}. \tag{7.2}$$

The Virasoro constraints require that $p^2 = -1$ and

$$2\sqrt{2}h^\mu p_\mu + \lambda_\mu{}^\mu = 0, \qquad h_\mu + \sqrt{2}\lambda_{\mu\nu}p^\nu = 0. \tag{7.3}$$

In view of the forthcoming VSFT construction it is important to notice that there is a certain arbitrariness in these formulas. One can rewrite them for instance as follows

$$2\sqrt{2}g^\mu p_\mu + a\theta_\mu{}^\mu = 0, \qquad bg_\mu + \sqrt{2}\theta_{\mu\nu}p^\nu = 0, \tag{7.4}$$

with a and b arbitrary (nonvanishing) constants, and h, λ related to g, θ as follows

$$\begin{aligned} h_\mu &= Ag_\mu + B(p\cdot g)p_\mu, \\ \lambda_{\mu\nu} &= C\theta_{\mu\nu} + D(p_\mu p^\rho\theta_{\rho\nu} + p_\nu p^\rho\theta_{\mu\rho}). \end{aligned} \tag{7.5}$$

Using the mass-shell condition it is easy to show that this simply requires

$$A = \frac{b}{2}\frac{3ab+2}{ab-1}D, \qquad B = bD, \qquad C = \frac{5}{2}\frac{ab}{ab-1}D.$$

According to the level n ansatz (7.1) the candidate to represent a level two state is

$$\begin{aligned} |\hat{\varphi}(\theta, 2, \mathbf{t}, p)\rangle &\equiv |\hat{\varphi}(\theta_1, \theta_2, \mathbf{t}, p)\rangle \\ &= \theta_1^{\mu_1}\langle a^\dagger_{\mu_1}\zeta_1^{(1)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle \\ &\quad + \theta_2^{\mu_1\mu_2}\langle a^\dagger_{\mu_1}\zeta_1^{(2)}\rangle\langle a^\dagger_{\mu_2}\zeta_2^{(2)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle. \end{aligned} \tag{7.6}$$

This ansatz has to be made more precise by specifying the vectors $|\zeta_j^{(i)}\rangle$. For generic vectors we do not get any on-shell open string state. In fact, on the basis of our attempts, it seems that only if the vectors $|\zeta_j^{(i)}\rangle$ probe the string midpoint will (7.6) be a cohomologically nontrivial solution to the LEOM. Therefore we make the choice $|\zeta_j^{(i)}\rangle \sim |0_\pm\rangle$; the latter states were introduced in the previous section and were designed to resolve the singularity at $k = 0$. But we must be more precise: the factors in front of $\lim_{\eta\to 0^+}|\eta_\pm\rangle$ play also a fundamental role and we must specify them. In summary, our ansatz will be

$$\begin{aligned} |\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle &= g^\mu\langle a^\dagger_\mu|s_+\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle \\ &\quad + \theta^{\mu\nu}\langle a^\dagger_\mu|\zeta_-\rangle\langle a^\dagger_\nu|\zeta_-\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle, \end{aligned} \tag{7.7}$$

where $|s_+\rangle = \lim_{\eta\to 0^+}|\eta_+\rangle s(\eta)$, $|\zeta_-\rangle = \lim_{\eta\to 0^+}|\eta_-\rangle\zeta(\eta)$, and, near $\eta = 0$,

$$\begin{aligned} s(\eta) &= \frac{s_{-1}}{\eta} + s_0 + s_1\eta + \ldots, \\ \zeta(\eta) &= \zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots \end{aligned} \tag{7.8}$$

As a consequence we have [see (6.9) and (6.10)]

$$\langle a^\dagger_\mu|s_+\rangle = -\sqrt{\frac{2}{\pi}}\langle a^\dagger_\mu|v^+_{\mathrm{RSZ}}\rangle(s_{-1} + s_0\eta + s_1\eta^2 + \ldots), \tag{7.9}$$

$$\langle a^\dagger_\mu|\zeta_-\rangle = \sqrt{\frac{2}{\pi}}\langle a^\dagger_\mu|v^-_{\mathrm{RSZ}}\rangle(\zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots). \tag{7.10}$$

These are well-defined expressions and it would seem that the terms proportional to η, η^2 play no role in the limit $\eta \to 0$. However this is not the case because the star product with the dressed sliver will take them back into the game. Only terms of order η^3 and higher will not play any role and can be disregarded.

It is time to pass to the explicit calculation of the LEOM. We have to find the conditions under which

$$|\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle = |\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle \tag{7.11}$$

The star products in (7.11) yield cumbersome formulas. In order not to clog our exposition with them we defer a full treatment to Appendix F, and use a technical simplification: we assume that the function $\xi(k)$, which represents the dressing vector ξ in the k-basis and which is nonvanishing only for negative k, is actually nonvanishing only for $k < k_0 < 0$ where k_0 is some small but finite negative constant. The consequences of this simplification will be commented upon in Sec. VIII. We can of course suppose that the regularization parameter $2\eta < |k_0|$. As a consequence all the quantities appearing in this computation which involve ξ can be neglected. On the other hand this restriction on the form of $\xi(k)$ does not imperil the properties we have requested for ξ in this and the previous paper [1]: this point is further developed in Sec. VIII. With this understanding we obtain

$$\begin{aligned}
&[\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle \\
&\quad * [\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] \\
&= e^{-Gp^2/2}\bigg[\frac{1}{2}\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle \\
&\quad + 2\theta_\mu^\mu\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle + 2\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_+\rangle p_\nu\mathcal{H}_+ \\
&\quad + 2\theta^{\mu\eta}p_\mu p_\eta\mathcal{H}_+^2\bigg]|\hat{\varphi}(\mathbf{t}, p)\rangle,
\end{aligned} \tag{7.12}$$

where we have used $|\zeta_+\rangle = (\rho_1 - \rho_2)|\zeta_-\rangle$, with (6.18), and

$$\begin{aligned}
&[g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * [g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] \\
&= e^{-Gp^2/2}\bigg[g^\mu\langle a_\mu^\dagger|s_+\rangle \\
&\quad - p\cdot g\langle t_0|\frac{1}{1+T}|s_+\rangle\bigg]\bigg|\hat{\varphi}(\mathbf{t}, p)\rangle.
\end{aligned} \tag{7.13}$$

The quantity $\mathcal{H}_+$ (see Appendix F) is a complicated expression of order η^{-1}, $\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle$, as well as $\langle t_0|\frac{1}{1+T}|s_+\rangle$, is of order η^{-2}, while, as we have already seen, $\langle a_\mu^\dagger|\zeta_+\rangle$ is of order η.

Now, from the first term in the RHS of Eq. (7.12) we see that the only way to satisfy the LEOM is to set $e^{-\frac{1}{2}Gp^2} = 2$, i.e., $p^2 = -1$, which reproduces the desired mass-shell condition. Next, in (7.13) we must split $g^\mu\langle a_\mu^\dagger|s_+\rangle$ (which is a finite term in η) in two halves. The first half reconstructs the first term in the RHS of (7.7), the second half must annihilate the linear term in $a^\dagger$ in the RHS of (7.12). This is the only way this unwanted term can be canceled. The latter operation on the other hand is only possible if

$$g^\mu \sim \theta^{\mu\nu}p_\nu. \tag{7.14}$$

Finally, the remaining unwanted terms in the above equations must cancel with one another order by order in η. Looking at the order -2 in η, one easily realizes that the only way to implement such cancellation is to require that

$$\theta_\mu{}^\mu \sim \theta^{\mu\nu}p_\mu p_\nu \sim p\cdot g, \tag{7.15}$$

with nonvanishing proportionality constants.

Equations (7.14) and (7.15) are not enough to conclude that the level two Virasoro constraints (7.4) are satisfied. However the accurate analysis of Appendix F proves that this is the case. In Appendix F it is also shown that the LEOM (7.11) is exactly satisfied together with the Virasoro constraints (7.4), provided some (not very restrictive) relations among the constants $a, b, \zeta_0, \zeta_1, \zeta_2, s_{-1}, s_0, s_1$ are satisfied. From the analysis in Appendix F it is clear that the coefficients a and b are regularization dependent, but, in turn, a and b can be absorbed via the redefinitions (7.5).

B. Level 3

The level 3 state in canonical quantization is

$$(h_\mu\alpha_3^{\mu\dagger} + \lambda_{\mu\nu}\alpha_2^{\mu\dagger}\alpha_1^{\nu\dagger} + \chi_{\mu\nu\rho}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger}\alpha_1^{\rho\dagger})|0\rangle e^{ipx}, \tag{7.16}$$

The Virasoro constraints require that $p^2 = -2$ and

$$3h^\mu p_\mu + \sqrt{2}\lambda_\mu{}^\mu = 0, \qquad 3h_\mu + \sqrt{2}\lambda_{\mu\nu}p^\nu = 0, \tag{7.17}$$

$$\begin{aligned}
\sqrt{2}(2\lambda_{\nu\mu}p^\nu - \lambda_{\mu\nu}p^\nu) + 3\chi_{\mu\nu}^\nu = 0,\\
\sqrt{2}\lambda_{(\mu\nu)} + 3\chi_{\mu\nu\rho}p^\rho = 0,
\end{aligned} \tag{7.18}$$

where $\lambda_{(\mu\nu)}$ is the symmetric part of $\lambda_{\mu\nu}$. It can be seen that the first equation is a consequence of the other three. It is however possible, as above, to redefine the polarizations as shown in Appendix G . In terms of the new ones $g_\mu, \omega_{\mu\nu}, \theta_{\mu\nu\rho}$ the Virasoro constraints become

$$3xg^\mu p_\mu + \sqrt{2}\omega_\mu{}^\mu = 0, \qquad 3g_\mu + \sqrt{2}y\omega_{\mu\nu}p^\nu = 0, \tag{7.19}$$

$$\begin{aligned}
2\sqrt{2}v\omega_{\nu\mu}p^\nu - \sqrt{2}u\omega_{\mu\nu}p^\nu + 3\theta_{\mu\nu}^\nu = 0,\\
\sqrt{2}\omega_{(\mu\nu)} + 3z\theta_{\mu\nu\rho}p^\rho = 0.
\end{aligned} \tag{7.20}$$

It is now easy to verify that the first condition is a consequence of the other three provided we set $x = \frac{z(2v-u)}{y}$. Therefore it need not be verified separately. The remaining constants y, u, v, z are arbitrary non−vanishing ones. From the general form (7.1), we select the following ansatz

$$|\hat{\varphi}(g,\omega,\theta,\mathbf{t},p)\rangle = (g^\mu\langle a_\mu^\dagger|r_-\rangle + \omega^{\mu\nu}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle + \theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle \times \langle a_\rho^\dagger|\zeta_-\rangle)|\hat{\varphi}(\mathbf{t},p)\rangle, \tag{7.21}$$

where $|r_-\rangle = \lim_{\eta\to0^+}|\eta_-\rangle r(\eta)$, and the same definition is understood for $|\zeta'_-\rangle, |\zeta_-\rangle$, while $|\lambda_+\rangle = \lim_{\eta\to0^+}|\eta_+\rangle\lambda(\eta)$. Near $\eta = 0$,

$$\lambda(\eta) = \frac{\lambda_{-1}}{\eta} + \lambda_0 + \lambda_1\eta + \ldots, \qquad \zeta(\eta) = \zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots \tag{7.22}$$

$\zeta'(\eta)$ and $r(\eta)$ have an expansion similar to $\zeta(\eta)$. Consequently, for the brackets inside (7.21), expansions similar to (7.9) and (7.10) hold.

The formulas involved in the evaluation of the linearized EOM are too large to be written down here. We can avoid such complications by introducing the simplifying assumption of the previous subsection. We render the dressing vector contributions evanescent in the limit $\eta \to 0$ so that we can simply avoid writing them down. The resulting formulas are as follows:

$$\begin{aligned}&[\theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] * |\hat{\Xi}\rangle \\ &\quad + |\hat{\Xi}\rangle * [\theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] \\ &= e^{-\frac{1}{2}Gp^2}\Big[3\theta_\mu{}^{\mu\rho}\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle + \theta^{\mu\nu\rho}\Big(\frac{1}{4} \\ &\quad \times \langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle + 3\langle a_\mu^\dagger|\zeta_-\rangle p_\nu p_\rho \mathcal{H}_+^2 \\ &\quad + 3\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\eta^\dagger|\zeta_+\rangle p_\rho \mathcal{H}_+\Big)\Big]|\hat{\varphi}(\mathbf{t},p)\rangle,\end{aligned} \tag{7.23}$$

$$\begin{aligned}&[\omega^{\mu\nu}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] * |\hat{\Xi}\rangle \\ &\quad + |\hat{\Xi}\rangle * [\omega^{\mu\nu}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] \\ &= e^{-Gp^2/2}\omega^{\mu\nu}\Big[\frac{1}{2}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle + \frac{1}{2}\langle a_\mu^\dagger|\zeta'_+\rangle\langle a_\nu^\dagger|\lambda_-\rangle \\ &\quad + \langle a_\mu^\dagger|\zeta'_-\rangle p_\nu\langle\mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle \\ &\quad + p_\mu\langle a_\nu^\dagger|\lambda_-\rangle\mathcal{H}_+\Big]|\hat{\varphi}(\mathbf{t},p)\rangle,\end{aligned} \tag{7.24}$$

and

$$[g^\mu\langle a_\mu^\dagger|r_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * [g^\mu\langle a_\mu^\dagger|r_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle] = e^{-Gp^2/2}g^\mu\langle a_\mu^\dagger|r_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle, \tag{7.25}$$

where $|\lambda_-\rangle = \lim_{\eta\to0^+}|\eta_-\rangle\lambda(\eta)$, and $|\zeta_+\rangle = \lim_{\eta\to0^+}|\eta_+\rangle\zeta(\eta)$, $\lambda(\eta)$ and $\zeta(\eta)$ being the same functions as above (7.22).

Now, in order for the LEOM to be satisfied the sum of these three terms, (7.23), (7.24), and (7.25), must reproduce (7.21). From the second term in the RHS of (7.24) we see that we must have $e^{-Gp^2/2} = 4$, i.e., $p^2 = -2$, the mass-shell condition for level 3 states. This implies that the RHS of the second equation $\omega^{\mu\nu}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle$ appears with a coefficient 2 in front, therefore half of this term will reproduce (7.21) and the other half must be canceled against the other terms. Similarly in the RHS of (7.25) the term $g^\mu\langle a_\mu^\dagger|r_-\rangle$ appears with a coefficient 4. So $1/4$ of it will reproduce (7.21) and $3/4$ will have to be canceled.

