

Ghost busting: \mathcal{PT} -symmetric interpretation of the Lee model

Carl M. Bender, Sebastian F. Brandt, Jun-Hua Chen, and Qinghai Wang

Department of Physics, Washington University, St. Louis Missouri 63130, USA

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The Lee model was introduced in the 1950s as an elementary quantum field theory in which mass, wave function, and charge renormalization could be carried out exactly. In early studies of this model it was found that there is a critical value of g^2 , the square of the renormalized coupling constant, above which g_0^2 , the square of the unrenormalized coupling constant, is *negative*. Thus, for g^2 larger than this critical value, the Hamiltonian of the Lee model becomes non-Hermitian. It was also discovered that in this non-Hermitian regime a new state appears whose norm is negative. This state is called a *ghost* state. It has always been assumed that in this ghost regime the Lee model is an unacceptable quantum theory because unitarity appears to be violated. However, in this regime while the Hamiltonian is not Hermitian, it does possess \mathcal{PT} symmetry. It has recently been discovered that a non-Hermitian Hamiltonian having \mathcal{PT} symmetry may define a quantum theory that is unitary. The proof of unitarity requires the construction of a new time-independent operator called C . In terms of C one can define a new inner product with respect to which the norms of the states in the Hilbert space are positive. Furthermore, it has been shown that time evolution in such a theory is unitary. In this paper the C operator for the Lee model in the ghost regime is constructed in the $V/N\theta$ sector. It is then shown that the ghost state has a positive norm and that the Lee model is an acceptable unitary quantum field theory for all values of g^2 .

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I. INTRODUCTION AND BRIEF REVIEW OF THE LEE MODEL

In 1954 the Lee model was introduced as a quantum field theory in which mass, wave function, and charge renormalization could be performed exactly and in closed form [1–4]. The Lee model describes a three-particle interaction of three spinless particles called V , N , and θ . The V and N particles are fermions and behave roughly like nucleons, and the θ particle is a boson and behaves roughly like a pion. The basic assumption of the model is that a V may emit a θ , but when it does so it becomes a N . Also, a N may absorb a θ , but when it does so it becomes a V . These two processes are summarized by

$$V \rightarrow N + \theta, \quad N + \theta \rightarrow V. \quad (1)$$

The Lee model is solvable because it does not respect crossing symmetry; that is, the crossed processes $V + \theta \rightarrow N$ and $N \rightarrow V + \theta$ are forbidden. Eliminating crossing symmetry makes the model solvable because it introduces two conservation laws. First, *the number of N quanta plus the number of V quanta is fixed*. Second, *the number of N quanta minus the number of θ quanta is fixed*.

As a result of these two highly constraining conservation laws, the Hilbert space of states decomposes into an infinite number of noninteracting sectors. The simplest sector is the vacuum sector. Because of the conservation laws there are no vacuum graphs and the bare vacuum is the physical vacuum. The next two sectors are the one- θ -particle and the one- N -particle sector. These two sectors are also trivial because the two conservation laws prevent any dynamical processes from occurring there. As a result, the masses of the N particle and the θ particle are not renormalized; that

is, the physical masses of these particles are the same as their bare masses.

The lowest nontrivial sector is the $V/N\theta$ sector. The physical states in this sector of the Hilbert space are linear combinations of the bare V and the bare $N\theta$ states and these states consist of the one-physical- V -particle state and the physical N - θ -scattering states. To find these states one can look for the poles and cuts of Green's functions. (The Feynman diagrams are merely chains of $N\theta$ bubbles connected by single V lines.) The renormalization in this sector is easy to perform. Following the conventional renormalization procedure, one finds that the mass of the V particle is renormalized; that is, the mass of the physical V particle is different from its bare mass. Most important, in the Lee model one can calculate the unrenormalized coupling constant as a function of the renormalized coupling constant in closed form. There are many ways to define the renormalized coupling constant. For example, in an actual scattering experiment one could define the square of the renormalized coupling constant g^2 as the value of the $N\theta$ scattering amplitude at threshold.

The higher sectors of the Lee model are difficult to study because the equations are complicated. However, many papers have been written over the years on various aspects of the Lee model. For example, Weinberg studied the VN sector because the V and N particles form a bound state whose properties resemble those of the deuteron [5]. Amado and Vaughn carried out detailed studies of scattering amplitudes in the $V\theta$ sector. These examinations required the solution of difficult integral equations [6,7]. Glaser and Källén studied the properties of the physical V particle for the case in which the mass parameters in the Hamiltonian are chosen so that this particle becomes un-

stable [8]. Bender and Nash examined the asymptotic freedom of the Lee model [9].

The most interesting aspect of the Lee model is the appearance of a ghost state in the $V/N\theta$ sector. To understand how this state appears, one must first perform coupling-constant renormalization. Expressing g_0^2 , the square of the unrenormalized coupling constant, in terms of g^2 , the square of the renormalized coupling constant, one obtains a graph like that in Fig. 1. In principle, the renormalized coupling constant is a physical parameter whose numerical value is determined by a laboratory experiment. If g^2 is measured to be near 0, then from Fig. 1 we can see that g_0^2 is also small. However, if the laboratory experiment gives a value of g^2 that is larger than this critical value, then the square of the unrenormalized coupling constant becomes negative. In this regime the unrenormalized coupling constant g_0 is imaginary and the Hamiltonian is non-Hermitian. Moreover, in this regime a new state appears in the $V/N\theta$ sector, and because its norm is negative, the state is called a ghost. There have been numerous attempts to make sense of the Lee model as a physical quantum theory in the ghost regime [2–4]. However, none of these attempts have been successful. To summarize the situation, in Ref. [4] Barton writes, “A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a nonunitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation.”

The purpose of this paper is to show that, contrary to the view expressed by Barton, it is indeed possible to give a physical interpretation to what is going on in the $V/N\theta$ sector of the Lee model when g_0 becomes imaginary and

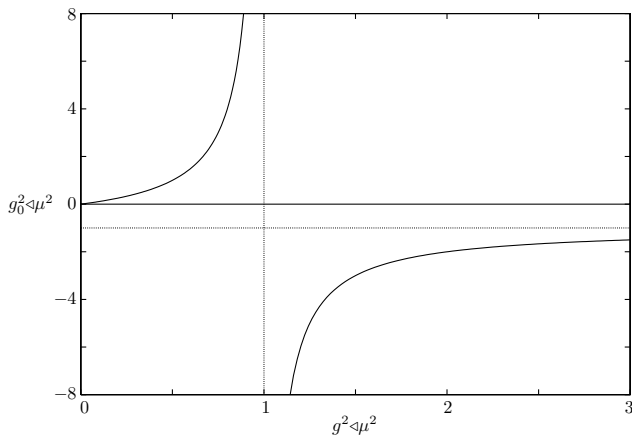


FIG. 1. Square of the unrenormalized coupling constant, g_0^2 , plotted as a function of the square of the renormalized coupling constant g^2 . When $g^2 = 0$ we have $g_0^2 = 0$, and as g^2 increases so does g_0^2 . However, as g^2 increases past a critical value, g_0^2 abruptly becomes negative. In this regime the unrenormalized coupling constant is negative and the Hamiltonian is non-Hermitian.

the Hamiltonian becomes non-Hermitian in the Dirac sense. (Dirac Hermitian conjugation, as indicated by the symbol \dagger , means combined transpose and complex conjugate.)

