PHYSICAL REVIEW D **71,** 025012 (2005)

Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts

Ruth Britto, Freddy Cachazo, and Bo Feng

School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540 USA

(Received 1 December 2004; published 31 January 2005)

We propose a systematic way to carry out the method introduced in F. Cachazo, hep-th/0410077 for computing certain unitarity cuts of one-loop $\mathcal{N} = 4$ amplitudes of gluons. We observe that the class of cuts for which the method works involves all next-to-MHV n-gluon one-loop amplitudes of any helicity configurations. As an application of our systematic procedure, we obtain the complete seven-gluon one-loop leading-color amplitude $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

DOI: 10.1103/PhysRevD.71.025012 PACS numbers: 11.15.Bt

I. INTRODUCTION

One-loop amplitudes of gluons in supersymmetric gauge theories possess many remarkable properties. One of them is that they are four-dimensional-cut constructible [1,2]. This means that the amplitudes are completely determined by their unitarity cuts.

Recently, a new method for computing certain unitarity cuts of one-loop amplitudes in $\mathcal{N} = 4$ gauge theories was proposed in [3]. The method uses the fact that unitarity cuts can be computed in two ways.

One is by a cut integral, where two tree-level amplitudes are connected by cut propagators. The other is by computing the imaginary part of the amplitude in a certain kinematical regime chosen in order to isolate the given cut.

In general, the amplitudes of interest are not known. However, they can be written as linear combinations of scalar box functions with unknown rational coefficients in the kinematical variables [4–6].[1] These functions are completely known in terms of logarithms and dilogarithms [10].

The key observation made in [3] is that if a given first-order differential operator acts on the cut integral to produce a rational function, then the operator must annihilate the coefficients that multiply the scalar box functions in the amplitude. This ensures that the result of applying the operator to the imaginary part of the amplitude is also a rational function.

The problem of finding the unknown coefficients in the amplitude is thus related to that of comparing two rational functions.

The rational function obtained from the action of the operator on the imaginary part of the amplitude naturally comes out as a sum over ‘‘simple fractions.’’ On the other hand, the rational function that comes from the action of the operator on the cut integral comes out in a compact form.

The aim of this paper is to provide a systematic method for carrying out the reduction of the latter into the form of the former. Once this is done, the unknown coefficients in the amplitude can simply be read off by directly comparing the two expressions.

In [3], a simple prescription was given for finding suitable operators for cuts where at least one of the tree-level amplitudes in the cut integral representation is maximally helicity violating (MHV). The idea is that when amplitudes are transformed to twistor space, they are localized on simple algebraic sets [11]. In particular, MHV tree-level amplitudes are localized on lines. In [11], differential operators for testing the localization of gluons on lines (collinear operators) were introduced. By using the holomorphic anomaly of unitarity cuts found in [12] by combining the results of [13,14], one can prove that these operators can only produce rational functions when acting on the cut integrals [3].

We also find that all unitarity cuts of next-to-MHV n-gluon one-loop amplitudes of any helicity configuration satisfy the requirements to be computable by our method. This extends the class of amplitudes given in [3] from $A_{n;1}(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$ to amplitudes with three negative-helicity gluons in arbitrary positions.

One-loop amplitudes of gluons that are known explicitly are very rare. The largest set is known for $\mathcal{N} = 4$ amplitudes, where all n-gluon MHV amplitudes are known [1]. In addition to this series of amplitudes, only the six-gluon next-to-MHV one-loop amplitude with any helicity configuration is known [2].

In this paper, we illustrate our general method by calculating the seven-gluon next-to-MHV amplitude $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$. This calculation involves the computation of the coefficients of 35 scalar box functions. This is the first amplitude where the three-mass scalar box function participates.

This paper is organized as follows. In Sec. II, we explain the systematic reduction procedure that produces the coefficients of the scalar box functions in the amplitude. In Sec. III, we apply our general method to the calculation of the seven-gluon amplitude $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$. In Sec. IV, we write down the explicit form of the coefficient of the 35 scalar box functions that participate in the seven-gluon amplitude. In Appendix A, we give the explicit form of the scalar

[1]This is strictly true in the spinor-helicity formalism of [7–9].

box functions and discuss their infrared singular behavior. In Appendix B, we prove that our method gives complete information about all next-to-MHV amplitudes with any helicity configuration.

Throughout the paper, we use the following notation and conventions. The external gluon labeled by i carries momentum p_i.

$$\begin{aligned} s_{ij} &\equiv 2p_i \cdot p_j = \langle i\ j\rangle[i\ j], \\ t_i^{[r]} &\equiv (p_i + p_{i+1} + \cdots + p_{i+r-1})^2, \\ \langle i|j_r &+ j_{r+1} + \cdots + j_s|k] \\ &\equiv \langle i\ j_r\rangle[j_r\ k] + \langle i\ j_{r+1}\rangle[j_{r+1}\ k] + \cdots + \langle i\ j_s\rangle[i\ j_s]. \end{aligned} \tag{1.1}$$

II. GENERAL REDUCTION TECHNIQUES

One-loop amplitudes of gluons in supersymmetric gauge theories are four-dimensional-cut constructible. This means that knowing the discontinuities of the amplitude is enough to fix the amplitude completely [1]. Having QCD computations in mind, one should consider one-loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills as well as one-loop amplitudes with an $\mathcal{N} = 1$ chiral super multiplet running in the loop.

Even though we concentrate on $\mathcal{N} = 4$ amplitudes, it should be kept in mind that everything is valid, with some minor modifications, for $\mathcal{N} = 1$ amplitudes.

The problem at hand is the computation of the leading-color n-gluon one-loop $\mathcal{N} = 4$ amplitudes. This is the part of the full amplitude proportional to $N\mathrm{Tr}(T^{a_1} \ldots T^{a_n})$.

These amplitudes can be written as linear combinations of scalar box functions, which are listed explicitly in Appendix A. (For $\mathcal{N} = 1$ one also has to include scalar triangle and bubble functions.)

$$A_{n;1}^{1-\mathrm{loop}} = \sum_{i=1}^{n}\Big(b_i F_{n:i}^{1m} + \sum_r c_{r,i} F_{n:r;i}^{2m} + \sum_r d_{r,i} F_{n:r;i}^{2m} + \sum_{r,r'} g_{r,r',i} F_{n:r:r';i}^{3m}\Big). \tag{2.1}$$

This means that computing the amplitude is equivalent to computing the coefficients. Note that we have not included four-mass scalar box functions. The reason is that for the classes of amplitudes considered in this paper these cannot appear, as proven in [3].

A new technique to compute these coefficients was proposed in [3]. The basic idea is to compute the unitarity cuts of (2.1) using the holomorphic anomaly found in [12]. Here we present a systematic procedure to carry out the proposal of [3] that is directly applicable to all cuts of next-to-MHV one-loop amplitudes.

Consider the unitarity cut in the $(i, i + 1, \ldots, j - 1, j)$ channel. This is given by the cut integral

$$C_{i,i+1,\ldots,j-1,j} = \int d\mu A^{\mathrm{tree}}[(-\ell_1), i, i + 1, \ldots, j - 1, j, (-\ell_2)]A^{\mathrm{tree}}(\ell_2, j + 1, j + 2, \ldots, i - 2, i - 1, \ell_1), \tag{2.2}$$

where $d\mu$ is the Lorentz-invariant phase space measure of two lightlike vectors (ℓ_1, ℓ_2) constrained by momentum conservation. We find it useful to define ℓ_1 and ℓ_2 as in Fig. 1. We follow the conventions of [3].

This cut can also be computed by taking the imaginary part of the full amplitude in the kinematical regime where $t_i^{[j-i+1]} = (p_i + p_{i+1} + \ldots + p_j)^2$ is positive and all other invariants are negative [1].

It is now clear that computing $C_{i,i+1,\ldots,j-1,j}$ provides information about the amplitude via

$$C_{i,i+1,\ldots,j-1,j} = \mathrm{Im}|_{t_i^{[j-i+1]}>0} A_{n;1}. \tag{2.3}$$

The class of cuts considered in [3] are those for which one of the tree-level amplitudes in (2.2) is a MHV amplitude. All next-to-MHV amplitudes have this property. If all three negative-helicity gluons appear on the same side of the cut, then the amplitude on the other side of the cut either vanishes or is MHV. If one side of the cut has exactly one negative-helicity gluon, there are three cases to consider for the helicities of the cut propagators on this side. If they are both positive, this tree amplitude vanishes. If exactly one is positive, then it is MHV. If both are negative, then their helicities are positive viewed from the other side of the cut, so that side is the MHV amplitude.

Let the left tree-level amplitude in (2.2) be the MHV amplitude [16],

$$A_{km}^{\mathrm{treeMHV}}[(-\ell_1), i, (i + 1), \ldots, j, (-\ell_2)] = \frac{\langle k\ m\rangle^4}{\langle \ell_1\ i\rangle\langle i\ i + 1\rangle \cdots \langle j - 1\ j\rangle\langle j\ \ell_2\rangle\langle \ell_2\ \ell_1\rangle}. \tag{2.4}$$

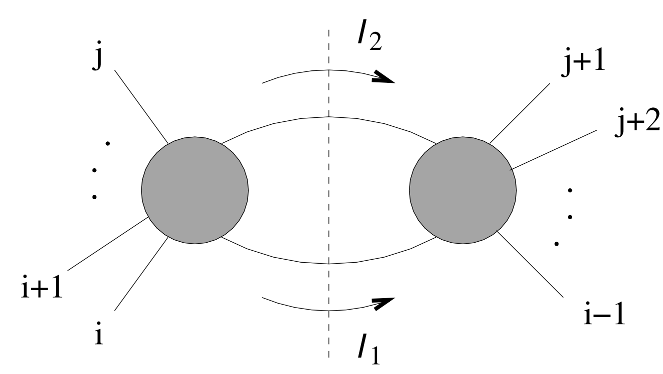

FIG. 1. Representation of the cut integral. Left and right tree-level amplitudes are on shell. Internal lines represent the legs coming from the cut propagators.

Using this in (2.2) we have

$$C_{i,i+1,\ldots,j-1,j} = \int d\mu \frac{\langle k\ m\rangle^4}{\langle i\ i+1\rangle \ldots \langle j-1\ j\rangle \langle \ell_2\ \ell_1\rangle} \times \frac{1}{\langle \ell_1\ i\rangle \langle j\ \ell_2\rangle} A^{\text{tree}}(\ell_2, j+1, j+2, \ldots, i-1, \ell_1). \tag{2.5}$$

The basic idea is to find a differential operator of first order that produces a rational function when acting on the cut (2.5). Let $\mathcal{O}$ be such an operator. Then $\mathcal{O}C_{i,i+1,\ldots,j-1,j}$ is a rational function. A simple prescription for finding such operators and for computing the rational function explicitly was given in [3]. We postpone this for the moment; we do not need the explicit form of the operator in what follows.

