

**Multigluon amplitudes,  $\mathcal{N} = 4$  constraints, and the Wess-Zumino-Witten model**Yasuhiro Abe,<sup>1,\*</sup> V. P. Nair,<sup>1,†</sup> and Mu-In Park<sup>2,‡</sup><sup>1</sup>*Physics Department, City College of the CUNY, New York, New York 10031, USA*<sup>2</sup>*Department of Physics, POSTECH, Pohang 790-784, Korea*

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Classical  $\mathcal{N} = 4$  Yang-Mills theory is defined by the superspace constraints. We obtain a solution of a subset of these constraints and show that it leads to the maximally helicity violating amplitudes. The action which leads to the solvable part of the constraints is a Wess-Zumino-Witten action on a suitably *extended* superspace. The non-maximally helicity violating tree amplitudes can also be expressed in terms of this action.

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**I. INTRODUCTION**

The construction of multigluon scattering amplitudes in the  $\mathcal{N} = 4$  super Yang-Mills theory has attracted a lot of attention recently. The calculation of some of these amplitudes, particularly the so-called maximally helicity violating (MHV) ones, was carried out quite some time ago [1]. Although the intermediate steps of the calculation were algebraically very complex, the final results were surprisingly simple. It was pointed out shortly afterward that the MHV amplitudes could be obtained in terms of the current correlators of a Wess-Zumino-Witten (WZW) theory and that there was a natural interpretation of this in super-twistor space [2]. (For some further developments along this direction, see [3]. For a discussion of twistor space, see [4]; for supertwistor space, see [5].) Recently, Witten showed that there is a deep connection of these results to string theory [6]. First of all, the supertwistor space  $\mathbf{CP}^{3|4}$ , as a supermanifold, is a Calabi-Yau space, so that it is possible to have a string theory with this target space. A topological version of such a string theory, the so-called topological  $B$ -model, can be constructed. The MHV amplitude is the restriction of a holomorphic function in  $\mathbf{CP}^{3|4}$  to a complex line. This complex line can be interpreted as a  $D$ -instanton in the string theory. The correlators of the  $B$ -model on this line become WZW correlators, reproducing the MHV amplitudes. One of the key observations in [6] was that the non-MHV amplitudes can be obtained as the correlators of the  $B$ -model restricted to algebraic curves of higher degree in  $\mathbf{CP}^{3|4}$ . This seems to be true by direct verification of many amplitudes [7]. Later, it was realized that one could perhaps simplify even more [8]. The amplitudes can be constructed by considering the MHV amplitudes, with a suitable off shell continuation, as the basic vertices. By connecting such vertices via propagators, one can obtain all the gauge theory amplitudes. This too seems to be born out by explicit calcula-

tions carried out so far [9]. It is a remarkable result, with all the tree amplitudes of the gauge theory obtained by a simple set of rules in twistor language.

An alternative string theory which leads to the same amplitudes has been proposed by Berkovits [10]. A number of other related works, including ramifications of these results in string theory are given in [11].

While these are remarkable developments, in this paper, we go back to the well-known formulation of supersymmetric gauge theories in terms of gauge potentials in superspace. Generally, such gauge potentials contain too many degrees of freedom, more than what is needed for the physical fields. One can then impose a set of constraints obeyed by the field strengths in superspace; these constraints can be solved in terms of some unconstrained fields and the latter can be used for the construction of the action for the theory. However, in the case of the  $\mathcal{N} = 4$  Yang-Mills theory, the constraints are too stringent and, in fact, imply the equations of motion via the Bianchi identities [12]. For the purpose of constructing an action with manifest  $\mathcal{N} = 4$  supersymmetry, this is bad news since we do not have fields which are off the mass shell. However, the good news is that this property shows that the classical equations of motion are equivalent to a set of first order equations in the appropriate superspace. One could then hope, in a way similar to the strategy for solving the first order self-duality (instanton) equations, that the constraints of the  $\mathcal{N} = 4$  can be solved. This is precisely what we attempt to do in this paper. Our approach has similarities to the use of the holomorphic Chern-Simons theory [6,13]. We introduce auxiliary bosonic variables to enlarge the space on which the constraints are written. A “gauge transformation” in this enlarged space is then made to eliminate some of the gauge potentials. The version of the constraints in the new gauge are then solved with one additional simplifying assumption. This leads to the formula for the MHV amplitudes. The suggestion made in [8] can be incorporated in this language rather neatly.

