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Field theory on noncommutative spacetimes: Quasiplanar Wick products

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We give a definition of admissible counterterms appropriate for massive quantum field theories on the noncommutative Minkowski space, based on a suitable notion of locality. We then define products of fields of arbitrary order, the so-called quasiplanar Wick products, by subtracting only such admissible counterterms. We derive the analogue of Wick's theorem and comment on the consequences of using quasiplanar Wick products in the perturbative expansion.

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I. INTRODUCTION

Interest in quantum field theories with nonlocal interactions has reemerged recently in the context of the analysis of quantum field theory on noncommutative spacetimes. Such spacetimes are studied for various reasons, one of them based on the observation that Heisenberg's uncertainty principle along with classical gravity suggests that the localization of an event in spacetime with an arbitrarily high precision should be impossible. Based on this argument, a noncommutative spacetime (called the noncommutative Minkowski space or quantum spacetime) was introduced in [1]. Here, the ordinary coordinates are replaced by noncommuting "quantum coordinates," i.e., self-adjoint operators q^μ, $\mu = 0, \ldots, 3$, with

$$[q^\mu, q^\nu] = iQ^{\mu\nu},$$

subject to certain "quantum conditions,"

$$[q^\rho, Q^{\mu\nu}] = 0, \qquad Q_{\mu\nu}Q^{\mu\nu} = 0,$$
$$(\tfrac{1}{2}Q_{\mu\nu}Q_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma})^2 = 16\lambda_P^8 I$$

where λ_P is the Planck length, such that for every state ω in the domain of $[q^\mu, q^\nu]$ the following relations hold among the uncertainties $\Delta(q^\mu) = \sqrt{\omega((q^\mu)^2) - \omega(q^\mu)^2}$:

$$\Delta q^0 \cdot (\Delta q^1 + \Delta q^2 + \Delta q^3) \geq \lambda_P^2,$$
$$\Delta q^1 \cdot \Delta q^2 + \Delta q^1 \cdot \Delta q^3 + \Delta q^2 \cdot \Delta q^3 \geq \lambda_P^2.$$

As shown in [1], the regular realizations of the quantum conditions, i.e., those satisfying

$$e^{i\alpha q}e^{i\beta q} = e^{i(\alpha+\beta)q}e^{-(i/2)\alpha Q\beta},$$
$$\alpha, \beta \in \mathbb{R}^4, \alpha q = \alpha_\mu q^\mu, \alpha Q\beta = \alpha_\mu Q^{\mu\nu}\beta_\nu,$$

are in one-to-one correspondence with the nondegenerate representations of a C^*-algebra which is isomorphic to the algebra $\mathcal{E} = C_0(\Sigma, \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators on a fixed separable infinite dimensional Hilbert space and Σ is the joint spectrum of the operators $Q^{\mu\nu}$. This spectrum, being fixed in a Poincaré-invariant way by the quantum conditions, is homeomorphic to two copies of the tangent bundle of the 2-sphere, the noncompact manifold $TS^2 \times \{1, -1\}$. The commutators $Q^{\mu\nu}$ are affiliated to the center $\mathcal{Z} = C_b(\Sigma)$ of the multiplier algebra $M(\mathcal{E})$ of $\mathcal{E}$.

In less technical terms this means that, given a function f on $\mathbb{R}^4$, a function $f(q)$ on quantum spacetime can be defined as an element of $M(\mathcal{E})$ by a generalized Weyl correspondence. The product of two such elements of $M(\mathcal{E})$ is given by the twisted convolution product

$$f(q)g(q) = (2\pi)^{-8} \times \int dk_1 dk_2 \hat{f}(k_1)\hat{g}(k_2)e^{-(i/2)k_1 Q k_2}e^{-i(k_1+k_2)q}.$$

Here, ˆ indicates the Fourier transform of a function on $\mathbb{R}^4$ and k_1, k_2 are elements of the ordinary Minkowski space. The exponential $\exp(-\frac{i}{2}k_1 Q k_2)$ is referred to as the twisting. In analogy, the free field $\phi(q)$ on quantum spacetime was formally given in [1] as $\phi(q) = (2\pi)^{-4}\int dk\hat{\varphi}(k)e^{-ikq}$ where φ is the free field on Minkowski space.

Different definitions of perturbative quantum field theory on noncommutative spacetimes have been discussed in the literature (cf., e.g., [2]). While these approaches are equivalent on the ordinary Minkowski space, they cease to be so on noncommutative spacetimes with noncommuting time variable.

One of the possible approaches is based on what is known as the Yang-Feldman approach in ordinary quantum field theory. As early as 1952, this approach was already employed in the context of theories with nonlocal interactions [3]. Here, the field equation is the starting point, which for a self-interacting bosonic field on the noncommutative Minkowski space may be given as follows:

$$(\Box_q + m^2)\phi(q) = -g\phi^{n-1}(q)$$

with derivatives ∂_{q^μ} defined as the infinitesimal generators of translations (see [1]). The field equation is then solved

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recursively in terms of a formal power series in the coupling constant,

$$\phi = \sum_{\kappa=0}^{\infty} g^{\kappa}\phi_{\kappa},$$

with
$$\phi_{\kappa}(q) = \int d^4 y G(y) \times \sum_{\sum_{i=1}^{n-1}\kappa_i=\kappa-1} \phi_{\kappa_1}(q-y)\ldots\phi_{\kappa_{n-1}}(q-y), \tag{1.1}$$

$y \in \mathbb{R}^4$, where G is one of the Green functions of the *ordinary* Klein-Gordon operator, chosen according to the given boundary conditions.

Unlike the modified Feynman rules [4] which are widely used for perturbative calculations on the noncommutative Minkowski space, neither the Yang-Feldman approach nor the Hamiltonian approach proposed in [1] entail a formal (i.e., before renormalization) violation of unitarity even for a noncommuting time variable (see [2]).

Already a second order calculation performed in [2] showed that the perturbation theory in these approaches is not free of ultraviolet divergences. As in ordinary quantum field theory this can be traced to the fact that products of fields $\phi^n(q)$ are ill-defined. Mimicking the renormalization procedure (in position space) of ordinary quantum field theory, the first aim thus should be to find well-defined products of fields. One of the conceptual problems we are faced with here is to find an adequate generalization of the locality principle on which the definition of such products on the ordinary Minkowski space is founded. Various approaches to address this question are possible.

In [5], which was based on the doctoral thesis of one of the authors [6], we used the best localized states introduced in [1] to replace the ordinary concept of locality by a notion of "approximate coincidence," compatible with the uncertainty relations. The limit of coinciding points, which usually entails the appearance of ultraviolet divergent expressions, is replaced by the evaluation of a conditional expectation, given by the so-called quantum diagonal map, which minimizes the difference variables while leaving the mean coordinates invariant. Employing this concept of approximate coincidence in the definition of the interaction term leads to a natural regularization in quantum field theory on the noncommutative Minkowski spacetime. No ultraviolet divergences appear. Unfortunately, only translation and rotation invariance are preserved in this approach, and the free theory is treated on a different footing than the interaction.

In the present paper we follow a different idea. Heuristically, a local functional of a field is an element of the algebra generated by the field and its derivatives. The obstruction that, as on commutative spacetime, the field is too singular for admitting pointwise products is circumvented by smearing the field over translations,

$$\phi_g(q) = \int dx g(x)\phi(q+x)$$

with a test function g. The smeared fields $\phi_g(q)$ are then well-defined elements of a topological algebra which depends continuously on the test functions. We are therefore led to algebra-valued distributions

$$\phi_g^n(q) = \int dx_1\cdots dx_n g(x_1,\ldots,x_n)\phi(q+x_1)\cdots\phi(q+x_n).$$

Now let $\mathcal{O}$ be a neighborhood of the origin of Minkowski space. We call $\phi_g^n(q)$ local of order $\mathcal{O}$ if supp$g \subset \mathcal{O}^n$. Our aim is to find suitable subtractions

$$\phi_g^n - \sum_{k=1}^{n}\phi^{n-k}_{\gamma_k^{(n)}(g)}$$

with continuous linear maps $\gamma_k^{(n)}$ from test functions with n variables to test functions with $n-k$ variables such that the limit $g \to \delta$ (limit of coinciding points) is a *well-defined* quantum field on quantum spacetime which is *local* of all orders.