Next, as in the previous subsection, we count the degrees of divergence for $\eta \to 0$ of the various terms in the above three equations, which is -2 for the first and third terms of the RHS of (7.23) and 0 for the remaining ones; it is 0 for the first two terms in the RHS of (7.24) and -2 for the other two; finally it is zero for the term in the RHS of (7.25). Now what we have to do is collect all the unwanted terms in the RHS and impose that the sum of the coefficients in front of them vanish. From what we just said, we can deduce that we must have

$$\omega^{\mu\nu} \sim \theta^{\mu\nu\rho}, \qquad p_\rho\theta_\mu^{\mu\rho} \sim \theta^{\mu\nu\rho}p_\mu p_\nu \sim a\omega^{\mu\rho}p_\mu + b\omega^{\rho\mu}p_\mu, \qquad \omega^{\mu\rho}p_\mu \sim g^\rho, \tag{7.26}$$

for some constants a and b. These are very close to (7.19) and (7.20). However it must be proven that the arbitrary constants we have at our disposal (i.e., x, y, u, v, z and the coefficients of $\zeta(\eta)$, $\lambda(\eta)$ and $r(\eta)$) are sufficient to satisfy all the conditions. This is an elementary algebraic problem. The straightforward calculations are carried out in Appendix G where it is shown that all the conditions are met. So we can conclude that

$$|\hat{\varphi}(g,\omega,\theta,\mathbf{t},p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(g,\omega,\theta,\mathbf{t},p)\rangle = |\hat{\varphi}(g,\omega,\theta,\mathbf{t},p)\rangle. \tag{7.27}$$

VIII. COHOMOLOGY

A solution to the LEOM is not automatically a solution fit to represent a physical string state. The reason for this is the huge gauge invariance which soaks all physical states in SFT. Any solution to the LEOM is in fact defined up to

$$\mathcal{Q}_0\Lambda \equiv \mathcal{Q}\Lambda + \Phi_0 * \Lambda - \Lambda * \Phi_0, \tag{8.1}$$

where Φ_0 is our reference classical solution (see Sec. III) and Λ is any string state of ghost number 0. Only string field solutions which cannot take the form of (8.1) are significant solutions and can represent physical states. Phrased another way, $\mathcal{Q}_0$ is nilpotent, therefore it defines a cohomology problem: only nontrivial cohomology classes are physically interesting. Unfortunately a systematic approach to this problem is missing (although some progress can be found in [31]), the more so for VSFT. Partial elaborations on the gauge freedom in VSFT can be found in [17,19]. In this section we will not try a systematic approach to the cohomology problem. Nevertheless it turns out to be rather easy to figure out Λ "counterterms" that "almost trivialize" the solutions we have found in the previous section, but actually do not kill them at all. This makes us confident that what we have found in the previous sections singles out nontrivial cohomology classes.

To simplify the problem as much as possible we will exclude all the Λ's with a nontrivial ghost content. If Λ is a matter state tensored with the ghost identity, see [17,19], then the gauge transformation (8.1) for a (pure matter) state ϕ can be written simply through Λ's matter part as follows:

$$\delta\phi = \hat{\Xi} *_m \Lambda - \Lambda *_m \hat{\Xi}, \tag{8.2}$$

where $\hat{\Xi}$ is the dressed sliver. Our problem is now to find matter states Λ such that (8.2) gives some of the solutions we found in the previous sections. Let us try the following one (we set $e = 1$ and drop the label m in $*_m$ throughout this section)

$$|\Lambda(g, \zeta)\rangle = g^\mu \langle (1 + C)\zeta a_\mu^\dagger \rangle |\hat{\varphi}_t(\mathbf{t}, p)\rangle, \tag{8.3}$$

where $|\hat{\varphi}_t(\mathbf{t}, p)\rangle$ is the tachyon wave function. The gauge transformation (8.2) becomes

$$\begin{aligned} &|\hat{\Xi}\rangle * |\Lambda(g, \zeta)\rangle - |\Lambda(g, \zeta)\rangle * |\hat{\Xi}\rangle \\ &= e^{-Gp^2/2}\Big\{ g^\mu \langle a^\dagger(\rho_1 - \rho_2)(1 + C)\zeta\rangle \\ &\quad + \frac{1}{\kappa + 1} g^\mu \langle a_\mu^\dagger (|\xi\rangle\langle\xi| - |C\xi\rangle\langle C\xi|) \frac{1}{1 - T} \\ &\quad \times |(1 + C)\zeta\rangle - 2\beta(p \cdot g)\Big[\langle\xi| \frac{T}{1 - T^2} |(1 + C)\zeta\rangle \\ &\quad - \kappa\langle\xi| \frac{1}{1 - T^2} |(1 + C)\zeta\rangle\Big]\Big\} |\hat{\varphi}_t(\mathbf{t}, p)\rangle, \end{aligned} \tag{8.4}$$

Now suppose that $\rho_2\zeta = \zeta$ and $\rho_1\zeta = 0$. We get

$$\begin{aligned} &|\hat{\Xi}\rangle * |\Lambda(g, \zeta)\rangle - |\Lambda(g, \zeta)\rangle * |\hat{\Xi}\rangle \\ &= e^{-Gp^2/2}\Big[-g^\mu \langle a^\dagger(1 - C)\zeta\rangle \\ &\quad - 2\beta(p \cdot g)\langle\xi| \frac{T - \kappa}{1 - T^2} |\zeta\rangle \Big] |\hat{\varphi}_t(\mathbf{t}, p)\rangle. \end{aligned} \tag{8.5}$$

Comparing now this with Eq. (5.3) we see that, if we choose the ζ's in the two equations to be the same, we set $g^\mu = d^\mu$ and suitably normalize $\Lambda(g, \zeta)$, the gauge transformation (8.5) gives back just the vector state eigenfunction (5.3), or, in other words, the latter belongs to the trivial cohomology class.

Therefore, if $\zeta(k)$ is a regular function for $k \sim 0$ (henceforth let us refer to such a $|\zeta\rangle$ as *regular* or *smooth* at $k = 0$) , the vector state we have constructed in Sec. VA is cohomologically trivial. In order to get something nontrivial we have to probe the string midpoint. Therefore let us try with $|\zeta\rangle \sim C|\eta\rangle$ (from now on let us refer to the latter as *singular* or *concentrated at* $k = 0$). It satisfies $\rho_2\zeta = \zeta$ and $\rho_1\zeta = 0$ and $|(1 + C)\zeta\rangle \sim |\eta_+\rangle$, $|(1 - C)\zeta\rangle \sim |\eta_-\rangle$ [see Eq. (6.6)]. Therefore, in this case too, as long as the parameter η remains finite, the vector state is trivial. One may be tempted to conclude that also in the limit $\eta \to 0$, such a conclusion persist and therefore the vector wave function we have defined be always trivial. But this would be a sloppy deduction. For in the process of taking the limit $\eta \to 0$, there emerges the true nature of cohomology.

For a cohomological problem to be well-defined it is not enough to have a nilpotent operator; one must also define the set of objects which such an operator acts upon, i.e., the space of cochains. In our case a precise definition of the cochain space has not been given so far, and it is time to fill in this gap. It is clear that the issue here is the distinction between the states that vanish and those that do not vanish in the limit $\eta \to 0$. For instance, [see (6.9) and (6.10)], $|0_+\rangle$ belongs to the former set (let us call it an *evanescent* state) while $|0_-\rangle$ belongs to the latter. We define the space of nonzero cochains as the space of states that are finite in the limit $\eta \to 0$, while the zero cochain is represented by 0. All this is well-defined and makes up a linear space and it is the only sensible choice to define a cohomology in this context (see Appendix H for a discussion of this point).

With the previous definition let us return to the vector eigenfunction. Thanks to the discussion following Eqs. (8.4) and (8.5), we see immediately that if ζ in (5.3) is smooth near $k = 0$, then the corresponding wave function is a coboundary. If, on the other hand, $\zeta \sim |C\eta\rangle$, i.e., is concentrated at $k = 0$, then the state is a nontrivial cocycle, because we cannot figure out any nonevanescent Λ which generate it via (8.2): the only one that does the job is evanescent. This same conclusion can be drawn for the level two and level three states we found above (which were formulated directly in terms of vectors concentrated at $k = 0$). All these states are cohomologically nontrivial.

At this point we can discuss also the implication of the simplifying assumption we introduced in Sec. VII, i.e., that the dressing function $\xi(k)$ is nonvanishing only from a certain finite negative point down to $-\infty$ in the k-axis. This assumption induced remarkable simplifications in our analysis, but that was the only reason why it was introduced: one can do without it. Anyhow let us ask ourselves what would have happened had we introduced this assumption in the vector case. In the case of ζ being concentrated at $k = 0$ the last two terms at the RHS of (5.4) would vanish and we would not need to impose the transversality condition (5.5). If, on the other hand, ζ is smooth at $k = 0$ then, in order to satisfy the LEOM we would have to impose the transversality condition (5.5) together with the additional condition (5.6), but in this case we would get a trivial solution. This conclusion seems to be paradoxical only if we forget the relation between cohomology and the Virasoro conditions. In fact it is perfectly logical. First of all we should remember that we have two ways of expressing the physicality of a given state. Either we say that this state is a nontrivial cocycle defined up to generic coboundaries (this is the cohomological way of putting it), or we impose conditions on the parameters of the state (polarizations) in such a way that its indeterminacy (coboundaries) get suppressed (and this is the gauge fixing way). Now, the above apparent paradox means that the simplifying assumption, which seems to suppress the transversality condition on the nontrivial cocycle (singular ζ), can be made up for by adding to the solution a trivial cocycle (regular ζ). In other words, the simplifying assumption corresponds to partially fixing the gauge freedom. It can be seen that this is true also in the more complicated cases of level 2 and level 3.

With these remarks we end our analysis of cohomology in VSFT. This problem would deserve of course a more thorough treatment, but we believe we have caught some of the essential features of it.

IX. PROLIFERATING SOLUTIONS

All the solutions to the LEOM considered so far depend on three parameters: e, α, β. As will be seen below, e has to be set equal to 1, but the other two parameters are free. We wish to show in this section that the solutions to the LEOM are even more general than this. In fact we can prove that, if $|\hat{\varphi}(\mathbf{t}, p)\rangle$ is the matter part of the tachyon solution to the linearized equation of motion, i.e., a solution to (3.6), then any state of the form

$$(\langle a^{\dagger}_{\mu_1}\xi_{\pm}\rangle \ldots \langle a^{\dagger}_{\mu_s}\xi_{\pm}\rangle)|\hat{\varphi}(\mathbf{t}, p)\rangle \tag{9.1}$$

where $\xi_{\pm} = (1 \pm C)\xi$, is also a solution for any s, with the same mass as the tachyon for any random choice of the $\pm$ signs. For

$$\begin{aligned}[\langle a^{\dagger}_{\mu}\xi\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * [\langle a^{\dagger}_{\mu}\xi\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle] \\ = \langle a^{\dagger}_{\mu}\xi\rangle[|\hat{\varphi}(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(\mathbf{t}, p)\rangle] \\ = \langle a^{\dagger}_{\mu}\xi\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle\end{aligned} \tag{9.2}$$

The derivation of the first equality is given in Appendix I. The same can be shown if we replace $\langle a^{\dagger}_{\mu}\xi\rangle$ with $\langle a^{\dagger}_{\mu}C\xi\rangle$. This proves the above claim for $s = 1$. But it is evident that now we can proceed recursively by replacing in (9.2) $|\hat{\varphi}(\mathbf{t}, p)\rangle$ with $\langle a^{\dagger}_{\nu}\xi\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle$ and $\langle a^{\dagger}_{\nu}C\xi\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle$, respectively, which are also solutions, thereby proving the statement for $s = 2$, and so on.

We refer to all these states as the descendants of $|\hat{\varphi}(\mathbf{t}, p)\rangle$, or *tachyon descendants*. We can easily define a generating state for them

$$|\hat{\varphi}(g, \mathbf{t}, p)\rangle = e^{(g^{\mu}_{+}\langle a^{\dagger}_{\mu}\xi_{+}\rangle + g^{\mu}_{-}\langle a^{\dagger}_{\mu}\xi_{-}\rangle)}|\hat{\varphi}(\mathbf{t}, p)\rangle \tag{9.3}$$

By differentiating with respect to $g^{\mu}_{\pm}$ we can generate all the solutions of the type (9.1).

A similar result holds also for the other (tensor) solutions of the LEOM. At level n such states take the form

$$\begin{aligned}|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle &\equiv |\hat{\varphi}(\theta_1, \ldots, \theta_n, \mathbf{t}, p)\rangle \\ &= \sum_{i=1}^{n} \theta_i^{\mu_1\ldots\mu_i}\langle a^{\dagger}_{\mu_1}\zeta_1^{(i)}\rangle \ldots \langle a^{\dagger}_{\mu_i}\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle\end{aligned} \tag{9.4}$$

where the polarizations θ_i must satisfy constraints similar to those found for level 1,2 and 3. As shown in Appendix I, we have a result similar to the above. The LEOM is satisfied with the same mass

$$\begin{aligned}&[h^{\nu}\langle a^{\dagger}_{\nu}\xi\rangle|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle] * |\hat{\Xi}\rangle \\ &\quad + |\hat{\Xi}\rangle * [h^{\nu}\langle a^{\dagger}_{\nu}\xi\rangle|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle] \\ &= h^{\nu}\langle a^{\dagger}_{\nu}\xi\rangle[|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle * |\hat{\Xi}\rangle \\ &\quad + |\hat{\Xi}\rangle * |n, \hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle] \\ &= h^{\nu}\langle a^{\dagger}_{\nu}\xi\rangle|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle\end{aligned} \tag{9.5}$$

but, now, under some conditions: either

$$\langle\xi|\frac{T-\kappa}{1-T^2}|C\zeta_j^{(i)}\rangle = 0, \tag{9.6}$$

(which is the case for instance when $\rho_2\zeta_j^{(i)} = \zeta_j^{(i)}$, $\rho_1\zeta_j^{(i)} = 0$ or, if this is not true (as is the case in our previous analysis), the polarization h is transverse to the θ_i's when contracted with the index μ_j:

$$h^{\nu}\eta_{\nu\mu_j}\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i} = 0, \tag{9.7}$$

and this must hold $\forall i, j, \quad 1 \le j \le i, \quad 1 \le i \le n$.