In the next section, we set up the connection between the MHV amplitudes and the WZW action. Section III is devoted to the solution of the constraints of the  $\mathcal{N} = 4$

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theory and the resulting  $S$  matrix. In Sec. IV we show how the non-MHV amplitudes, along the lines of [8], can be phrased in our language.

## II. MULTIGLUON AMPLITUDES

We start our discussion by writing the WZW action in a form suitable for our purpose [14]. As is well known, the WZW action is related to the chiral Dirac determinant and so, taking  $A_{\bar{z}}(z, \bar{z})$ 's in the fundamental representation of  $SU(N)$ , we can write

$$\begin{aligned} S_{\text{WZW}}(M^\dagger) &= \text{Tr} \log D_{\bar{z}} - \text{Tr} \log \partial_{\bar{z}} \\ &= \text{Tr} \log [1 + (\partial_{\bar{z}})^{-1} A_{\bar{z}}] \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \\ &\quad \times \int \frac{d^2 z_1}{\pi} \dots \frac{d^2 z_n}{\pi} \text{Tr} \left[ \frac{A_{\bar{z}}(1) A_{\bar{z}}(2) \dots A_{\bar{z}}(n)}{z_{12} z_{23} \dots z_{n1}} \right], \end{aligned} \quad (1)$$

where  $A_{\bar{z}} = M^{\dagger-1} \partial_{\bar{z}} M^\dagger$ ,  $D_{\bar{z}}$  is the covariant derivative  $\partial_{\bar{z}} + A_{\bar{z}}$ , and we have used the fact that the inverse of  $\partial_{\bar{z}}$  is given by  $[\pi(z - z')]^{-1}$  and  $z_{mn} = z_m - z_n$ . Also  $A_{\bar{z}}(n)$  denotes  $A_{\bar{z}}(z_n, \bar{z}_n)$  and  $d^2 z$  is the real two-dimensional volume element, equal to  $dz d\bar{z}/(-2i)$ , in the complex coordinates  $z, \bar{z}$  for the Riemann surface.

The derivative of the action with respect to  $A_{\bar{z}}$  defines the expectation value of the fermion current  $J$  which minimally couples to  $A_{\bar{z}}$ ; we can, therefore, express the above equation as a series of current correlators,

$$\begin{aligned} \langle J^{a_1}(1) J^{a_2}(2) \dots J^{a_n}(n) \rangle &= \frac{(-1)^{n+1}}{n \pi^n} \left[ \frac{\text{Tr}(t^{a_1} t^{a_2} \dots t^{a_n})}{z_{12} z_{23} \dots z_{n1}} \right. \\ &\quad \left. + \text{permutations} \right]. \end{aligned} \quad (2)$$

We now introduce a spinor variable  $u^A$ ,  $A = 1, 2$ ,

$$u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (3)$$

The complex projective space  $\mathbf{CP}^1$  is defined by making the identification  $u \sim \lambda u$ , for any complex number  $\lambda$  which is not zero,  $\lambda \in \mathbf{C} - \{0\}$ . This reduces the space to one complex dimension. Utilizing this identification, we can take  $\beta/\alpha = z$  as the local complex coordinate of  $\mathbf{CP}^1$  except in the neighborhood of  $\alpha = 0$ ; a convenient normalization is to take  $\bar{\alpha}\alpha = (1 + z\bar{z})^{-1}$ . (Near  $\alpha = 0$ , we can use  $\alpha/\beta$  as the local coordinate.)

There is a natural  $SL(2, \mathbf{C})$  action on  $u$  given by  $u \rightarrow gu$ ,  $g \in SL(2, \mathbf{C})$ . The scalar product of two  $u$ 's by  $(u_1 u_2) = \epsilon_{AB} u_1^A u_2^B$ , where  $u^1 = \alpha$ ,  $u^2 = \beta$ . (The lower index numbers represent the numberings of  $u$ 's and the upper ones represent their components.) This scalar product is invariant under the  $SL(2, \mathbf{C})$  action. The current correlators (2) may be written for  $\mathbf{CP}^1$  by writing  $z_{12} = -(\alpha_1 \beta_2 -$

$\alpha_2 \beta_1)/\alpha_1 \alpha_2 = -(u_1 u_2)/\alpha_1 \alpha_2$ . Introducing  $J$  by  $\alpha^2 J = J$ , we find

$$\begin{aligned} \langle J^{a_1}(1) \dots J^{a_n}(n) \rangle &= -\frac{1}{n \pi^n} \left[ \frac{\text{Tr}(t^{a_1} \dots t^{a_n})}{(u_1 u_2)(u_2 u_3) \dots (u_n u_1)} \right. \\ &\quad \left. + \text{permutations} \right]. \end{aligned} \quad (4)$$

We can take this expression as something globally valid on  $\mathbf{CP}^1$ , (2) being the local version valid in a neighborhood which does not include  $\alpha = 0$ .