The Lee model is a cubic interaction and there have been several studies of theories having a cubic interaction multiplied by an imaginary coupling constant and in all these studies it was found that the spectrum is real and positive. In two especially important cases it was noticed that the spectrum of Reggeon field theory is real and positive [10] and that the spectrum of Lee-Yang edge-singularity models is also positive [11].

The first explanation of how a complex Hamiltonian could have a positive real spectrum was given by Bender and Boettcher [12], who studied the family of quantum-mechanical Hamiltonians

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \geq 0). \quad (2)$$

In this study it was shown that these Hamiltonians all have only real, positive, and discrete energy eigenvalues and it was argued that the unbroken \mathcal{PT} (space-time reflection) symmetry of these Hamiltonians might account for the reality of the spectrum.

While it is necessary for the spectrum of a Hamiltonian to be real and positive in order for a Hamiltonian to define a physically acceptable theory of quantum mechanics, it is not sufficient. One must also have a Hilbert space of states and an associated inner product whose norm is positive. Positivity of the norm is essential in order to have a probabilistic interpretation. Moreover, time evolution must be unitary (probability-conserving). In early attempts to define the Hilbert space for non-Hermitian \mathcal{PT} -symmetric Hamiltonians it was conjectured that since the Hermiticity of the Hamiltonian ($H = H^\dagger$) was replaced by \mathcal{PT} symmetry ($H = H^{\mathcal{PT}}$), the Hermitian inner product

$$\langle A|B \rangle \equiv |A\rangle^\dagger \cdot |B\rangle$$

should be replaced by the \mathcal{PT} inner product

$$\langle A|B \rangle \equiv |A\rangle^{\mathcal{PT}} \cdot |B\rangle.$$

However, it was quickly observed that while the \mathcal{PT} norm of a state is real, the sign of the norm can be either positive or negative, depending on the state.

A negative norm is physically unacceptable, so it is necessary to find an alternative definition of the inner product. This definition is given in the work of Bender, Brody, and Jones [13]. The key discovery in this paper was that a \mathcal{PT} -symmetric Hamiltonian that has an unbroken \mathcal{PT} symmetry possesses a new symmetry represented by a linear operator that was called C . The C operator has three crucial properties. First, the square of the C operator is unity,

$$C^2 = 1, \quad (3)$$

and therefore its eigenvalues are ± 1 . Second, C commutes with \mathcal{PT} ,

$$[C, \mathcal{PT}] = 0, \quad (4)$$

and therefore C is \mathcal{PT} symmetric. Third, C commutes with the Hamiltonian,

$$[C, H] = 0, \quad (5)$$

and therefore the eigenstates of the Hamiltonian are also eigenstates of C . In fact, states of H having a negative \mathcal{PT} norm have eigenvalue -1 under C , and eigenstates of H having a positive \mathcal{PT} norm have eigenvalue $+1$ under C . From these three properties one can use the C operator to construct the correct inner product for the Hilbert space of states of the Hamiltonian:

$$\langle A|B \rangle \equiv |A\rangle^{C\mathcal{PT}} \cdot |B\rangle. \quad (6)$$

This $C\mathcal{PT}$ inner product is associated with a *positive* norm. Furthermore, the usual time translation operator e^{iHt} preserves the inner product. Thus, with respect to this inner product, time evolution is unitary.

The implication of Ref. [13] is that a non-Hermitian \mathcal{PT} -symmetric Hamiltonian determines its own Hilbert space of state vectors and the associated inner product. One can thus regard a non-Hermitian \mathcal{PT} -symmetric Hamiltonian as a *bootstrap* theory because one must find the eigenvectors and eigenvalues of the Hamiltonian in order to discover what the Hilbert space is.

The key step in understanding non-Hermitian \mathcal{PT} -symmetric quantum theories is constructing the C operator. Several papers have been published regarding the construction of this operator. A perturbative calculation of C for cubic quantum-mechanical theories was given in Ref. [14] and a nonperturbative WKB calculation of C for the more general quantum-mechanical theory in (2) was given in Ref. [15]. A perturbative calculation of the C operator in D -dimensional cubic quantum-field theories was done [16,17]. From these papers it is evident that the best way to calculate C in a theory with a cubic interaction is to solve the system of operator equations in (3)–(5).

This paper is organized very simply. In Sec. II we consider first the case of a *quantum-mechanical* Lee model. It is shown that when the renormalized coupling constant is larger than the critical value shown in Fig. 1, the Hamiltonian becomes non-Hermitian and a ghost state appears in the $V/N\theta$ sector of the theory. In this regime the theory is \mathcal{PT} symmetric. Kleefeld was the first to point out this transition to \mathcal{PT} symmetry [18]; this reference gives a beautiful and comprehensive review of the history of non-Hermitian Hamiltonians. The \mathcal{PT} norm of the physical V particle is positive, but the \mathcal{PT} norm of the

ghost is negative. We must then calculate the C operator for this theory, and we do this *exactly and in closed form*. We verify that the $C\mathcal{PT}$ norms of the V and the ghost states are positive. Because the $C\mathcal{PT}$ norm of the ghost state is positive the term “ghost” is actually inappropriate. With this metric the Lee model in this sector is a fully consistent and unitary quantum theory.

Next, in Sec. III we generalize the results in Sec. II to the case of a *quantum-field-theoretic* Lee model. Again we verify that when the renormalized coupling constant is larger than a critical value, the Hamiltonian becomes non-Hermitian and a ghost state appears in the $V/N\theta$ sector. The \mathcal{PT} norms of the physical V and the $N\theta$ scattering states are positive but the \mathcal{PT} norm of the ghost state is negative. As in Sec. II, we again calculate the C operator in the $V/N\theta$ sector and we verify that the $C\mathcal{PT}$ norms of all states in the $V/N\theta$ sector are positive. These calculations demonstrate that there is no ghost and that the Lee model in the $V/N\theta$ sector is a fully consistent unitary quantum theory.

II. QUANTUM-MECHANICAL LEE MODEL

The quantum-mechanical Lee model describes the interaction of the three particles V , N , and θ , but in this model there is only a time variable and no space variable. Correspondingly, there is only energy and no momentum. Particles do not move but simply sit at one point and evolve in time. To create states of these particles we apply the creation operators V^\dagger , N^\dagger , and a^\dagger to the vacuum state $|0\rangle$. The bare states in this model are given by

$$\begin{aligned} |1, 0, 0\rangle &\equiv V^\dagger |0\rangle, \\ |0, 1, 0\rangle &\equiv N^\dagger |0\rangle, \\ |0, 0, 1\rangle &\equiv a^\dagger |0\rangle, \\ |1, 0, n\rangle &\equiv V^\dagger \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (n \in \mathbb{Z}, n \geq 0), \\ |0, 1, n\rangle &\equiv N^\dagger \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (n \in \mathbb{Z}, n \geq 0), \\ |1, 1, n\rangle &\equiv V^\dagger N^\dagger \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (n \in \mathbb{Z}, n \geq 0). \end{aligned} \quad (7)$$