Consider now the action of $\mathcal{O}$ on (2.3), i.e.,

$$\mathcal{O}\, C_{i,i+1,\ldots,j-1,j} = \mathcal{O}\mathrm{Im}|_{t_i^{[j-i+1]}>0} A_{n;1}. \tag{2.6}$$

Since the operator $\mathcal{O}$ is of first order, it produces two terms for each term in the amplitude (2.1): one term when it acts on the scalar box function and one more when it acts on the coefficient. It turns out that the imaginary part of each scalar box function is the logarithm of a rational function R of the kinematical invariants.[2] Therefore, when $\mathcal{O}$ acts on the logarithms it produces rational functions. However, when it acts on the coefficients, the logarithms survive. In [3] it was proven that the only way this can be consistent with the fact that $\mathcal{O}C_{i,i+1,\ldots,j-1,j}$ is a rational function is that $\mathcal{O}$ annihilates the coefficients.

This means that we can write $\mathcal{O}C_{i,i+1,\ldots,j-1,j}$ schematically as follows:

$$\mathcal{O}\, C_{i,i+1,\ldots,j-1,j} = \sum_k a_k \frac{\mathcal{O}(R_k)}{R_k}, \tag{2.7}$$

where a_k stands for a general coefficient in (2.1), and the sum runs over the terms produced by all box functions that develop an imaginary part in the kinematical regime of interest for this cut.

Now we can clearly describe the mathematical problem involved in the calculations of the coefficients a_k.

From the action of the operator on the cut integral we find a rational function

$$\mathcal{O}\, C_{i,i+1,\ldots,j-1,j} = \frac{P}{Q\prod_k G_k}, \tag{2.8}$$

where P, Q, and G_k are polynomials. Generically P is not annihilated by $\mathcal{O}$. On the other hand, we have defined Q such that $\mathcal{O}Q = 0$. All other factors in the denominator that are not annihilated by $\mathcal{O}$ become one of the G_k.

[2]This is not true for the four-mass scalar box function, but as proven in [3] these do not contribute to the cuts we consider.

The problem is to find a way of writing (2.8) in the form (2.7) in order to read off the coefficients. It is important to mention that every a_k is annihilated by $\mathcal{O}$; this was proven in [3].

The way to deal with this problem is to realize that for any two functions G_1 and G_2 satisfying $\mathcal{O}^2(G_k) = 0$, the following combination,

$$H(G_1, G_2) = \mathcal{O}(G_1)G_2 - \mathcal{O}(G_2)G_1, \tag{2.9}$$

is annihilated by $\mathcal{O}$. In the calculations we have done, the factors G_k arising from the cut integrals all satisfy $\mathcal{O}^2(G_k) = 0$, and we believe that this property is satisfied generally.

Therefore, any rational function with both factors in the denominator "splits" as follows,

$$\frac{P}{QG_1G_2\prod_k' G_k} = \frac{P}{Q\prod_k' G_k}\left(\frac{\mathcal{O}(G_1)}{G_1} - \frac{\mathcal{O}(G_2)}{G_2}\right) \times \frac{1}{H(G_1, G_2)}, \tag{2.10}$$

where $\prod'$ means a product not including G_1 or G_2.

It is clear that this procedure can be repeated as many times as necessary until the original rational function (2.8) is written in the form

$$\frac{P}{Q\prod_k G_k} = \sum_k \frac{P_k}{Q_k}\frac{\mathcal{O}(G_k)}{G_k}. \tag{2.11}$$

This formula is very similar to what we want (2.7). However, the procedure just described only guarantees that $\mathcal{O}Q_k = 0$ but, in general, the same is not true of P_k. Recall that the coefficients a_k, which we are after, are annihilated by $\mathcal{O}$.

The way out of this problem is to realize that near a kinematical region[3] where a given $G_l = 0$ we should find

$$\frac{P_l}{Q_l} \to a_l. \tag{2.12}$$

Since P_l is a polynomial, this implies that P_l admits an expansion of the form

$$P_l = Q_l a_l + \sum_{m=1}^{\infty} h_m (G_l)^m, \tag{2.13}$$

where most terms in the sum are zero because P_l has a finite degree. Note that $P_l - Q_l a_l$ is a polynomial divisible by G_l. Therefore it can be written as $P_l - Q_l a_l = G_l X_l$, where X_l is some polynomial. We think of this as a kind of "polynomial division."

The decomposition of P_l in the form (2.13) is easily done by introducing coordinates where G_l is one variable and all other variables are kinematical invariants which are

[3]We thank Oleg Lunin for suggesting to look at this particular regime.

annihilated by $\mathcal{O}$. This guarantees that $Q_l a_l$ is annihilated by $\mathcal{O}$, as it should be.

After this is done for each P_k in (2.11), we are left with

$$\mathcal{O}\, C_{i,i+1,\ldots,j-1,j} = \sum_k a_k \frac{\mathcal{O}(G_k)}{G_k} + \sum_k \frac{X_k}{Q_k} \mathcal{O}(G_k). \quad (2.14)$$

Comparing (2.14) to (2.7) we find that a miraculous cancellation must take place, namely,

$$\sum_k \frac{X_k}{Q_k} \mathcal{O}(G_k) = 0. \quad (2.15)$$

Indeed, we find this cancellation in all the cuts considered in the next section.

In practice, the splitting procedure is done most efficiently as follows. The operation performed in (2.10) splits the rational function into two terms, such that G_1 appears only in the denominator of one term and G_2 appears only in the denominator of the other. To determine the coefficient P_1/Q_1 in (2.11), all we need is to isolate the factor G_1 from all other factors G_k, one factor at a time. That is, if k runs from 1 to r, we apply the operation (2.10) $r-1$ times, and each time, we keep only the term with G_1 remaining in the denominator. The result is that

$$\frac{P_1}{Q_1} \frac{\mathcal{O}(G_1)}{G_1} = \mathcal{O} C_{i,i+1,\ldots,j-1,j} \times \prod_{k=2}^{r} \frac{\mathcal{O}(G_1) G_k}{H(G_1, G_k)}. \quad (2.16)$$

Thus, computing all r coefficients (before performing the polynomial division) requires a total of only $r(r-1)$ operations. The point is that it is most efficient to obtain first the coefficient of one factor, dropping terms that do not contain it, and then start over for the next factor.

A. Collinear operators

The question is now how to construct differential operators that produce rational functions when acting on the cut integral (2.2). In [3], a simple prescription was given. Consider any operator F_{ijk} that tests whether gluons i, j, and k are localized on a line in twistor space. (These operators were originally introduced in Sec. 3 of [11]. For a short review see Sec. 2 of [3].)

These are defined in the spinor-helicity formalism of [7–9] as follows:

$$F_{ijk;\dot{a}} = \langle i\ j\rangle \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} + \langle k\ i\rangle \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}} + \langle j\ k\rangle \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}}, \quad (2.17)$$

where $\dot{a}$ is a negative-chirality spinor index. Therefore $F_{ijk;\dot{a}}$ is a spinor-valued differential operator.

In the following, it will be convenient to introduce a fixed, arbitrary, negative-chirality spinor $\eta^{\dot{a}}$ and consider

$$[F_{ijk}, \eta] = \epsilon^{\dot{a}\dot{b}} \eta_{\dot{a}} F_{ijk;\dot{b}}. \quad (2.18)$$

Note that the brackets in (2.18) are meant to indicate the inner product of two negative-chirality spinors and not the commutator of operators.

Naively, any operator that tests the collinearity in twistor space of three gluons in the MHV amplitude of (2.2) annihilates the cut integral. This is because tree-level MHV amplitudes are localized on a line [11]. However, it was found in [12] that the cut integral has a holomorphic anomaly that spoils this result. Instead, the collinear operator produces a delta function that localizes the integral completely when ℓ_1 or ℓ_2 participates in it. Therefore, it produces a rational function.

Going back to the particular cut integral (2.2), it turns out that the only collinear operators that localize the integral are those of the form $[F_{ikl}, \eta]$ and $[F_{klj}, \eta]$, where k, l are any gluons participating on the left side of the cut.[4]

Consider, for example, the action of the collinear operator $[F_{i,i+1,i+2}, \eta]$ on the cut integral $C_{i,i+1,\ldots,j}$.

In order to describe the rational function very explicitly, we have to exhibit the explicit dependence on the spinors λ_{ℓ_1} and $\tilde{\lambda}_{\ell_1}$ of the tree-level amplitude on the right in (2.2):

$$A^{\text{tree}}(\ell_2, j+1, j+2, \ldots, i-1, \ell_1) = A^{\text{tree}}(\ell_2, j+1, j+2, \ldots, i-1, \{\lambda_{\ell_1}, \tilde{\lambda}_{\ell_1}\}). \quad (2.19)$$

Now we are ready to write the action of the operator [3][5]:

$$[F_{i,i+1,i+2}, \eta] C_{i,i+1,\ldots,j-1,j} = \frac{t}{(2p_i \cdot P_L)} \frac{\langle k\ m\rangle^4}{\langle i\ i+1\rangle \ldots \langle j-1\ j\rangle} \frac{\langle i+1\ i+2\rangle [i\ \eta]}{\langle \ell_2\ i\rangle \langle j\ \ell_2\rangle} \times A^{\text{tree}}(\ell_2, j+1, j+2, \ldots, i-1, \{\lambda_i, t\tilde{\lambda}_i\}), \quad (2.20)$$

with

$$\ell_2 = P_L - t p_i, \qquad t = \frac{P_L^2}{(2p_i \cdot P_L)}, \qquad P_L = p_i + p_{i+1} + \ldots + p_j. \quad (2.21)$$

All we need is to put the explicit form of the tree-level amplitude on the right, make the substitutions, and apply the procedure described above with the generic operator $\mathcal{O}$ replaced by $[F_{i,i+1,i+2}, \eta]$. To reconstruct the whole amplitude, we need to know that the coefficient of every scalar box function in (2.1) can be calculated from one of the cuts. This is proven in Appendix B.

To illustrate this technique, we compute the full next-to-MHV leading-color $\mathcal{N} = 4$ seven-gluon amplitude $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

[4] Of course, if k or l is equal to i (j) then the operator $[F_{ikl}, \eta]$ ($[F_{klj}, \eta]$) vanishes trivially.

[5] A similar formula was obtained for MHV one-loop amplitudes in [17].

III. COMPUTATION OF $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$

In this paper, we compute the seven-gluon amplitude with the particular helicity configuration $(- - - + + + +)$. All other helicity configurations of seven gluons could be computed in just the same way, with no new ingredients.

The amplitude $A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ is expressed in terms of 35 box functions. We abbreviate the indices on the coefficients of (2.1) for simplicity.

$$A_{7:1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) = \sum_{i=1}^{7}(b_i F^{1m}_{7:i} + c_i F^{2m}_{7:2;i} + d_{2,i} F^{2m}_{7:2;i} + d_{3,i} F^{2m}_{7:3;i} + g_i F^{3m}_{7:2:2;i}). \tag{3.1}$$

Ten of these were already computed in [3] from the C_{123} cut, namely,[6]

$$b_4 = c_5 = d_{2,2} = d_{3,5} = \frac{(t_1^{[3]})^3}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 4|2+3|1]\langle 7|1+2|3]},$$
$$c_1 = c_2 = d_{2,6} = d_{3,1} = g_2 = g_4 = 0. \tag{3.2}$$

Here we have defined

$$\langle i|j_r + j_{r+1} + \cdots + j_s|k] \equiv \langle i\, j_r\rangle[j_r\, k] + \langle i\, j_{r+1}\rangle[j_{r+1}\, k] + \cdots + \langle i\, j_s\rangle[j_s\, k].$$

We apply our reduction technique first by applying $[F_{456}, \eta]$ on the cut C_{456}. This yields five more coefficients. Next, we apply $[F_{712}, \eta]$ on the cut C_{712}. This calculation is slightly more involved, because here it is possible for fermions and scalars to circulate in the loop. We find seven more coefficients. We can obtain corresponding results for the cuts C_{567} and C_{234} simply by permuting the labels, for nine new coefficients. At this point we have found 31 of the 35 coefficients. The remaining four are easily determined by the known infrared behavior of the amplitude.