Also here we can replace $\langle a^{\dagger}_{\nu}\xi\rangle$ with $\langle a^{\dagger}_{\nu}C\xi\rangle$ and obtain a new solution with the same mass, and therefore we can define the $\pm$ combination, as above. Inductively we can prove that

$$(h_1^{\nu_1}\langle a^{\dagger}_{\nu_1}\xi\rangle \ldots h_s^{\nu_s}\langle a^{\dagger}_{\nu_s}\xi\rangle)|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle \tag{9.8}$$

satisfy the LEOM with the same mass provided each h_j is transverse to each θ_i on all indices. We can then introduce C in every $\langle a^{\dagger}|\xi\rangle$ factor and obtain new independent solutions. It is evident that the most general state with the same mass takes the form

$$\begin{aligned}&\langle a^{\dagger}_{\nu_1}\xi_{\pm}\rangle \ldots \langle a^{\dagger}_{\nu_s}\xi_{\pm}\rangle \\ &\quad \times \sum_{i=1}^{n} \theta_i^{\nu_1\ldots\nu_s;\mu_1\ldots\mu_i}\langle a^{\dagger}_{\mu_1}\zeta_1^{(i)}\rangle \ldots \langle a^{\dagger}_{\mu_i}\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle,\end{aligned} \tag{9.9}$$

with generic s, provided the tensor θ_i are traceless when any index ν is contracted with any index μ. However, any state of the type (9.9) is a finite linear combination of states of type (9.4). A generating function for the latter is

$$|\hat{\varphi}(h, \mathbf{t}, p)\rangle = e^{(h_+^\mu \langle a_\mu^\dagger \xi_+\rangle + h_-^\mu \langle a_\mu^\dagger \xi_-\rangle)} |\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle. \quad (9.10)$$

Differentiating with respect to $h_\pm$ the required number of times, we can construct any state of the type (9.4). A generating function is particularly useful in computing norms or amplitudes.

To finish this section a comment is in order concerning the enormous proliferation of solutions to the linearized equations of motion. All the states we have found seem to be cohomologically nontrivial on the basis of the analysis in the previous section. The existence of an infinite tower of descendants of a given solution is, generically speaking, hardly a surprise. We notice that a similar phenomenon is familiar in field theory. If $\phi_0(x)$ is a solution to the Klein-Gordon equation $(\partial^2 + m^2)\phi = 0$, then all the derivatives of ϕ_0 are solutions with the same mass. We conjecture that here we are coming across something similar, although the difference among different states of each tower is given here not by the application of the space derivatives (i.e., by powers of $\hat{p}$), but rather by the application of the creation operators $a_n^\dagger$, $n > 0$.

But now, the important question is: what is the nature of these states? They seem to be physical, so it is important to clarify whether they are simple copies of the first state of the tower (the *parent* state, not containing $\langle a^\dagger \xi\rangle$ factors in their $\mathcal{P}$ polynomial) or have a different physical meaning. Looking at the generating state (9.3) one can see that, if $g_\pm \sim p$, this turns into a redefinition of the arbitrary constants α and β (see Sec. IVA). Therefore, since these constants do not enter into physical quantities, such as G, (they might appear in quantities like H, see below, which is not by itself physical) we conclude that the states of this type are copies of the tachyon eigenfunction, without any physical differentiation from it. It is possible to see that this is true for any other tower of solutions. So the proliferation we find seems to be a proliferation of representatives of physical states (much in the same way as in the Coulomb representation of CFT we have two representatives for any vertex). This redundancy of representatives, which, it should be stressed, is due to dressing, may be a residue of the gauge symmetry of VSFT.

X. ON THE D25-BRANE TENSION

One of the unsatisfying aspects of the sliver in operator formalism was the disagreement between the energy density of the classical solution and the brane tension computed via the 3-tachyon on-shell coupling. In this section we would like to show that our approach can lead to a solution of this problem.

A. 3-tachyon on-shell coupling

The cubic term of the VSFT action evaluated for three on-shell tachyon fields should be equal to $g_T/3$, where g_T is the 3-tachyon coupling constant for the open string, i.e.,

$$g_T = \frac{1}{g_0^2}\langle \varphi_t(\mathbf{t}, p_1)|\varphi_t(\mathbf{t}, p_2) * \varphi_t(\mathbf{t}, p_3)\rangle \Big|_{p_1^2=p_2^2=p_3^2=-m_t^2=1} \quad (10.1)$$

Here $|\varphi_t(\mathbf{t}, p)\rangle$ must be normalized so as to give the canonical kinetic term in the low-energy action (see [9], Sec. 5.2). Using (5.1), an explicit calculation gives

$$g_T^2 = \frac{8g_0^2}{G^3} A^{13}\tilde{A}^{-1}\exp(-6H) \quad (10.2)$$

where (see Appendix J)

$$H = H_0 - \frac{(f_e - 1)^2(\kappa + f_e)^2}{2(f_e + 1)(f_e^3 - 1)} \times \left[\left(\frac{1}{\kappa + f_e} - \alpha\right)^2 \langle \mathbf{t}_0| \frac{1}{1 + T}|\xi\rangle^2 - \beta^2\right] \quad (10.3)$$

and

$$A = \frac{[\det(1 - \hat{T}_{e_1}\hat{T}_{e_2})]^3}{[\det(1 - \hat{\mathcal{T}}_{e_1e_2e_3}\mathcal{M}_3)]^2} = \frac{(f_1f_2 - 1)^6}{(f_1f_2f_3 - 1)^4} \frac{[\det(1 - T^2)]^3}{[\det(1 - T\mathcal{M}_3)]^2}. \quad (10.4)$$

$\tilde{A}$ is obtained from A by replacing all the relevant objects with tilded ones (ghost part). H_0 is a naively vanishing "bare" term. However in level truncation it turns out to be nonvanishing due to the so-called "twist anomaly" [27,28].

It was shown by Okuyama that the ratio of determinants in the RHS of (10.4) diverges like $L^{5/18}$ as $L \to \infty$. Similarly, the corresponding term in $\tilde{A}$ behaves as $L^{11/18}$. Now, in order for g_T to be finite, the only possibility is to tune the "dressing" parameter e to the value one in some suitable way. This is the reason why, as anticipated many times in the previous sections, we have to set $e = 1$. But in the formula (10.4) this has to be done with an appropriate scaling of e to 1, in such a way as to get an overall finite result. This is very close to what we did in [1] to make the dressed sliver action finite. Following the same prescription, we render separately finite A and $\tilde{A}$ (the matter and ghost part). This entails that H must be finite too. It is easy to see that the only way to implement this is to let $f_e \to 1$ (i.e., $e \to 1$) in such a way that

$$f_e - 1 = s_t L^{-5/36} \qquad \text{and} \qquad f_{\tilde{e}} - 1 = \tilde{s}_t L^{-11/36} \quad (10.5)$$

where s_t and $\tilde{s}_t$ are constants. We note that f_e and $f_{\tilde{e}}$ scale the same way as f_ϵ and $f_{\tilde{\epsilon}}$ in [1].

Using $f_e \to 1$ in (10.3) we obtain $H = H_0$. From (10.2) it then follows that g_T is independent of the dressing parameters α and β. We expect this to be true for all physical quantities.

As in the case of the energy of the dressed sliver, the precise value of g_T depends not only on the value of the (so far undetermined) scaling parameter s_t, but also on the way in which the multiple limit $f_1, f_2, f_3 \to 1$ is taken. Now we would like to argue that, with the proper choice of limit prescriptions, two problems, which affect the approach with the standard sliver, may be solved:

(i) Validity of EOM and LEOM when contracted with the solutions themselves.

(ii) Correct value of the product of the sliver energy density and g_T^2.

B. Scaling limit

In general, observables contain such terms as $(f_1 f_2 - 1)$ and/or $(f_1 f_2 f_3 - 1)$. In the scaling limit $f_i - 1 \approx s_i L^x$, where $x < 0$ and $L \to \infty$, one expects

$$(f_1 f_2 - 1) \approx s_{12} L^x, \qquad (f_1 f_2 f_3 - 1) \approx s_{123} L^x \quad (10.6)$$

but the scaling coefficients s_{12} and s_{123} are *a priori* not unique. They depend on the precise prescription for taking the multiple limits (see [1], Sec. 5 and Appendix C).

In Sec. 5 of [1] it was shown that there is a connection between the prescription for taking limits and the validity of the EOM. Considering the EOM for the dressed sliver contracted with the dressed sliver, we have

$$\langle \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \mathcal{Q} | \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} \rangle = \left(1 - \frac{1}{f_1 f_2}\right)^{-26} \left(1 - \frac{1}{\tilde{f}_1 \tilde{f}_2}\right)^2 \langle \Xi | \mathcal{Q} | \Xi \rangle \quad (10.7)$$

$$\langle \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} * \hat{\Xi}_{\epsilon_3 \tilde{\epsilon}_3} \rangle = \left(1 - \frac{1}{f_1 f_2 f_3}\right)^{-26} \left(1 - \frac{1}{\tilde{f}_1 \tilde{f}_2 \tilde{f}_3}\right)^2 \langle \Xi | \Xi * \Xi \rangle, \quad (10.8)$$

where $|\Xi\rangle = |\hat{\Xi}_0\rangle$ is Hata and Kawano's sliver. Let us denote

$$\zeta_{cc} = - \frac{\langle \Xi | \mathcal{Q} | \Xi \rangle}{\langle \Xi | \Xi * \Xi \rangle}. \quad (10.9)$$

If the EOM holds for this sliver solution one gets $\zeta_{cc} = 1$. However, it was argued in [16] that this may not be the case in the level truncation regularization. We believe that this "anomaly" should be resolved within the level truncation scheme and we expect (see below) that the result should be $\zeta_{cc} = 1$. However we would like to point out that the formalism we have presented in this paper can also allow for values of $\zeta_{cc} \neq 1$. So, to keep this possibility into account, we leave ζ_{cc} generic. In fact, as we will see, this variable can be absorbed by the dressing.

From the requirement that "contracted" EOM be satisfied

$$\lim_{\epsilon_i, \tilde{\epsilon}_j \to 1} \langle \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \mathcal{Q} | \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} \rangle = - \lim_{\epsilon_i, \tilde{\epsilon}_j \to 1} \langle \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} * \hat{\Xi}_{\epsilon_3 \tilde{\epsilon}_3} \rangle \quad (10.10)$$

we obtain the following condition on the scaling parameters

$$\left(\frac{s_{ccc}}{s_{cc}}\right)^{-26} \left(\frac{\tilde{s}_{ccc}}{\tilde{s}_{cc}}\right)^2 = \zeta_{cc}. \quad (10.11)$$

We see that a possible anomaly in the contracted EOM can be cured by an appropriate limit prescription. However it should be noticed that the limit prescription to be used in such a case is not *a priori* clear and far from simply describable. We recall that in [1] (see Sec. 2) we defined a privileged way of taking this kind of limit: the nested limits prescription. In the light of the analysis of [1] this looked like the most natural prescription. Any other way seems to be artificial. This is the reason why we tend to believe that there should not be any anomalous ζ_{cc}.

In the case of the contracted LEOM for our tachyon solution (5.9)

$$\langle \hat{\phi}_e(\mathbf{t}, p) | \mathcal{Q}_0 | \hat{\phi}_e(\mathbf{t}, p) \rangle = 0 \quad (10.12)$$

the possible anomaly [14,16] is cured by taking

$$\zeta_{tt} \equiv - \frac{\langle \phi_t | \mathcal{Q} | \phi_t \rangle}{2 \langle \phi_t | \phi_t * \Xi \rangle} = \left(\frac{s_{ttc}}{s_{tt}}\right)^{-26} \left(\frac{\tilde{s}_{ttc}}{\tilde{s}_{tt}}\right)^2 \quad (10.13)$$

where ϕ_t is the undressed tachyon $e = 0$ (from the symmetry of 3-string vertex for cyclic permutations it follows $s_{ttc} = s_{tct} = s_{ctt}$).

C. D25-brane energy

Let us now calculate the product of the dressed sliver energy density and g_T^2, which if our dressed sliver represents the D25-brane, should be

$$(E_c g_T^2)_{\mathrm{OST}} = \frac{1}{2\pi^2}. \quad (10.14)$$

From (10.7) and (10.2) we obtain

$$E_c g_T^2 = \left(\frac{s_{tt}}{s_{cc}}\right)^{26} \left(\frac{\tilde{s}_{tt}}{\tilde{s}_{cc}}\right)^{-2} \left(\frac{s_{ttt}}{s_{tt}}\right)^{-52} \left(\frac{\tilde{s}_{ttt}}{\tilde{s}_{tt}}\right)^4 (E_c g_T^2)_0 \quad (10.15)$$

where $(E_c g_T^2)_0$ is the result for the standard sliver[4]. In [14,28] it was shown that $(E_c g_T^2)_0$ is given by

$$(E_c g_T^2)_0 = \frac{\pi^2}{3} \left(\frac{16}{27 \ln 2}\right)^3 \quad (10.16)$$

which is obviously different from (10.14).

Note that scaling parameters s_{ttt} and $\tilde{s}_{ttt}$ do not appear in any LEOM and so are not affected by the analysis of the previous subsection. Therefore they can take values such that (10.14) is satisfied for the dressed sliver.

The possibility we have just pointed out is important because it removes a sort of no-go theorem [16] that seemed to exist in the operator treatment of the sliver

[4]It should be mentioned that the calculation of g_T using the definite–twist tachyon (5.9) shows a dependence on the parameter β. This dependence may of course be absorbed within the scaling parameters.

solution. However we should point out that there is a difference between the limiting/tuning procedure used in [1] to define a finite energy density of the dressed sliver and the same procedure used here in order to obtain the matching between the RHS and LHS of (10.14). In the first case the critical dimension was behind the argument we used and supported it (see Appendix K); in the latter case we have not been able to find a similar argument in favor of our tuning procedure. Without this the theory has apparently lost some of the predictability: see, for instance, (10.15) which is undetermined without knowing s_{ttt} and $\tilde{s}_{ttt}$. However we believe that such an argument should exist which relates tuning to the consistency of the whole theory (of which we have explored only a minute part).