We also note that the variation of the WZW action can be written as

$$\begin{aligned} \delta S_{\text{WZW}} &= -\frac{1}{\pi} \int d^2 z \text{Tr} [M^{\dagger-1} \partial_{\bar{z}} M^\dagger \delta(M^{\dagger-1} \delta M^\dagger)] \\ &= \frac{1}{\pi} \int \text{Tr} (D_{\bar{z}} \mathcal{A}_z M^{\dagger-1} \delta M^\dagger) \\ &= -\frac{1}{\pi} \int \text{Tr} [\mathcal{A}_z D_{\bar{z}} (M^{\dagger-1} \delta M^\dagger)] \\ &= -\frac{1}{\pi} \int \text{Tr} (\mathcal{A}_z \delta A_{\bar{z}}), \end{aligned} \quad (5)$$

where  $D_{\bar{z}}$  is the covariant derivative in the adjoint representation,  $D_{\bar{z}} \mathcal{A}_z = \partial_{\bar{z}} \mathcal{A}_z + [A_{\bar{z}}, \mathcal{A}_z]$ .  $\mathcal{A}_z$  is defined by

$$\mathcal{A}_z = M^{\dagger-1} \partial_z M^\dagger. \quad (6)$$

Notice that this obeys the equation

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} \mathcal{A}_z + [\mathcal{A}_z, A_{\bar{z}}] = 0. \quad (7)$$

Putting these considerations aside for the moment and turning to the Yang-Mills theory, the maximally helicity violating tree amplitudes correspond to the scattering of  $n - 2$  gluons of negative helicity and two gluons of positive helicity (or the other way) and are given by [1]

$$\begin{aligned} \mathcal{A}(+ + - - \dots -) &= i g^{n-2} (u_I u_J)^4 \\ &\quad \times \frac{\text{Tr}(t^{a_1} t^{a_2} \dots t^{a_n})}{(u_1 u_2)(u_2 u_3) \dots (u_n u_1)}, \end{aligned} \quad (8)$$

where  $g$  is the coupling constant. The gluons are all massless described by null momenta  $p_\mu$  with  $p^2 = 0$ .  $u$ 's are the spinor momenta of particles given by  $p_{A\dot{A}} = p_\mu (\sigma^\mu)_{A\dot{A}} = u_A \bar{u}_{\dot{A}}$ , where  $\sigma^\mu = (1, \sigma)$  and  $\sigma$  are Pauli matrices. The labels  $I$  and  $J$  refer to the positive helicity gluons. For simplicity of presentation, all gluons are taken as incoming. The expression  $\mathcal{A}$  in (8) is actually a subamplitude; the full amplitude is obtained by summing over such subamplitudes with all noncyclic permutations. This subamplitude has cyclic symmetry, so we also can sum over all permutations and divide by  $n$ . There is also a momentum conservation  $\delta$ -function which we have not displayed.

The MHV amplitude (8), (with momentum conservation inserted), can now be written as

$$\mathcal{A}(u, \bar{u}) = \int \prod_n d^2 v_n e^{(i/2)v_n \cdot \bar{u}_n} \tilde{\mathcal{A}}(u, v),$$

$$\tilde{\mathcal{A}}(u, v) = \int d^4 x \prod_n \delta(v_{\dot{A}n} - x_{AA} u_n^A) i g^{n-2} (u_I u_J)^4 \quad (9)$$

$$\times \frac{\text{Tr}(t^{a_1} t^{a_2} \dots t^{a_n})}{(u_1 u_2)(u_2 u_3) \dots (u_n u_1)}.$$

The Fourier-transformed amplitude  $\tilde{\mathcal{A}}$  is holomorphic in the twistor variable  $Z_\alpha = (v_{\dot{A}}, u_A)$ . The  $\delta$ -functions in  $\tilde{\mathcal{A}}$  show that it has support at various points  $u_n^A$  (and corresponding  $v$ 's) on a line  $v_{\dot{A}} = x_{AA} u^A$ . This is a complex line in the space of  $Z$ 's,  $x_{AA}$  specifying the choice of this line. Equation (9) was the form used in [6] to relate these amplitudes to the topological  $B$ -model.