We treat θ as a boson, but treat V and N as fermions, so there are no multi- V or multi- N states. These creation and annihilation operators satisfy the following commutation and anticommutation relations:

$$\begin{aligned} [a, a] &= [a^\dagger, a^\dagger] = 0, & [a, a^\dagger] &= 1, \\ [a, N] &= [a, V] = [a, N^\dagger] = [a, V^\dagger] = [a^\dagger, N] = [a^\dagger, V] = [a^\dagger, N^\dagger] = [a^\dagger, V^\dagger] = 0, \\ [N, N]_+ &= [N^\dagger, N^\dagger]_+ = [V, V]_+ = [V^\dagger, V^\dagger]_+ = 0, \\ [V, N]_+ &= [V^\dagger, N^\dagger]_+ = [N, V^\dagger]_+ = [N^\dagger, V]_+ = 0, \\ [N, N^\dagger]_+ &= [V, V^\dagger]_+ = 1. \end{aligned} \quad (8)$$

The Hamiltonian for the quantum-mechanical Lee model has the form

$$H = H_0 + g_0 H_1, \quad (9)$$

where

$$\begin{aligned} H_0 &= m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a, \\ H_1 &= V^\dagger N a + a^\dagger N^\dagger V. \end{aligned} \quad (10)$$

The bare states (7) are the eigenstates of H_0 and the physical states are the eigenstates of the full Hamiltonian H . Note that the mass parameters m_N and m_θ represent the *physical* masses of the one- N -particle and one- θ -particle states because these states do not undergo mass renormalization. However, m_{V_0} is the *bare* mass of the V particle.

The V , N , and θ particles are all treated as pseudoscalars. To understand why this is so, recall that in quantum mechanics the position operator

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger)$$

and the momentum operator

$$p = \frac{1}{i\sqrt{2}}(a - a^\dagger)$$

both change sign under parity reflection:

$$\mathcal{P} x \mathcal{P} = -x, \quad \mathcal{P} p \mathcal{P} = -p. \quad (11)$$

Thus, we conclude that

$$\begin{aligned} \mathcal{P} V \mathcal{P} &= -V, & \mathcal{P} N \mathcal{P} &= -N, \\ \mathcal{P} a \mathcal{P} &= -a, & \mathcal{P} V^\dagger \mathcal{P} &= -V^\dagger, \\ \mathcal{P} N^\dagger \mathcal{P} &= -N^\dagger, & \mathcal{P} a^\dagger \mathcal{P} &= -a^\dagger. \end{aligned} \quad (12)$$

Under time reversal p and i change sign but x does not:

$$\mathcal{T} p \mathcal{T} = -p, \quad \mathcal{T} i \mathcal{T} = -i, \quad \mathcal{T} x \mathcal{T} = x. \quad (13)$$

Thus,

$$\begin{aligned} \mathcal{T} V \mathcal{T} &= V, & \mathcal{T} N \mathcal{T} &= N, & \mathcal{T} a \mathcal{T} &= a, \\ \mathcal{T} V^\dagger \mathcal{T} &= V^\dagger, & \mathcal{T} N^\dagger \mathcal{T} &= N^\dagger, & \mathcal{T} a^\dagger \mathcal{T} &= a^\dagger. \end{aligned} \quad (14)$$

Note that when the bare coupling constant g_0 is real, H in (9) is Hermitian: $H^\dagger = H$. When g_0 is imaginary,

$$g_0 = i\lambda_0 \quad (\lambda_0 \text{ real}), \quad (15)$$

H is not Hermitian, but by virtue of the transformation properties in (12) and (14), H is $\mathcal{P}\mathcal{T}$ -symmetric: $H^{\mathcal{P}\mathcal{T}} = H$.

Let us first assume that g_0 is real so that H is Hermitian and let us examine the simplest nontrivial sector of the quantum-mechanical Lee model, namely, the $V/N\theta$ sector. To do so, we look for the eigenstates of the Hamiltonian H in the form of linear combinations of the bare

one- V -particle and the bare one- N -one- θ -particle states. We find that there are two eigenfunctions and two eigenvalues. We interpret the eigenfunction corresponding to the lower-energy eigenvalue as the physical one- V -particle state, and we interpret the eigenfunction corresponding to the higher-energy eigenvalue as the physical one- N -one- θ -particle state. (In the field-theoretic version of the Lee model this higher-energy state corresponds to the continuum of physical N - θ scattering states.) Thus, we make the *ansatz*

$$\begin{aligned} |V\rangle &= c_{11}|1, 0, 0\rangle + c_{12}|0, 1, 1\rangle, \\ |N\theta\rangle &= c_{21}|1, 0, 0\rangle + c_{22}|0, 1, 1\rangle, \end{aligned} \quad (16)$$

and demand that these states be eigenstates of H with eigenvalues m_V (the renormalized V -particle mass) and $E_{N\theta}$. The eigenvalue problem reduces to a pair of elementary algebraic equations:

$$\begin{aligned} c_{11}m_{V_0} + c_{12}g_0 &= c_{11}m_V, \\ c_{21}g_0 + c_{22}(m_N + m_\theta) &= c_{22}E_{N\theta}. \end{aligned} \quad (17)$$

The solutions to (17) are

$$\begin{aligned} m_V &= \frac{1}{2} \left(m_N + m_\theta + m_{V_0} - \sqrt{\mu_0^2 + 4g_0^2} \right), \\ E_{N\theta} &= \frac{1}{2} \left(m_N + m_\theta + m_{V_0} + \sqrt{\mu_0^2 + 4g_0^2} \right), \end{aligned} \quad (18)$$

where

$$\mu_0 \equiv m_N + m_\theta - m_{V_0}. \quad (19)$$

Notice that m_V , the mass of the physical V particle, is different from m_{V_0} , the mass of the bare V particle, because the V particle undergoes mass renormalization.

Next, we perform wave-function renormalization. Following Barton we define the wave-function renormalization constant Z_V by the relation [4]

$$1 = \langle 0 | \frac{1}{\sqrt{Z_V}} V | V \rangle. \quad (20)$$

This gives

$$Z_V = \frac{2g_0^2}{\sqrt{\mu_0^2 + 4g_0^2} \left(\sqrt{\mu_0^2 + 4g_0^2} - \mu_0 \right)}. \quad (21)$$

Finally, we perform coupling-constant renormalization. Again, following Barton we note that $\sqrt{Z_V}$ is the ratio between the renormalized coupling constant g and the bare coupling constant g_0 [4]. Thus,

$$\frac{g^2}{g_0^2} = Z_V. \quad (22)$$

After some elementary algebra we find that in terms of the renormalized mass and coupling constant, the bare coupling constant satisfies:

$$g_0^2 = \frac{g^2}{1 - \frac{g^2}{\mu^2}}, \quad (23)$$

where μ is defined as

$$\mu \equiv m_N + m_\theta - m_V. \quad (24)$$

We cannot freely choose the value of g because the value of g is in principle taken from experimental data. Once the value of g has been determined experimentally, we can use (23) to determine g_0 . The relation in (23) is plotted in Fig. 1. Figure 1 reveals an extremely surprising property of the Lee model: If g is larger than the critical value μ , then the square of g_0 is negative, and g_0 is imaginary.

As g approaches its critical value from below, the two energy eigenvalues in (18) vary accordingly. The energy eigenvalues are the two zeros of the secular determinant $f(E)$ obtained from applying Cramer's rule to (17). We have plotted $f(E)$ as a function of E in Figs. 2–5. Figure 2 shows $f(E)$ for very small g and Fig. 3 shows $f(E)$ for a larger value of g , but one for which g is still smaller than its critical value. As g (and g_0) increase, the energy of the physical $N\theta$ state increases. The energy of the $N\theta$ state becomes infinite as g reaches its critical value. As g increases past its critical value, the upper energy eigenvalue goes around the bend: It abruptly jumps from being large and positive to being large and negative (see Fig. 4). Then, as g continues to increase, this energy eigenvalue approaches the energy of the physical V particle from below.