XI. DISCUSSION

In this paper we have addressed the problem of finding the open string spectrum on a D25-brane in the operator formulation of VSFT. From previous works [9,16,17,32] it looked like the (standard) sliver solution is unable to capture such a spectrum with the expected physicality conditions (Virasoro constraints); in particular, it was not possible to derive the transversality condition for the U(1)-gauge field and to describe lower spin components of massive excitations.

To begin, in this paper we have proved that the dressing creates an extra structure whereby the photon transversality can be accounted for. Then we have coped with the problem of higher level states, where we have come to terms with the crucial role played by the $k \sim 0$ region. Our analysis implies that one has to take into account a twist-even $k = 0$ mode of the k-spectrum. This mode is usually disregarded because it vanishes when contracted with any Fock-space operator. Because of the star product, however, it becomes essential for the consistency of the formulas necessary in our calculations. In particular, it preserves the validity of many of the properties of the 3-strings vertex Neumann coefficients when the $k \sim 0$ region is probed.

We have also analyzed part of the cohomology problem implicit in the LEOM which generates on-shell fluctuations of the background D25-brane. In view of our analysis we find that two sets of modes are not gauge trivial. The first set covers the whole open string spectrum (our analysis stopped at level 3, i.e., $m^2 = 2$, but we do not see any conceivable obstruction in going further). The second set is "orthogonal" to the first and consists of states constructed by applying powers of $\xi_n a_n^\dagger$ factors to the states of the first set. They give rise to an infinite tower of descendants for every physical state, but they seem to describe the same observables as their parent states, thus creating a degeneracy of "representatives." A possibility is that this (numerable) redundancy is inherited by the gauge invariance of SFT and, thus, that it may be gauge fixed via some more refined study of the gauge structure of VSFT that takes into account the ghost sector.[5]

Finally we turned our attention to the well-known problem of matching the energy density of the classical solution with the D25-brane tension computed via the three on-shell tachyons coupling; although this problem was resolved in [15] in the BCFT formulation of VSFT, it remained an open puzzle for the operator formalism [16]. By extending the analysis started in [1], we showed that the arbitrariness we have in tuning the dressing parameters e to the level truncation cutoff can be used to satisfy the LEOM, when contracted with the solutions themselves. Appropriately choosing the scaling limit, we can obtain the correct value for the energy density expressed through the 3-tachyons coupling constant. However, at present this value can only be accommodated, not uniquely determined.

Because of the extensive matter dealt with in this paper it was inevitable to leave aside or only partially treat a number of issues. We have already pointed out that our analysis of the dressed sliver spectrum lacks an algorithm to find general solutions to the LEOM. Our cohomological analysis of the spectrum is also incomplete. Finally a comment about regularizations is in order. It looks like we are using three different regularizations of the k-spectrum: the level truncation to evaluate G_0 and H_0, Okuyama's prescription to evaluate determinants and the η-regulator to compute LEOM's solutions. As we have shown in Sec. VI, the η-regularization is based on the density of eigenvalues formula (6.1) in the $L \to \infty$ limit. Okuyama's regularization [10] and the twist anomaly computation of [28] are also based on the same formula and use $1/\log L$ as a regulator. In Sec. VI we have shown that, for large L, the identification (6.12) is justified. Therefore the three regularization procedures are based on the same regulator and they must lead to the same regularized quantities. A direct comparison of the results obtained with the three different procedures is, however, in general, not easy because they are devised to compute different objects in different contexts (G_0 and H_0 with the large L level truncation, infinite matrix determinants with Okuyama's prescription and state polarizations with the η regulator). It would certainly be desirable to have a unified prescription.

Even with these cautionary remarks, we believe we have produced strong evidence that if VSFT is properly regularized, it can consistently describe the physical content (both perturbative and nonperturbative) of bosonic string theory. It may therefore be a useful tool in tackling more challenging problems like the search for time-dependent solutions [33] and open-closed string duality [18].

[5] Some analysis of this kind was performed in [32] in order to implement the U(1) gauge transformations on the massless vector.

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APPENDIX A: A COLLECTION OF USEFUL FORMULAS

In this appendix we collect some useful results and formulas involving the matrices of the three strings vertex coefficients.

To start with, we recall that

(i) V^{rs}_{nm} are symmetric under simultaneous exchange of the two couples of indices;

(ii) they are endowed with the property of cyclicity in the r, s indices, i.e., $V^{rs} = V^{r+1,s+1}$, where $r, s = 4$ is identified with $r, s = 1$.

Next, using the twist matrix C ($C_{mn} = (-1)^m \delta_{mn}$), we define

$$X^{rs} \equiv CV^{rs}, \qquad r, s = 1, 2. \tag{A1}$$

These matrices are often rewritten in the following way $X^{11} = X, X^{12} = X_+, X^{21} = X_-$. They commute with one another

$$[X^{rs}, X^{r's'}] = 0, \tag{A2}$$

moreover

$$CV^{rs} = V^{sr}C, \qquad CX^{rs} = X^{sr}C. \tag{A3}$$

Next we quote some useful identities:

$$\begin{gathered} X^{11} + X^{12} + X^{21} = 1, \qquad X^{12}X^{21} = (X^{11})^2 - X, \\ (X^{12})^2 + (X^{21})^2 = 1 - (X^{11})^2, \\ (X^{12})^3 + (X^{21})^3 = 2(X^{11})^3 - 3(X^{11})^2 + 1, \end{gathered} \tag{A4}$$

and

$$\frac{1 - TX}{1 - X} = \frac{1}{1 - T}, \qquad \frac{X}{1 - X} = \frac{T}{(1 - T)^2}. \tag{A5}$$

Using these one can show, for instance, that

$$\begin{aligned} \mathcal{K}^{-1} &= \frac{1}{(1+T)(1-X)} \begin{pmatrix} 1 - TX & TX_+ \\ TX_- & 1 - TX \end{pmatrix} \\ &= \frac{1}{1 - T^2} \begin{pmatrix} 1 & T(\rho_1 - T\rho_2) \\ T(\rho_2 - T\rho_1) & 1 \end{pmatrix}, \\ \mathcal{M}\mathcal{K}^{-1} &= \frac{1}{(1+T)(1-X)} \begin{pmatrix} (1-T)X & X_+ \\ X_- & (1-T)X \end{pmatrix} \\ &= \frac{1}{1 - T^2} \begin{pmatrix} T & \rho_1 - T\rho_2 \\ \rho_2 - T\rho_1 & T \end{pmatrix}. \end{aligned} \tag{A6}$$

Where we have defined the left/right Fock-space projectors,

$$\rho_1 = \frac{1}{(1+T)(1-X)}[X^{12}(1 - TX) + T(X^{21})^2], \tag{A7}$$

$$\rho_2 = \frac{1}{(1+T)(1-X)}[X^{21}(1 - TX) + T(X^{12})^2]. \tag{A8}$$

They satisfy

$$\begin{gathered} \rho_1^2 = \rho_1, \qquad \rho_2^2 = \rho_2, \\ \rho_1 + \rho_2 = 1, \qquad \rho_1\rho_2 = 0, \end{gathered} \tag{A9}$$

i.e., they project onto orthogonal subspaces. Moreover,

$$\rho_1^T = \rho_1 = C\rho_2 C, \qquad \rho_2^T = \rho_2 = C\rho_1 C, \tag{A10}$$

where T represents matrix transposition. As was shown in [7], ρ_1, ρ_2 project out half the string modes. Using these projectors one can prove that

$$\begin{gathered} (X_+, X_-)\mathcal{K}^{-1} = (\rho_1, \rho_2), \\ \mathcal{M}\mathcal{K}^{-1}\mathcal{T}\begin{pmatrix} X_- \\ X_+ \end{pmatrix} = \begin{pmatrix} TX\rho_2 + TX_+\rho_1 \\ TX_-\rho_2 + TX\rho_1 \end{pmatrix}, \end{gathered} \tag{A11}$$

which are used throughout the paper.

The following relations are often useful

$$\begin{gathered} \rho_1 X_+ + \rho_2 X_- = 1 - XT, \\ \rho_1 X_- + \rho_2 X_+ = X(T - 1). \end{gathered} \tag{A12}$$

The next set of equations involve $\mathbf{v}_0, \mathbf{v}_\pm$. We start with

$$\begin{gathered} \mathbf{v}_+ + \mathbf{v}_- + \mathbf{v}_0 = 0, \qquad \mathbf{v}_0^2 + \mathbf{v}_+^2 + \mathbf{v}_-^2 = \frac{4}{3}V_{00}, \\ \mathbf{v}_0\mathbf{v}_- + \mathbf{v}_0\mathbf{v}_+ + \mathbf{v}_-\mathbf{v}_+ = -\frac{2}{3}V_{00}. \end{gathered} \tag{A13}$$

Next we have the representation in terms of $\mathbf{v}_0$

$$\begin{aligned} \mathbf{v}_+ &= \frac{1}{1+T}[(T-2)\rho_2 + (1-2T)\rho_1]\mathbf{v}_0, \\ \mathbf{v}_- &= \frac{1}{1+T}[(T-2)\rho_1 + (1-2T)\rho_2]\mathbf{v}_0, \end{aligned}$$

from which we get

$$\begin{aligned} \mathbf{v}_+ - \mathbf{v}_0 &= -\frac{3}{1+T}(\rho_2 + T\rho_1)\mathbf{v}_0, \\ \mathbf{v}_+ - \mathbf{v}_- &= -3\frac{1-T}{1+T}(\rho_2 - \rho_1)\mathbf{v}_0, \\ \mathbf{v}_- - \mathbf{v}_0 &= -\frac{3}{1+T}(\rho_1 + T\rho_2)\mathbf{v}_0. \end{aligned} \tag{A14}$$

Using these equations in (A13) it is easy to obtain in particular

$$\frac{2}{3}V_{00} = 3\langle \mathbf{v}_0|\frac{T^2 - T + 1}{(1+T)^2}|\mathbf{v}_0\rangle = \langle \mathbf{t}_0|\frac{1}{1+T}|\mathbf{v}_0\rangle, \tag{A15}$$

where $\mathbf{t}_0 = 3\frac{T^2-T+1}{T+1}|\mathbf{v}_0\rangle$.

In this work we use the continuous basis to evaluate various brackets which appear in the computations. We therefore need the matrices and vectors that define the 3 strings vertex in the k-basis. We use normalized k-vectors, see [34],

$$|k\rangle = \sum_{n=1}^{\infty}\frac{1}{k}\sqrt{\frac{nk}{2\sinh\frac{\pi k}{2}}}\oint\frac{dz}{2\pi i}\frac{1}{z^{n+1}}[1 - \exp(-k\tan^{-1}z)]|n\rangle,$$

so that

$$\langle k|k'\rangle = \delta(k - k').$$

With these conventions we have

$$\begin{aligned} X &= \int_{-\infty}^{\infty} dk X(k)|k\rangle\langle k|, \quad X(k) = -\frac{1}{1 + 2\cosh\frac{\pi k}{2}}, \\ T &= \int_{-\infty}^{\infty} dk T(k)|k\rangle\langle k|, \quad T(k) = -e^{-\pi|k|/2}, \\ \rho_1 &= \int_0^{\infty} dk|k\rangle\langle k|, \quad \rho_2 = \int_{-\infty}^{0} dk|k\rangle\langle k|, \end{aligned} \tag{A16}$$

and

$$\begin{aligned} |\mathbf{v}_0\rangle &= \int_{-\infty}^{\infty} dk v_0(k)|k\rangle, \\ v_0(k) &= -\frac{4}{3k}\sqrt{\frac{k}{\sinh\frac{\pi k}{2}}}\frac{\sinh^2\frac{\pi k}{4}}{1 + 2\cosh\frac{\pi k}{2}}|, \\ \mathbf{t}_0\rangle &= \int_{-\infty}^{\infty} dk t_0(k)|k\rangle, \\ t_0(k) &= -\frac{4}{k(e^{\pi|k|/2} - 1)}\sqrt{\frac{k}{\sinh\frac{\pi k}{2}}}\sinh^2\frac{\pi k}{4}. \end{aligned} \tag{A17}$$

All other matrices and vectors can be easily obtained using the properties (A12) and (A14). Notice that, since $C|k\rangle = -|-k\rangle$, twist-even vectors are represented by odd functions and vice-versa.

Notice also that $\mathbf{t}_0$ has a jump discontinuity in $k = 0$

$$t_0(0^+) = -t_0(0^-) = -\sqrt{\frac{\pi}{2}}.$$

APPENDIX B: SOLVING FOR $\mathbf{t}_+$ AND $\mathbf{t}_-$

To solve for $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$ in the LEOM in full generality, we reintroduce the parameter ϵ in the equation of motion (3.6). This means deforming it as follows

$$\begin{aligned} \mathrm{e\,xp}[-\mathbf{t}'a^\dagger\hat{p}]|\hat{\Xi}_{e*\epsilon}\rangle &= |\hat{\Xi}\epsilon\rangle * (\exp[-\mathbf{t}a^\dagger\hat{p}]|\hat{\Xi}_e\rangle) \\ &\quad + (\exp[-\mathbf{t}a^\dagger\hat{p}]|\hat{\Xi}_e\rangle) * |\hat{\Xi}\epsilon\rangle. \end{aligned} \tag{B1}$$

This seems to be a sensible deformation of (3.6), since we know that, as $\epsilon \to 1$, $\hat{\Xi}_{e*\epsilon} \to \hat{\Xi}_e$. As for $\mathbf{t}'$, this deformation makes sense only if $\mathbf{t}' \to \mathbf{t}$ as $\epsilon \to 1$. This is indeed what happens.