The generator of Lorentz transformations for the  $u$ 's is given by

$$J_{AB} = \frac{1}{2} \left( u_A \frac{\partial}{\partial u^B} + u_B \frac{\partial}{\partial u^A} \right), \quad (10)$$

where  $u_A = \epsilon_{AB} u^B$ . The spin operator is given by  $S_\mu = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\nu\alpha} p^\beta$ , where  $J^{\mu\nu}$  is the full Lorentz generator. This works out to  $S_{AA} = J_{AB} u^B \bar{u}^{\dot{A}} = s p_{AA}$  identifying the helicity as

$$s = \frac{1}{2} u^A \frac{\partial}{\partial u^A}. \quad (11)$$

Thus  $s$  is half the degree of homogeneity in the  $u$ 's.

One of the basic observations made in [2] was that the subamplitude could be written as

$$\mathcal{A}(1, 2, \dots, n) = i g^{n-2} \int d^4 x d^2 \theta_1 d^2 \theta_2 d^2 \theta_3 d^2 \theta_4$$

$$\times \prod_i e^{i p_i \cdot x} \langle A^{a_1}(p_1) A^{a_2}(p_2) \dots A^{a_n}(p_n) \rangle. \quad (12)$$

In this formula

$$A^a(p) = J^a \phi(u, \bar{u}), \quad (13)$$

where  $J^a$  is the current of a WZW theory on  $\mathbf{CP}^1$ , which satisfies the current correlators (4) and hence has degree of homogeneity in  $u$  equals  $-2$ , and  $\phi(u, \bar{u})$  is the  $\mathcal{N} = 4$  superfield

$$\phi(u, \bar{u}) = a_- + \xi^i a_i + \frac{1}{2} \xi^i \xi^j a_{ij} + \frac{1}{3!} \xi^i \xi^j \xi^k \epsilon_{ijkl} \bar{a}^l$$

$$+ \xi^1 \xi^2 \xi^3 \xi^4 a_+, \quad (14)$$

where  $\xi^i = (u\theta)^i = \epsilon_{AB} u^A \theta^{Bi}$  ( $i, j = 1, 2, \dots, \mathcal{N}$  are the supersymmetry indices). We can interpret  $a_+$  as the classical value of the annihilation operator for a positive helicity gluon and  $a_-$  as the annihilation operator for a

negative helicity gluon. The components  $a_i, \bar{a}^i$  correspond to four spin- $\frac{1}{2}$  particles and  $a_{ij}$  correspond to six spin-zero particles—in agreement with the particle content of  $\mathcal{N} = 4$  theory.<sup>1</sup> Notice that the assignment of helicity is consistent with Eq. (11). The expectation value in (12) is taken as in the WZW theory, which means that we can use (4). Formula (12) also includes the momentum conservation  $\delta$ -function; it is generated by the  $x$ -integration. Further, it includes similar amplitudes for the superpartners, namely, the fermions and the scalars, though these do not contribute to the *classical* scattering of gluons.

We now want to carry out one more step of consolidation by defining an action for these amplitudes. The WZW action is defined in two dimensions. There are three independent components for a null momentum vector; therefore, a slight generalization is needed. The Lorentz-invariant volume element in momentum space can be written as

$$d\mu(p) = \frac{d^3 p}{2p_0} = -\frac{1}{4i} [(udu)d^2 \bar{u} - (\bar{u}d\bar{u})d^2 u]$$

$$= \frac{1}{2} (\bar{\alpha}\alpha) d(\bar{\alpha}\alpha) \frac{dz d\bar{z}}{(-2i)}. \quad (15)$$

In terms of the spinor  $u$ , we have still kept the identification of the local  $\mathbf{CP}^1$  coordinate, in the coordinate patch we are working with, as  $z = \beta/\alpha$ .