When g increases past its critical value, the Hamiltonian H in (9) becomes non-Hermitian, but its eigenvalues in the $V/N\theta$ sector remain real, as we can see from Figs. 4 and 5. The eigenvalues remain real because H becomes \mathcal{PT}

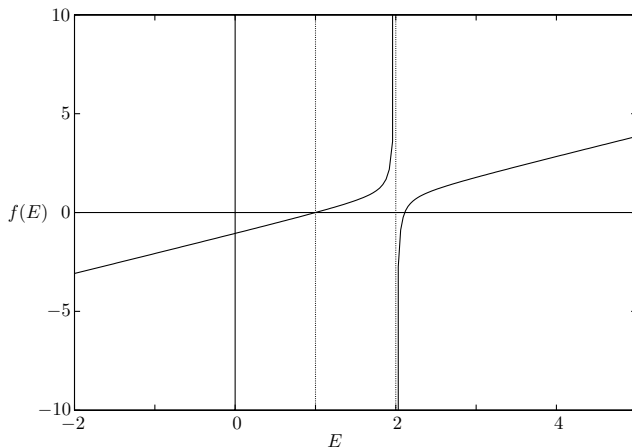


FIG. 2. Plot of the secular determinant $f(E)$ obtained by applying Cramer's rule to (17) for g_0 real and small. Values of E for which $f(E) = 0$ correspond to eigenvalues of the Hamiltonian (9). Observe that $f(E)$ has two zeros, the lower one corresponding to the energy of the physical V particle and the upper one corresponding to the energy of the physical $N\theta$ state.

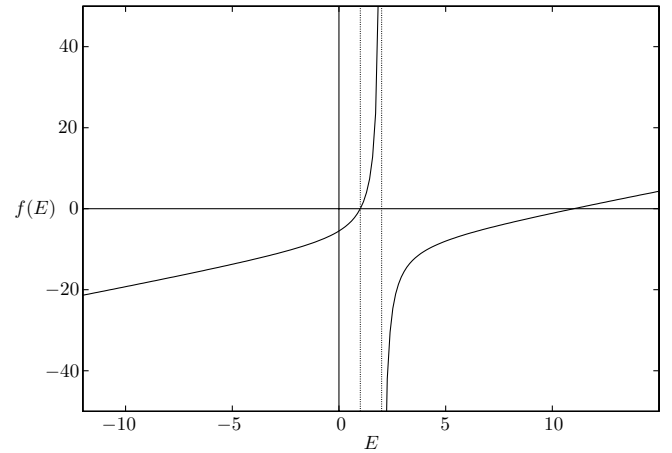


FIG. 3. Same as in Fig. 2 except that the value of g is larger. Note that the larger eigenvalue E (the larger value of E for which $f(E) = 0$), which corresponds to the physical $N\theta$ state, has moved up the real- E axis.

symmetric, and cubic \mathcal{PT} -symmetric Hamiltonians that have been studied in the past have been shown to have real spectra. However, in the \mathcal{PT} -symmetric regime it is no longer appropriate to interpret the lower eigenvalue as the energy of the physical $N\theta$ state. Rather, it is the energy of a new kind of state $|G\rangle$ called a *ghost*. As is shown in Refs. [2–4], the Hermitian norm of this state is *negative*. Until the writing of this paper, a satisfactory physical interpretation of the ghost state had not been found.

We will now show how a physical interpretation of the ghost state emerges easily when we use the methods developed in Ref. [13]. We begin by verifying that in the \mathcal{PT} -symmetric regime, where g_0 is imaginary, the states

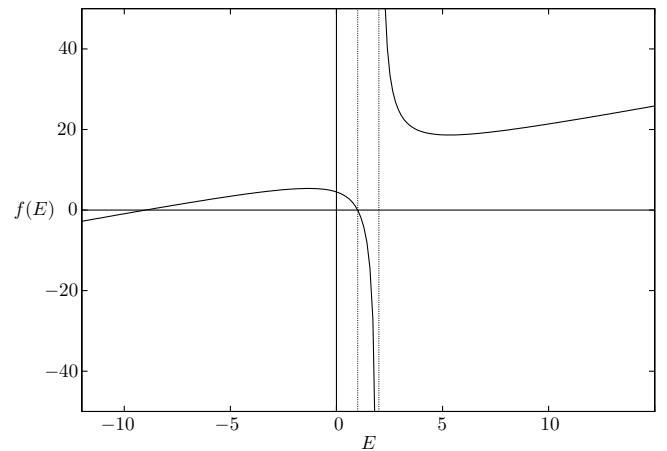


FIG. 4. Same as in Fig. 3 except that g is larger than its critical value and the unrenormalized coupling constant g_0 is imaginary. In this regime the Hamiltonian is non-Hermitian. Observe that the larger zero of $f(E)$ has moved out to infinity and is now moving up the negative real- E axis below the energy of the physical V particle. The $N\theta$ state has disappeared and has been replaced by a ghost state.

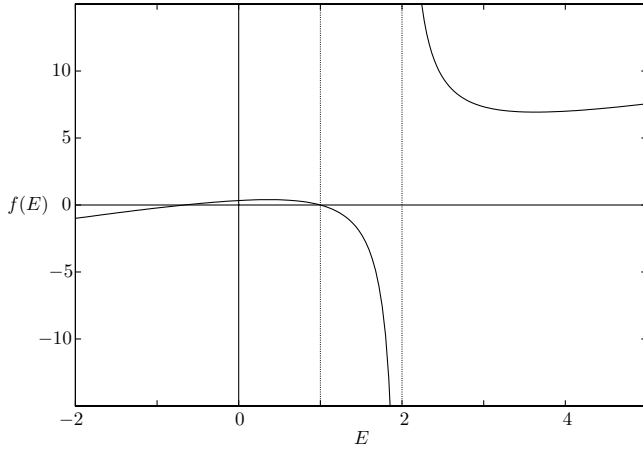


FIG. 5. Same as in Fig. 4 except that g has an even larger value. Note that as g continues to increase the ghost energy continues to move up towards the energy of the physical V particle.

of the Hamiltonian are eigenstates of the \mathcal{PT} operator, and we then choose the multiplicative phases of these states so that their \mathcal{PT} eigenvalues are unity:

$$\mathcal{PT}|G\rangle = |G\rangle, \quad \mathcal{PT}|V\rangle = |V\rangle.$$

It is then straightforward to verify that the \mathcal{PT} norm of the V state is positive, while the \mathcal{PT} norm of the ghost state is negative.

We then follow the procedures explained in Refs. [16,17] to calculate the C operator. In these papers it is shown that the C operator can be expressed as an exponential of a function Q multiplying the parity operator:

$$C = e^{Q(V^\dagger, V; N^\dagger, N; a^\dagger, a)} \mathcal{P}. \quad (25)$$

We then impose the algebraic operator equations in (3)–(5). The condition $C^2 = 1$ gives

$$Q(V^\dagger, V; N^\dagger, N; a^\dagger, a) = -Q(-V^\dagger, -V; -N^\dagger, -N; -a^\dagger, -a). \quad (26)$$

Thus, $Q(V^\dagger, V; N^\dagger, N; a^\dagger, a)$ is an odd function in total powers of $V^\dagger, V, N^\dagger, N, a^\dagger$, and a .