In the following we will find a solution to (B1) and then take the limit for $\epsilon \to 1$.

$$\begin{aligned} \mathbf{t}'_+ &= \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} \\ &\quad + (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix} \end{aligned} \tag{B2}$$

$$\mathbf{t}'_- = (X_+, X_-)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} \tag{B3}$$

We rewrite Eq. (B2) in a more explicit form, using the methods and results of Appendix B of [1]. In particular we need the formula

$$\begin{aligned} &(1 - \mathcal{P}_{\epsilon e}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon e} \\ &\quad = \frac{1}{B_{e\epsilon}}\begin{pmatrix} e f_e & \epsilon(\rho_1 - \kappa\rho_2) \\ e(\rho_2 - \kappa\rho_1) & \epsilon f_\epsilon \end{pmatrix}\mathcal{P}_{\epsilon e}, \end{aligned} \tag{B4}$$

where

$$\mathcal{P}_{\epsilon e} = \begin{pmatrix} \epsilon & 0 \\ 0 & e \end{pmatrix}P, \qquad B_{e\epsilon} = 1 + (1 - e)(1 - \epsilon)\kappa.$$

Then Eq. (4.5) can be rewritten as follows

$$\mathbf{t}'_+ = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\mathcal{K}^{-1}\mathcal{T}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\mathcal{K}^{-1}(\mathbf{t}_+) + \frac{1}{B_{\epsilon e}}(\rho_1, \rho_2)\begin{pmatrix} ef_e & \epsilon(\rho_1 - \kappa\rho_2) \\ e(\rho_2 - \kappa\rho_1) & \epsilon f_\epsilon \end{pmatrix}\mathcal{P}_{\epsilon e}\left[\begin{pmatrix} \frac{1}{1-T^2} & \frac{TX_+}{(1+T)(1-X)} \\ \frac{TX_-}{(1+T)(1-X)} & \frac{1}{1-T^2} \end{pmatrix}\begin{pmatrix} 3\frac{1-T}{1+T}(\rho_2 - \rho_1)|\mathbf{v}_0\rangle \\ -\frac{3}{1+T}(\rho_2 + T\rho_1)|\mathbf{v}_0\rangle \end{pmatrix} + \begin{pmatrix} \frac{T}{1-T^2} & \frac{X_+}{(1+T)(1-X)} \\ \frac{X_-}{(1+T)(1-X)} & \frac{T}{1-T^2} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix}\right]. \tag{B5}$$

Carrying out the algebra one finds

$$\mathbf{t}'_+ = \rho_2\mathbf{t}_+ + \rho_1\mathbf{t}_0 + \frac{1}{\kappa + f_\epsilon}f_e\Big[(1 - f_\epsilon)|\xi\rangle \times \langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle - |C\xi\rangle \times \langle\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_0\rangle + (f_\epsilon - 1)|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle + |C\xi\rangle\langle\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_+\rangle\Big].$$

Applying now C to both sides of this equation and summing the two we get a C-symmetric equation.

$$2\mathbf{t}'_+ = \mathbf{t}_+ + \mathbf{t}_0 + \frac{1}{\kappa + f\epsilon}f_e\Big[(1 - f_\epsilon)|\xi + C\xi\rangle \times \langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle - |\xi + C\xi\rangle\langle C\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_0\rangle + (f_\epsilon - 1)|\xi + C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle + |\xi + C\xi\rangle \times \langle\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_+\rangle\Big]. \tag{B6}$$

Taking the difference we get instead

$$0 = (\rho_2 - \rho_1)(\mathbf{t}_+ - \mathbf{t}_0) + \frac{1}{\kappa + f_\epsilon f_e}\Big[(1 - f_\epsilon)|\xi - C\xi\rangle \times \langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle + |\xi - C\xi\rangle\langle C\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_0\rangle + (f_\epsilon - 1)|\xi - C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle - |\xi - C\xi\rangle \times \langle\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_+\rangle\Big]. \tag{B7}$$

Recalling that $(\rho_1 - \rho_2)^2 = 1$, we multiply the last equation by $\rho_1 - \rho_2$ and obtain

$$\mathbf{t}_+ = \mathbf{t}_0 - \frac{1}{\kappa + f_\epsilon f_e}\Big[(1 - f_\epsilon)\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle + \langle C\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_0\rangle + (f_\epsilon - 1)\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle - \langle\xi|\frac{f_e + T}{1-T^2}|\mathbf{t}_+\rangle\Big]|\xi + C\xi\rangle. \tag{B8}$$

The solution to this equation is clearly of the form $\mathbf{t} = \mathbf{t}_0 + H|\xi + C\xi\rangle$, for some constant H. The latter can be determined by plugging this ansatz in (B8). One easily gets

$$\mathbf{t}_+ = \mathbf{t}_0 + \frac{1}{\kappa + f_e}|\xi + C\xi\rangle\langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle. \tag{B9}$$

Now we can replace this solution back into (B6). One easily obtains

$$\mathbf{t}'_+ = \mathbf{t}_0 + \frac{1}{\kappa + f_\epsilon f_e}|\xi + C\xi\rangle\langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle. \tag{B10}$$

We see that as $\epsilon \to 1$, $\mathbf{t}'_+ \to \mathbf{t}_+$.

As for (B3) we proceed in the same way. From the difference equation we obtain

$$M_-|\mathbf{t}_-\rangle \equiv \left\{1 + \frac{1}{\kappa + f_\epsilon f_e}|\xi - C\xi\rangle\Big[(f_\epsilon - 1)\langle\xi|\frac{T}{1-T^2} - \langle\xi|\frac{f_e + T}{1-T^2}\Big]\right\}|\mathbf{t}_-\rangle = 0. \tag{B11}$$

The solution must be in the kernel of the operator M_- and must have the form

$$|\mathbf{t}_-\rangle = \beta|(1 - C)\xi\rangle, \tag{B12}$$

for some constant β. Plugging this in the previous equation we find

$$M_-|\mathbf{t}_-\rangle = \beta\frac{(f_\epsilon - 1)(f_e + \kappa)}{\kappa + f_\epsilon f_e}|\xi - C\xi\rangle.$$

Therefore, (B12) solves (B11) either when $f_\epsilon = 1$ ($\epsilon = 1$), or when $f_e = -\kappa$ ($e \to \infty$) and $f_\epsilon \neq 1$. We are interested here in the first case. Putting $f_\epsilon = 1$ and (B12) in (B3) we obtain $\mathbf{t}'_- = \mathbf{t}_-$ for any β and f_e.

APPENDIX C: CALCULATING G

Let us first compute G with $\mathbf{t} = \mathbf{t}_+$ starting from Eq. (4.12). Our procedure consists of separating the ξ-independent part from the rest. The latter corresponds to Hata *et al.*'s calculation [9,27,28]. For instance

$$
\begin{aligned}
&(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} \\
&\quad = (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\mathcal{K}^{-1}\mathcal{T}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} \\
&\qquad + (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0) \\
&\qquad \times \mathcal{K}^{-1}\frac{1}{B_{\epsilon e}}\begin{pmatrix} ef_e & \epsilon(\rho_1 - \kappa\rho_2) \\ e(\rho_2 - \kappa\rho_1) & \epsilon f_\epsilon \end{pmatrix} \\
&\qquad \times \mathcal{P}_{\epsilon e}(1 + \mathcal{M}\mathcal{K}^{-1}\mathcal{T}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix},
\end{aligned}
\tag{C1}
$$

where again $B_{\epsilon e} = 1 + (1 - e)(1 - \epsilon)\kappa$. The first piece in the RHS is the ξ-independent part. Carrying out the algebra one gets the following result (C1)

$$
\begin{aligned}
&(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} \\
&\quad = 3\langle\mathbf{t}_0|\frac{T(2T-1)}{(T+1)^2(T-1)}|\mathbf{v}_0\rangle + \frac{2}{B_{\epsilon e}}\Big\{\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle \\
&\qquad \times\Big[e(1-\epsilon)\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_0\rangle - \epsilon\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle\Big]\Big\}.
\end{aligned}
\tag{C2}
$$

Proceeding in the same way with the third term in (4.12) we find

$$
\begin{aligned}
&(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix} \\
&\quad = \frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1-T}|\mathbf{t}_0\rangle + \frac{1}{B_{\epsilon e}}\Big[e(\epsilon-1)\langle\mathbf{t}_0|\frac{T}{1-T^2}|\xi\rangle \\
&\qquad \times\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle + \langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle([\epsilon - e(1-\epsilon)] \\
&\qquad \times\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle + \epsilon\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_+\rangle)\Big].
\end{aligned}
\tag{C3}
$$

Similarly for the last term on the RHS of (4.12) we find

$$
\begin{aligned}
&(0, \mathbf{t}_+)\mathcal{M}\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix} \\
&\quad = \langle\mathbf{t}_0|\frac{T}{1-T^2}|\mathbf{t}_0\rangle + \frac{2}{B_{\epsilon e}}\Big[e(1-\epsilon)\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle \\
&\qquad \times\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle - \epsilon\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle \\
&\qquad \times\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle\Big].
\end{aligned}
\tag{C4}
$$

Now we turn to the terms containing the twist-odd part. We need

$$
\begin{aligned}
&- 2(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} \\
&\quad - (0, \mathbf{t}_+)\mathcal{M}\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} + (0, \mathbf{t}_-)\mathcal{M}\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} \\
&\quad = -\frac{\beta(1-\epsilon)}{1 + (1-\epsilon)(1-e)\kappa}\Big[(2 + 2\kappa - \epsilon\kappa) \\
&\qquad \times\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle + e\kappa\langle\mathbf{t}_0|\frac{1}{1-T}|\xi\rangle\Big],
\end{aligned}
\tag{C5}
$$

and also

$$
(0, \mathbf{t}_-)\mathcal{M}\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} = 2\beta^2\kappa\frac{(1-\epsilon)(\kappa+1)}{1 + (1-\epsilon)(1-e)\kappa}.
\tag{C6}
$$

Using above formulae in (4.12) and (4.13) one obtains (4.14) and (4.15), respectively.

APPENDIX D: FORMULAS FOR STAR PRODUCTS IN LEOM

In this appendix we explicitly write down some formulas which are needed in order to evaluate the star products in the LEOM when the involved state is of the type (3.4) with a nontrivial polynomial $\mathcal{P}$, or, in other words, is the product of a tachyonlike state times a polynomial of the creation operators like (5.2). The best course in this case is to introduce the state (3.11), which depends on the variable vector β^μ, compute the star products of this state with the dressed sliver and then differentiate with respect to β^μ, setting $\beta^\mu = 0$ afterwards, in such a way as to "pull down" the desired monomials of the type (5.2). The calculation is straightforward and the relevant results for the matter part are recorded in the following formulas (where, for simplicity, we have set $\epsilon = 1$)

$$
\begin{aligned}
&(\ldots\langle\zeta a_\mu^\dagger\rangle\ldots)|\hat{\varphi}_e(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle \\
&\quad = (\ldots\langle-\zeta\frac{\partial}{\partial\beta^\mu}\rangle\ldots)\exp\Big[-\frac{1}{2}\mathcal{A}_1 - \mathcal{B}_1 \\
&\qquad - p\cdot(C_1 + \mathcal{D}_1)\Big]|\hat{\varphi}_e(\mathbf{t}, p)\rangle\Big|_{\beta=0},
\end{aligned}
\tag{D1}
$$

$$
\begin{aligned}
&(\ldots\langle\zeta a_\mu^\dagger\rangle\ldots)|\hat{\varphi}_e(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle \\
&\quad = (\ldots\langle-\zeta\frac{\partial}{\partial\beta^\mu}\rangle\ldots)\exp\Big[-\frac{1}{2}\mathcal{A}_2 - \mathcal{B}_2 \\
&\qquad - p\cdot(C_2 + \mathcal{D}_2)\Big]|\hat{\varphi}_e(\mathbf{t}, p)\rangle\Big|_{\beta=0},
\end{aligned}
\tag{D2}
$$

where

$$\mathcal{A}_1 \equiv (\beta, 0)\mathcal{M}\hat{\mathcal{K}}_{e1}^{-1}\begin{pmatrix} C\beta \\ 0 \end{pmatrix} = \langle\beta|\frac{T}{1-T^2}|C\beta\rangle - \langle\beta|\frac{1}{1-T^2}|C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\beta\rangle - \langle\beta|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|C\beta\rangle, \tag{D3}$$

$$\mathcal{A}_2 \equiv (0, \beta)\mathcal{M}\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix} 0 \\ C\beta \end{pmatrix} = \langle\beta|\frac{T}{1-T^2}|C\beta\rangle - \langle\beta|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|C\beta\rangle - \langle C\beta|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\beta\rangle, \tag{D4}$$

$$\mathcal{B}_1 \equiv a^\dagger(V_+, V_-)\hat{\mathcal{K}}_{e1}^{-1}\begin{pmatrix} C\beta \\ 0 \end{pmatrix} = \langle a^\dagger\rho_2\beta\rangle + \frac{1}{\kappa + f_e}\langle a^\dagger C\xi\rangle\langle\xi|\frac{T+f_e}{1-T^2}|C\beta\rangle, \tag{D5}$$

$$\mathcal{B}_2 \equiv a^\dagger(V_+, V_-)\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix} 0 \\ C\beta \end{pmatrix} = \langle a^\dagger\rho_1\beta\rangle + \frac{1}{\kappa + f_e}\langle a^\dagger \xi\rangle\langle\xi|\frac{T+f_e}{1-T^2}|\beta\rangle, \tag{D6}$$

$$\mathcal{C}_1 \equiv (\mathbf{t}, 0)\mathcal{M}\hat{\mathcal{K}}_{e1}^{-1}\begin{pmatrix} C\beta \\ 0 \end{pmatrix} = \langle\mathbf{t}|\frac{T}{1-T^2}|C\beta\rangle - \langle\mathbf{t}|\frac{1}{1-T^2}|C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\beta\rangle - \langle\mathbf{t}|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|C\beta\rangle, \tag{D7}$$

$$\mathcal{C}_2 \equiv (0, \mathbf{t})\mathcal{M}\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix} 0 \\ C\beta \end{pmatrix} = \langle\mathbf{t}|\frac{T}{1-T^2}|C\beta\rangle - \langle\mathbf{t}|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|C\beta\rangle - \langle C\mathbf{t}|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\beta\rangle, \tag{D8}$$

and

$$\mathcal{D}_1 \equiv (\mathbf{v}_+ - \mathbf{v}_0, \mathbf{v}_- - \mathbf{v}_+)\hat{\mathcal{K}}_{e1}^{-1}\begin{pmatrix} C\beta \\ 0 \end{pmatrix} = -\langle\mathbf{t}_0|\frac{\rho_1 T + \rho_2}{1-T^2}|C\beta\rangle + \langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle \times\left[\langle\xi|\frac{1}{1-T^2}|C\beta\rangle + \langle\xi|\frac{T}{1-T^2}|\beta\rangle\right],$$

$$\mathcal{D}_2 \equiv (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix} 0 \\ C\beta \end{pmatrix} = -\langle\mathbf{t}_0|\frac{\rho_1 T + \rho_2}{1-T^2}|\beta\rangle + \langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle \times\left[\langle\xi|\frac{1}{1-T^2}|\beta\rangle + \langle\xi|\frac{T}{1-T^2}|C\beta\rangle\right]. \tag{D10}$$

APPENDIX E: CALCULATIONS FOR THE VECTOR STATE

Applying the formulas of the previous section in the particular case of the vector excitation (5.3) we get

$$|\hat{\varphi}_{e,v}\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}_{e,v}\rangle = e^{-Gp^2/2}\Big\{d^\mu\langle a^\dagger(1-C)\zeta\rangle + \frac{1}{\kappa + f_e}d^\mu\langle a_\mu^\dagger(1-C)\xi\rangle\langle\xi|\frac{f_e+T}{1-T^2}|\zeta\rangle + p\cdot d\Big[-2\langle\mathbf{t}|\frac{T}{1-T^2}|(1-C)\zeta\rangle + \langle\mathbf{t}|\frac{T}{1-T^2}|(1-C)\xi\rangle\langle\xi|\frac{1}{1-T^2}|\zeta\rangle + \langle\mathbf{t}|\frac{1}{1-T^2}|(1-C)\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta\rangle\Big]\Big\}\mathcal{N}_v|\hat{\varphi}_e(\mathbf{t}, p)\rangle. \tag{E1}$$

A necessary condition to satisfy the LEOM is

$$\langle\xi|\frac{f_e+T}{1-T^2}|\zeta\rangle = 0.$$

On the other hand, the presence of the operator $1 - C$ in all the terms of the second line tells us that only the $\mathbf{t}_-$ part of $\mathbf{t}$ contributes to this terms. Inserting the explicit form of $\mathbf{t}_-$ one easily finds the result (5.4).