A. The cut C_{456}

The cut C_{456} is given by

$$C_{456} = \mathrm{Im}|_{t_4^{[3]}>0}(c_1 F^{2m}_{7:2;1} + d_{2,2} F^{2m}_{7:2;2} + d_{3,4} F^{2m}_{7:3;4} + b_7 F^{1m}_{7:7} + c_4 F^{2m}_{7:2;4} + c_5 F^{2m}_{7:2;5} + d_{2,5} F^{2m}_{7:2;5} + d_{3,1} F^{2m}_{7:3;1} + g_5 F^{3m}_{7:2:2;5} + g_7 F^{3m}_{7:2:2;7}) \tag{3.3}$$

or by the cut integral

$$C_{456} = \int d\mu A^{\mathrm{tree}}[(-\ell_1)^-, 4^+, 5^+, 6^+, (-\ell_2)^-] \times A^{\mathrm{tree}}(\ell_1^+, \ell_2^+, 7^+, 1^-, 2^-, 3^-). \tag{3.4}$$

Note that in this case only gluons can run in the loop and that the five-gluon tree-level amplitude is a MHV amplitude. According to the general discussion of Sec. II, we should consider the action of the collinear operator $[F_{456}, \eta]$ on both (3.3) and (3.4).

The first step is to calculate the action of the collinear operator $[F_{456}, \eta]$ on C_{456} given by (3.3). Note that the three box functions in the top line of (3.3) are annihilated by the operator, so we cannot calculate those coefficients directly using this operator. Let us list the imaginary parts of the relevant scalar box functions in the kinematical regime where $t_4^{[3]} > 0$ and all other invariants are negative.[7]

$$\begin{aligned}
\mathrm{Im}|_{t_4^{[3]}>0} F^{1m}_{7;7} &= -\ln\Big(1 - \frac{t_4^{[3]}}{t_4^{[2]}}\Big) - \ln\Big(1 - \frac{t_4^{[3]}}{t_5^{[2]}}\Big),\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{2m}_{7:2;5} &= \ln\Big(1 - \frac{t_5^{[2]}}{t_4^{[3]}}\Big) - \ln\Big(1 - \frac{t_5^{[2]} t_1^{[3]}}{t_4^{[3]} t_5^{[3]}}\Big) + \ln\Big(-\frac{t_4^{[3]}}{t_5^{[3]}}\Big) + \ldots,\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{2m}_{7:2;4} &= \ln\Big(1 - \frac{t_4^{[2]}}{t_4^{[3]}}\Big) - \ln\Big(1 - \frac{t_4^{[2]} t_7^{[3]}}{t_3^{[3]} t_4^{[3]}}\Big) + \ln\Big(-\frac{t_4^{[3]}}{t_3^{[3]}}\Big) + \ldots,\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{2m}_{7:2;5} &= \ln\Big(-\frac{t_4^{[3]}}{t_3^{[2]}}\Big) + \ln\Big(1 - \frac{t_5^{[2]}}{t_4^{[3]}}\Big) + \ldots,\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{2m}_{7:3;1} &= \ln\Big(-\frac{t_4^{[3]}}{t_6^{[2]}}\Big) + \ln\Big(1 - \frac{t_4^{[2]}}{t_4^{[3]}}\Big) + \ldots,\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{3m}_{7:2:2;5} &= \ln\Big(-\frac{t_4^{[3]}}{t_5^{[4]}}\Big) + \ln\Big(1 - \frac{t_5^{[2]}}{t_4^{[3]}}\Big) - \ln\Big(1 - \frac{t_5^{[2]} t_2^{[2]}}{t_4^{[3]} t_5^{[4]}}\Big) + \ldots,\\
\mathrm{Im}|_{t_4^{[3]}>0} F^{3m}_{7:2:2;7} &= \ln\Big(-\frac{t_4^{[3]}}{t_6^{[3]}}\Big) + \ln\Big(1 - \frac{t_4^{[2]}}{t_4^{[3]}}\Big) - \ln\Big(1 - \frac{t_7^{[2]} t_4^{[2]}}{t_6^{[3]} t_4^{[3]}}\Big) + \ldots.
\end{aligned} \tag{3.5}$$

The ellipses represent terms that are annihilated by the collinear operator $[F_{456}, \eta]$. In other words, the terms

[6]We conjugate the coefficients of [3], which were derived for the seven-gluon one-loop amplitude $A_{7:1}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-, 7^-)$ with the opposite helicity assignments.

[7]In these expressions we suppress an overall factor of π.

represented by ellipses depend on p_4, p_5, and p_6 only through the combination $p_4 + p_5 + p_6$.

Now we can compute the action of the collinear operator on C_{456} given by the imaginary part of the amplitude (3.3). Here we denote $[F_{456}, \eta]$ by $\mathcal{O}$ in order to make contact with the general discussion of Sec. II and to avoid cluttering the equations.

$$\begin{aligned}\mathcal{O}C_{456} = {}& b_7 \frac{\mathcal{O}(t_4^{[2]} t_5^{[2]})}{t_4^{[2]} t_5^{[2]}} - c_5 \frac{\mathcal{O}(t_5^{[2]} t_1^{[3]} - t_4^{[3]} t_5^{[3]})}{t_5^{[2]} t_1^{[3]} - t_4^{[3]} t_5^{[3]}} - c_4 \frac{\mathcal{O}(t_4^{[2]} t_7^{[3]} - t_3^{[3]} t_4^{[3]})}{t_4^{[2]} t_7^{[3]} - t_3^{[3]} t_4^{[3]}} + d_{2,5} \frac{\mathcal{O}(t_3^{[2]})}{t_3^{[2]}} + d_{3,1} \frac{\mathcal{O}(t_6^{[2]})}{t_6^{[2]}} \\ & - g_5 \frac{\mathcal{O}(t_5^{[2]} t_2^{[2]} - t_4^{[3]} t_5^{[4]})}{t_5^{[2]} t_2^{[2]} - t_4^{[3]} t_5^{[4]}} - g_7 \frac{\mathcal{O}(t_7^{[2]} t_4^{[2]} - t_6^{[3]} t_4^{[3]})}{t_7^{[2]} t_4^{[2]} - t_6^{[3]} t_4^{[3]}} + (-b_7 + c_4 + d_{3,1} + g_7) \frac{\mathcal{O}(t_4^{[3]} - t_4^{[2]})}{t_4^{[3]} - t_4^{[2]}} \\ & + (-b_7 + c_5 + d_{2,5} + g_5) \frac{\mathcal{O}(t_4^{[3]} - t_5^{[2]})}{t_4^{[3]} - t_5^{[2]}}.\end{aligned} \tag{3.6}$$

We have written in the first seven terms the contributions from the poles that uniquely identify a given scalar box function. This is manifest from the fact that only one coefficient appears in front of each of them. On the other hand, the poles in the last two terms are common to several box functions and so their coefficients are linear combinations of the scalar box function coefficients.

We now turn to the computation of the action of the collinear operator on the cut integral representation of C_{456}. The cut integral (2.5) is written as

$$\begin{aligned}C_{456} = \int d\mu \frac{\langle \ell_2\ \ell_1 \rangle^3}{\langle \ell_1\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ \ell_2\rangle} \\ \times A_6^{\text{tree}}(\ell_1^+, \ell_2^+, 7^+, 1^-, 2^-, 3^-),\end{aligned} \tag{3.7}$$

where for the tree-level six-gluon amplitude we use a result from [18,19]:

$$\begin{aligned}A_6^{\text{tree}}(1^-, 2^-, 3^-, \ell_1^+, \ell_2^+, 7^+) &= \left[\frac{\beta^2}{t_{\ell_2 7 1} s_{\ell_2 7} s_{71} s_{23} s_{3\ell_1}} + \frac{\gamma^2}{t_{712} s_{71} s_{12} s_{3\ell_1} s_{\ell_1 \ell_2}} + \frac{\beta\gamma t_{\ell_1 \ell_2 7}}{s_{\ell_1\ell_2} s_{\ell_2 7} s_{71} s_{12} s_{23} s_{3\ell_1}}\right], \\ \beta &= [\ell_2 7]\langle 2\ 3\rangle\langle 1|\ell_2 + 7|\ell_1], \\ \gamma &= [\ell_1 \ell_2]\langle 1\ 2\rangle\langle 3|\ell_1 + \ell_2|7], \\ s_{ij} &= \langle i\ j\rangle[i\ j], \\ t_{ijk} &= \langle i\ j\rangle[i\ j] + \langle i\ k\rangle[i\ k] + \langle j\ k\rangle[j\ k].\end{aligned} \tag{3.8}$$

This amplitude could be written in terms of the MHV diagrams of [20]. In this case, the formula in (3.8) is simpler, but for more gluons we expect the MHV diagrams to be most efficient.

The integral (3.7) is of the form analyzed in Sec. II. Here we want to compute the action of $[F_{456}, \eta]$ to C_{456}. We can simply apply the general formula (2.20) to get the result. Note that (2.20) is the result of the action of the operator on a single pole. In the case at hand, the operator $[F_{456}, \eta]$ acts nontrivially on two poles, namely $1/\langle \ell_1 4\rangle$ and $1/\langle 6 \ell_2\rangle$. This only means that we have to apply (2.20) twice and add the results.