The crucial fact now is that the usual Wick ordering is not of this type when applied to fields on quantum spacetime, as some of the subtracted terms are not local. We would therefore like to refrain from subtracting them and therefore introduce a modified Wick product, the so-called *quasiplanar Wick product*, which is obtained by admitting only such maps $\gamma_k^{(n)}$ in the subtraction procedure which do not decrease the order of locality. Fortunately, the terms which remain unsubtracted compared to the ordinary Wick product turn out to be finite in the limit of coinciding points such that our procedure yields a well-defined product in this case.

We then postulate that only quasiplanar Wick products are admissible as counterterms in perturbative renormalization. While this seems to be necessary from the point of view of locality (and, as far as we checked up to now, also sufficient for the absorption of ultraviolet divergences), it seriously modifies the asymptotic behavior of the theory. It turns out that in the Yang-Feldman approach the asymptotic outgoing and incoming free fields are neither local nor Lorentz invariant, although the subtraction procedure itself is fully Lorentz covariant. We find that the notorious infrared-ultraviolet mixing shows up in our framework not as an inconsistency of the theory but in a drastic change of the dispersion relation which we compute to first order in ϕ^4 theory. This may allow new tests of the theory.

It is noteworthy that the formalism presented here may formally also be applied in the Hamiltonian approach.

It should be stressed that in our setting the Planck length λ_P is kept fixed at its physical value. If one adopts the point of view that in the limit "$\lambda_P \to 0$" the theory should reduce to the usual renormalized theory on Minkowski space, one has to find additional counterterms, which for $\lambda_P \neq 0$ correspond to finite renormalizations and in the limit "$\lambda_P \rightarrow 0$" produce the missing ordinary counterterms needed on Minkowski space. So far, we have not been able to find a local and Lorentz-invariant definition of such counterterms.

Also in view of the modified dispersion relation, it seems that in all our attempts to introduce interactions of fields on quantum spacetime, Lorentz invariance is sooner or later lost—although the underlying geometry of our model of quantum spacetime as well as the theory of free fields on quantum spacetime are fully Lorentz (and Poincaré) invariant. This point calls for a deeper understanding we still lack at the moment.

We would like to emphasize that results regarding the renormalization of field theories on a noncommutative Euclidean spacetime [7] cannot be directly applied to field theories on the noncommutative Minkowski space. We will see explicitly in an example that a tadpole which is finite in the Euclidean setting fails to be so on the noncommutative Minkowski spacetime. This is not very surprising as no generalization of Osterwalder-Schrader positivity seems to be available and not even the Wick rotation itself has been given proper meaning in a space/time noncommutative setting.

We will furthermore see that a theory of self-interacting scalar fields with commuting time variable cannot be renormalized by local counterterms.

This paper focuses on the combinatorial aspects and the physical consequences of the idea to admit only local counterterms. The full proof that quasiplanar Wick products are well defined at coinciding points ($g \rightarrow \delta$) turned out to be rather technical and is merely sketched in this paper. Details regarding domains of definition and appropriate test function spaces will be the subject of a forthcoming publication.

The results presented here are based to a large extent on the doctoral thesis of one of the authors [8] where further details may be found.

II. FIELDS ON THE NONCOMMUTATIVE MINKOWSKI SPACE

In [1], the quantization of a function $f(x)$ on ordinary spacetime was defined in terms of the Weyl correspondence

$$W(f) \equiv f(q) := \int dk e^{ikq} \check{f}(k) = (2\pi)^{-4} \int dk e^{-ikq} \hat{f}(k),$$

where $\check{f}(k) = (2\pi)^{-4} \int dx f(x) e^{-ikx}$, $\hat{f}(k) = (2\pi)^4 \check{f}(-k)$. By analogy, the free field ϕ on the quantum spacetime was defined by the heuristic formula

$$\phi(q) = (2\pi)^{-4} \int dk \hat{\varphi}(k) \otimes e^{-ikq},$$

where φ is the free field on Minkowski space and $\hat{\varphi}$ is its Fourier transform. $\phi(q)$ is to be thought of as a (formal) element of the tensor product $\mathfrak{F} \otimes \mathcal{E}$, where $\mathfrak{F}$ is the algebra of polynomials of the free field. Roughly speaking, this means that, after evaluation in a suitable state ω on $\mathcal{E}$, we obtain an element of $\mathfrak{F}$. A precise definition can be given in terms of the dual W^* of the Weyl quantization (known as the Wigner transform), which is defined by

$$\check{\psi}_\omega(k) \equiv (W^* \omega)\check{}(k) = (2\pi)^{-4} \omega(e^{-ikq}),$$

where ω is a state on $\mathcal{E}$. Note that $k \mapsto \omega(e^{ikq})$ defines a function in the Schwartz space $S(\mathbb{R}^4)$, provided that ω is in the domain of all monomials in the q^μ's [since $\frac{\partial}{\partial \alpha_\mu} e^{i\alpha q} = i e^{i\alpha q}[q^\mu + \frac{1}{2}(Q\alpha)^\mu]$, and $i\alpha_\mu e^{i\alpha q} = [(Q^{-1}q)_\mu, e^{i\alpha q}]$]. For such ω we may set

$$\phi(\omega) \equiv (W\varphi)(\omega) := \varphi(W^* \omega),$$

and, with this definition, a quantum field on quantum spacetime is an affine functional on a suitable *-weakly dense subset of the state space $S(\mathcal{E})$ of $\mathcal{E}$, taking values in $\mathfrak{F}$. In this sense, we may now write

$$\phi(\omega) = \int dx \varphi(x) \psi_\omega(x) = \int dk \hat{\varphi}(k) \check{\psi}_\omega(k),$$

and thus recover an expression which is well known from field theory on Minkowski space.

The positivity property of the state ω implies that the field ϕ respects the Heisenberg uncertainty relations for the simultaneous determination of the coordinates. Nevertheless, the field is still too singular to admit (pointwise) products: Indeed,

$$(k_1, k_2) \mapsto \check{\psi}^{(2)}_\omega(k_1, k_2) \equiv (2\pi)^{-8} \omega(e^{-ik_1 q} e^{-ik_2 q})$$

fails to be strongly decreasing.

Therefore, as mentioned in the introduction, we smear the quantum field over translations. Let $f \in S(\mathbb{R}^4)$. Then we set

$$\phi_f(\omega) \equiv \varphi(\psi_\omega \times f),$$

where $\times$ denotes the ordinary convolution product; eventually, we will be interested in the limit $f \rightarrow \delta$. According to the above discussion, $\phi_f(\omega)$ can be written as

$$\begin{aligned}\phi_f(\omega) &= \int dx \varphi(x)(\psi_\omega \times f)(x) \\ &= \int dk \hat{\varphi}(k) \check{f}(k) (2\pi)^4 \check{\psi}_\omega(k),\end{aligned}$$

and in order to establish the connection with the heuristic formula from the introduction, we note that, formally, this can be understood as the evaluation of

$$\int dx \phi(q + x) f(x) = \int dk \hat{\varphi}(k) \check{f}(k) \otimes e^{-ikq}$$

in a state ω, since $\omega(e^{-ikq}) = (2\pi)^4 \check{\psi}_\omega(k)$ by definition.

Now, the nth power of ϕ_f exists and is given by

$$(\phi_f)^n(\omega) = \varphi^{\otimes n}(\psi^{(n)}_\omega \times f^{\otimes n}),$$

with $f \in S(\mathbb{R}^4)$,

$$\check{\psi}_{\omega}^{(n)}(k_1, \ldots, k_n) = (2\pi)^{-4n}\omega(e^{-ik_1 q}\ldots e^{-ik_n q}), \quad (2.1)$$

and where $\varphi^{\otimes n}$ is the operator valued distribution

$$\varphi^{\otimes n}(x_1, \ldots, x_n) = \varphi(x_1)\cdots\varphi(x_n).$$

More generally, for $f \in S(\mathbb{R}^{4n})$, we may define regularized products of fields by

$$\phi_f^n(\omega) = \varphi^{\otimes n}(\psi_\omega^{(n)} \times f),$$

so that

$$(\phi_f)^n = \phi_{f^{\otimes n}}^n.$$

Products of regularized fields are defined by

$$\phi_f^n \phi_g^m = \phi_{f\otimes g}^{n+m}, \qquad f \in S(\mathbb{R}^{4n}), \qquad g \in S(\mathbb{R}^{4m}),$$

and the adjoint is given by

$$\phi_f^{n*} = \phi_{f^*}^n,$$

where $f^*(x_1, \ldots, x_n) = \overline{f(x_n, \ldots, x_1)}$.