Next, we impose the condition $[C, \mathcal{PT}] = 0$ and obtain the result that

$$Q(V^\dagger, V; N^\dagger, N; a^\dagger, a) = Q^*(-V^\dagger, -V; -N^\dagger, -N; -a^\dagger, -a), \quad (27)$$

where $*$ denotes complex conjugation.

Finally, we impose the condition that the operator C commutes with H : $[C, H] = 0$, which requires that

$$[e^Q, H_0] = g_0[e^Q, H_1]_+. \quad (28)$$

Although in Refs. [16,17] we were only able to find the C operator to leading order in perturbation theory, for the Lee

model it is possible to calculate the C operator exactly and in closed form. To do so, we seek a solution for Q as a formal Taylor series in powers of g_0 :

$$Q = \sum_{n=0}^{\infty} g_0^{2n+1} Q_{2n+1}. \quad (29)$$

Only odd powers of g_0 appear in this series and Q_{2n+1} are all anti-Hermitian: $Q_{2n+1}^\dagger = -Q_{2n+1}$. From (28) we get

$$Q_1 = \frac{2}{\mu_0} (V^\dagger N a - a^\dagger N^\dagger V),$$

$$Q_3 = -\frac{8}{3\mu_0^3} (V^\dagger N a n_\theta - n_\theta a^\dagger N^\dagger V),$$

$$Q_5 = \frac{32}{5\mu_0^5} (V^\dagger N a n_\theta^2 - n_\theta^2 a^\dagger N^\dagger V), \quad \vdots$$

$$Q_{2n+1} = (-1)^n \frac{2^{2n+1}}{(2n+1)\mu_0^{2n+1}} (V^\dagger N a n_\theta^n - n_\theta^n a^\dagger N^\dagger V), \quad (30)$$

and so on, where $n_\theta = a^\dagger a$ is the number operator for θ -particle quanta.

Finally, we sum over all Q_{2n+1} and obtain the *exact* result that

$$Q = V^\dagger N a \frac{1}{\sqrt{n_\theta}} \arctan\left(\frac{2g_0\sqrt{n_\theta}}{\mu_0}\right) - \frac{1}{\sqrt{n_\theta}} \arctan\left(\frac{2g_0\sqrt{n_\theta}}{\mu_0}\right) a^\dagger N^\dagger V. \quad (31)$$

We must now exponentiate this result to obtain the C operator. Fortunately, the exponential of Q simplifies considerably because we are treating the V and N particles as fermions and therefore we can use the identity $n_{V,N}^2 = n_{V,N}$. Our exact result for e^Q is

$$e^Q = \left[1 - n_V - n_N + n_V n_N + \frac{\mu_0 n_N (1 - n_V)}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} + \frac{\mu_0 n_V (1 - n_N)}{\sqrt{\mu_0^2 + 4g_0^2 (n_\theta + 1)}} + V^\dagger N a \frac{2g_0\sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} - \frac{2g_0\sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} a^\dagger N^\dagger V \right]. \quad (32)$$

We can also express the parity operator \mathcal{P} in terms of number operators:

$$\mathcal{P} = e^{i\pi(n_V + n_N + n_\theta)} = (1 - 2n_V)(1 - 2n_N)e^{i\pi n_\theta}. \quad (33)$$

Combining e^Q and \mathcal{P} , we obtain the exact expression for

C:

$$C = \left[1 - n_V - n_N + n_V n_N + \frac{\mu_0 n_N (1 - n_V)}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} \right. \\ \left. + \frac{\mu_0 n_V (1 - n_N)}{\sqrt{\mu_0^2 + 4g_0^2 (n_\theta + 1)}} + V^\dagger N a \frac{2g_0 \sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} \right. \\ \left. - \frac{2g_0 \sqrt{n_\theta}}{\sqrt{\mu_0^2 + 4g_0^2 n_\theta}} a^\dagger N^\dagger V \right] (1 - 2n_V)(1 - 2n_N) e^{i\pi n_\theta}. \quad (34)$$

Using this C operator we calculate the $C\mathcal{PT}$ norm of the V state and the ghost state and find that they are both positive. Furthermore, as is shown in Ref. [13], time evolution is unitary. Thus, we have verified that with the proper definition of the inner product, the quantum-mechanical Lee model is a physically acceptable and fully consistent quantum theory even in the ghost regime where the unrenormalized coupling constant is imaginary.

III. QUANTUM-FIELD-THEORETIC LEE MODEL

The field-theoretic Lee model is more general than the quantum-mechanical Lee model discussed in Sec. II in that it allows the V , N , and θ particles to move. Thus, the Hamiltonian for the field-theoretic Lee model is composed of operators that create and annihilate V , N , and θ particles of a given momentum. This Hamiltonian has the form

$$H = H_0 + H_I. \quad (35)$$

The free Hamiltonian is given by

$$H_0 = m_{V_0} \int d\vec{p} V_{\vec{p}}^\dagger V_{\vec{p}} + m_N \int d\vec{p} N_{\vec{p}}^\dagger N_{\vec{p}} + \int d\vec{k} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}, \quad (36)$$

where $\omega_{\vec{k}} = \sqrt{k^2 + m_\theta^2}$, and the interaction Hamiltonian is given by

$$H_I = \int d\vec{k} h_{\vec{k}} \int d\vec{p} [V_{\vec{p}}^\dagger N_{\vec{p}-\vec{k}} a_{\vec{k}} + a_{\vec{k}}^\dagger N_{\vec{p}-\vec{k}}^\dagger V_{\vec{p}}], \quad (37)$$

where

$$h_{\vec{k}} = \frac{g_0 \rho(\omega_{\vec{k}})}{(2\pi)^{3/2} \sqrt{2\omega_{\vec{k}}}}. \quad (38)$$

The function $\rho(\omega_{\vec{k}})$ is a cutoff that ensures that the process of renormalization can be carried out using finite quantities. Note that when the unrenormalized coupling constant g_0 is imaginary, $g_0 = i\lambda_0$ (λ_0 real), H is not Hermitian but is \mathcal{PT} symmetric.