Consider first the action on the pole $1/\langle \ell_1 4\rangle$. We find

$$\begin{aligned}([F_{456}, \eta]C_{456})^{\text{first}} = {}& \frac{[4\ \eta](t_4^{[3]})^2}{\langle 5\ 6\rangle t_4^{[2]}}\left[\frac{\beta_1^2}{(t_5^{[2]} t_2^{[2]} - t_4^{[3]} t_2^{[3]})(t_5^{[2]} t_4^{[4]} - t_4^{[3]} t_5^{[3]}) t_7^{[2]} t_2^{[2]} t_3^{[2]}} + \frac{\gamma_1^2}{t_7^{[3]} t_7^{[2]} t_1^{[2]} t_3^{[2]} t_4^{[3]}} \right. \\ & \left. + \frac{\beta_1 \gamma_1 t_4^{[4]}}{t_4^{[3]}(t_5^{[2]} t_4^{[4]} - t_4^{[3]} t_5^{[3]}) t_7^{[2]} t_1^{[2]} t_2^{[2]} t_3^{[2]}}\right], \\ \beta_1 = {}& -\langle 2\ 3\rangle\langle 4|5 + 6|7]\langle 1|5 + 6|4], \\ \gamma_1 = {}& \langle 1\ 2\rangle\langle 3|4 + 5 + 6|7].\end{aligned} \tag{3.9}$$

We identify the four poles important to this cut as those factors in the denominator not annihilated by $[F_{456}, \eta]$. These are $t_4^{[2]}$, $t_3^{[2]}$, $(t_5^{[2]}t_4^{[4]} - t_4^{[3]}t_5^{[3]})$, $(t_5^{[2]}t_2^{[2]} - t_4^{[3]}t_2^{[3]})$, which appear, respectively (and uniquely), in the box functions $F^{1m}_{7:7}$, $F^{2m}_{7:2;5}$, $F^{2m}_{7:2;5}$, $F^{3m}_{7:2:2;5}$. These four poles are the G_k of the previous section. Now we apply our procedure to separate the cut into simple fractions. For example, to isolate the particular pole $G_0 = (t_5^{[2]}t_2^{[2]} - t_4^{[3]}t_2^{[3]})$, we evaluate

$$([F_{456}, \eta]C_{456})^{\text{first}} \times \left(\frac{t_4^{[2]}\mathcal{O}(G_0)}{H(G_0, t_4^{[2]})}\right)\left(\frac{t_3^{[2]}\mathcal{O}(G_0)}{H(G_0, t_3^{[2]})}\right) \times \left(\frac{(t_5^{[2]}t_4^{[4]} - t_4^{[3]}t_5^{[3]})\mathcal{O}(G_0)}{H[G_0, (t_5^{[2]}t_4^{[4]} - t_4^{[3]}t_5^{[3]})]}\right). \tag{3.10}$$

Perform the "polynomial division" of Sec. II on the numerator to separate the "extra" part proportional to G_0. It simplifies computations to perform the operations (3.10) on each term of (3.9) separately, for only the poles that appear in that term. As long as the procedure is consistent for all poles in each term, it is valid. After all, we are multiplying by factors that appear in pairs that sum to 1. As long as the arguments G_k of H satisfy $\mathcal{O}^2(G_k) = 0$, we can use any ones we like.

The first check that our procedure is working is that (2.15) is satisfied: the extra parts from each of the four poles sum to zero.

The remainder of (3.10) is found to be of the form

$$-c_5\frac{\mathcal{O}(G_0)}{G_0}. \tag{3.11}$$

We now have our first coefficient, c_5, and our second consistency check, because its conjugate was already computed in [3]. Indeed, our result agrees:

$$c_5 = \frac{(t_1^{[3]})^3}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 4|2+3|1]\langle 7|1+2|3]}. \tag{3.12}$$

The other three coefficients calculated from $([F_{456}, \eta]C_{456})^{\text{first}}$ are b_7, $d_{2,5}$, and g_5.

$$\begin{aligned} d_{2,5} &= \frac{\langle 1\,2\rangle^3(t_4^{[3]})^3}{\langle 4\,5\rangle\langle 5\,6\rangle\langle 7\,1\rangle t_7^{[3]}\langle 7|1+2|3]\langle 6|4+5|3](\langle 4\,2\rangle t_4^{[3]} + \langle 2\,3\rangle\langle 4|5+6|3])}, \\ g_5 &= \frac{\langle 23\rangle^3\langle 4|5+6|7]^3}{\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle[7\,1]\langle 4|2+3|1](\langle 4\,2\rangle t_4^{[3]} + \langle 2\,3\rangle\langle 4|5+6|3])(\langle 5\,6\rangle\langle 4|2+3|5] - \langle 4\,6\rangle t_2^{[3]})}. \end{aligned} \tag{3.13}$$

The expression for b_7 was found, but by itself is too complicated to write here. We will have more to say on this presently.

The action of $\mathcal{O}$ on the second pole $1/\langle 6\ell_2\rangle$ similarly yields four coefficients:

$$\begin{aligned} c_4 &= \frac{\langle 3|1+2|7]^3}{[7\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle t_7^{[3]}\langle 6|7+1|2]}, \\ d_{3,1} &= 0, \\ g_7 &= \frac{(\langle 6\,1\rangle t_4^{[3]} - \langle 7\,1\rangle\langle 6|4+5|7])^3}{[2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle\langle 6|4+5|3]\langle 6|7+1|2](\langle 4\,6\rangle t_6^{[3]} - \langle 4\,5\rangle\langle 6|7+1|5])}. \end{aligned} \tag{3.14}$$

The coefficient b_7 appears here too and agrees with the expression computed from the other term. Moreover, we can check two more relations among these coefficients. The box functions participating in this cut have some poles that do not appear in the integral. These are $(t_4^{[3]} - t_4^{[2]})$ and $(t_4^{[3]} - t_5^{[2]})$. Equation (3.6) then implies the two relations $-b_7 + c_4 + d_{3,1} + g_7 = 0$ and $-b_7 + c_5 + d_{2,5} + g_5 = 0$. We have checked that our coefficients do indeed satisfy these relations. In Sec. IV, we will use the first relation to list b_7 in terms of c_4 and g_7, but we must stress that we have computed it independently.

To summarize, the cut C_{456} involves the ten coefficients seen in (3.3). We have computed the seven that appear on the second and third lines. Two of the coefficients of the first line are known from (3.1): $c_1 = d_{2,2} = 0$. The last coefficient, $d_{3,4}$, will show up in the cut we compute next.

B. The cut C_{712}

The cut C_{712} is given by

$$\begin{aligned} C_{712} = \text{Im}|_{t_7^{[3]}>0}(&c_4F^{2m}_{7:2;4} + d_{2,5}F^{2m}_{7:2;5} + d_{3,7}F^{2m}_{7:3;7} \\ &+ b_3F^{1m}_{7:3} + c_1F^{2m}_{7:2;1} + c_7F^{2m}_{7:2;7} + d_{2,1}F^{2m}_{7:2;1} \\ &+ d_{3,4}F^{2m}_{7:3;4} + g_1F^{3m}_{7:2:2;1} + g_3F^{3m}_{7:2:2;3}). \end{aligned} \tag{3.15}$$

For this cut, there are three possible helicity assignments for ℓ_1, ℓ_2. If we denote the helicity of (ℓ_1, ℓ_2) by the assignment on the amplitude $A^{\text{tree}}(\ell_1, 7^+, 1^-, 2^-, \ell_2)$, these three cases are: (a) $(\ell_1, \ell_2) = (+, -)$; (b) $(\ell_1, \ell_2) = (+, -)$; (c) $(\ell_1, \ell_2) = (+, +)$. Notice that the assignment $(\ell_1, \ell_2) = (-, -)$ does not contribute, because the amplitude $A^{\rm tree}(\ell_1^-, 7^+, 1^-, 2^-, \ell_2^-)$ vanishes.

Now let us discuss these three assignments. For cases (a) and (b), the particle circulating in the loop can be a gluon, fermion, or complex scalar of the $\mathcal{N} = 4$ multiplet. Thus the expression will be[8]

$$\begin{aligned} C_{n12}^{(a/b)} &= \int d\mu A^{\rm tree,V}[(-\ell_1)^{\pm}, n^+, 1^-, 2^-, (-\ell_2)^{\mp}]A^{\rm tree,V}[\ell_2^{\pm}, 3^-, 4^+, \ldots, (n-1)^+, \ell_1^{\mp}] \\ &\quad + (-4)\int d\mu A^{\rm tree,F}[(-\ell_1)^{\pm}, n^+, 1^-, 2^-, (-\ell_2)^{\mp}]A^{\rm tree,F}[\ell_2^{\pm}, 3^-, 4^+, \ldots, (n-1)^+, \ell_1^{\mp}] \\ &\quad + (+3)\int d\mu A^{\rm tree,S}[(-\ell_1)^{\pm}, n^+, 1^-, 2^-, (-\ell_2)^{\mp}]A^{\rm tree,S}[\ell_2^{\pm}, 3^-, 4^+, \ldots, (n-1)^+, \ell_1^{\mp}], \end{aligned} \tag{3.16}$$

where (-4) counts the four fermions and $(+3)$ counts the three complex scalars in the $\mathcal{N} = 4$ multiplet. The supersymmetric Ward identity relates fermion and scalar MHV amplitudes to gluon MHV amplitudes by [19,21]

$$\begin{aligned} &A(F_1^-, g_2^+, \ldots, g_j^-, \ldots, F_n^+) \\ &\quad = \frac{\langle j\ n\rangle}{\langle j\ 1\rangle} A^{\rm MHV}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+), \\ &A(S_1^-, g_2^+, \ldots, g_j^-, \ldots, S_n^+) \\ &\quad = \frac{\langle j\ n\rangle^2}{\langle j\ 1\rangle^2} A^{\rm MHV}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+). \end{aligned} \tag{3.17}$$

We need to be careful about the ordering when ℓ_1, ℓ_2 are fermions. They should be ordered according to (3.16). If F^+ and F^- exchange positions in (3.17), there is an extra $(-)$ sign. Having taken care of the $\mathcal{N} = 4$ multiplet we have[9]

$$\begin{aligned} &C_{n12}^{(a)+(b)} \\ &\quad = \frac{(-)^5}{[n\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle \ldots \langle (n-2)(n-1)\rangle} \\ &\qquad \times \int d\mu \frac{\rho^2[\ell_1\ n]^2[\ell_2\ n]^2\langle 3\ \ell_1\rangle^2\langle 3\ \ell_2\rangle^2}{[\ell_1\ n][2\ \ell_2][\ell_2\ \ell_1]\langle \ell_2\ 3\rangle\langle (n-1)\ell_1\rangle\langle \ell_1\ \ell_2\rangle}, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \rho^2 &= \left(\frac{\langle 3\ \ell_2\rangle^2[\ell_2\ n]^2}{\langle 3\ \ell_1\rangle^2[\ell_1\ n]^2}\right)^2 + 4\left(\frac{\langle 3\ \ell_2\rangle^2[\ell_2\ n]^2}{\langle 3\ \ell_1\rangle^2[\ell_1\ n]^2}\right) + 6 \\ &\quad + 4\left(\frac{\langle 3\ \ell_2\rangle^2[\ell_2\ n]^2}{\langle 3\ \ell_1\rangle^2[\ell_1\ n]^2}\right)^{-1} + \left(\frac{\langle 3\ \ell_2\rangle^2[\ell_2\ n]^2}{\langle 3\ \ell_1\rangle^2[\ell_1\ n]^2}\right)^{-2} \\ &= \frac{\langle 3|(n+1+2)|n]^4}{[\ell_1\ n]^2[\ell_2\ n]^2\langle 3\ \ell_1\rangle^2\langle 3\ \ell_2\rangle^2}. \end{aligned} \tag{3.19}$$

Making the substitution for ρ^2, we get

$$\begin{aligned} C_{n12}^{(a)+(b)} &= \frac{\langle 3|(n+1+2)|n]^4}{(t_n^{[3]})^4} \\ &\quad \times \frac{(-)(-)^5(t_n^{[3]})^3}{[n\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle \ldots \langle (n-2)(n-1)\rangle} \\ &\quad \times \int d\mu \frac{1}{[\ell_1\ n][2\ \ell_2]\langle \ell_2\ 3\rangle\langle (n-1)\ell_1\rangle} \\ &= \frac{\langle 3|(n+1+2)|n]^4}{(t_n^{[3]})^4}[C_{123}^{\dagger}]|_{j\to j-1}. \end{aligned} \tag{3.20}$$

Using the result of [3] for $C_{123}^{\dagger}$, we can read out the contribution of the $(a) + (b)$ part to the following coefficients (with $n = 7$):

$$\begin{aligned} b_3^{(a)+(b)} &= c_4^{(a)+(b)} = d_{2,1}^{(a)+(b)} = d_{3,4}^{(a)+(b)} \\ &= \frac{\langle 3|(1+2)|7]^3}{(t_7^{[3]})[7\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6|7+1|2]}. \end{aligned} \tag{3.21}$$

Now we discuss the assignment (c) given by[10]

$$\begin{aligned} C_{712}^{(c)} &= \int d\mu A^{\rm tree}[(-\ell_1)^+, 7^+, 1^-, 2^-, (-\ell_2)^+] \\ &\quad \times A^{\rm tree}(4^+, 5^+, 6^+, \ell_1^-, \ell_2^-, 3^-). \end{aligned}$$

Notice that for the assignment (c), only gluons can propagate along internal lines. The first factor is again a MHV amplitude, so we can directly apply the general method of Sec. II. The second factor has the same helicity structure $(+++---)$ that we saw in the previous cut, making this computation very similar to the previous one. The collinear operator acts on two poles, namely $1/\langle 7\ell_1\rangle$ and $1/\langle 2\ell_2\rangle$. Each of the terms thus obtained involves four unique poles of the scalar box functions in (3.15). We apply the reduction procedure to produce the following coefficients (after again confirming (2.15), that all the extra pieces sum to zero).