Let  $S_{\text{WZW}}$  be the action for the WZW theory defined in (1). But, one can easily embed it on a more general space by generalizing  $A_{\bar{z}}(z, \bar{z})$  to  $A_{\bar{z}}(z, \bar{z}, \dots)$  in some proper way. We now use a specific form for  $A_{\bar{z}}$  given by

$$A_{\bar{z}}(z, \bar{z}; x^\mu, \theta^{Ai}) = \pi \int d(\bar{\alpha}\alpha) \frac{\bar{\alpha}}{2\alpha} \tilde{A}. \quad (16)$$

The WZW action is expressed in terms of this potential as  $[\tilde{A}(n)$  denotes  $\tilde{A}(u_n, \bar{u}_n, x^\mu, \theta^{Ai})$ ]

$$S[\tilde{A}] = - \sum_{n=2}^{\infty} \frac{1}{n} \int d\mu(p_1) \dots d\mu(p_n)$$

$$\times \text{Tr} \left[ \frac{\tilde{A}(1) \dots \tilde{A}(n)}{(u_1 u_2)(u_2 u_3) \dots (u_n u_1)} \right]. \quad (17)$$

We choose  $\tilde{A}$  to be given by

$$\tilde{A}(u_n, \bar{u}_n, x^\mu, \theta^{Ai}) = t^a \left( a_-^a + \xi^i a_i^a + \frac{1}{2} \xi^i \xi^j a_{ij}^a \right.$$

$$+ \frac{1}{3!} \xi^i \xi^j \xi^k \epsilon_{ijkl} \bar{a}^{al}$$

$$\left. + \xi^1 \xi^2 \xi^3 \xi^4 a_+^a \right) e^{i p \cdot x}. \quad (18)$$

<sup>1</sup>The particle content of  $\mathcal{N} = 3$  is the same as that of  $\mathcal{N} = 4$ , but the MHV amplitude is more naturally expressed for  $\mathcal{N} = 4$ . See the paper of Rosly and Selivanov [3] for a comparison.

Notice that a part of this field is the same as the superfield  $\phi$  of (14) except for the extra color index we added. The scattering amplitude can now be written as

$$\mathcal{A} = \left[ \left( \frac{\delta}{\delta a^{a_1}(p_1)} \right) \cdots \left( \frac{\delta}{\delta a^{a_n}(p_n)} \right) \exp(i\Gamma[\tilde{A}]) \right]_{A=0} \sim, \quad (19)$$

$$\Gamma[\tilde{A}] = \int d^4x d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4 \frac{1}{g^2} S[g\tilde{A}].$$

If we consider  $n$  external gluons one must consider two positive helicity and  $n - 2$  negative gluons in order to saturate the Grassmann integration; in this construction,  $\mathcal{N} = 4$  is crucial to get the right MHV amplitude (8), and moreover the vanishing of amplitudes  $\mathcal{A}(- \cdots -) = 0$ ,  $\mathcal{A}(+ - \cdots -) = 0$  is automatically satisfied.

Let us now recall that the  $S$  matrix can be expressed in terms of the action as follows. Let  $\Gamma(\varphi)$  denote the effective quantum action of a set of fields, generically denoted by  $\varphi$ . The quantum equations of motion are the critical points of  $\Gamma$  defined by

$$\frac{\delta\Gamma}{\delta\varphi} = 0. \quad (20)$$

The functional which gives the  $S$  matrix is then given by

$$\mathcal{F} = [\exp(i\Gamma)]_{(\delta\Gamma/\delta\varphi)=0}. \quad (21)$$

The solutions of the equations of motion depend on a number of free parameters, which define the phase space of the theory; the  $S$  matrix is a functional of this free data in the solutions. Thus, for example, in perturbation theory, the solution is obtained as an expansion around the free field  $\varphi = \sum_k a_k u_k(x) + a_k^* u_k^*(x)$ , where  $u_k(x)$  are plane wave modes. The free data are the mode coefficients  $a_k, a_k^*$ . The amplitude for a process  $k_1, k_2, \cdots \rightarrow p_1, p_2, \cdots$  is given by

$$\mathcal{A} = \left[ \frac{\delta}{\delta a_{k_1}} \frac{\delta}{\delta a_{k_1}} \cdots \frac{\delta}{\delta a_{p_1}^*} \frac{\delta}{\delta a_{p_2}^*} \cdots \mathcal{F} \right]_{a_k = a_k^* = 0}. \quad (22)$$

In the classical theory, we can use the classical action  $S_{\text{cl}}$  in place of  $\Gamma$ .

Notice that the expression (19) is very similar to (22). In fact, if we identify  $\int d^4x d^8\theta S[\tilde{A}]$  in (19) as some sort of classical action for the theory, this is exactly the expected expression. We shall see below how this can emerge from the constraints of  $\mathcal{N} = 4$  Yang-Mills theory.