The $V/N\theta$ sector of the Lee model is spanned by the one-bare- V -particle state and the one-bare- N -one-bare--states. Because of the conservation laws of the Lee model, this sector is an invariant subspace. The physical energy eigenstates in this sector have the general form $|E, \vec{p}\rangle$,

where

$$|E, \vec{p}\rangle = \varepsilon \left(V_{\vec{p}}^\dagger + \int d\vec{k} \varphi_{\vec{k}} N_{\vec{p}-\vec{k}}^\dagger a_{\vec{k}}^\dagger \right) |0\rangle. \quad (39)$$

Under the action of the Hamiltonian this state has the energy eigenvalue E :

$$H|E, \vec{p}\rangle = E|E, \vec{p}\rangle. \quad (40)$$

The eigenvalue problem (40) has the form of two coupled simultaneous equations:

$$(m_N + \omega_{\vec{k}}) \varphi_{\vec{k}} + h_{\vec{k}} = E \varphi_{\vec{k}}, \quad (41)$$

$$m_{V_0} + \int d\vec{k} h_{\vec{k}} \varphi_{\vec{k}} = E. \quad (42)$$

Eqs. (41) and (42) are the analog of (17). We solve (41) for $\varphi_{\vec{k}}$,

$$\varphi_{\vec{k}} = - \frac{h_{\vec{k}}}{\omega_{\vec{k}} + m_N - E},$$

and substitute the result into (42) to obtain

$$\int d\vec{k} \frac{h_{\vec{k}}^2}{\omega_{\vec{k}} + m_N - E} = m_{V_0} - E. \quad (43)$$

Next, we define $f(E)$ by

$$f(E) = E - m_{V_0} + \int d\vec{k} \frac{h_{\vec{k}}^2}{\omega_{\vec{k}} + m_N - E}. \quad (44)$$

The function $f(E)$ is the field-theory analog of the function $f(E)$ that is plotted in Figs. 2–5 and the roots of $f(E)$ are the physical energy levels of the Lee model Hamiltonian (35). Note that the first derivative of $f(E)$ is

$$f'(E) = 1 + \int d\vec{k} \frac{h_{\vec{k}}^2}{(\omega_{\vec{k}} + m_N - E)^2}. \quad (45)$$

To show how to use the function $f(E)$ to find the energy levels, we discretize the integral over \vec{k} to convert it to a summation. We take $\rho(\omega_{\vec{k}})$ in (38) to be a sharp cutoff function and we plot in Fig. 6 the function $f(E)$ versus E for small real g_0 . Observe that there is an isolated energy level that corresponds to the mass of the physical V particle. We will call this energy level m_V . Above this energy level there is a gap and then a bunch of closely associated energy levels that correspond to the physical N - θ scattering states. In the continuum limit the discrete sum over \vec{k} becomes an integral over \vec{k} , and this bunch of discrete states becomes a continuous cut on the real axis in the complex- E plane.

As g_0 increases (but still remains real), the highest-energy N - θ scattering state separates from the rest and moves up towards positive infinity. This separation is illustrated in Fig. 7. Just as the renormalized coupling constant g increases passes its critical value and the un-

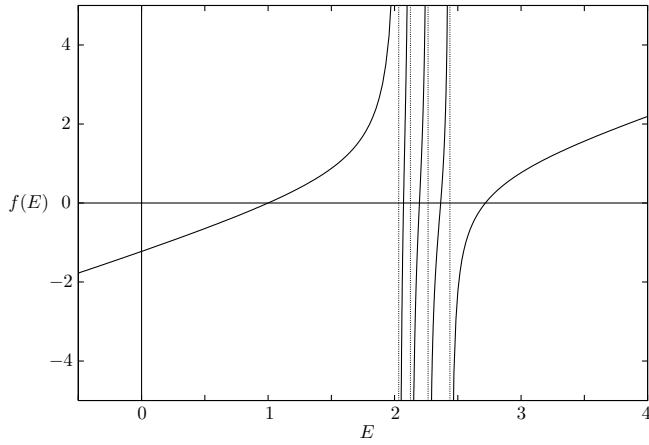


FIG. 6. Field theoretic version of Fig. 2. The function $f(E)$ in (44) is plotted as a function of E for the case in which the integral over \vec{k} is replaced by a discrete sum over \vec{k} . The energy eigenvalues of H in (35) are the zeros of $f(E)$. The graph is constructed for a small real value of g_0 . The lowest eigenvalue, which corresponds to the physical V particle, is isolated from the other energies, which correspond to physical N - θ scattering states. In the continuum limit in which the discrete sum is replaced by an integral, the physical N - θ scattering states become dense and form a cut on the real axis in the complex- E plane.

renormalized coupling constant becomes imaginary, the isolated high-energy state abruptly becomes large and *negative* and lies below the physical V state. This jump, which occurs just as the Hamiltonian becomes non-Hermitian, is characterized by the appearance of the ghost state (see Fig. 8). We will call this energy level E_G . As the renormalized coupling constant continues to increase, the ghost energy E_G increases towards the V -particle energy m_V (see Fig. 9).

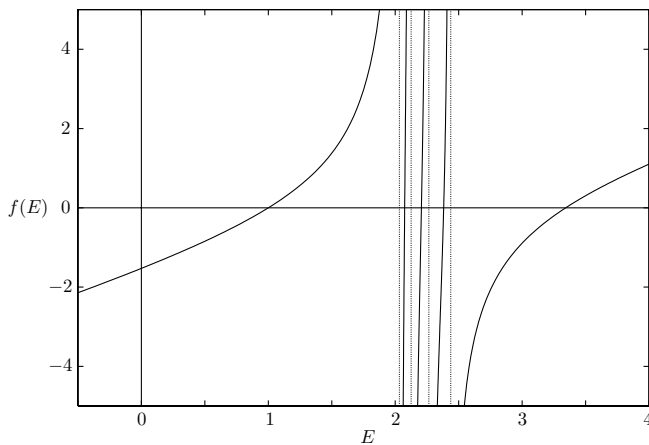


FIG. 7. Same as in Fig. 6 but with g_0 having a larger real value. Observe that as g_0 increases, the highest energy physical N - θ scattering state separates from all the rest and moves off towards positive infinity.

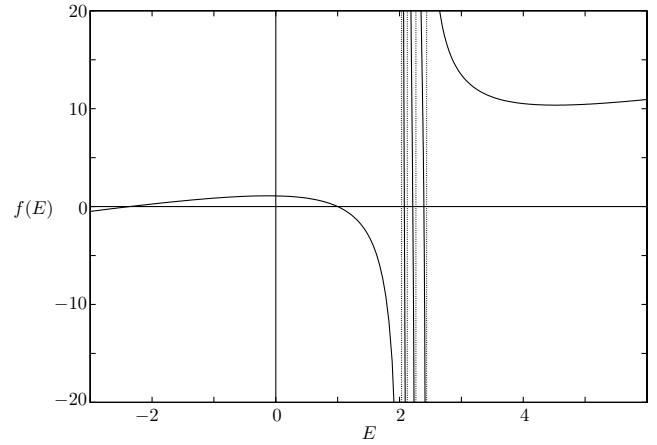


FIG. 8. Same as in Fig. 7 except that the renormalized coupling constant is larger than its critical value and the unrenormalized coupling constant g_0 is imaginary. Observe that the largest zero of $f(E)$ has abruptly become large and negative. This state is referred to as the ghost state. The ghost state appears when the Hamiltonian becomes non-Hermitian.