[8]We use n for generality. In our particular example, $n = 7$.

[9]The $(-)^5$ sign comes from the left hand part since it is $\overline{\rm MHV}$. The rule to go from MHV to $\overline{\rm MHV}$ is to exchange $\langle\rangle \leftrightarrow []$ and multiply by $(-)^n$.

[10]Relative to assignments (a) and (b), there is an extra $(-)$ sign. The reason is that for the assignment (c) the left-hand side is MHV already, so we do not have the $(-)^5$ factor here.

$$
\begin{aligned}
d_{2,1}^{(c)} &= -\frac{\langle 3|4+5|6]^3\langle 1\,2\rangle^3}{\langle 7\,1\rangle\langle 3\,4\rangle\langle 4\,5\rangle t_3^{[3]}t_7^{[3]}\langle 2|7+1|6](\langle 6\,5\rangle\langle 7|1+2|6]-\langle 7\,5\rangle t_7^{[3]})},\\
g_1^{(c)} &= \frac{\langle 1\,2\rangle^3\langle 7|5+6|4]^3}{\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle[3\,4]\langle 7|1+2|3](\langle 7\,2\rangle t_5^{[3]}+\langle 2\,1\rangle\langle 7|5+6|1])(\langle 6\,5\rangle\langle 7|1+2|6]-\langle 7\,5\rangle t_7^{[3]})},\\
c_7^{(c)} &= -\frac{\langle 1\,2\rangle^3[5\,4]^3}{t_3^{[3]}\langle 6\,7\rangle\langle 7\,1\rangle[3\,4]\langle 2|3+4|5]\langle 6|4+5|3]},\\
g_3^{(c)} &= -\frac{\langle 1\,2\rangle^3\langle 2\,3\rangle^3[5\,6]^3}{\langle 7\,1\rangle\langle 3\,4\rangle\langle 2|3+4|5]\langle 2|7+1|6](\langle 7\,1\rangle\langle 2|3+4|1]-t_2^{[3]}\langle 7\,2\rangle)(t_3^{[4]}\langle 2\,4\rangle-\langle 3\,4\rangle\langle 2|7+1|3])},\\
d_{3,4}^{(c)} &= -\frac{\langle 1\,2\rangle^3(t_4^{[3]})^3}{\langle 4\,5\rangle\langle 5\,6\rangle\langle 7\,1\rangle t_7^{[3]}\langle 7|1+2|3]\langle 6|4+5|3](t_3^{[4]}\langle 2\,4\rangle-\langle 3\,4\rangle\langle 2|7+1|3])},
\end{aligned}
\tag{3.22}
$$

and a complicated expression for $b_3^{(c)}$. Here, the analog of (3.6) from the previous case is the same equation but with all indices shifted by $+3$. This is because box functions are oblivious to helicity. As before, there are two relations derived from the poles present in the box functions that do not appear in the cut integral. They are $-b_3 + c_7 + d_{3,4} = 0$ and $-b_3 + c_1 + d_{2,1} + g_1 = 0$. We have confirmed that both of these relations are satisfied.

We now have explicit expressions for nine of the ten coefficients appearing in (3.15). The seven coefficients appearing in the second and third lines have just been computed by our reduction method, and c_4 and $d_{2,5}$ were evaluated in the previous cut. (We did find a contribution to c_4 again in (3.21). But remember that the operator $[F_{712}, \eta]$ gives no information about the coefficients in the first line of (3.15), because those box functions are annihilated. Therefore $c_4^{(c)}$ is undetermined, and we must take the result for c_4 from the previous cut.) It is possible to find the single remaining coefficient, $d_{3,7}$, by imposing the finiteness of this cut. All cuts in three-particle channels are finite. This condition is discussed and derived in Appendix A.

$$
d_{3,7} = -2b_3 + 2c_7 - 2c_4 + 2d_{3,4} - d_{2,5} + 2d_{2,1} + g_3 + g_1. \tag{3.23}
$$

Incidentally, now that we have computed $d_{3,4}$ explicitly, it is possible to test the finiteness of the cut C_{456} as a consistency check. This condition, derived similarly, is

$$
0 = -b_7 - c_1 + c_4 + c_5 - \frac{1}{2}d_{2,2} + d_{2,5} + d_{3,1} - \frac{1}{2}d_{3,4} + \frac{1}{2}g_5 + \frac{1}{2}g_7. \tag{3.24}
$$

C. The cuts C_{567} and C_{234}: reflection of indices

Knowing the contributions from the cuts C_{456} and C_{712}, we can use reflection symmetry of the indices to get the contributions from cuts C_{567} and C_{234} without further calculations. Under the reflection of indices

$$
\sigma:\quad 1\leftrightarrow 3,\quad 4\leftrightarrow 7,\quad 5\leftrightarrow 6,\quad \ell_1\leftrightarrow\ell_2, \tag{3.25}
$$

every possible helicity assignment of ℓ_1, ℓ_2 of, for example, cut C_{456} is mapped to a unique corresponding helicity assignment of ℓ_1, ℓ_2 of cut C_{567} where the ordering is reversed. Recalling that the cut is given by multiplication of two tree-level amplitudes, where one has five legs and the other has six, and using the identity

$$
A_n^{\rm tree}(1, 2, \ldots, n) = (-)^n A_n^{\rm tree}(n, \ldots, 2, 1), \tag{3.26}
$$

we immediately get the following results. If the cut C_{456} is given by some function $f(1, 2, 3, 4, 5, 6, 7)$, then the cut C_{567} is given by $-f(3, 2, 1, 7, 6, 5, 4)$. Since the cut structure determines the amplitude completely, the same reflection property holds for the amplitude as well. Now, remember that the amplitude can be expanded into box functions as $\sum_j a_j F_j$, where F_j represents all the box functions. If the action of σ on indices transforms $F_k \rightarrow F_t$, we find immediately that $a_t| = -a_k|_\sigma$, where $|_\sigma$ means to act σ on the gluon labels in the function a_k. For our example, we have

$$\begin{aligned} &b_1 = -b_7|_\sigma, \quad b_2 = -b_6|_\sigma, \quad b_3 = -b_5|_\sigma, \quad b_4 = -b_4|_\sigma, \\ &c_1 = -c_2|_\sigma, \quad c_3 = -c_7|_\sigma, \quad c_4 = -c_6|_\sigma, \quad c_5 = -c_5|_\sigma, \\ &g_1 = -g_5|_\sigma, \quad g_2 = -g_4|_\sigma, \quad g_3 = -g_3|_\sigma, \quad g_6 = -g_7|_\sigma, \\ &d_{2,1} = -d_{3,6}|_\sigma, \quad d_{2,2} = -d_{3,5}|_\sigma, \quad d_{2,3} = -d_{3,4}|_\sigma, \quad d_{2,4} = -d_{3,3}|_\sigma, \\ &d_{2,5} = -d_{3,2}|_\sigma, \quad d_{2,6} = -d_{3,1}|_\sigma, \quad d_{2,7} = -d_{3,7}|_\sigma. \end{aligned} \tag{3.27}$$

Applying this transformation to the coefficients we have already computed yields expressions for the following previously undetermined coefficients:

$$b_1,\ b_5,\ c_3,\ c_6,\ d_{2,3},\ d_{2,7}d_{3,2},\ d_{3,6},\ g_3.$$

The explicit expressions are listed in Sec. IV.

D. Completion and consistency checks

At this point we have succeeded in computing 31 of the coefficients. In principle, we could compute the remaining four coefficients by applying the same general method of Sec. II to the remaining two cuts, i.e., C_{345} and C_{671}.

The four coefficients we are missing are b_2, b_6, $d_{2,4}$, and $d_{3,3}$.

From the condition that both C_{345} and C_{671} are finite, we obtain two equations:

$$\begin{aligned} -b_6 + d_{2,4} - \frac{1}{2}d_{3,3} &= -c_3 - c_4 + c_7 + d_{3,7} - \frac{1}{2}(-d_{2,1} + g_4 + g_6), \\ -b_2 - \frac{1}{2}d_{2,4} + d_{3,3} &= c_3 - c_6 - c_7 - d_{2,7} - \frac{1}{2}(-d_{3,6} + g_2 + g_7). \end{aligned} \tag{3.28}$$

Therefore we are left with the problem of determining two coefficients, say b_2 and b_6.

Before we derive the remaining coefficients, let us make some observations about the known infrared singular behavior of one-loop amplitudes [22,23]. We have already found that in the final form of the amplitude all singular terms of the form

$$-\frac{1}{\epsilon^2}(-t_i^{[3]})^{-\epsilon} \tag{3.29}$$

cancel for all $i = 1, \ldots, 7$. This is the statement that cuts in three-particle channels are finite. However, up to now we have not considered cuts in two-particle channels. It turns out that the singular behavior in these cuts is universal and produces a term in the amplitude of the form

$$A_{7:1}^{1-\mathrm{loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)|_{\mathrm{IR}} = \left[-\frac{1}{\epsilon^2}\sum_{i=1}^{7}(-t_i^{[2]})^{-\epsilon}\right]A_7^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+). \tag{3.30}$$

Note that this translates into seven equations our coefficients have to satisfy.

Taking the terms of (3.30) involving the $i = 5$ singularity, we find that our coefficients have to satisfy the following equation (see Appendix A for details of the derivation):

$$b_1 + b_7 - c_5 + \frac{1}{2}(-d_{2,5} + d_{2,7} - d_{3,2} + d_{3,7} - g_1 - g_5) = A_7^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+). \tag{3.31}$$

This equation only involves known coefficients and is therefore a consistency check.