Given a regular representation of $\mathcal{E}$ on some Hilbert space $\mathcal{H}$, the (formal) elements ϕ_f^n of $\mathfrak{F} \otimes \mathcal{E}$ can be represented by operators on a dense domain in $\mathcal{H}_\varphi \otimes \mathcal{H}$, where $\mathcal{H}_\varphi$ is the Fock space of the free field.

We now look for suitably subtracted products of fields

$$\vdots\phi_f^n\vdots = \sum_{k=0}^{n} \phi_{\gamma_k^{(n)}(f)}^{n-k} = \phi_f^n + \sum_{k=1}^{n} \phi_{\gamma_k^{(n)}(f)}^{n-k},$$

where $\gamma_k^{(n)}: S(\mathbb{R}^{4n}) \to S(\mathbb{R}^{4(n-k)})$, $k = 0, \ldots, n$, are continuous linear maps, such that

(1) when $f \to \delta$, the limit of $\vdots\phi_f^n\vdots$ exists as an affine $\mathfrak{F}$-valued functional on some dense subset of $S(\mathcal{E})$;

(2) the maps $\gamma_k^{(n)}$ can be chosen to be local in the sense that

$$\mathrm{supp}\gamma_k^{(n)}(f) \subset \overline{\bigcup_{\substack{U\subset\{1,\ldots,n\}\\ |U|=n-k}} P_U \mathrm{supp} f},$$

where P_U is the projection $\mathbb{R}^{4n} \mapsto \mathbb{R}^{4|U|}$ given by

$$P_U(x_1, \ldots, x_n) = (x_u)_{u\in U}.$$

Note that condition (2) ensures that, in the limit where $f \to \delta$, the product of fields (if it exists) is local of all orders.

In order to clarify the above idea, let us first discuss the ordinary Wick product $:\varphi^{\otimes n}:$ on Minkowski space. It is obtained from the product $\varphi^{\otimes n}$ by "putting all annihilation operators to the right," or equivalently, given by an alternating sum over all possible contractions of n fields. To put this latter definition into a compact form, we now introduce the following notation.

Let N be a finite ordered set. A contraction in N is a pair consisting of a subset $A \subset N$ and an injective map $\alpha: A \to N \setminus A$ such that $\alpha(a) > a$ for all $a \in A$ (with respect to the order of N). The set of all contractions in N, including the empty contraction with $A = \emptyset$, is denoted by $C(N)$. A is considered as an ordered subset of N (with its natural order) and $\alpha(A)$ is an ordered set which inherits its order $<^\alpha$ from A via the map α [i.e., $\alpha(a) <^\alpha \alpha(a')$ if $a < a'$]. In what follows, the letter U will denote the set of uncontracted indices, $U = N \setminus [A \cup \alpha(A)]$. If different contractions C are involved, we label A, α, U by a lower index C.

To every contraction $C \in C(N)$ we associate a linear continuous map, the so-called contraction map

$$\gamma_0^C: S(\mathbb{R}^{4|N|}) \to S(\mathbb{R}^{4|U|}),$$

by

$$\gamma_0^C(f)(x_U) = \int dx_A dx_{\alpha(A)} \prod_{a\in A} \Delta_+(x_a - x_{\alpha(a)}) f(x_N).$$

Here, Δ_+ denotes the ordinary two-point function of the free field and we have used the convention that, for a finite ordered set B, x_B denotes the tuple $x_B = (x_{b_1}, \ldots, x_{b_{|B|}})$ with $b_1 < b_2 < \cdots < b_{|B|}$.

In Fourier space, the contraction map assumes the form

$$[\gamma_0^C(f)]\check{}(k_U) = (2\pi)^{8|A|} \int d\mu_A(k_A) \check{f}(k_N)|_{k_{\alpha(A)}=-k_A},$$

where $d\mu_A(k_A) = \prod_{a\in A} d\mu(k_a)$ with $d\mu(k)$ denoting the Lorentz-invariant measure on the mass shell

$$d\mu(k) = (2\pi)^{-3}\frac{d\mathbf{k}}{2\omega_\mathbf{k}}\bigg|_{k_0=\omega_\mathbf{k}}, \qquad \omega_\mathbf{k} = \sqrt{m^2 + \mathbf{k}^2}.$$

Making use of the contraction maps γ_0^C, we can now write the Wick products on commutative spacetime as the alternating sum

$$:\varphi^{\otimes|N|}: = \sum_{C\in C(N)} (-1)^{|A|} \varphi^{\otimes|U|} \circ \gamma_0^C,$$

where for $U = \emptyset$, $\varphi^{\otimes|U|} = 1$. Roughly speaking, for coinciding arguments, the right-hand side of the above consists of a vertex with n legs plus (or minus) all possible tadpoles.

A first attempt to define suitably subtracted products of fields on the noncommutative Minkowski space was to generalize the ordinary Wick products to the noncommutative spacetime [1]. However, while this prescription fulfills condition (1), it violates condition (2), as we shall see below.

Before proceeding, we observe that any state $\tilde{\omega} \in S(\mathcal{E})$ can be decomposed as $\tilde{\omega} = \mu \circ \omega$, where μ is a probability measure on Σ and ω is a positive, unital Z-linear map taking values in $\overline{Z} = L^\infty(\Sigma, \mu)$("a $\overline{Z}$-valued state") with

$$\omega\bigg(\prod_{j\in N} e^{ik_j q}\bigg) = \exp\bigg(-\frac{i}{2}\sum_{j<l} k_j Q k_l\bigg)\omega\bigg(\exp\bigg(i\sum_{j\in N} k_j q\bigg)\bigg) \in \overline{Z}.$$

Unfortunately, no Lorentz-invariant choice of μ exists. Particular choices of μ are the measure which is supported on the rotation and translation invariant subset $\Sigma_1 \subset \Sigma$ (see [1]) and the point measure. The latter choice can equivalently be understood as the case where a fixed non-commutativity matrix $[q^\mu, q^\nu] = i\theta^{\mu\nu} \in \Sigma$ is used and $\mathcal{Z}$ is trivial. This special case is therefore included in our more general setting. In the considerations which follow, the integration over Σ will for the most part be irrelevant, and we therefore refrain from performing it until the very last. Note that the formalism is fully covariant, but that we will frequently replace the operators $Q^{\mu\nu}$ by generic spectral values $\sigma^{\mu\nu}$, $\sigma \in \Sigma$, in the sense of the joint functional calculus of the $Q^{\mu\nu}$. If necessary, we will furthermore consider $\overline{\mathcal{Z}}$-valued test functions, distributions, Hilbert space vectors and operators.