Observe that in Figs. 6–9, E_G and m_V lie in the interval $(-\infty, m_N + m_\theta)$. Also, note that $f'(E_G) > 0$ and that $Z_V = f'(m_V) < 0$. When $E > m_N + m_\theta$, $f'(E) < 0$. Note also that for any two eigenvalues $E_1 \neq E_2$, we have

$$\begin{aligned} & \int d\vec{k} \frac{h_{\vec{k}}^2}{(\omega_{\vec{k}} + m_N - E_1)(\omega_{\vec{k}} + m_N - E_2)} \\ &= \frac{1}{E_1 - E_2} \int d\vec{k} h_{\vec{k}}^2 \left[\frac{1}{\omega_{\vec{k}} + m_N - E_1} - \frac{1}{\omega_{\vec{k}} + m_N - E_2} \right] \\ &= \frac{1}{E_1 - E_2} [(m_{V_0} - E_1) - (m_{V_0} - E_2)] = -1. \end{aligned} \quad (46)$$

In the ghost regime, where the Hamiltonian is non-Hermitian, we follow the approach in Sec. II and choose

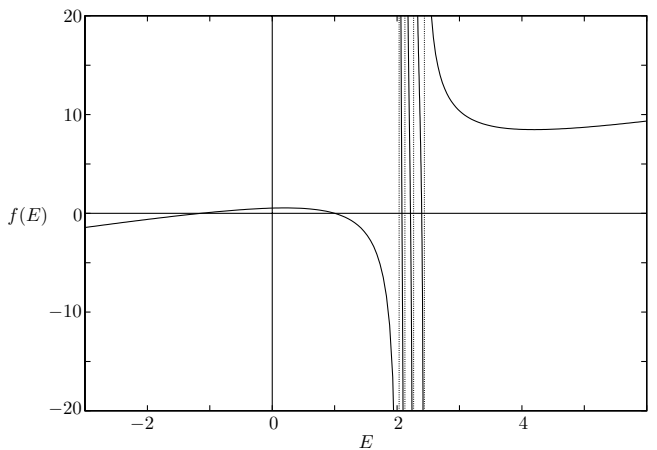


FIG. 9. Same as in Fig. 8 except that the renormalized coupling constant has a larger value. Observe that the energy of the ghost always lies below the energy of the physical V particle, but that as g increases, the ghost energy moves up towards the V energy.

the phases ε of the states in (39) so that the eigenvalues of the \mathcal{PT} operator are all +1. That is, we require that

$$\begin{aligned} \mathcal{PT}|E, \vec{p}\rangle &= -\varepsilon^* \left(V_{\vec{p}}^\dagger - \int d\vec{k} \frac{h_{\vec{k}}}{\omega_{\vec{k}} + m_N - E} N_{\vec{p}-\vec{k}}^\dagger a_{\vec{k}}^\dagger \right) |0\rangle \\ &= |E, \vec{p}\rangle. \end{aligned} \quad (47)$$

$$\begin{aligned} |E_1, \vec{p}_1\rangle^{\mathcal{PT}} \cdot |E_2, \vec{p}_2\rangle &= \varepsilon_{E_1} \varepsilon_{E_2} \left(1 + \int d\vec{k} \frac{h_{\vec{k}}^2}{(\omega_{\vec{k}} + m_N - E_1)(\omega_{\vec{k}} + m_N - E_2)} \right) \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \\ &= \varepsilon_{E_1}^2 f'(E_1) \delta_{E_1, E_2} \delta^{(3)}(\vec{p}_1 - \vec{p}_2). \end{aligned} \quad (48)$$

For the physical V state and the physical N - θ scattering states, we have $f'(E) < 0$, so by choosing ε_E , we can normalize these states as

$$|E_1, \vec{p}_1\rangle^{\mathcal{PT}} \cdot |E_2, \vec{p}_2\rangle = \delta_{E_1, E_2} \delta^{(3)}(\vec{p}_1 - \vec{p}_2). \quad (49)$$

However, for the ghost state $f'(E_G) > 0$, and we must normalize the state as

$$|E_G, \vec{p}_1\rangle^{\mathcal{PT}} \cdot |E_G, \vec{p}_2\rangle = -\delta^{(3)}(\vec{p}_1 - \vec{p}_2) \quad (50)$$

Thus the ghost state has a negative \mathcal{PT} norm, while the V state and the N - θ scattering states have positive \mathcal{PT} norms.

We must introduce the C operator in order to repair the negative sign of the norm of the ghost state. Following the approach in Sec. II we seek the C operator in the form

$$C = e^Q \mathcal{P}_I, \quad (51)$$

where in this equation \mathcal{P}_I represents the *intrinsic* parity operator rather than the conventional parity operator \mathcal{P} . The intrinsic parity operator changes the sign of a pseudoscalar field, but unlike the ordinary parity operator \mathcal{P} , \mathcal{P}_I does not change the sign of the spatial variable in the argument of the field. The exact formula for \mathcal{P}_I is

$$\mathcal{P}_I = e^{i\pi(n_V + n_N + n_\theta)}. \quad (52)$$

Using \mathcal{P} instead of \mathcal{P}_I in this formula would give the wrong C operator, as we will see in (75) and (77).

In the $V/N\theta$ sector, the conditions $C^2 = 1$ in (3) and $[C, \mathcal{PT}] = 0$ in (4) require Q to have the form

$$Q = \int d\vec{p} d\vec{k} (\gamma_{\vec{k}} V_{\vec{p}}^\dagger N_{\vec{p}-\vec{k}} a_{\vec{k}} + \gamma_{\vec{k}}^* a_{\vec{k}}^\dagger N_{\vec{p}-\vec{k}}^\dagger V_{\vec{p}}), \quad (53)$$

where the function $\gamma_{\vec{k}}$ is as yet unknown. In this sector, Q^2 can be expressed as

$$\begin{aligned} Q^2 &= \int d\vec{k} |\gamma_{\vec{k}}|^2 \int d\vec{p} V_{\vec{p}}^\dagger V_{\vec{p}} \\ &\quad + \int d\vec{p} d\vec{k}_1 d\vec{k}_2 \gamma_{\vec{k}_1}^* \gamma_{\vec{k}_2} a_{\vec{k}_1}^\dagger N_{\vec{p}-\vec{k}_1}^\dagger N_{\vec{p}-\vec{k}_2} a_{\vec{k}_2}, \end{aligned} \quad (54)$$

where we have ignored the terms which will vanish when applied to a state in this sector. In this sector, Q^3 is

This requirement implies that ε must be imaginary: $\varepsilon^* = -\varepsilon$.

Having chosen the phases of states in this way, we can now calculate the \mathcal{PT} norms of the states in the $V/N\theta$ sector. To do so we use the general formula for the \mathcal{PT} norm of the state $|E_1, \vec{p}_1\rangle$:

proportional to Q :

$$\begin{aligned} Q^3 &= \int d\vec{l} |\gamma_{\vec{l}}|^2 \int d\vec{p} d\vec{k} (\gamma_{\vec{k}} V_{\vec{p}}^\dagger N_{\vec{p}-\vec{k}} a_{\vec{k}} + \gamma_{\vec{k}}^* a_{\vec{k}}^\dagger N_{\vec{p}-\vec{k}}^\dagger V_{\vec{p}}) \\ &= \beta^2 Q, \end{aligned} \quad (55)$$

where

$$\beta^2 \equiv \int d\vec{k} |\gamma_{\vec{k}}|^2 > 0. \quad (56)$$

Using these formulas, we can expand e^Q in terms of Q and Q^2 :

$$e^Q = 1 + \frac{\sinh\beta}{\beta} Q + \frac{\cosh\beta - 1}{\beta^2} Q^2. \quad (57)$$