The tree-level seven-gluon amplitude is given by [24]

$$\begin{aligned} &A^{\mathrm{tree}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) \\ &= \left[\frac{\langle 2\ 3\rangle\langle 1|6+7|5]\langle 1|2+3|4]^2}{[2\ 3]\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle t_3^{[2]}t_2^{[3]}t_6^{[3]}} - \frac{\langle 2\ 1\rangle\langle 3|5+4|6]\langle 3|2+1|7]^2}{[2\ 1]\langle 6\ 5\rangle\langle 5\ 4\rangle\langle 4\ 3\rangle t_7^{[2]}t_7^{[3]}t_3^{[3]}}\right] \\ &+ \left[\frac{[4\ 5]\langle 1\ 2\rangle\langle 3|1+2|7](\langle 5\ 6\rangle\langle 3|1+2|6] + \langle 5\ 7\rangle\langle 3|1+2|7])}{[1\ 2]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle t_3^{[2]}t_7^{[2]}t_3^{[3]}}\right. \\ &\left. - \frac{[7\ 6]\langle 3\ 2\rangle\langle 1|3+2|4](\langle 6\ 5\rangle\langle 1|3+2|5] + \langle 6\ 4\rangle\langle 1|3+2|4])}{[3\ 2]\langle 7\ 6\rangle\langle 6\ 5\rangle\langle 5\ 4\rangle t_7^{[2]}t_3^{[2]}t_6^{[3]}}\right] \\ &+ \left[\frac{\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][6\ 7][(\langle 3\ 4\rangle[6\ 4]\langle 1\ 6\rangle - \langle 1\ 7\rangle[5\ 7]\langle 3\ 5\rangle) + (\langle 3\ 4\rangle[7\ 4]\langle 1\ 7\rangle) + (\langle 1\ 6\rangle[6\ 5]\langle 3\ 5\rangle)]}{\langle 4\ 5\rangle\langle 6\ 7\rangle t_3^{[2]}t_7^{[2]}t_3^{[3]}t_6^{[3]}}\right] \\ &+ \left[\frac{\langle 1|2+3|4]\langle 3|2+1|7]t_1^{[3]}}{[1\ 2][2\ 3]t_7^{[2]}t_3^{[2]}\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle}\right]. \end{aligned} \tag{3.32}$$

With the help of a symbolic manipulation program, we have analytically verified the relation (3.31). From the form of the seven-gluon tree amplitude (3.32) it is clear that this is an impressive check of our coefficients.

Now that we have checked our previous calculations, we can use two of the equations in (3.30) that involve the unknown coefficients, i.e., b_2 and b_6, in order to find them. Take, for example, the equations derived from looking at the $i = 4$ and $i = 7$ terms in (3.30),

$$\begin{aligned} &b_6 + b_7 - c_4 - \frac{1}{2}(d_{2,4} - d_{2,6} + d_{3,1} - d_{3,6} + g_4 + g_7) \\ &\quad = A_7^{\rm tree}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+), \\ &b_2 + b_3 - c_7 + \frac{1}{2}(d_{2,2} - d_{2,7} + d_{3,2} - d_{3,4} - g_3 - g_7) \\ &\quad = A_7^{\rm tree}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+). \end{aligned} \tag{3.33}$$

These two equations give b_6 and b_2 in terms of known coefficients, respectively. They are expressed as

$$\begin{aligned} b_6 &= A_7^{\rm tree}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) - b_7 + c_4 \\ &\quad + \frac{1}{2}(d_{2,4} - d_{2,6} + d_{3,1} - d_{3,6} + g_4 + g_7), \\ b_2 &= A_7^{\rm tree}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) - b_3 + c_7 \\ &\quad - \frac{1}{2}(d_{2,2} - d_{2,7} + d_{3,2} - d_{3,4} - g_3 - g_7). \end{aligned} \tag{3.34}$$

Finally, using these expressions for b_2 and b_6 in the two equations in (3.28), we solve for $d_{2,4}$ and $d_{3,3}$ to find

$$\begin{aligned} d_{2,4} &= 2A_7^{\rm tree} - 2b_4 - 2b_5 + d_{2,2} - d_{3,4} + d_{3,6} + g_5, \\ d_{3,3} &= 2A_7^{\rm tree} - 2b_4 - 2b_3 + d_{3,5} - d_{2,3} + d_{2,1} + g_1. \end{aligned} \tag{3.35}$$

This completes the list of all 35 coefficients in the one-loop seven-gluon amplitude.

Now we use the remaining equations derived from the infrared structure (3.30) as further consistency checks of our coefficients. We successfully checked that the equations for $i = 1, 2, 3, 6$ are satisfied.

In the next section we summarize our results.

IV. THE FULL AMPLITUDE $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$

Here we summarize all results for $A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ that were scattered through the previous sections into one complete form, so that a reader interested only in results can skip all derivations. The amplitude is

$$\begin{aligned} &A_{7;1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+) \\ &\quad = \sum_{j=1}^{7} b_j F_{7:j}^{1m} + \sum_{j=1}^{7} c_j F_{7:2;j}^{2m} + \sum_{j=1}^{7} d_{2,j} F_{7:2;j}^{2m} \\ &\qquad + \sum_{j=1}^{7} d_{3,j} F_{7:3;j}^{2m} + \sum_{j=1}^{7} g_j F_{7:2:2;j}^{3m}. \end{aligned} \tag{4.1}$$

A few remarks must be made before we list the 35 coefficients. Twenty-five of them have explicit forms. Four of them (b_1, b_3, b_5, b_7) are expressed in terms of the 25 explicit ones. We stress that we calculated them independently but are abbreviating them for convenience only. The last six coefficients were derived in terms of the others in the following order: $d_{2,7}$, $d_{3,7}$, b_2, b_6, $d_{2,4}$, $d_{3,3}$.

First we recall our conventions and make a couple of convenient definitions:

$$\begin{aligned} 2p_i \cdot p_j &= \langle i\, j\rangle [i\, j], \\ t_i^{[r]} &= (p_i + p_{i+1} + \cdots + p_{i+r-1})^2, \\ \langle i | j_r + j_{r+1} + \cdots + j_s | k] &\equiv \langle i\, j_r\rangle [j_r\, k] + \langle i\, j_{r+1}\rangle [j_{r+1}\, k] + \cdots \\ &\quad + \langle i\, j_s\rangle [i\, j_s], \\ S_1 &\equiv \frac{\langle 3|1+2|7]^3}{t_7^{[3]}[7\,1][1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6|7+1|2]}, \\ S_2 &\equiv -\frac{\langle 1|3+2|4]^3}{t_2^{[3]}[4\,3][3\,2]\langle 1\,7\rangle\langle 7\,6\rangle\langle 6\,5\rangle\langle 5|4+3|2]}. \end{aligned} \tag{4.2}$$

Here is the list of the 35 coefficients.

$$\begin{aligned} &b_1 = c_6 + g_6, \qquad b_2 = A^{\rm tree} - b_3 + c_7 - \frac{1}{2}d_{2,2} + \frac{1}{2}d_{2,7} - \frac{1}{2}d_{3,2} + \frac{1}{2}d_{3,4} + \frac{1}{2}g_3 + \frac{1}{2}g_7, \\ &b_3 = g_1 + d_{2,1}, \qquad b_4 = \frac{(t_1^{[3]})^3}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 4|2+3|1]\langle 7|1+2|3]}, \qquad b_5 = g_5 + d_{3,6}, \\ &b_6 = A^{\rm tree} - b_5 + c_3 - \frac{1}{2}d_{3,5} + \frac{1}{2}d_{3,7} - \frac{1}{2}d_{2,5} + \frac{1}{2}d_{2,3} + \frac{1}{2}g_3 + \frac{1}{2}g_6, \qquad b_7 = c_4 + g_7. \end{aligned} \tag{4.3}$$

$$c_1 = 0, \qquad c_2 = 0, \qquad c_3 = \frac{\langle 2\ 3\rangle^3[6\ 7]^3}{t_6^{[3]}\langle 3\ 4\rangle\langle 4\ 5\rangle[7\ 1]\langle 2|7+1|6]\langle 5|6+7|1]}, \qquad c_4 = S_1,$$
$$c_5 = \frac{(t_1^{[3]})^2}{[1\ 2][3\ 2]\langle 4\ 5\rangle\langle 7\ 6\rangle\langle 5\ 6\rangle\langle 4|2+3|1]\langle 7|2+1|3]}, \qquad c_6 = S_2, \tag{4.4}$$
$$c_7 = -\frac{\langle 2\ 1\rangle^3[5\ 4]^3}{t_3^{[3]}\langle 1\ 7\rangle\langle 7\ 6\rangle[4\ 3]\langle 2|4+3|5]\langle 6|5+4|3]}.$$

$$d_{2,1} = S_1 - \frac{\langle 3|4+5|6]^3\langle 1\ 2\rangle^3}{\langle 7\ 1\rangle\langle 3\ 4\rangle\langle 4\ 5\rangle t_3^{[3]}t_7^{[3]}\langle 2|7+1|6](\langle 6\ 5\rangle\langle 7|1+2|6] - \langle 7\ 5\rangle t_7^{[3]})},$$
$$d_{2,2} = \frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 4|2+3|1]\langle 7|1+2|3]},$$
$$d_{2,3} = S_2 - \frac{\langle 3\ 2\rangle^3(t_5^{[3]})^3}{\langle 3\ 4\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle t_2^{[3]}\langle 4|2+3|1]\langle 5|6+7|1](t_5^{[4]}\langle 2\ 7\rangle - \langle 1\ 7\rangle\langle 2|3+4|1])}, \tag{4.5}$$
$$d_{2,4} = 2A^{\text{tree}} - 2b_4 - 2b_5 + d_{2,2} - d_{3,4} + d_{3,6} + g_5,$$
$$d_{2,5} = \frac{\langle 1\ 2\rangle^3(t_4^{[3]})^3}{\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle t_7^{[3]}\langle 7|1+2|3]\langle 6|4+5|3](\langle 4\ 2\rangle t_4^{[3]} + \langle 2\ 3\rangle\langle 4|5+6|3])}, \qquad d_{2,6} = 0,$$
$$d_{2,7} = -2b_5 + 2c_3 - 2c_6 + 2d_{2,3} - d_{3,2} + 2d_{3,6} + g_3 + g_5.$$

$$d_{3,1} = 0, \qquad d_{3,2} = -\frac{\langle 3\ 2\rangle^3(t_5^{[3]})^3}{\langle 7\ 6\rangle\langle 6\ 5\rangle\langle 4\ 3\rangle t_2^{[3]}\langle 4|3+2|1]\langle 5|7+6|1](\langle 7\ 2\rangle t_5^{[3]} + \langle 2\ 1\rangle\langle 7|6+5|1])},$$
$$d_{3,3} = 2A^{\text{tree}} - 2b_4 - 2b_3 + d_{3,5} - d_{2,3} + d_{2,1} + g_1,$$
$$d_{3,4} = S_1 - \frac{\langle 1\ 2\rangle^3(t_4^{[3]})^3}{\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 7\ 1\rangle t_7^{[3]}\langle 7|1+2|3]\langle 6|4+5|3](t_3^{[4]}\langle 2\ 4\rangle - \langle 3\ 4\rangle\langle 2|7+1|3])},$$
$$d_{3,5} = \frac{(t_1^{[3]})^3}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 4|2+3|1]\langle 7|1+2|3]}, \tag{4.6}$$
$$d_{3,6} = S_2 + \frac{\langle 1|7+6|5]^3\langle 3\ 2\rangle^3}{\langle 4\ 3\rangle\langle 1\ 7\rangle\langle 7\ 6\rangle t_6^{[3]}t_2^{[3]}\langle 2|4+3|5](\langle 5\ 6\rangle\langle 4|3+2|5] - \langle 4\ 6\rangle t_2^{[3]})},$$
$$d_{3,7} = -2b_3 + 2c_7 - 2c_4 + 2d_{3,4} - d_{2,5} + 2d_{2,1} + g_3 + g_1.$$