We now set

$$:\phi_f^n:(\omega) = :\varphi^{\otimes n}:(\psi_\omega^{(n)} \times f),$$

with $f \in \mathcal{S}(\mathbb{R}^{4n})$ and with $(\psi_\omega^{(n)})^\vee$ given by (2.1). From the above it then follows that

$$\begin{aligned} :\phi_f^{|N|}:(\omega) &= :\varphi^{\otimes|N|}:(\psi_\omega^{(|N|)} \times f) \\ &= \sum_{C \in \mathcal{C}(N)} (-1)^{|A|} \varphi^{\otimes|U|}[\gamma_0^C(\psi_\omega^{(|N|)} \times f)]. \end{aligned}$$

We now define the *quantum contraction* γ^C by requiring (for $\overline{\mathcal{Z}}$-valued states ω)

$$\gamma_0^C(\psi_\omega^{(|N|)} \times f) = \psi_\omega^{(|U|)} \times \gamma^C(f),$$

such that

$$:\phi_f^{|N|}:(\omega) = \sum_{C \in \mathcal{C}(N)} (-1)^{|A|} \phi_{\gamma^C(f)}^{|U|}(\omega).$$

To compute γ^C we use the fact that due to the commutation relations of coordinates on quantum spacetime we have

$$\prod_{j \in N} e^{-ik_j q}\Big|_{k_{\alpha(A)} = -k_A} = \prod_{j \in U} e^{-ik_j q} e^{-i\langle k_A, I k_A\rangle - i\langle k_A, E k_U\rangle}$$

where I is a $4|A| \times 4|A|$ matrix (called the *intersection matrix*) and E a $4|A| \times 4|U|$ matrix (called the *enclosure matrix*) with 4×4 blocks, where [with respect to the natural order of both A and $\alpha(A)$ as subsets of N]

$$I_{aa'} = \begin{cases} Q, & \text{if } a < a' < \alpha(a) < \alpha(a') \\ 0, & \text{otherwise} \end{cases}$$

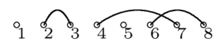
$$E_{au} = \begin{cases} Q, & \text{if } a < u < \alpha(a) \\ 0, & \text{otherwise} \end{cases}$$

and where for two momenta k, k' the contraction with Q is defined by $kQk' = k_\mu Q^{\mu\nu} k'_\nu$.

We thus obtain

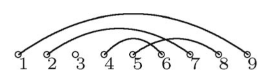
$$[\gamma^C(f)]^\vee(k_U) = (2\pi)^{8|A|} \int d\mu_A(k_A) e^{-i\langle k_A, I k_A\rangle - i\langle k_A, E k_U\rangle} \times \check{f}(k_N)|_{k_{\alpha(A)} = -k_A}. \quad (2.2)$$

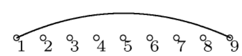
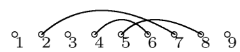
In terms of graphs, these definitions can be visualized as follows: For the ordered set $N = (1, \ldots, n)$ draw a number of n points in a horizontal line. For a contraction C, connect each point $a \in A$ with its respective partner $\alpha(a)$ by a curve in the upper half plane (called an *internal line*). Then the entry $I_{aa'}$ of the intersection matrix is nonzero if and only if their connecting curves intersect and $a < a'$, and the entry E_{au} of the enclosure matrix vanishes if and only if the vertical line from u to $+\infty$ (called an *external line*) crosses the internal line connecting a and $\alpha(a)$.

Example: Consider the contraction C in $N = (1, \ldots, 8)$ where $A = (2, 4, 6)$ and $\alpha(A) = (3, 7, 8)$. The corresponding graph then is , and it allows to directly read off the intersection and the enclosure matrix: $I_{46} = Q$, $E_{45} = Q$, all others 0.

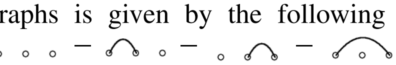
Note moreover that every contraction may be naturally decomposed into connected components as illustrated by the following example.

Example: The contraction $C \in \mathcal{C}(\{1, \ldots, 9\})$ where $A = (1, 2, 4, 5)$, and $\alpha(A) = (9, 7, 6, 8)$ has two connected components C_1 and C_2 with $A_{C_1} = \{1\}$ and $A_{C_2} = \{2, 4, 5\}$. In terms of graphs, the connected components of



are given by  (C_1) and  (C_2).

III. QUASIPLANAR WICK PRODUCTS

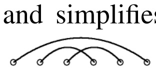
According to the program outlined in the previous section, we now want to introduce subtracted products of fields on the noncommutative Minkowski space which are defined in terms of local contractions only. This condition is not satisfied by ordinary Wick products. To see this, consider the third Wick power $:\phi_f^3:$ which in terms of graphs is given by the following sum of contractions . The last contraction yields

$$\gamma^C(f)(x_2) = \int dx_1 dx_3 \int d\mu(k) e^{-ik(x_1 - x_3)} \times f(x_1, x_2 + Qk, x_3)$$

where we have performed the fiberwise-defined coordinate transformation $x_2 \to x_2 + \sigma k$. This expression clearly violates the locality condition [Condition (2) of Sec. II]. For $f(x_1, x_2, x_3) = \delta(x_1 - x_2)\delta(x_1 - x_3)g(x_1)$ (which renders a well-defined expression, as we shall see below) it was shown in [8] that this nonlocality cannot be cured by adding a correction term from the range of the Klein-Gordon operator.

It is easy to see that a contraction is local if its enclosure matrix vanishes, since in this case the uncontracted variables decouple from the contracted variables and we find

$$\mathrm{supp}\gamma^C(f) \subset \overline{P_U \mathrm{supp} f}.$$

The contractions with vanishing enclosure matrix may be represented by graphs whose external lines are not crossed by internal lines. We call these graphs (and the corresponding contractions) quasiplanar. The set of contractions for which *all* connected components are quasiplanar will be denoted by $C_{qp}(N)$. Note that due to the definition of connected components used here (which differs from the one in [8] and simplifies the combinatorics below), the contraction  is quasiplanar but not in $C_{qp}(N)$.

We now define the quasiplanar Wick products by the following formula [$f \in S(\mathbb{R}^{4|N|})$]:

$$\vdots\phi_f^{|N|}\vdots = \sum_{C \in C_{qp}(N)} (-1)^\kappa \phi^{|U|}_{\gamma^C(f)}, \tag{3.1}$$

where κ is the number of connected components of C. For an example see Appendix A. It is clear that by definition quasiplanar Wick products fulfill the locality condition.

With the initial conditions $\vdots 1 \vdots = 1$ and $\vdots\phi\vdots = \phi$, the quasiplanar Wick products can be uniquely characterized by the recursion relation [$f \in S(\mathbb{R}^4)$, $g \in S(\mathbb{R}^{4|N|})$]

$$\vdots\phi^{|\{1\}\sqcup N|}_{f\otimes g}\vdots = \phi_f \vdots\phi_g^{|N|}\vdots - \sum_{\substack{C \in C_{qp}(\{1\}\sqcup N)_{1\in A} \\ C \text{ connected}}} \vdots\phi^{|U|}_{\gamma^C(f\otimes g)}\vdots. \tag{3.2}$$

Here, the symbol $\sqcup$ denotes the disjoint union of two ordered sets, where the second set is appended to the first set, such that for all $n \in N$, $m \in M$, $n < m$ in $N \sqcup M$. For an example of (3.2) see Appendix B.

Instead of directly proving the recursion relation (3.2), we prove the analogue of Wick's theorem of which (3.2) is a corollary. Let N and M be ordered finite sets. We let $C(N, M)$ denote the set of all quasiplanar contractions $C \in C(N \sqcup M)$ which have the property that every connected component C' of C connects N and M, in the sense that $\alpha(A_{C'} \cap N) \cap M \neq \emptyset$. Note that $C \in C(N, M)$ is in general not in $C_{qp}(N \sqcup M)$.

Theorem 1 "Wick's theorem for quasiplanar Wick products": *Let* $f \in S(\mathbb{R}^{4n})$ and $g \in S(\mathbb{R}^{4m})$. *Then*

$$\vdots\phi_f^n\vdots\vdots\phi_g^m\vdots = \sum_{C \in C(N,M)} \vdots\phi^{|U|}_{\gamma^C(f\otimes g)}\vdots \tag{3.3}$$

where $N = \{1, \ldots, n\}$ *and* $M = \{n+1, \ldots, n+m\}$

Proof.—After inserting the definition of quasiplanar Wick products (3.1), the left-hand side is

$$\sum_{\substack{C \in C_{qp}(N) \\ C' \in C_{qp}(M)}} (-1)^{\kappa_C + \kappa_{C'}} \phi^{|U_C| + |U_{C'}|}_{\gamma^C(f)\otimes\gamma^{C'}(g)}.$$

For the right-hand side we find

$$\sum_{C \in C(N,M)} \sum_{C' \in C_{qp}(U_C)} (-1)^{\kappa_{C'}} \phi^{|U_{C'}|}_{\gamma^{C'}\circ\gamma^C(f\otimes g)}.$$

In the latter expression, C' may be decomposed into three mutually disconnected contractions C_1, C_2 and C_3 where $C_1 \in C_{qp}(N \cap U_C)$, $C_3 \in C_{qp}(M \cap U_C)$ and $C_2 \in C(N \cap U_C, M \cap U_C)$. Note that C_2 is connected since $C' \in C_{qp}(U_C)$. We may now combine C and C_2 to a single contraction $C_4 \in C(N, M)$. We observe that every nonempty contraction $C_4 \in C(N, M)$ appears twice in the sum, but with opposite signs. Hence all these contributions cancel, and only the empty contraction remains which yields the theorem. ■

Two concrete applications of Wick's theorem for quasiplanar Wick products [formula (3.3)] may be found in Appendix C.