APPENDIX F: LEVEL 2 CALCULATIONS

Using the results of Appendix D, and keeping in mind the formulas

$$\rho_1|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle + \frac{1}{2}|0_\mp\rangle, \qquad \rho_2|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle - \frac{1}{2}|0_\mp\rangle,$$

the explicit formulas for the level two state are as follows

$$
\begin{aligned}
&[\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle]*|\hat\Xi\rangle+|\hat\Xi\rangle*[\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle]\\
&\quad=e^{-Gp^2/2}\bigg\{\frac{1}{2}\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle+2\theta^\mu_\mu\bigg(\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle+2\langle\zeta_-|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta_-\rangle\bigg)\\
&\qquad+2\theta^{\mu\nu}\bigg[\langle a_\mu^\dagger|\zeta_+\rangle p_\nu\mathcal{H}_++\langle a_\mu^\dagger|\zeta_-\rangle p_\nu\mathcal{H}_-+\frac{1}{\kappa+1}\langle a_\mu^\dagger(1+C)|\xi\rangle\langle\xi|\frac{1}{1-T}|\zeta_-\rangle p_\nu\mathcal{H}_++p_\mu p_\nu(\mathcal{H}_+^2\\
&\qquad+\mathcal{H}_-^2)\bigg]\bigg\}|\hat\varphi(\mathbf{t},p)\rangle,
\end{aligned}\tag{F1}
$$

where we have used $|\zeta_+\rangle=(\rho_1-\rho_2)|\zeta_-\rangle$ and we have disregarded terms that explicitly vanish when $\eta\to0$, i.e., evanescent terms like (6.18). Moreover

$$
\begin{aligned}
\mathcal{H}_+={}&-\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\zeta_-\rangle+\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta_-\rangle+\frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1+T}|\zeta_+\rangle+\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\\
&\times\langle\xi|\frac{1}{1+T}|\zeta_-\rangle,
\end{aligned}\tag{F2}
$$

and

$$
\mathcal{H}_-=-\beta\langle\xi|\frac{T-\kappa}{1-T^2}|\zeta_-\rangle.\tag{F3}
$$

The other relevant star product is

$$
\begin{aligned}
&[g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle]*|\hat\Xi\rangle+|\hat\Xi\rangle*[g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle]\\
&\quad=e^{-Gp^2/2}[g^\mu\langle a_\mu^\dagger|s_+\rangle+\frac{1}{\kappa+1}g^\mu\langle a_\mu^\dagger(1+C)|\xi\rangle\langle\xi|\frac{1}{1-T}|s_+\rangle-(p\cdot g)(\langle\mathbf{t}_0|\frac{1}{1-T}|s_+\rangle-2\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\\
&\qquad\times\langle\xi|\frac{1}{1-T}|s_+\rangle-2\langle\mathbf{t}_+|\frac{T}{1-T^2}|s_+\rangle+2\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|s_+\rangle+2\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\\
&\qquad\times\langle\xi|\frac{1}{1-T^2}|s_+\rangle)]|\hat\varphi(\mathbf{t},p)\rangle.
\end{aligned}\tag{F4}
$$

In order for the LEOM to be satisfied the sum of (F1) and (F4) must reproduce (7.7). A first condition for this to be true can be easily recognized: the coefficient in front of the $\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle$ term in the RHS of (F1) must be 1, which implies $p^2=-1$. This identifies the mass of the solution with the level 2 mass. Next, many terms in the RHS of (F1),(F3) diverge as $\eta\to0$. Therefore another condition for LEOM to be satisfied is that the corresponding coefficients vanish. Every bracket in the previous formulas are calculated by going to the k-basis, i.e., by inserting a completeness $\int dk|k\rangle\langle k|$ and then evaluating the k integral. The brackets that contain $|s_+\rangle, |\eta_-\rangle, |\zeta_+\rangle$ involve integrals evaluated essentially at $k=0$; the other brackets are finite. Remembering (7.22), (7.9), and (7.10), and moreover that $\mathbf{t}_0$ is finite at $k=0$ (see Appendix A), while $\frac{1}{1+T(k)}\sim1/k$ and $\xi(k)\to0$ as $k\to0$ and $|k_0|>2\eta$, it is easy to determine the degrees of divergence for $\eta\approx0$. To simplify the analysis we introduce an auxiliary assumption which was already mentioned in the text. We assume that $\xi(k)\neq0$ only for $k<k_0<0$. This makes all terms containing ξ in the previous formulas irrelevant as far as the LEOM is concerned. Under this hypothesis Eq. (F1) reduces to (7.12) and Eq. (F4) to (7.13). The surviving quantities are as follows

$$
\begin{aligned}
\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle={}&-\frac{\zeta_0^2\ln3}{\pi}\frac{1}{\eta^2}-2\frac{\zeta_0\zeta_1\ln3}{\pi}\frac{1}{\eta}+\pi\zeta_0^2\\
&-\frac{\ln3}{\pi}(\zeta_1^2+2\zeta_0\zeta_2)+\ldots,
\end{aligned}\tag{F5}
$$

$$
\begin{aligned}
\mathcal{H}_+&=\frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1+\mathcal{T}}|\zeta_+\rangle+\ldots\\
&=-\frac{\zeta_0\ln3}{\sqrt{\pi}}\frac{1}{\eta}-\frac{\zeta_1\ln3}{\sqrt{\pi}}+\ldots,
\end{aligned}\tag{F6}
$$

$$
\begin{aligned}
\langle\mathbf{t}_0|\frac{1}{1+T}|s_+\rangle={}&-\frac{2s_{-1}\ln3}{\sqrt{\pi}}\frac{1}{\eta^2}-\frac{2s_0\ln3}{\sqrt{\pi}}\frac{1}{\eta}\\
&-\bigg(\frac{1}{24}\sqrt{\pi^3}s_{-1}+\frac{2s_1\ln3}{\sqrt{\pi}}\bigg)+\ldots
\end{aligned}\tag{F7}
$$

It is important to notice that the numbers (in particular ln3) that appear in this expansion depends heavily on the particular regulator state $|\eta\rangle$ (6.4) we are using. Therefore they should not be attributed any particular significance. This also imply that the conditions we will obtain below are regularization dependent (see comment at the end of Sec. VII A).

Now we can impose the necessary cancellations. We must have

$$2\theta^{\mu\nu}\langle a^\dagger_\mu|\zeta_+\rangle p_\nu \mathcal{H}_+ + \frac{1}{2}g^\mu\langle a^\dagger_\mu|s_+\rangle = 0 \qquad \text{(F8)}$$

in the limit $\eta \to 0$. This implies that $g_\mu \sim \theta_{\mu\nu}p^\nu$. Assuming (7.4) we find

$$s_{-1} = -2\sqrt{\frac{2}{\pi}}b\zeta_0^2 \ln 3. \qquad \text{(F9)}$$

The next requirement is that

$$2\theta_\mu{}^\mu\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle + 2\theta^{\mu\nu}p_\mu p_\nu\mathcal{H}_+^2 - p\cdot g\langle \mathbf{t}_0|\frac{1}{1+T}|s_+\rangle = 0. \qquad \text{(F10)}$$

All three terms diverge like η^{-2} as $\eta \to 0$. The most divergent contribution vanishes if $5ab\ln 3 = 4$. The vanishing of the $1/\eta$ term requires

$$\sqrt{2}\zeta_0\zeta_1(ab\ln 3 - 4) + \sqrt{\pi}as_0 = 0. \qquad \text{(F11)}$$

This equation binds together the values of s_0, ζ_0, ζ_1. Finally we must impose that also the η^0 term vanishes. This results in an equation of the same type as (F11), involving also ζ_2 and s_1. It is not very illuminating and therefore we will not write it explicitly.

After imposing these (mild) conditions we see that the linearized EOM is satisfied provided $p^2 = -1$ and the Virasoro constraints in the form (7.4) are satisfied.

To end this appendix, let us add a few lines on how one can do without the auxiliary assumption made before Eq. (F5). In this case we give up this assumption and simply take $\xi(k) \sim k$ as $k \to 0$ (this satisfies (2.20) in a far less restrictive way than the auxiliary condition). Then all the terms in the RHS of (F1) and (F4) are nonvanishing. Two types of terms are dangerous: the term containing $\langle a^\dagger|\zeta_-\rangle$ in the RHS of (F1) and the two terms proportional to $\langle a^\dagger(1+C)|\xi\rangle$, which are present in both equations. These terms cannot be canceled within the present ansatz for the level 2 state. To deal with the first term we can add to the ansatz (7.7) a term $g^\mu\langle a^\dagger_\mu|r_-\rangle|\hat{\varphi}(\mathbf{t},p)\rangle$, where $|r_-\rangle$ is similar to $|\zeta_-\rangle$, and $r(\eta) = r_0 + r_1\eta + \ldots$. Adjusting the parameter r_0 we can easily cancel the first dangerous term. As for the other two, we can simply add to the ansatz two terms formally equal to the two terms of (7.7), where $|\zeta_-\rangle$ and $|s_+\rangle$ are replaced by $|(1-C)\zeta'\rangle$ and $|(1+C)s'\rangle$, with $\rho_2\zeta' = \zeta'$, $\rho_1\zeta' = 0$ and $\rho_2 r' = r'$, $\rho_1 r' = 0$. We can easily take $\zeta'(k)$, $r'(k)$ to cancel the above two terms as well as all the remaining terms not containing string oscillators $a^\dagger$.

APPENDIX G: LEVEL 3 CALCULATIONS

The first part of this appendix is devoted to redefining the polarizations as mentioned at the beginning of Sec. VII B. Such redefinitions are as follows

$$\begin{aligned}
h_\mu &= Ag_\mu + Bp\cdot gp_\mu,\\
\lambda_{\mu\nu} &= C\omega_{\mu\nu} + (D_+p_\mu\omega_{\rho\nu} + D_-p_\nu\omega_{\mu\rho})p^\rho\\
&\quad + D'p_\mu p_\nu\omega^{\rho\sigma}p_\rho p_\sigma,\\
\chi_{\mu\nu\rho} &= E\theta_{\mu\nu\rho} + F(p_\mu\theta_{\sigma\nu\rho} + p_\nu\theta_{\mu\sigma\rho} + p_\rho\theta_{\mu\nu\sigma})p^\sigma\\
&\quad + H(p_\mu p_\nu\theta_{\sigma\tau\rho} + p_\mu p_\rho\theta_{\sigma\nu\tau} + p_\nu p_\rho\theta_{\mu\sigma\tau})p^\sigma p^\tau\\
&\quad + H'p_\mu p_\nu p_\rho\theta^{\lambda\sigma\tau}p_\lambda p_\sigma p_\tau.
\end{aligned} \qquad \text{(G1)}$$

Inserting the above redefinitions into (7.17) and (7.18) we get

$$\begin{gathered}
3\sqrt{2}\Big(\frac{A-2B}{C} - 2\frac{A}{C}\frac{D_+ + D_- - D'}{D_+ - 2D'}\Big)g\cdot p + 2\omega_\mu{}^\mu = 0,\\
3g_\mu + \sqrt{2}\frac{C-2D_-}{A}\omega^\nu_\mu p_\nu = 0,\\
2\sqrt{2}\frac{C}{E}\omega_{\nu\mu}p^\nu - \sqrt{2}\Big(\frac{C+4D_+-2D_-}{E} + \frac{(D_++D_-)(F-H)}{E(F-2H)}\Big)\omega_{\mu\nu}p^\nu + 3\theta^\nu_{\mu\nu} = 0,\\
2\omega_{(\mu\nu)} + 3\sqrt{2}\frac{E-2F}{C}\theta_{\mu\nu\rho}p^\rho = 0.
\end{gathered}$$

These equations are of the same form as (7.19) and (7.20), with an obvious identification of the coefficients x, y, u, v, z. The coefficients $A, \ldots, H'$ are subject to the conditions

$$\begin{gathered}
\frac{E-2F}{C} = \frac{H-2H'}{D'}, \qquad \frac{E-2F}{2C} = \frac{F-2H}{D_+ + D_-},\\
\frac{C-2D_-}{A} = \frac{D_+-2D'}{B}F\Big(C + 2D_- - 4D_+ - \frac{(F-H)(D_++D_-)}{F-2H}\Big)\\
= E\Big(2D_- - D_+ - 2D' - \frac{2D'(H-H')}{H-2H'}\Big).
\end{gathered}$$