$$g_1 = \frac{\langle 1\ 2\rangle^3\langle 7|5+6|4]^3}{\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 1\rangle[3\ 4]\langle 7|1+2|3](\langle 7\ 2\rangle t_5^{[3]} + \langle 2\ 1\rangle\langle 7|5+6|1])(\langle 6\ 5\rangle\langle 7|1+2|6] - \langle 7\ 5\rangle t_7^{[3]})}, \qquad g_2 = 0,$$
$$g_3 = -\frac{\langle 1\ 2\rangle^3\langle 2\ 3\rangle^3[5\ 6]^3}{\langle 7\ 1\rangle\langle 3\ 4\rangle\langle 2|3+4|5]\langle 2|7+1|6](\langle 7\ 1\rangle\langle 2|3+4|1] - t_2^{[3]}\langle 7\ 2\rangle)(t_3^{[4]}\langle 2\ 4\rangle - \langle 3\ 4\rangle\langle 2|7+1|3])}, \qquad g_4 = 0,$$
$$g_5 = -\frac{\langle 3\ 2\rangle^3\langle 4|6+5|7]^3}{\langle 6\ 5\rangle\langle 5\ 4\rangle\langle 4\ 3\rangle[1\ 7]\langle 4|3+2|1](\langle 4\ 2\rangle t_4^{[3]} + \langle 2\ 3\rangle\langle 4|6+5|3])(\langle 5\ 6\rangle\langle 4|3+2|5] - \langle 4\ 6\rangle t_2^{[3]})}, \tag{4.7}$$
$$g_6 = \frac{(\langle 5\ 3\rangle t_5^{[3]} - \langle 4\ 3\rangle\langle 5|6+7|4])^3}{[1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 5|6+7|1]\langle 5|3+4|2](\langle 7\ 5\rangle t_3^{[3]} - \langle 7\ 6\rangle\langle 5|3+4|6])},$$
$$g_7 = -\frac{(\langle 6\ 1\rangle t_4^{[3]} - \langle 7\ 1\rangle\langle 6|5+4|7])^3}{[3\ 2]\langle 1\ 7\rangle\langle 7\ 6\rangle\langle 6\ 5\rangle\langle 5\ 4\rangle\langle 6|5+4|3]\langle 6|1+7|2](\langle 4\ 6\rangle t_6^{[3]} - \langle 4\ 5\rangle\langle 6|1+7|5])}.$$

We repeat here the tree-level amplitude [24] for the reader's convenience.

$$\begin{aligned}
&A^{\text{tree}}(1^-,2^-,3^-,4^+,5^+,6^+,7^+)\\
&=\left[\frac{\langle 2\,3\rangle\langle 1|6+7|5]\langle 1|2+3|4]^2}{[2\,3]\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle t_3^{[2]}t_2^{[3]}t_6^{[3]}}-\frac{\langle 2\,1\rangle\langle 3|5+4|6]\langle 3|2+1|7]^2}{[2\,1]\langle 6\,5\rangle\langle 5\,4\rangle\langle 4\,3\rangle t_7^{[2]}t_7^{[3]}t_3^{[3]}}\right]\\
&+\left[\frac{[4\,5]\langle 1\,2\rangle\langle 3|1+2|7](\langle 5\,6\rangle\langle 3|1+2|6]+\langle 5\,7\rangle\langle 3|1+2|7])}{[1\,2]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle t_3^{[2]}t_7^{[2]}t_3^{[3]}}\right.\\
&\left.-\frac{[7\,6]\langle 3\,2\rangle\langle 1|3+2|4](\langle 6\,5\rangle\langle 1|3+2|5]+\langle 6\,4\rangle\langle 1|3+2|4])}{[3\,2]\langle 7\,6\rangle\langle 6\,5\rangle\langle 5\,4\rangle t_7^{[2]}t_3^{[2]}t_6^{[3]}}\right]\\
&+\left[\frac{\langle 1\,2\rangle\langle 2\,3\rangle[4\,5][6\,7][(\langle 3\,4\rangle[6\,4]\langle 1\,6\rangle-\langle 1\,7\rangle[5\,7]\langle 3\,5\rangle)+(\langle 3\,4\rangle[7\,4]\langle 1\,7\rangle)+(\langle 1\,6\rangle[6\,5]\langle 3\,5\rangle)]}{\langle 4\,5\rangle\langle 6\,7\rangle t_3^{[2]}t_7^{[2]}t_3^{[3]}t_6^{[3]}}\right]\\
&+\left[\frac{\langle 1|2+3|4]\langle 3|2+1|7]t_1^{[3]}}{[1\,2][2\,3]t_7^{[2]}t_3^{[2]}\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle}\right]. \qquad (4.8)
\end{aligned}$$

We have written the tree amplitude so that every bracketed expression changes sign under the index shift $1\leftrightarrow 3$, $4\leftrightarrow 7$, $5\leftrightarrow 6$. This is the reflection symmetry made manifest.

ACKNOWLEDGMENTS

We thank O. Lunin and P. Svrček for helpful discussions and D. Kosower for a question prompting us to add Appendix B. R. B. and B. F. were supported by NSF Grant No. PHY-0070928. F. C. was supported in part by the Martin A. and Helen Chooljian Membership at the Institute for Advanced Study and by DOE Grant No. DE-FG02-90ER40542.

Note added.—The reader will be interested to know that the seven-gluon amplitude with the helicity configuration $(---++++)$ has now also been computed in [15], along with all other helicity configurations, using the direct unitarity method. (Please be warned that the first version of our paper contained a typo in the coefficient $d_{3,4}$ and a corresponding typo in $d_{2,3}$, which was obtained by a permutation of labels.) It is interesting to note that, according to [15], the reduction techniques of the direct unitarity method give ‘‘quite large’’ formulas for the coefficients. One advantage of our method is that we derive the coefficients analytically in a simple form. The authors of [15] were able to produce similarly simple formulas by postulating Ansätze that were checked numerically at random kinematic points.

APPENDIX A: BOX FUNCTIONS AND DIVERGENCE ANALYSIS

The scalar box functions used in this paper are the following:

$$\begin{aligned}
F^{1m}_{n:i} &= -\frac{1}{\epsilon^2}[(-t_{i-3}^{[2]})^{-\epsilon}+(-t_{i-2}^{[2]})^{-\epsilon}-(-t_{i-3}^{[3]})^{-\epsilon}]\\
&\quad+\mathrm{Li}_2\left(1-\frac{t_{i-3}^{[3]}}{t_{i-3}^{[2]}}\right)+\mathrm{Li}_2\left(1-\frac{t_{i-3}^{[3]}}{t_{i-2}^{[2]}}\right)+\frac{1}{2}\ln^2\left(\frac{t_{i-3}^{[2]}}{t_{i-2}^{[2]}}\right)\\
&\quad+\frac{\pi^2}{6}, \qquad (A1)
\end{aligned}$$

$$\begin{aligned}
F^{2m}_{n:r;i} &= -\frac{1}{\epsilon^2}[(-t_{i-1}^{[r+1]})^{-\epsilon}+(-t_{i}^{[r+1]})^{-\epsilon}-(-t_{i}^{[r]})^{-\epsilon}\\
&\quad-(-t_{i-1}^{[r+2]})^{-\epsilon}]+\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right)\\
&\quad+\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_i^{[r+1]}}\right)+\mathrm{Li}_2\left(1-\frac{t_{i-1}^{[r+2]}}{t_{i-1}^{[r+1]}}\right)\\
&\quad+\mathrm{Li}_2\left(1-\frac{t_{i-1}^{[r+2]}}{t_i^{[r+1]}}\right)-\mathrm{Li}_2\left(1-\frac{t_i^{[r]}t_{i-1}^{[r+2]}}{t_{i-1}^{[r+1]}t_i^{[r+1]}}\right)\\
&\quad+\frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right), \qquad (A2)
\end{aligned}$$

$$\begin{aligned}
F^{2m}_{n:r;i} &= -\frac{1}{\epsilon^2}[(-t_{i-2}^{[2]})^{-\epsilon}+(-t_{i-1}^{[r+1]})^{-\epsilon}-(-t_{i}^{[r]})^{-\epsilon}\\
&\quad-(-t_{i-2}^{[r+2]})^{-\epsilon}]-\frac{1}{2\epsilon^2}\frac{(-t_i^{[r]})^{-\epsilon}(-t_{i-2}^{[r+2]})^{-\epsilon}}{(-t_{i-2}^{[2]})^{-\epsilon}}\\
&\quad+\frac{1}{2}\ln^2\left(\frac{t_{i-2}^{[2]}}{t_{i-1}^{[r+1]}}\right)+\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right)\\
&\quad+\mathrm{Li}_2\left(1-\frac{t_{i-2}^{[r+2]}}{t_{i-1}^{[r+1]}}\right), \qquad (A3)
\end{aligned}$$

$$F^{3m}_{n:r:r';i} = -\frac{1}{\epsilon^2}[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+r']})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[r']})^{-\epsilon} - (-t_{i-1}^{[r+r'+1]})^{-\epsilon}] - \frac{1}{2\epsilon^2}\frac{(-t_i^{[r]})^{-\epsilon}(-t_{i+r}^{[r']})^{-\epsilon}}{(-t_i^{[r+r']})^{-\epsilon}}$$
$$- \frac{1}{2\epsilon^2}\frac{(-t_{i+r}^{[r']})^{-\epsilon}(-t_{i-1}^{[r+r'+1]})^{-\epsilon}}{(-t_{i-1}^{[r+1]})^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+r']}}\right) + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i-1}^{[r+r'+1]}}{t_i^{[r+r']}}\right) - \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}t_{i-1}^{[r+r'+1]}}{t_{i-1}^{[r+1]}t_i^{[r+r']}}\right), \tag{A4}$$

The dilogarithm function is defined by $\mathrm{Li}_2(x) = -\int_0^x \ln(1-z)dz/z$. Now we specialize to seven gluons and discuss the infrared singular structure of the one-loop amplitude. Recall that the seven-gluon amplitude is written as a sum of scalar box functions as in (3.1). The box functions contain divergences when $\epsilon \to 0$ of the form

$$\frac{1}{\epsilon^2}(-t_i^{[2]})^{-\epsilon}, \qquad \frac{1}{\epsilon^2}(-t_i^{[3]})^{-\epsilon}, \tag{A5}$$

remembering that $t_i^{[r]} = t_{i+r}^{[7-r]}$ for seven gluons, by momentum conservation.

Now it is clear that the divergent structure of the seven-gluon amplitude takes the form

$$A_{7:1}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}\sum_{i=1}^{7}[\alpha_i(-t_i^{[2]})^{-\epsilon} + \beta_i(-t_i^{[3]})^{-\epsilon}], \tag{A6}$$

where α_i and β_i are linear combinations of the coefficients in (3.1). The α_i and β_i appear in the body of the paper. Here we describe how to compute them from the box functions, taking α_5 as an example.