We will now give a closed formula specifying the relation between quasiplanar Wick products and ordinary Wick products. In fact, we show that quasiplanar Wick polynomials can be expressed in terms of Wick polynomials via the formula

$$\vdots\phi_f^{|N|}\vdots = \sum_{C \in C_{ap}(N)} :\phi^{|U|}_{\gamma^C(f)}:. \tag{3.4}$$

Here, $C_{ap}(N)$ is the set of all *aplanar* contractions of N. A contraction is called aplanar if for every connected component the corresponding part of the enclosure matrix is nontrivial. Note that the empty contraction is quasiplanar and aplanar, and that contractions may be neither in C_{qp} nor in C_{ap}. For an example of (3.4) see Appendix D.

We prove formula (3.4) by showing that it satisfies the recursion relation (3.2). The initial conditions are obviously fulfilled. Now for the first term on the right-hand side of the recursion relation we find, using (3.4) and Wick's theorem (for ordinary Wick products),

$$\phi_f \vdots\phi_g^{|N|}\vdots = \sum_{\substack{C \in C_{ap}(\{1\}\sqcup N) \\ 1 \notin A}} \left(:\phi^{|U|}_{\gamma^C(f\otimes g)}: + \sum_{u \in U\setminus\{1\}} :\phi^{|U|-2}_{\gamma^{(1,u)}\circ\gamma^C(f\otimes g)}: \right) \tag{3.5}$$

where $(1, u)$ is the contraction with $A = \{1\}$ and $\alpha(1) = u$. Applying (3.4) also to the second term in the recursion relation yields

$$-\sum_{C \in C(\{1\},N)} \sum_{C' \in C_{ap}(U_C)} :\phi^{|U_{C'}|}_{\gamma^{C'}\circ\gamma^C(f\otimes g)}:.$$

The combined contractions from (3.5) may be decomposed into connected components. Now, those contractions for which the component containing 1 has a vanishing enclosure matrix cancel with the second term in the recursion relation. Hence, only the sum over all aplanar contractions of $\{1\} \sqcup N$ remains, which proves the claim.

Formula (3.4) shows explicitly that the limit "$\lambda_P \to 0$" does not yield the ordinary Wick products, since

$$\sum_{C \in C_{ap}(N)} :\phi^{|U|}_{\gamma^C(f)}: = :\phi^{|N|}_f: + \sum_{\substack{C \in C_{ap}(N)\\ U \neq N}} :\phi^{|U|}_{\gamma^C(f)}:$$

and the terms which compared to the ordinary Wick product remain unsubtracted do not vanish in this limit.

IV. QUASIPLANAR WICK PRODUCTS AT COINCIDING POINTS (SKETCH)

Let us now consider a quasiplanar Wick product at coinciding points, i.e., an expression of the form $\vdots\phi^n_g\vdots(q)$ where $g(x_N) = \prod_{j=1}^n \delta(x_j)$, $N = \{1, \ldots, n\}$. We will sketch an argument showing that such a product is well defined. The mathematical details will be treated in a forthcoming publication. The proof is based on the idea that using (3.4) we may rewrite the quasiplanar Wick product in terms of ordinary Wick products and that for a suitable test function h, the normal ordered product of fields at coinciding points,

$$\int dk_U :\prod_{i \in U} \hat{\varphi}(k_i): \check{h}\Big(\sum_{j \in U} k_j\Big) = \int dx :\varphi(x)^{|U|}: h(x)$$

is a well-defined element of $\mathfrak{F}$.

We therefore apply (3.4) to $\vdots\phi^n_g\vdots$ and evaluate the resulting expression in a suitable state $\tilde{\omega} = \mu \circ \omega$ to obtain

$$\vdots\phi^n_g\vdots(\tilde{\omega}) = \sum_{C \in C_{ap}(N)} \mu\bigg(\int dk_U :\prod_{i \in U} (2\pi)^{4|U|} \check{\varphi}(k_i): \times [\gamma^C(g)]\check{}(k_U) \hat{\psi}^{(|U|)}_\omega(-k_U)\bigg), \tag{4.1}$$

where $[\gamma^C(g)]\check{}(k_U)$ is a bounded (not rapidly decreasing) function of k_U given by (2.2) with $\check{g}(k_N) \equiv (2\pi)^{-4n}$, and where $\check{\psi}^{(|U|)}_\omega$ which is given by (2.1) is quickly decreasing only in the sum of the momenta. Let us now pick an arbitrary contribution to the right-hand side of (4.1). Using $\varphi(x) = (2\pi)^{3/2} \int d\mu(k)[a(k)e^{-ikx} + a^*(k)e^{+ikx}]$, we then decompose the Wick polynomial into a sum of normal ordered products of creation and annihilation operators $\prod_{u \in U \setminus U'} a^*(k_u) \prod_{u' \in U'} a(k_{u'})$ with $U' \subset U$.

We now consider the pure creation part ($U' = \varnothing$), since we know from ordinary field theory that it is the term requiring the most care in a product of fields at coinciding points. From (2.2) we conclude that, in this case, $[\gamma^C(g)]\check{}(k_U)$ is of the form

$$(2\pi)^{-4|U|} \int d\mu_A(k_A) e^{-i\langle k_A, Ik_A\rangle - i\langle k_A, E(-k_U)\rangle},$$

where all momenta are on the *positive* mass shell. We now parametrize the mass shell in coordinates in which σ has the standard form $\sigma^{(0)}$ by $k = (w\cosh\theta, v_1, w\sinh\theta, v_2)$ with $\theta \in \mathbb{R}$, $v = (v_1, v_2) \in \mathbb{R}^2$ and $w = \sqrt{v^2 + m^2}$, such that the measure on the mass shell assumes the form $\frac{1}{2}\int d^2 v d\theta$. This may be done without loss of generality, since for any $\sigma \in \Sigma$, there is an element Λ of the full Lorentz group such that $\sigma = \Lambda\sigma^{(0)}\Lambda^t$, and thus $k\sigma p = (\Lambda^t k)\sigma^{(0)}(\Lambda^t p)$. If Λ is proper, all $\Lambda^t k_j$, $j \in U \cup A$ can obviously be parametrized by such coordinates as above, and if Λ is improper, we use $k\sigma p = (-\Lambda^t k)\sigma^{(0)}(-\Lambda^t p)$ and parametrize $-\Lambda^t k_j$, $j \in U \cup A$ by the above coordinates. In (4.1), this amounts to simply renaming the arguments. Up to numerical constants, $[\gamma^C(g)]\check{}(k_U)$ is therefore given by

$$\int dk(\theta, v)_A \exp\Big(-i\sum_{s<t} J_{st}[w_s w_t \sinh(\theta_s - \theta_t) + v_s \wedge v_t]\Big), \tag{4.2}$$

where $v_s \wedge v_t = v_{s,1}v_{t,2} - v_{s,2}v_{t,1}$ and where the indices s, t are elements of the index set $U \sqcup A$. $J_{st} = 1$ if the corresponding block of the intersection or enclosure matrix is nonzero and $J_{st} = 0$ otherwise. The integrals over k_A are not absolutely convergent but oscillatory. To evaluate them, we shift the integrations over the rapidity variables θ_A into the complex plane, $\theta_a + i\eta_a$ such that for $a < a'$, $a, a' \in A$,

$$0 < \eta_a < \eta_{a'} < \pi.$$

Using the formulas

$$\sinh(\theta + i\eta) = \sinh\theta\cos\eta + i\cosh\theta\sin\eta$$
$$\text{and} \quad \cosh(\theta + i\eta) = \cosh\theta\cos\eta + i\sinh\theta\sin\eta,$$

and setting $\theta_{st} = \theta_s - \theta_t$, $\eta_{st} = \eta_s - \eta_t$ we may now replace the integral appearing in (4.2) by the following expression:

$$\int d^2 v_A \int d\theta_A \exp\Big(-i\sum_{s<t} J_{st}(w_s w_t \sinh\theta_{st}\cos\eta_{st} + v_s \wedge v_t) + \sum_{s<t} J_{st} w_s w_t \cosh\theta_{st}\sin\eta_{st}\Big) \tag{4.3}$$

where we put $\eta_u = 0$ for $u \in U$. The integrand decreases fast in the variables $(\theta_{st}, J_{st} = 1)$, since by construction

$$\sin\eta_{st} < 0 \qquad \text{for all } s < t \in U \sqcup A.$$

Since by definition C is aplanar, all connected components of the contraction have a nontrivial enclosure matrix, and we infer that $\exp(+\sum J_{st} w_s w_t \cosh\theta_{st}\sin\eta_{st})$ is also fast decreasing in θ_A: Connectedness ensures that all θ_A appear at least once and aplanarity ensures that the exponential does not only depend on the difference variables $\theta_{aa'}$, $a, a' \in A$. Hence, the integrations over $d\mu_A(k_A)$ are well defined. Since furthermore, $\check{\psi}^{(U)}_\omega(-k_U)$ is fast decreasing in $-\sum_{u \in U} k_u$ (all on the positive mass shell), we may conclude that the pure creation parts appearing on the right-hand side of (4.1) yield well-defined operators in $\mathfrak{F}$.

An analogous argument shows that the pure annihilation parts ($U' = U$) are well defined. In this case, we find

$$[\gamma(g)]\check{}(k_{U'}) = \int d\mu_A(k_A) e^{-i\langle k_A, Ik_A\rangle - i\langle k_A, Ek_{U'}\rangle}$$

and an analytic continuation $\theta_a + i\eta_a$, $a \in A$, with $-\pi < \eta_a < \eta_{a'} < 0$ for $a < a'$, would yield the desired result since $\check{\psi}_\omega^{(U')}(-k_U)$ is fast decreasing in $\sum_{u \in U'} k_u$. More generally, for contributions with $U' \neq \emptyset$, we have

$$[\gamma(g)]\check{}(k_U) = \int d\mu_A(k_A) e^{-i\langle k_A, Ik_A\rangle - i\langle k_A, E(\epsilon_U k_U)\rangle},$$

where $\epsilon_U k_U$ is the tuple $(\epsilon_u k_u)_{u \in U}$, with $\epsilon_u = +1$ for $u \in U'$ and $\epsilon_u = -1$ for $u \in U \setminus U'$. In this case, $\check{\psi}_\omega(-k_U)$ is fast decreasing in $-\sum_{u \in U} \epsilon_u k_u$. We evaluate the expression on a suitable vector in Fock space to get rid of the annihilation operators and shift the integrations over the rapidity variables $\theta_{A \sqcup U'}$ into the complex plane, $\theta_s + i\eta_s$ for $s \in A \sqcup U'$, such that $0 < \eta_s < \eta_t < \pi$ for $s < t$, $s, t \in A \sqcup U'$. Note that in this case, also some of the arguments of $\check{\psi}_\omega$ will be analytically continued.

In a similar manner as in the above discussion, we can give meaning to expressions of the form

$$\prod_{i=1}^{m} \vdots\phi^{n_i}(q - x_i)\vdots$$

which appear in the perturbative solution of the Yang-Feldman Eq. (1.1). Here, we may formally write

$$\int dx \vdots\phi^n(q - x)\vdots G(x) \stackrel{\text{def}}{=} \vdots\phi_g^n\vdots(q)$$

where $g(x_N) = G(x_1)\prod_{j=2}^{n} \delta(x_1 - x_j)$, $N = \{1, \ldots, n\}$ with a suitable test function G. Applying Wick's theorem for quasiplanar Wick products (Theorem 1) to an expression $\vdots\phi_g^n\vdots\vdots\phi_f^m\vdots$, with $g(x_N)$ as above and for $M = \{n+1, \ldots, n+m\}$, $f(x_M) = F(x_{n+1})\prod_{j=n+2}^{n+m} \delta(x_{n+1} - x_j)$, we obtain integrals of the form

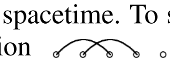
$$[\gamma^C(g \otimes f)]\check{}(k_U) = \int d\mu_A(k_A) e^{-i\langle k_A, Ik_A\rangle} \times \check{G}\Big(\sum_{i \in N} k_i\Big)\check{F}\Big(\sum_{j \in M} k_j\Big)\Big|_{k_{\alpha(A)} = -k_A},$$

where $C \in C(N, M)$. Again, we use coordinates $k(\theta, v)$ such that the twisting is given by $\sigma^{(0)}$ and shift the integration over θ_A into the complex plane, $\theta_a + i\eta_a$ such that $0 < \eta_a < \eta_{a'} < \pi$ for $a < a'$, $a, a' \in A$. We now observe that from the analytic continuation we obtain the factor $\exp(+\sum_{a<b} I_{ab} w_a w_b \cosh\theta_{ab} \sin\eta_{ab})$, $a, b \in A$, which strongly decreases in $(\theta_{ab}, I_{ab} \neq 0)$. By definition, we have $\alpha(A_{C'} \cap N) \cap M \neq \emptyset$ for any connected component C' of $C \in C(N, M)$. Therefore, in any connected component at least one internal momentum $k_{\tilde{A}}$ appears *both* in $\check{G}$ and (with opposite sign) in $\check{F}$ and we conclude that the integrand is strongly decreasing in θ_A such that the integrals are well defined.

In order to make the above discussion mathematically sound, several details are missing. Since they turned out to be quite complicated, we shall treat them in the forthcoming publication mentioned above, and only name the necessary steps here. First of all, we will specify the space of suitable test functions on which the analytic continuation as performed above is well defined. In this test function space, sequences of functions have to exist which converge to δ-distributions in an appropriate topology, such that the integrals in question, evaluated in such sequences, converge to the expressions discussed above (in the appropriate topology). We will moreover show that the Fock space vectors with wave functions from this set of functions form a Lorentz-invariant stable domain for the quasiplanar Wick products and specify the set of admissible states $\tilde{\omega}$ on $\mathcal{E}$.

V. CONSEQUENCES

In this section we would like to point out some of the consequences of our analysis. In particular, we comment on the modified dispersion relation resulting from the use of quasiplanar Wick products in the perturbative expansion. While these remarks are not yet conclusive, they provide a hint as to how the ultraviolet-infrared mixing problem appears in our framework.

The first conclusion we may draw from the previous sections is that the divergences discussed here are not compatible with those arising in a theory on a Euclidean noncommutative spacetime. To see this, consider the quasiplanar contraction . As is well known, on a Euclidean noncommutative spacetime this contribution yields a finite result in the limit of coinciding points (i.e., as a tadpole contribution). This can be understood as follows: Consider a test function f in the relative coordinates $x_1 - x_3$, $x_2 - x_4$ and in x_5 which tends to a product of a test function g in x_5 and δ-distributions in the relative coordinates. Then on a Euclidean noncommutative spacetime, we have

$$\gamma_{\text{euc}}^C(f)(x_5) \propto \int dp \int dk \frac{1}{k^2 + m^2} \frac{1}{p^2 + m^2} e^{-ikQp} \times (\mathcal{F}_{1,2}^{-1} f)(k, p, x_5),$$

where $\mathcal{F}_{1,2}^{-1}$ indicates the inverse Fourier transform with respect to the first and the second argument. Introducing Schwinger parameters and swapping the order of integration, we then find

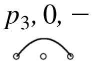
$$\int_0^\infty d\alpha d\beta \int dk d\xi e^{-(\alpha+\beta)m^2} e^{\alpha k^2} e^{-\xi^2/4\beta} \pi^2 \beta^{-2} \times (\mathcal{F}_1^{-1} f)(k, Qk - \xi, x_5).$$

For Q being of maximal rank, this yields a well-defined expression even for $(\mathcal{F}_1^{-1}f)(k, y, x_5)$ tending to $c\delta(y)g(x_5)$, namely,

$$g(x_5)\int_0^\infty d\alpha d\beta e^{-(\alpha+\beta)m^2}\frac{1}{(\alpha\beta+\frac{\lambda_P^4}{4})^2},$$

where, without loss of generality, we have set $(Qp)^2 = \lambda_P^4(p_2^2 + p_1^2 + p_0^2 + p_3^2)$.