The second part of the appendix concerns the equations that must be verified among the terms of Eqs. (7.23), (7.24), and (7.25) for the LEOM (7.27) to be satisfied. As explained in the text we have to impose that all the terms in the RHS of Eqs. (7.23), (7.24), and (7.25) that do not reproduce the level 3 state vanish. There are two such terms: one linear in $a^\dagger$

$$\begin{aligned}
&3\theta_\mu{}^{\mu\rho}\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle\langle a^\dagger_\rho|\zeta_-\rangle\\
&\quad + 3\theta^{\mu\nu\rho}\langle a^\dagger_\rho|\zeta_-\rangle p_\mu p_\nu\mathcal{H}_+^2\\
&\quad + \omega^{\mu\nu}\Big(\langle a^\dagger_\mu|\zeta'_-\rangle P_\nu\langle\mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle\\
&\quad + \langle a^\dagger_\nu|\lambda_-\rangle P_\mu\mathcal{H}_+\Big) + \frac{3}{4}G^\mu\langle a^\dagger_\mu|r_-\rangle = 0,
\end{aligned} \qquad \text{(G3)}$$

and another quadratic in $a^\dagger$

$$3\theta^{\mu\nu\rho}\langle a^\dagger_\mu|\zeta_-\rangle\langle a^\dagger_\nu|\zeta_+\rangle p_\rho \mathcal{H}_+ + \omega^{\mu\nu}\Big(\frac{1}{4}\langle a^\dagger_\mu|\zeta'_-\rangle\langle a^\dagger_\nu|\lambda_+\rangle + \frac{1}{2}\langle a^\dagger_\mu|\zeta'_+\rangle\langle a^\dagger_\nu|\lambda_-\rangle\Big) = 0. \qquad \text{(G4)}$$

Now we use the η-expansions (F5) and (F6), together with

$$\langle \mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle = \frac{\lambda_{-1}\ln 3}{\sqrt{\pi}}\frac{1}{\eta^2} + \Big(\frac{\lambda_0 \ln 3}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{4}\lambda_{-1}\Big)\frac{1}{\eta} - \Big(\frac{\sqrt{\pi}^3}{48}\lambda_{-1} - \frac{\lambda_1\ln 3}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{4}\lambda_0\Big) + \ldots$$

Equation (G4) implies $2\omega_{(\mu\nu)} + 3\sqrt{2}z\theta^\rho_{\mu\nu}p_\rho = 0$ for some z. The terms in the RHS of (G4) are of overall order 0 in η, therefore only one condition is requested:

$$3\sqrt{2}z\zeta'_0\lambda_{-1} = 8\frac{\zeta_0^3}{\sqrt{\pi}}\ln 3. \qquad \text{(G5)}$$

The RHS of (G3) contains terms of order -2, -1, and 0 in η as $\eta \to 0$. We must therefore satisfy three conditions. Using that $\theta^{\mu\nu\rho}p_\mu p_\nu \sim \omega^{(\rho\mu)}p_\mu$, we see that the condition involving the term of order -2takes exactly the form of the first Eq. (7.20) with

$$u = \sqrt{\frac{\pi}{2}}\frac{\zeta'_0\lambda_{-1}}{\zeta_0^3} - \frac{\ln 3}{6z}, \qquad \text{(G6)}$$

$$v = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\lambda_1}{\zeta_0^2} + \frac{\ln 3}{12z}. \qquad \text{(G7)}$$

For generic values of ζ_0, ζ'_0, λ_{-1}, Eqs. (G5)–(G7) fix u, v, and z to some specific (nonvanishing) values. Now the vanishing of the term $\sim\eta^{-1}$ leads to an equation similar to the first Eq. (7.20), with identifications for u and v different from (G6) and (G7),

$$u = -\frac{2}{3}\sqrt{\frac{\pi}{2}}\frac{\zeta'_0\lambda_0 + \zeta'_1\lambda_{-1}}{\zeta_0^2\zeta_1} - \frac{\ln 3}{6z}, \qquad \text{(G8)}$$

$$v = \frac{1}{6\sqrt{2}}\frac{\lambda_1\zeta_1 + \lambda_0\zeta_0}{\zeta_1\zeta_0^2} + \frac{\ln 3}{12z}. \qquad \text{(G9)}$$

These equations, however, involve three additional parameters ζ_1, λ_0, ζ'_1. So it is easy to tune them to the same specific values for u and v. Finally the term of order η^0 involves also g^μ. In this case there are several possibilities.[6] One of these is that $g_\mu \sim \omega_{\nu\mu}p^\nu$. In the latter case also the constant y in (7.19) gets determined in terms of all the parameters, which include now also ζ_2, ζ'_2, λ_1, r_0. Since the relevant equations are cumbersome and not particularly illuminating we do not write them down. In conclusion, the LEOM for the state (7.21) is satisfied together with the Virasoro constraints (7.19) and (7.20) (the first is a consequence of the other three), provided some mild conditions on the various parameters that enter the game are complied with.

[6] In order to restrict the number of these possibilities and obtain more binding conditions one should give up the simplifying assumption and treat the level 3 in full generality.

APPENDIX H: THE COCHAIN SPACE

In this appendix we would like to explain in more detail the definition of the space of cochains given in Sec. VIII.

From Eq. (8.4), it would seem that, should we keep η finite throughout the cohomological analysis, all the states we have constructed would be trivial. This is due to the fact that in the gauge transformed expressions there appears the operator $\rho_1 - \rho_2$, which has the property that $(\rho_1 - \rho_2)|\eta_\pm\rangle = |\eta_\mp\rangle$. However this would be misleading, since in this argument we forget all the corrections to the LEOM that vanish only when $\eta \to 0$. In addition, one should not forget that the η dependence is an artifact of our regularization, it does not correspond to anything that has to do with the physical string modes. It can only appear at an intermediate step in our calculations. Therefore, the space of cochains should not contain any reference to the η dependence. There are only two ways to implement this. We can say that every cochain is defined up to evanescent states, but this would lead to incurable inconsistencies. For instance, the 0 state would be defined up to evanescent states, but we know that by applying, for instance, a gauge transformation to $|\eta_+\rangle$, which is evanescent, we get $|\eta_-\rangle$, which is in a nonzero class; so we would get the paradoxical result that applying the BRST operator to 0, we get something different from zero. This possibility has consequently to be excluded.

The only consistent possibility is the one put forward in the text. The nonzero cochains are those obtained by explicitly taking the limit $\eta \to 0$ for nonevanescent states, that is taking the limit in expressions of the type $\langle a^\dagger\zeta\rangle$ both for regular and singular ζ's [see Sec. VI, in particular, formula (6.8)]. It is clear that one gets in this way well-defined expressions for the states. This will form the set of nonzero cochains. To this we have to add the zero cochain, which is simply the zero state. All together they form a linear space. By definition this is the space of cochains where we want to compute the cohomology of the VSFT fluctuations. The η-regularization enters into the game when we come to compute the star products of the LEOM or of (8.2). Without such regularization these star products are ambiguous. From this point of view we see that the η-regularization concerns the star product rather than the states themselves. The cohomological problem at this point is well-defined.

APPENDIX I: TOWERS OF SOLUTIONS

In this appendix we prove the statements used in Sec. VIII to show that for any solutions to the LEOM we can construct an infinite tower of solutions with the same mass. We begin with the calculation of the star product $[h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle] * |\hat{\Xi}\rangle$. Written down explicitly this becomes

$$\left[h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle\right] * |\hat{\Xi}\rangle$$
$$= h^\nu\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle -\xi\frac{\partial}{\partial\beta^\nu}\rangle\langle -\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle -\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\exp\left[-\frac{1}{2}\mathcal{A}_1 - \mathcal{B}_1 - p(\mathcal{C}_1 + \mathcal{D}_1)\right]|\hat{\varphi}(\mathbf{t}, p)\rangle\Big|_{\beta=0}. \quad (I1)$$

Now we set $\mathcal{F}_1 = -\frac{1}{2}\mathcal{A}_1$ and $\mathcal{G}_1 = -\mathcal{B}_1 - p(\mathcal{C}_1 + \mathcal{D}_1)$. Then the RHS of (I1) becomes

$$= h^\nu\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle -\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle -\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\langle\xi\frac{\partial(\mathcal{F}_1+\mathcal{G}_1)}{\partial\beta^\nu}\rangle e^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat{\varphi}(\mathbf{t}, p)\rangle$$
$$= h^\nu\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle -\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle -\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\langle -\xi\frac{\partial\mathcal{G}_1}{\partial\beta^\nu}\rangle e^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat{\varphi}(\mathbf{t}, p)\rangle + h^\nu\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\sum_{j=1}^{i}\Big(\langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle$$
$$\times\langle -\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\widetilde{\langle -\zeta_j^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle}\ldots\langle -\zeta_i^{(i)}\frac{\partial}{\partial\beta}\rangle\Big)e^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat{\varphi}(\mathbf{t}, p)\rangle$$
$$= h^\nu\langle -\xi\frac{\partial\mathcal{G}_1}{\partial\beta_\nu}\rangle\left[\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle\right]$$
$$+ h^\nu\sum_{j=1}^{i}\langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle[\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle].$$

Tilded quantities denote omitted ones. Now, using the formulas of Appendix D, and the fact that $\rho_2\xi = \xi$, $\rho_1\xi = 0$, it is easy to prove that

$$\langle -\xi\frac{\partial\mathcal{G}_1}{\partial\beta_\nu}\rangle = \langle a_\nu^\dagger\xi\rangle, \qquad \langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle = \eta_{\nu\mu_j}\langle\xi|\frac{\kappa - T}{1-T^2}|C\zeta_j^{(i)}\rangle.$$

Inserting this back in the previous equations, we obtain

$$\left[h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle\right] * |\hat{\Xi}\rangle$$
$$= h^\nu\langle a_\nu^\dagger\xi\rangle\left[\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle\right] + h^\nu\sum_{j=1}^{i}\eta_{\nu\mu_j}\langle\xi|\frac{\kappa - T}{1-T^2}|C\zeta_j^{(i)}\rangle$$
$$\times\left[\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle * |\hat{\Xi}\rangle\right]. \quad (I2)$$

Now we repeat the calculation for the commuted product

$$|\hat{\Xi}\rangle * \left[h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t}, p)\rangle\right]$$
$$= h^\nu\sum_{i=1}^{n}\theta_i^{\mu_1\ldots\mu_i}\langle -\xi\frac{\partial}{\partial\beta^\nu}\rangle\langle -\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle -\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\exp\left[-\frac{1}{2}\mathcal{A}_2 - \mathcal{B}_2 - p(\mathcal{C}_2 + \mathcal{D}_2)\right]|\hat{\varphi}(\mathbf{t}, p)\rangle\Big|_{\beta=0}. \quad (I3)$$

Now, to simplify notation, we set $\mathcal{F}_2 = -\frac{1}{2}\mathcal{A}_2$ and $\mathcal{G}_2 = -\mathcal{B}_2 - p(\mathcal{C}_2 + \mathcal{D}_2)$, and proceeding as before (I3) becomes

$$= h^\nu\langle -\xi\frac{\partial \mathcal{G}_2}{\partial\beta_\nu}\rangle\Big(|\hat{\Xi}\rangle * \Big[\sum_{i=1}^n \theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t},p)\rangle\Big]\Big)$$

$$+ h^\nu\sum_{j=1}^i\langle\xi\frac{\partial^2\mathcal{F}_2}{\partial\beta^\nu\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle(|\hat{\Xi}\rangle * [\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t},p)\rangle])$$

$$= h^\nu\langle a_\nu^\dagger\xi\rangle\Big\{|\hat{\Xi}\rangle * \Big[\sum_{i=1}^n \theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t},p)\rangle\Big]\Big\}, \tag{I4}$$

since

$$\langle -\xi\frac{\partial\mathcal{G}_2}{\partial\beta^\nu}\rangle = \langle a_\nu^\dagger\xi\rangle, \qquad \langle\xi\frac{\partial^2\mathcal{F}_2}{\partial\beta^\nu\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle = 0.$$

Collecting the above results we have

$$[h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * [h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle] =$$
$$= h^\nu\langle a_\nu^\dagger\xi\rangle[|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle]$$
$$+ h^\nu\sum_{j=1}^i\eta_{\nu\mu_j}\langle\xi|\frac{\kappa - T}{1-T^2}|C\zeta_j^{(i)}\rangle[\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat{\varphi}(\mathbf{t},p)\rangle * |\hat{\Xi}\rangle]. \tag{I5}$$

The last line vanishes if $\langle\xi|\frac{T-\kappa}{1-T^2}|C\zeta_j^{(i)}\rangle = 0$ or if, for those j's for which this is not true, h is transverse to the tensor θ. In this case, if $|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle$ is a solution to the linearized equation of motion,

$$[h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle] * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * [h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle] = h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle, \tag{I6}$$

i.e., also $h^\nu\langle a_\nu^\dagger\xi\rangle|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle$ is a solution. All the results similar to this used in Sec. VIII can be obtained by obvious extensions of the previous argument.

APPENDIX J: CALCULATING H

The number H comes from the three-point tachyon vertex. If we take (5.1) as the tachyon solution, the three-tachyon vertex is given by

$$_1\langle\phi_e(\mathbf{t},p_1)|_2\langle\phi_e(\mathbf{t},p_2)|_3\langle\phi_e(\mathbf{t},p_3)|V_3\rangle_{123} = (\det\hat{\mathcal{K}}_3)^{-D/2}\hat{\mathcal{N}}_e^3\exp[-\mathcal{H}_1(p_1,p_2,p_3)]. \tag{J1}$$

$\mathcal{H}_1$ is given by

$$\mathcal{H}_1(p_1,p_2,p_3) = \chi^T\hat{\mathcal{K}}_3^{-1}\lambda + \frac{1}{2}\lambda^T\mathcal{V}\hat{\mathcal{K}}_3^{-1}{}_3\lambda + \frac{1}{2}\chi^T\hat{\mathcal{K}}_3^{-1}\hat{\Sigma}_3\chi + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)V_{00}, \tag{J2}$$

with $p_1 + p_2 + p_3 = 0$. In this equation the various symbols are as follows

$$\lambda^T = (\lambda_1,\lambda_2,\lambda_3), \qquad \lambda_i = -p_i\mathbf{t}C, \quad i = 1,2,3,$$
$$\chi = \begin{pmatrix}\mathbf{v}_0p_1 + \mathbf{v}_-p_2 + \mathbf{v}_+p_3\\ \mathbf{v}_+p_1 + \mathbf{v}_0p_2 + \mathbf{v}_-p_3\\ \mathbf{v}_-p_1 + \mathbf{v}_+p_2 + v_0p_3\end{pmatrix} = \begin{pmatrix}\mathbf{v}_0 & \mathbf{v}_- & \mathbf{v}_+\\ \mathbf{v}_+ & \mathbf{v}_0 & \mathbf{v}_-\\ \mathbf{v}_- & \mathbf{v}_+ & \mathbf{v}_0\end{pmatrix}\begin{pmatrix}1\\ p_2\\ p_3\end{pmatrix},$$
$$\hat{\Sigma}_3 = \begin{pmatrix}\hat{S}_e & 0 & 0\\ 0 & \hat{S}_e & 0\\ 0 & 0 & \hat{S}_e\end{pmatrix}, \qquad \mathcal{V}_3 = \begin{pmatrix}V & V_+ & V_-\\ V_- & V & V_+\\ V_+ & V_- & V\end{pmatrix}. \tag{J3}$$

Finally $\hat{\mathcal{K}}_3 = 1 - \hat{\Sigma}_3\mathcal{V}_3$. Since

$$\mathcal{N}_e = \frac{g_0}{\sqrt{G}}\sqrt{\frac{\det(1-\hat{S}_e^2)^{D/2}}{\det(1-\hat{\tilde{S}}^2)}}\exp\Big[-\frac{1}{2}p^2\mathbf{t}\frac{1}{1+\hat{T}_e}C\mathbf{t}\Big], \tag{J4}$$

the total exponential in (J1) is given by

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$
$$\mathcal{H}_2(p_1,p_2,p_3) = (p_1^2 + p_2^2 + p_3^2)H_2, \tag{J5}$$
$$H_2 = -\frac{1}{2}\langle\mathbf{t}|\frac{1}{1+\hat{T}_e}C|\mathbf{t}\rangle.$$

Similarly one can show that $\mathcal{H}_1(p_1,p_2,p_3) = (p_1^2 + p_2^2 + p_3^2)H_1$. Let us set $H = H_1 + H_2$.