The infrared behavior of the box functions contributing to α_5 are as follows.

$$\begin{aligned}
&F^{1m}_{7:1}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}[(-t_5^{[2]})^{-\epsilon} + (-t_6^{[2]})^{-\epsilon} - (-t_5^{[3]})^{-\epsilon}], \qquad F^{1m}_{7:7}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}[(-t_4^{[2]})^{-\epsilon} + (-t_5^{[2]})^{-\epsilon} - (-t_4^{[3]})^{-\epsilon}],\\
&F^{2m}_{7:2;5}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}[(-t_4^{[3]})^{-\epsilon} + (-t_5^{[3]})^{-\epsilon} - (-t_5^{[2]})^{-\epsilon} - (-t_1^{[3]})^{-\epsilon}],\\
&F^{2mh}_{7:2;5}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_3^{[2]})^{-\epsilon} + (-t_4^{[3]})^{-\epsilon} - \frac{1}{2}(-t_5^{[2]})^{-\epsilon} - \frac{1}{2}(-t_7^{[3]})^{-\epsilon}\right],\\
&F^{2m\,h}_{7:2;7}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_5^{[2]})^{-\epsilon} + (-t_6^{[3]})^{-\epsilon} - \frac{1}{2}(-t_7^{[2]})^{-\epsilon} - \frac{1}{2}(-t_2^{[3]})^{-\epsilon}\right],\\
&F^{3m}_{7:2:2;1}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_7^{[3]})^{-\epsilon} + \frac{1}{2}(-t_5^{[3]})^{-\epsilon} - \frac{1}{2}(-t_1^{[2]})^{-\epsilon} - \frac{1}{2}(-t_5^{[2]})^{-\epsilon}\right],\\
&F^{3m}_{7:2:2;5}|_{\mathrm{IR}} = -\frac{1}{\epsilon^2}\left[\frac{1}{2}(-t_4^{[3]})^{-\epsilon} + \frac{1}{2}(-t_2^{[3]})^{-\epsilon} - \frac{1}{2}(-t_5^{[2]})^{-\epsilon} - \frac{1}{2}(-t_2^{[2]})^{-\epsilon}\right].
\end{aligned} \tag{A7}$$

Collecting all the terms with $t_5^{[2]}$, we find that

$$\alpha_5 = b_1 + b_7 - c_5 - \frac{1}{2}d_{2,5} + \frac{1}{2}d_{2,7} - \frac{1}{2}d_{3,2} + \frac{1}{2}d_{3,7} - \frac{1}{2}g_1 - \frac{1}{2}g_5. \tag{A8}$$

Similar calculations give expressions for the β_i.

APPENDIX B: CONSTRUCTING A GENERAL NEXT-TO-MHV AMPLITUDE

Here we flesh out the claim that our method can compute all next-to-MHV amplitudes (i.e., those with exactly three negative helicities, in arbitrary positions). In Sec. II we already argued that all next-to-MHV amplitudes have the property, required for our method, that one of the tree-level amplitude factors in the cut integral (2.2) is MHV. But to calculate the amplitude, we must be sure that we can determine each coefficient in (2.1) from one of these cuts.

To see that this is correct, consider the scalar box function associated with each coefficient. To be able to determine the coefficient by our method, the box function must have the property that it appears in some cut $C_{i,i+1,\ldots,j}$, where the amplitude $A^{\mathrm{tree}}[(-\ell_1), i, i+1, \ldots, j-1, j, (-\ell_2)]$ is MHV, but is not annihilated by all operators $[F_{klm}, \eta]$ where $i \leq k, l, m \leq j$. (The operator $[F_{klm}, \eta]$ annihilates box functions where gluons k, l, m are attached to the same corner of the box.)

One-mass scalar box functions appear in only one cut (disregarding cuts in two-particle channels). See Fig. 2. The cut has three gluons, say k, l, m, on one side. Since the tree-level amplitude on that side has five particles, it is MHV (unless it vanishes). The box function appears in the cut C_{klm} and is not annihilated by the operator $[F_{klm}, \eta]$, so

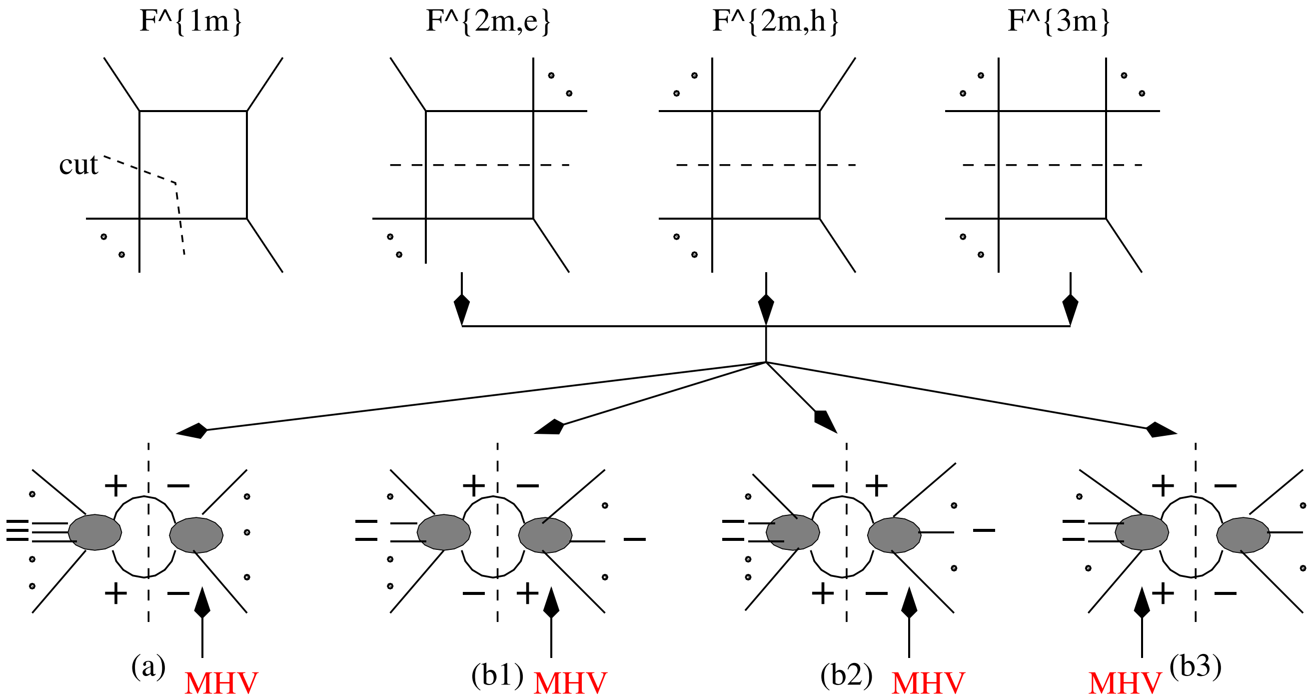

FIG. 2 (color online). Cuts of a general next-to-MHV amplitude. The first row shows cuts, for each type of scalar box function, that are guaranteed to have at least three gluons on each side. The second row illustrates, for two-mass and three-mass box functions, how to identify one side or the other as a MHV tree amplitude, so that a suitable operator can be chosen to calculate the coefficient.

the coefficient can be calculated by this operator acting on this cut.

The remaining scalar box functions can be analyzed as a group. For this general analysis, we should consider the cuts indicated in Fig. 2, to be sure that there are at least three gluons on each side. There are two cases. Case (a): If all three of the negative-helicity gluons appear on the same side of the cut, then the opposite side must be MHV (or vanish). We can choose k, l, m from that side such that they are not all on the same corner of the box. Case (b): If there are two negative-helicity gluons on one side of the cut, and one on the other, then the sides will alternately be MHV, depending on the helicity assignments of the cut propagators. In any case it is possible to choose three gluons k, l, m from the MHV side that are not all on the same corner of the box. The operator $[F_{klm}, \eta]$ then can be used to analyze the cut in question without annihilating the scalar box function. The three cases (b1), (b2), (b3) would suggest using two separate operators, depending on which side of the cut is MHV. In fact, a single one of them will suffice to determine the coefficient of a particular box function.

[1] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[2] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[3] F. Cachazo, hep-th/0410077.
[4] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952); G. Passarino and M. Veltman, Nucl. Phys. **B160**, 151 (1979); G. 't Hooft and M. Veltman, Nucl. Phys. **B153**, 365 (1979); R. G. Stuart, Comput. Phys. Commun. **48**, 367 (1988); R. G. Stuart and A. Gongora, Comput. Phys. Commun. **56**, 337 (1990).
[5] W. van Neerven and J. A. M. Vermaseren, Phys. Lett. **137B**, 241 (1984).
[6] D. B. Melrose, Nuovo Cimento A **40**, 181 (1965); G. J. van Oldenborgh and J. A. M. Vermaseren, Z. Phys. C **46**, 425 (1990); G. J. van Oldenborgh, PhD thesis, University of Amsterdam, 1990; A. Aeppli, PhD thesis, University of Zurich, 1992.
[7] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981); P. De Causmaeker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. **B206**, 53 (1982); R. Kleiss and W. J. Stirling, Nucl. Phys. **B262**, 235 (1985); R. Gastmans and T. T. Wu, *The Ubiquitous Photon: Heliclity Method for QED and QCD* (Clarendon Press, Oxford, 1990).

[8] Z. Xu, D.-H. Zhang, and L. Chang, Nucl. Phys. **B291**, 392 (1987).
[9] J. F. Gunion and Z. Kunszt, Phys. Lett. **161B**, 333 (1985).
[10] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B412**, 751 (1994).
[11] E. Witten, Commun. Math. Phys.**252**, 189 (2004).
[12] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 10 (2004) 077.
[13] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 10 (2004) 074.
[14] A. Brandhuber, B. Spence, and G. Travaglini, hep-th/0407214.
[15] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, hep-th/0410224.
[16] S. Parke and T. Taylor, Phys. Rev. Lett. **56**, 2459 (1986); F. A. Berends and W. T. Giele, Nucl. Phys. **B306**, 759 (1988).
[17] I. Bena, Z. Bern, D. A. Kosower, and R. Roiban, hep-th/0410054.
[18] M. Mangano, S. J. Parke, and Z. Xu, Nucl. Phys. **B298**, 653 (1988).
[19] M. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991).
[20] F. Cachazo, P. Svrček, and E. Witten, J. High Energy Phys. 09 (2004) 006.
[21] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 996 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).
[22] W. T. Giele and E. W. N. Glover, Phys. Rev. D **46**, 1980 (1992); W. T. Giele, E. W. N. Glover, and D. A. Kosower, Nucl. Phys. **B403**, 633 (1993).
[23] Z. Kunszt and D. Soper, Phys. Rev. D **46**, 192 (1992); Z. Kunszt, A. Signer, and Z. Trócsányi, Nucl. Phys. **B420**, 550 (1994).
[24] F. A. Berends, W. T. Giele, and H. Kuijf, Nucl. Phys. **B333**, 120 (1990).