In contrast to this, the same contraction is ill-defined on the noncommutative Minkowski space in the limit of coinciding points. In order to keep the calculation simple, we consider a test function f in the relative variable $x_1 - x_3$ with $\check{f}$ tending to a constant. We then find $\gamma^C(f) \propto \int d\mu(p)d\mu(k)e^{-ipQk}\check{f}(p) \propto \int d\mu(p)\check{f}(p)\Delta_+(Qp)$ and, while $\Delta_+(Qp)$ is a bounded function for p on the positive mass shell, it is not integrable. To see this, we choose coordinates on the mass shell such that the twisting is given by $\sigma^{(0)}$ and the argument of the two-point function is $-\lambda_P^4(p_1^2 + p_3^2)$. It follows that, in the limit where $\check{f}$ tends to a constant, the integration over p_2 diverges logarithmically. This means that results on renormalization gained in a Euclidean theory may not be directly applied in the Minkowskian regime.

Furthermore, we would like to emphasize that, for commuting time variable, the quasiplanar Wick products are in general no longer well defined. To see this, we first use the fact that an antisymmetric 3×3 matrix has determinant zero. We can therefore set, without loss of generality, $Qp = \lambda_P^2(0, p_3, 0, -p_1)$. Already the simplest aplanar contraction,  becomes ill-defined in the limit of coinciding points, since (contrary to the case where Q is nondegenerate) it contains the ill-defined integral $\int d\mu(p)\exp[-i\lambda_P^2(p_3k_1 - p_1k_3)]$. Since the contraction still violates the locality condition, it follows that such a theory is not renormalizable by local counterterms.[1] See also [9].

The application of quasiplanar Wick products in the framework of the Yang-Feldman equation is straightforward. In the rules spelled out explicitly in [8] for ordinary Wick products, one only has to replace the Wick products by quasiplanar Wick products. From preliminary calculations we have performed at lower orders of the perturbative expansion, it is reasonable to hope that quasiplanar counterterms suffice as counterterms to render the theory ultraviolet finite. However, if we employ the quasiplanar Wick products and thus refrain from subtracting nonlocal counterterms, we encounter a serious modification of the dispersion relation. Similar discussions in the context of space-space-noncommutativity, which are not founded on the general construction of quasiplanar Wick products, may be found in [10,11].

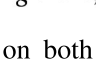
Let us assume that all ultraviolet divergent terms can be absorbed in quasiplanar (thus local) counterterms, leading, in particular, to a finite mass m in the renormalized field equation,

$$(\Box_q + m^2)\phi(q) = -g\phi^{n-1}(q) + \underbrace{(m^2 - m_0^2)}_{=\delta m^2}\phi(q) + \cdots$$

where m_0 is the bare mass and the dots indicate the remaining counterterms (starting with order g^2). If we now insert the renormalized field as a formal power series in g, we find at lowest order, for $n = 4$,

$$(\Box + m^2)[\phi_0(q) + \cdots] = -g\phi_0^3(q) + \delta m_1^2\phi_0(q) + \cdots \tag{5.1}$$

Now according to our program,

$$-g\,\phi_0^3(q) + \delta m_1^2\,\phi_0(q) = -g\,\vdots\,\phi_0^3(q)\,\vdots = -g\,{:}\phi_0^3(q){:} - g$$  ,

such that taking the expectation value $\langle 0|\cdot|p\rangle$ on both sides of Eq. (5.1), we find a modification of the ordinary dispersion relation of the following form:

$$-p^2 + m^2 = -g\Delta_+(Qp) + \cdots,$$

where Δ_+ is the two-point function at mass m.

Allowing for additional counterterms, α and βp^2, we thus find at this order

$$p^2 - m^2 - g[\Delta_+(Qp) + \alpha_1 + \beta_1 p^2] = 0.$$

We now choose the fixed value $\sigma^{(0)}$ for Q. Then the transversal velocity $v_\perp = (v_1, v_3)$ is

$$v_\perp = \nabla_{p_\perp}p_0 = \frac{p_\perp}{p_0}\frac{1+\frac{g}{1-g\beta_1}\eta(p)}{1-\frac{g}{1-g\beta_1}\eta(p)},$$

where

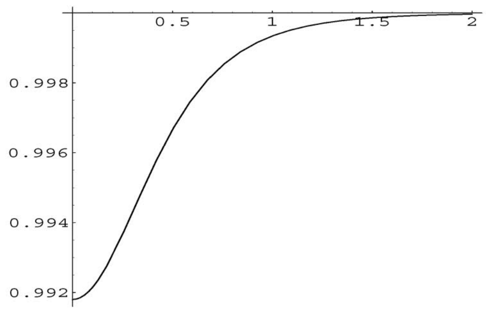
$$\eta(p) = (2p_1)^{-1}\partial_{p_1}\Delta_+(\sigma^{(0)}p) = (2p_3)^{-1}\partial_{p_3}\Delta_+(\sigma^{(0)}p)$$
$$= -\frac{m^2K_2(\lambda_P^2m\sqrt{p_0^2 - p_2^2 + p_\perp^2})}{8\pi^2(p_0^2 - p_2^2 + p_\perp^2)},$$

and therefore depends only on $(\sigma^{(0)}p)^2$. Now assume that p is on the physical mass shell, $p^2 = M^2$, where M is allowed to be different from m (though the latter is finite), then

$$\eta(p)|_{p^2=M^2} = -\frac{m^2K_2(\lambda_P^2mM\sqrt{1+\frac{2p_\perp^2}{M^2}})}{8\pi^2M^2(1+\frac{2p_\perp^2}{M^2})}.$$

If the masses m and M are both assumed to be of the order of the Planck mass, the factor $[1+\frac{g}{1-g\beta_1}\eta(p)]/[1-\frac{g}{1-g\beta_1}\eta(p)]$ as a function of the transversal component $p_\perp$ is of the following form:

[1]The ill definedness of the contraction may also be understood by the fact that for the Q under consideration, $\Delta_+(x + Qk)$ cannot be multiplied (as a distribution) with $\delta(x_0)\delta(x_2)$.



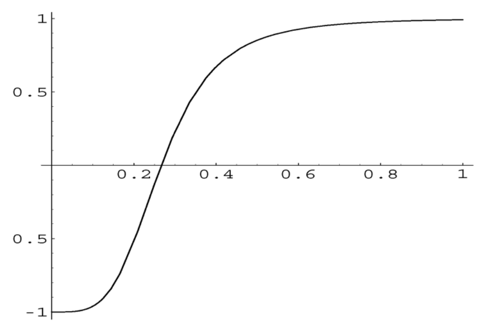
$m = M = 1,\ \beta_1 = 0,\ g = 1/5,\ \lambda_P = 1$

plotted with `Mathematica`.

Surprisingly, the maximal deviation does not occur at high momenta but at $p_\perp = 0$. In the above numerical setting, this point of maximal deviation is of the order of 1%,

$$[1 + g\tilde{\eta}(m)]/[1 - g\tilde{\eta}(m)]|_{m^2=1} \simeq 0.99, \qquad \tilde{\eta}(m) \stackrel{\text{def}}{=} \eta(p)|_{\substack{p_\perp=0\\ p^2=m^2}} = -(8\pi^2)^{-1}K_2(\lambda_P^2 m^2).$$

Using smaller masses $m = M < m_P$, the deviation becomes even larger, as we can see in the following plot, where $\frac{1+g\tilde{\eta}(m)}{1-g\tilde{\eta}(m)}$ (i.e., the maximal deviation from 1) is plotted as a function of the mass m, ranging from 0 to 1:



$M = m,\ \beta_1 = 0,\ g = 1/5,\ \lambda_P = 1$

plotted with `Mathematica`.

We see that the group velocity may even become negative. Integrating over, say, Σ_1 would not improve the situation: Since the scale λ_P remains fixed, the behavior sketched above would qualitatively remain the same.