Next, we impose the condition in (5) that C commutes with H . This gives the formula

$$[e^Q, H_0] = [e^Q, H_I]_+. \quad (58)$$

Substituting (57) into (58), we obtain the equations

$$\frac{\sinh\beta}{\beta} \mu_{\vec{k}} \gamma_{\vec{k}} = 2h_{\vec{k}} + \frac{\cosh\beta - 1}{\beta^2} (\beta^2 h_{\vec{k}} + \gamma_{\vec{k}} \int d\vec{l} \gamma_{\vec{l}}^* h_{\vec{l}}), \quad (59)$$

$$\frac{\cosh\beta - 1}{\beta \sinh\beta} (\mu_{\vec{k}_2} - \mu_{\vec{k}_1}) = \frac{h_{\vec{k}_2}}{\gamma_{\vec{k}_2}} + \frac{h_{\vec{k}_1}}{\gamma_{\vec{k}_1}^*}, \quad (60)$$

$$\int d\vec{k} (\gamma_{\vec{k}} + \gamma_{\vec{k}}^*) h_{\vec{k}} = 0, \quad (61)$$

where $\mu_{\vec{k}} \equiv \omega_{\vec{k}} + m_N - m_{V_0}$. Equation (59) gives

$$\gamma_{\vec{k}} = \frac{(\cosh\beta + 1)h_{\vec{k}}}{\frac{\sinh\beta}{\beta} \mu_{\vec{k}} - \frac{\cosh\beta - 1}{\beta^2} \beta^2}, \quad (62)$$

where

$$\beta_2 = \int d\vec{k} \gamma_{\vec{k}}^* h_{\vec{k}}. \quad (63)$$

Substituting (62) into (60), we get

$$\beta_2^* = \beta_2. \quad (64)$$

Thus, β_2 is real. Combining this result with (62), we find that $\gamma_{\vec{k}}$ is imaginary. Hence, (61) is satisfied.

The numbers β and β_2 satisfy two constraints:

$$\beta^2 = - \int d\vec{k} \frac{(\cosh\beta + 1)^2 h_k^2}{\left(\frac{\sinh\beta}{\beta} \mu_{\vec{k}} - \frac{\cosh\beta - 1}{\beta^2} \beta_2\right)^2}, \quad (65)$$

$$\beta_2 = - \int d\vec{k} \frac{(\cosh\beta + 1) h_k^2}{\frac{\sinh\beta}{\beta} \mu_{\vec{k}} - \frac{\cosh\beta - 1}{\beta^2} \beta_2} > 0. \quad (66)$$

Let us define

$$E_0 \equiv m_{V_0} + \frac{\cosh\beta - 1}{\beta \sinh\beta} \beta_2. \quad (67)$$

Then (65) and (66) become

$$\left(\frac{\sinh\beta}{\cosh\beta + 1}\right)^2 = - \int d\vec{k} \frac{h_k^2}{(\omega_{\vec{k}} + m_N - E_0)^2}, \quad (68)$$

$$E_0 - m_{V_0} = - \int d\vec{k} \frac{h_k^2}{\omega_{\vec{k}} + m_N - E_0}. \quad (69)$$

Equation (69) shows that E_0 is an *eigenvalue* of H . From (68) we have

$$f'(E_0) = 1 + \int d\vec{k} \frac{h_k^2}{(\omega_{\vec{k}} + m_N - E_0)^2} = \frac{2}{\cosh\beta + 1} > 0. \quad (70)$$

We know that only the ghost state has $f'(E) > 0$, so

$$E_0 = E_G. \quad (71)$$

We can now apply the operator C to the eigenstates of H :

$$\begin{aligned} C|E, \vec{p}\rangle &= \varepsilon_E \left(-1 + \frac{\sinh\beta}{\beta} \beta_3 - \frac{\cosh\beta - 1}{\beta^2} \beta^2 \right) V_{\vec{p}}^\dagger |0\rangle \\ &\quad - \varepsilon_E \int d\vec{k} \left(\frac{h_{\vec{k}}}{\omega_{\vec{k}} + m_N - E} - \frac{\sinh\beta}{\beta} \gamma_{\vec{k}} \right. \\ &\quad \left. + \frac{\cosh\beta - 1}{\beta^2} \gamma_{\vec{k}} \beta_3 \right) N_{\vec{p}-\vec{k}}^\dagger a_{\vec{k}}^\dagger |0\rangle, \end{aligned} \quad (72)$$

where

$$\begin{aligned} \beta_3 &\equiv \int d\vec{k} \frac{\gamma_{\vec{k}}^* h_{\vec{k}}}{\omega_{\vec{k}} + m_N - E} \\ &= - \frac{\beta(\cosh\beta + 1)}{\sinh\beta} \\ &\quad \times \int d\vec{k} \frac{h_k^2}{(\omega_{\vec{k}} + m_N - E)(\omega_{\vec{k}} + m_N - E_0)}. \end{aligned} \quad (73)$$

If $E \neq E_0$, then from (46) we have

$$\beta_3 = \frac{\beta(\cosh\beta + 1)}{\sinh\beta}. \quad (74)$$

Substituting this result into (72), we obtain

$$C|E, \vec{p}\rangle = |E, \vec{p}\rangle. \quad (75)$$

If $E = E_0$, we have

$$\begin{aligned} \beta_3 &= - \frac{(\cosh\beta + 1)\beta}{\sinh\beta} \int d\vec{k} \frac{h_k^2}{(\omega_{\vec{k}} + m_N - E_0)^2} \\ &= \frac{\beta(\cosh\beta - 1)}{\sinh\beta}. \end{aligned} \quad (76)$$

In this case (72) is

$$C|E, \vec{p}\rangle = -|E, \vec{p}\rangle. \quad (77)$$

Equations (75) and (77) show that it was necessary to use \mathcal{P}_I and not \mathcal{P} in the *ansatz* for C in (51). Had we used the *ansatz* $C = e^Q \mathcal{P}$ we would find that on the right sides of (75) and (77), the state has the form $|E, -\vec{p}\rangle$. Thus, the vector $|E, \vec{p}\rangle$ is *not* an eigenstate of C .

Finally, we combine these results with the \mathcal{PT} norm results in (49) and (50) and demonstrate the positivity of the $C\mathcal{PT}$ norm:

$$|E_1, \vec{p}_1\rangle^{C\mathcal{PT}} \cdot |E_2, \vec{p}_2\rangle = \delta_{E_1, E_2} \delta^{(3)}(\vec{p}_1 - \vec{p}_2). \quad (78)$$

This shows that in the $V/N\theta$ sector the $C\mathcal{PT}$ norms of all states are positive even when the Hamiltonian is non-Hermitian. Thus, the Lee model remains unitary in the $V/N\theta$ sector and the ghost state does not interfere with the unitarity because the Lee model Hamiltonian becomes \mathcal{PT} symmetric when it ceases to be Hermitian. Indeed, the use of the term ghost is not appropriate because, as we have shown, this state has a positive norm [19].

ACKNOWLEDGMENTS

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- [19] In previous works on \mathcal{PT} symmetry, it is assumed that in order to construct the C operator the theory must have an *unbroken* \mathcal{PT} symmetry. However, in the quantum-mechanical Lee model it is clear that the \mathcal{PT} symmetry is broken because for sufficiently many θ quanta there exist sectors in the Hilbert space whose energy eigenvalues are complex. Presumably, such sectors also exist in the quantum-field-theoretic Lee model. Fortunately, the strict decomposition of the Hilbert space into decoupled sectors allows us to establish unitarity in the crucial $V/N\theta$ sector. We assume that the underlying reason for the breaking of \mathcal{PT} symmetry is the violation of crossing symmetry. Note that this violation gives rise to a nonlocal interaction term in the Hamiltonian for the quantum-field-theoretic Lee model.