All expressions can be straightforwardly computed once we explicitly determine the quantity

$$\hat{\mathcal{K}}_3^{-1} = (1 - \hat{\Sigma}_3 \mathcal{V})^{-1}.$$

It turns out that

$$\hat{\mathcal{K}}_3^{-1} = \mathcal{K}_3^{-1}[1 + (1 - \mathcal{P}_{\mathcal{E}}\mathcal{M}_3\mathcal{K}_3^{-1})^{-1}\mathcal{P}_{\mathcal{E}}\mathcal{M}_3\mathcal{K}_3^{-1}]. \tag{J6}$$

Moreover we have

$$\begin{aligned}(1 - \mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1})^{-1}\mathcal{P}_e \sum_{n=0}^{\infty}(\mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1})^n\mathcal{P}_e \\ = \frac{\kappa + f_e}{f_e^3 - 1} \\ \times \begin{pmatrix} f_e^2 & (f_e\rho_1 + \rho_2) & (f_e\rho_2 + \rho_1) \\ (f_e\rho_2 + \rho_1) & f_e^2 & (f_e\rho_1 + \rho_2) \\ (f_e\rho_1 + \rho_2) & (f_e\rho_2 + \rho_1) & f_e^2 \end{pmatrix}\mathcal{P}_{e.}\end{aligned} \tag{J7}$$

With the use of this formula one can directly compute all the contributions in (J2) given the general tachyon solution

$$\begin{aligned}\mathbf{t} &= \mathbf{t}_+ + \mathbf{t}_- \\ \mathbf{t}_+ &= \mathbf{t}_0 + \alpha W(\xi + C\xi), \qquad W = \langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle, \\ \mathbf{t}_- &= \beta(\xi - C\xi).\end{aligned} \tag{J8}$$

When momentum conservation holds we have the following identity

$$\begin{aligned}(p_1, p_2, p_3)\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a\end{pmatrix}\begin{pmatrix} 1 \\ p_2 \\ p_3\end{pmatrix}\Bigg|_{\sum_i p_i = 0} \\ = \left[a - \frac{1}{2}(b + c)\right]\sum_i p_i^2.\end{aligned} \tag{J9}$$

Let us begin analyzing the contribution coming from the twist-even part of the tachyon. With lengthy but straightforward manipulations we get

$$\begin{aligned}\chi^T\hat{\mathcal{K}}_3^{-1}\lambda = -\frac{1}{2}(p_1^2 + p_2^2 + p_3^2)\bigg\{\langle\mathbf{t}_0|\frac{1}{1-T^2}|\mathbf{t}_+\rangle \\ - \frac{2}{f_e^2 + f_e + 1}\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\bigg[(f_e - 1) \\ \times \langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle \\ - (1 + 2f_e)\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_+\rangle\bigg]\bigg\},\end{aligned} \tag{J10}$$

$$\begin{aligned}\lambda^T\mathcal{V}_3\hat{\mathcal{K}}_3^{-1}\lambda = (p_1^2 + p_2^2 + p_3^2)\bigg\{\langle\mathbf{t}_+|\frac{T}{1-T^2}|\mathbf{t}_+\rangle \\ - \frac{1}{2}\langle\mathbf{t}_+|\frac{1}{1-T^2}|\mathbf{t}_+\rangle + \frac{1}{f_e^2 + f_e + 1} \\ \times\bigg[(f_e - 1)\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle^2 - (2f_e + 1) \\ \times\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle^2 + 2(f_e + 2) \\ \times\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\bigg]\bigg\},\end{aligned} \tag{J11}$$

$$\begin{aligned}\chi^T\hat{\mathcal{K}}_3^{-1}\hat{\Sigma}_3\chi = \frac{3}{2}(p_1^2 + p_2^2 + p_3^2) \\ \times\bigg(\langle\mathbf{t}_0|\frac{T(1-2T)}{(1-T^2)(1+T)}|\mathbf{v}_0\rangle \\ + \frac{f_e + 2}{f_e^2 + f_e + 1}\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle^2\bigg).\end{aligned} \tag{J12}$$

Plugging inside the expression for $\mathbf{t}_+$ we get

$$\begin{aligned}&\mathcal{H}_1^+(p_1, p_2, p_3) \\ &= (p_1^2 + p_2^2 + p_3^2)\bigg\{\frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1+T}|\mathbf{t}_0\rangle + 2\alpha W^2 \\ &+ \alpha^2\Big(\kappa - \frac{1}{2}\Big)W^2 - \frac{1}{f_e^2 + f_e + 1}\frac{1}{2}[\alpha^2 \\ &\times(f_e(1 + 2\kappa - 2\kappa^2) - (1 - 4\kappa + \kappa^2)) \\ &+ 2\alpha(f_e(2\kappa - 1) + (\kappa - 2)) - (2f_e + 1)]W^2\bigg\}.\end{aligned} \tag{J13}$$

The second contribution to H comes from the normalization in front of the tachyon state (J4), that is

$$\begin{aligned}\mathcal{H}_2^+(p_1, p_2, p_3) = -\frac{1}{2}(p_1^2 + p_2^2 + p_3^2) \\ \times\langle\mathbf{t}_+|\frac{1}{1+\hat{T}_e}C|\mathbf{t}_+\rangle.\end{aligned}$$

We have

$$\begin{aligned}\langle\mathbf{t}_+|\frac{1}{1+\hat{T}_e}C|\mathbf{t}_+\rangle = \langle\mathbf{t}_+|\frac{1}{1+T}|\mathbf{t}_+\rangle + \frac{2}{\kappa + f_e} \\ \times\langle\mathbf{t}_+|\frac{1}{1+T}|\xi\rangle^2\sum_{n=0}^{\infty}\Big(\frac{\kappa - 1}{\kappa + f_e}\Big)^n\end{aligned} \tag{J14}$$

$$\begin{aligned}= \langle\mathbf{t}_0|\frac{1}{1+T}|\mathbf{t}_0\rangle + 4\alpha W^2 + 2\alpha^2(\kappa - 1)W^2 \\ + \frac{2}{f_e + 1}[\alpha(\kappa - 1) + 1]^2W^2.\end{aligned} \tag{J15}$$

The total twist-even contribution in H, let us call it H^+, is then

$$H^+ = H_1^+ + H_2^+ \tag{J16}$$

$$= H_0 - \frac{(f_e - 1)^2(\kappa + f_e)^2}{2(f_e + 1)(f_e^3 - 1)}\left(\frac{1}{\kappa + f_e} - \alpha\right)^2 \times \langle \mathbf{t}_0| \frac{1}{1 + T}|\xi\rangle^2. \tag{J17}$$

The bare contribution H_0 is naively zero, but in level truncation regularization it acquires a nonvanishing value [28]. We stress once more that the dressing contribution is not affected by the twist anomaly as the half string vector ξ does not excite the $k = 0$ (zero momentum) midpoint mode.

Now we turn to the twist-odd contributions which, for $e \neq 1$, do not vanish identically for any solution to the LEOM. Let us analyze first the purely imaginary contribution linear in β. It is easy to see that the part coming from H_2 is identically zero by twist symmetry, and the same is true for the term $\lambda_-^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda_+$ in H_1. So the only potential contributions arise from the tachyon linear term $\chi^T \hat{\mathcal{K}}_3^{-1} \lambda_-$. It is straightforward to compute these terms by plugging $\mathbf{t}_- = \beta(\xi - C\xi)$ and to show again that twist symmetry requires this contribution to vanish. So there are no imaginary contributions in H. The quadratic terms in β come out from $\lambda_-^T \mathcal{V} \hat{\mathcal{K}}_3^{-1}{}_3 \lambda_-$ in H_1 and $\langle \mathbf{t}_-| \frac{1}{1+\hat{T}_e} C|\mathbf{t}_-\rangle$ in H_2. They can be directly computed plugging the explicit expression for $\mathbf{t}_-$. The result is

$$\lambda_-^T \mathcal{V} \hat{\mathcal{K}}_3^{-1} \lambda_- = \beta^2 \frac{(f_e + \kappa)[f_e(2\kappa - 1) + (\kappa - 2)]}{f_e^2 + f_e + 1} \sum_i p_i^2 \tag{J18}$$

$$-\frac{1}{2}\langle \mathbf{t}_-| \frac{1}{1 + \hat{T}_e} C|\mathbf{t}_-\rangle = -\beta^2 \frac{(\kappa - 1)(\kappa + f_e)}{f_e + 1}. \tag{J19}$$

Together they sum up to

$$\mathcal{H}^- \equiv H^- \sum_i p_i^2 = \frac{1}{2}\lambda_-^T \mathcal{V} \hat{\mathcal{K}}_3^{-1}{}_3 \lambda_- - \frac{1}{2}\langle \mathbf{t}_-| \frac{1}{1 + \hat{T}_e} C|\mathbf{t}_-\rangle \sum_i p_i^2, \tag{J20}$$

$$H^- = \beta^2 \frac{(f_e - 1)^2(\kappa + f_e)^2}{2(f_e + 1)(f_e^3 - 1)}. \tag{J21}$$

APPENDIX K: ROLE OF THE CRITICAL DIMENSION

This appendix is devoted to the role played by the critical dimension ($D = 26$) in VSFT, see Sec. X. Let us start from the normalized action

$$S[\hat{\psi}] = -\frac{1}{g_0^2}\left(\frac{1}{2}\langle \hat{\psi}| \mathcal{Q}|\hat{\psi}\rangle + \frac{1}{3}\langle \hat{\psi}|\hat{\psi} * \hat{\psi}\rangle\right). \tag{K1}$$

By means of the operator field redefinition [13]

$$\psi = e^{-1/4 \ln\gamma(K_2 - 4)}\hat{\psi}, \tag{K2}$$

it can be brought to the form

$$S'[\psi] = -\frac{1}{g_0^2\gamma^3}\left(\frac{1}{2}\langle \psi| \mathcal{Q}|\psi\rangle + \frac{1}{3}\langle \psi|\psi * \psi\rangle\right) = -\frac{1}{g_0^2}\left(\frac{1}{2\gamma}\langle \tilde{\psi}| \mathcal{Q}|\tilde{\psi}\rangle + \frac{1}{3}\langle \tilde{\psi}|\tilde{\psi} * \tilde{\psi}\rangle\right), \tag{K3}$$

where $\tilde{\psi} = \gamma\psi$. Both forms of the action have been considered previously in the literature [8,9] in the limit $\gamma \to 0$, implying a singular normalization of the action. What we have shown above is that free effective parameters appear in the process of regularizing the classical action so that a singular normalization of the latter can be avoided. This remark is of more consequence than it looks at first sight. The point is that the redefinition (K2) can harmlessly be implemented only in $D = 26$. In noncritical dimensions, as a consequence of such a redefinition, an anomaly appears [20]. In the course of our derivation above the critical dimension has never featured, but this remark brings it back into the game. This has an important consequence: setting $\gamma = g_0^{2/3}$ in the middle term of Eq. (K3), it is evident that in critical dimensions we can make any parameter completely disappear from the action by means of a field redefinition. So, in $D = 26$, the value of the brane tension is dynamically produced and not put in by hand.

The very reason for this is that the family of operators $K_n = L_n - (-1)^n L_{-n}$ leaves the action cubic term invariant (only in $D = 26$) while it acts linearly on the kinetic term as [13]

$$[K_{2n}, \mathcal{Q}] = -4n(-1)^n \mathcal{Q}. \tag{K4}$$

In other words $\mathcal{Q}$ is an "eigenvector" of K_{2n}, so every parameter can be absorbed by a field redefinition. In Open String Field Theory, on the other hand, one cannot implement a redefinition like (K2) since Q_B does not transform as an eigenvector of K_{2n}, so the coupling constant there is really a free parameter in the action.

Let us elaborate more on this aspect. We remark that both the string fields ψ and $\hat{\psi}$ above satisfy the same EOM. Therefore, there seems to exist different solutions of the EOM corresponding to the same energy, and, on the other hand, a given solution can be attributed different tensions (depending on what constant we put in front of the action, which does not affect the EOM). Since any constant put in front of the action in VSFT in critical dimension can be absorbed via a field redefinition, it is illusory to try to cure this problem by multiplying the action by some constant. This is a fact of VSFT in critical dimension and we have to come to terms with it. It is apparent from the above that VSFT does not predict the exact value of the D-brane tension, but rather makes room for it to emerge dynamically. It is at this point that dressing comes handy. We showed in [1] that in the theory there naturally arise scaling constants s and $\tilde{s}$ (see eq. (6.25) there) that can be adjusted to the physical value of the D-brane tension. Therefore the answer to the above puzzle is that if we redefine the string field in the action as in Eq. (2.38), the parameters s and $\tilde{s}$ should be scaled accordingly in such a way as to preserve the physical value of the brane tension. Of course, in this way, we are left with a multiplicity of solutions corresponding to the same tension. This has to be attributed to an invariance of the type discussed in Secs. IX and XI(perhaps a remnant of the original gauge invariance of the theory).

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