If we take into account that m and M may differ from one another, it is possible to allow for small physical masses M while taking m to be very large. To see this, observe that at $p_\perp = 0$ and $p^2 = M^2$,

$$\eta(p) = -\frac{1}{8\pi^2}\frac{m^2}{M^2}K_2\left(\lambda_P^2 M^2 \frac{m}{M}\right),$$

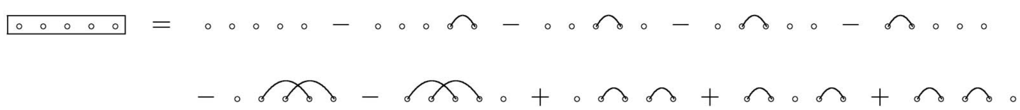
and since $\alpha^2 K_2(\beta\alpha) \longrightarrow 0$ for α large enough, it is possible to make the deviation arbitrarily small even for small masses by choosing m large enough. It remains to be investigated whether this scheme can be applied consistently to all orders, but in any case it would be a "fine-tuning" procedure which does not seem to be very natural.

However small, the modification of the dispersion relation has serious consequences. In ordinary local quantum field theory, the Hilbert space of the asymptotic fields is the Fock space of the free fields with fixed (constant) mass. The above analysis shows that this cannot be true for the asymptotic fields in the framework considered here, since their mass will in general depend on the momentum. In a realistic model such as quantum electrodynamics, the modified dispersion relation could provide predictions which by comparison with experiment might seriously restrict the scale of noncommutativity. In the above, this scale was taken to be of the order of the Planck length. The effect being larger for a smaller parameter λ_P (i.e., for a higher energy), it is not impossible that in a realistic model such as quantum electrodynamics, where phenomenological calculations so far have provided lower bounds for the energy scale of noncommutativity, an *upper* bound for the energy scale could be derived in this way—depending on how questions concerning renormalization can be solved.

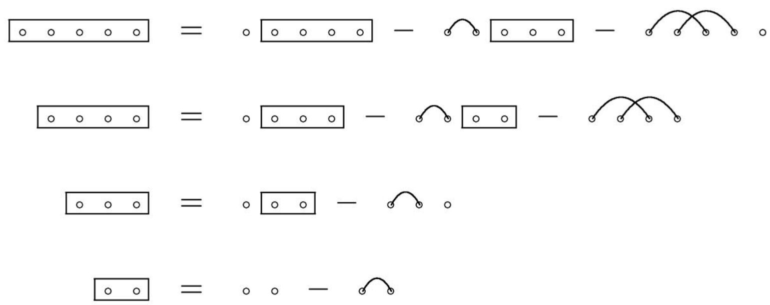
APPENDIX: EXAMPLES

In the following examples, quasiplanar Wick products are symbolized by boxes, and contractions by connecting lines as explained at the end of Sec. II.

A. Formula (3.1)



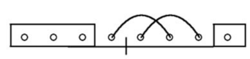
B. Formula (3.2)



C. Formula (3.3)

In what follows, the underscore symbolizes quasiplanar Wick ordering of fields which are not direct neighbors. For instance, for the contraction $C \in \mathcal{C}[(1, \dots, 4) \sqcup (5, \dots, 8)]$ with $U_C = (1, 2, 3, 8)$, $A_C = (4, 5)$, $\alpha(4) = 6$ and $\alpha(5) = 7$, we write

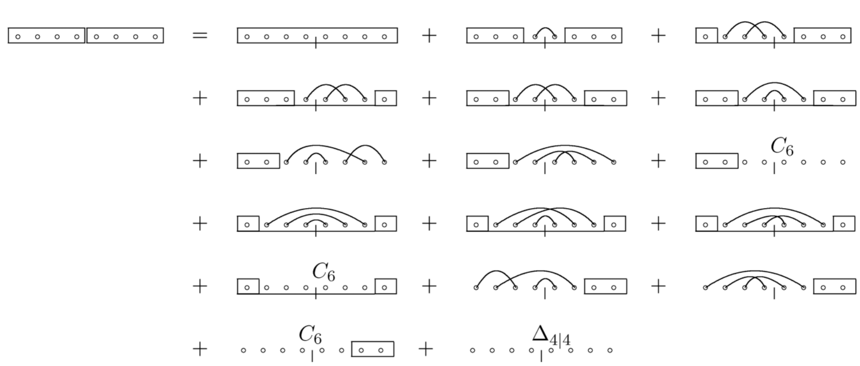
$$\vdots\, \phi^{|U|}_{\gamma^C(f)} \,\vdots \; =$$



,

where the small vertical line serves to separate the sets $(1, \dots, 4)$ and $(5, \dots, 8)$ from one another.

Example 1:



where

$$C_6 = \sum_{\substack{C \in \mathcal{C}(N) \\ \text{connected} \\ U_C = \emptyset}} \gamma^C(f) \; =$$

with $N = (1, \dots, 6)$, and where

$$\Delta_{4|4} = \sum_{\substack{C \in \mathcal{C}(N \sqcup M) \\ U_C = \varnothing}} \gamma^C(f) = \sum_{\substack{C \in \mathcal{C}(N \sqcup M) \\ A_C = (1,2,3,4)}} \gamma^C(f) + \sum_{i=1}^{9} \sum_{C_i \in \mathcal{C}(N \sqcup M)} \gamma^{C_i}(f)$$

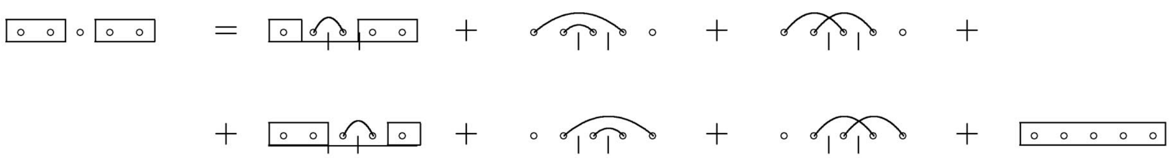
with $N = (1, 2, 3, 4)$, $M = (5, 6, 7, 8)$, and with the pairs (A_i, α_i) of the contractions C_i determined by

$$A_1 = (1, 2, 4, 5),\ \alpha_1(1) = 3,\ \alpha_1(5) = 7 \qquad A_2 = (1, 2, 4, 5),\ \alpha_2(1) = 3,\ \alpha_2(5) = 8$$
$$A_3 = (1, 2, 4, 6),\ \alpha_3(1) = 3,\ \alpha_3(6) = 8 \qquad A_4 = (1, 2, 3, 5),\ \alpha_4(2) = 4,\ \alpha_4(5) = 7$$
$$A_5 = (1, 2, 3, 5),\ \alpha_5(2) = 4,\ \alpha_5(5) = 8 \qquad A_6 = (1, 2, 3, 6),\ \alpha_6(2) = 4,\ \alpha_6(6) = 8$$
$$A_7 = (1, 2, 3, 5),\ \alpha_7(1) = 4,\ \alpha_7(5) = 7 \qquad A_8 = (1, 2, 3, 5),\ \alpha_8(1) = 4,\ \alpha_8(5) = 8$$
$$A_9 = (1, 2, 3, 6),\ \alpha_9(1) = 4,\ \alpha_9(6) = 8$$

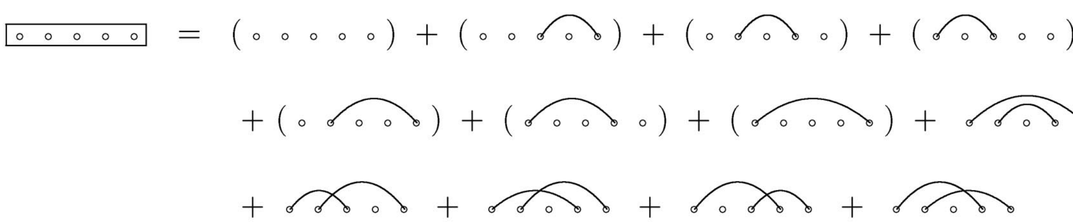
such that for instance,

$$\sum_{C_1 \in \mathcal{C}(N \sqcup M)} \gamma^{C_1}(f) =$$

Example 2:



D. Formula (3.4)



Here, the round brackets denote ordinary Wick ordering (of all uncontracted fields in an expression).

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