### III. A SOLUTION TO THE CONSTRAINTS OF $\mathcal{N} = 4$ YANG-MILLS THEORY

In the  $\mathcal{N} = 4$  super Yang-Mills theory, superspace is described by  $(x^\mu, \theta^{Ai}, \bar{\theta}^{\dot{A}})$  and we introduce the standard spinorial derivatives

$$D_{Ai} = \frac{\partial}{\partial \theta^{Ai}} + i(\sigma^\mu)_{AA} \bar{\theta}^{\dot{A}} \frac{\partial}{\partial x^\mu},$$

$$D_{\dot{A}}^i = -\frac{\partial}{\partial \bar{\theta}^{\dot{A}}} - i\theta^{Ai}(\sigma^\mu)_{AA} \frac{\partial}{\partial x^\mu}. \quad (23)$$

We also have the usual derivative  $\partial/\partial x^\mu$ . We then introduce gauge potentials  $A_{Ai}, \bar{A}_{\dot{A}}^i, A_\mu$ , which are functions of  $x^\mu, \theta^{Ai}, \bar{\theta}^{\dot{A}}$ , corresponding to these derivatives. Generally speaking this will give too many degrees of freedom and one has to impose constraints which reduce them to the required number of fields for the chosen value of  $\mathcal{N}$ . For  $\mathcal{N} = 4$ , the constraints are

$$F_{AiBj} + F_{BiAj} = 0, \quad F_{\dot{A}\dot{B}}^{ij} + F_{\dot{B}\dot{A}}^{ij} = 0, \quad F_{Ai\dot{B}}^j = 0, \quad (24)$$

along with a subsidiary condition

$$W_{ij} = \frac{1}{2} \epsilon_{ijkl} \bar{W}^{kl}, \quad (25)$$

where  $F_{AiBj} = \epsilon_{AB} W_{ij}$ ,  $F_{\dot{A}\dot{B}}^{ij} = \epsilon_{\dot{A}\dot{B}} \bar{W}^{ij}$ . These constraints have long been known to be rather stringent and lead to the equations of motion via the Bianchi identity [12]. This property shows that the second order classical equations of motion of the theory are equivalent to a set of first order equations in an appropriate superspace, suggesting a certain integrability for the  $\mathcal{N} = 4$  Yang-Mills theory [15].

We have seen that the MHV amplitudes have a natural interpretation in twistor space where there are additional bosonic variables. This leads to a possible strategy for solving the constraints. We will first write them in a larger space which is, more or less, a variant of supertwistor space. We will then do a gauge transformation (depending on the additional variables) to eliminate some of the usual gauge potentials. In the new gauge, the solution to the constraints is simpler. Such a method has been used to construct superfields for  $\mathcal{N} = 2$  Yang-Mills theory; that construction was based on harmonic superspace, which is a close relative of twistor space [16].

We start by introducing a complex spinor  $u^A$ . (This time we are not thinking of it as a spinor momentum—not yet.) The complex conjugate of  $u^A$  transforms as a dotted spinor  $\bar{u}_{\dot{A}} = (u^A)^*$ . So, to get something that transforms in a similar way to  $u^A$ , we introduce a vector  $K_{A\dot{A}}$  and write  $\bar{w}^A = K^{A\dot{A}} \bar{u}_{\dot{A}}$ . Thus for a fixed choice of  $K$ ,  $\bar{w}^A$  has the same information as the conjugate of  $u^A$ . Using these variables, we can take combinations of the derivatives on superspace for the undotted-sector as

$$D_i^+ = u^A D_{Ai}, \quad D_i^- = -\bar{w}^A D_{Ai}. \quad (26)$$

[We will take  $K$  such that the scalar product  $(\bar{w}u)$  is not zero.] We also have similar combinations for the gauge potentials. The constraints of the  $\mathcal{N} = 4$  theory can now be written as

$$\begin{aligned} F_{ij}^{++} &= F_{ij}^{+-} + F_{ij}^{-+} = F_{ij}^{--} = 0, \\ F_{\dot{A}\dot{B}}^{ij} + F_{\dot{B}\dot{A}}^{ij} &= 0, \quad F_{i\dot{B}}^{\pm j} = 0. \end{aligned} \quad (27)$$

The components which are not zero are  $F_{ij}^{+-} = (u\bar{w})W_{ij}$ ,  $F_{\dot{A}\dot{B}}^{ij} = \epsilon_{\dot{A}\dot{B}}\bar{W}^{ij}$ .

Let  $\mathcal{D}_i^+$ ,  $\mathcal{D}_i^-$ ,  $\mathcal{D}_{\dot{A}}^i$  denote the gauged versions of the spinorial derivatives  $\mathcal{D} = D + A$  with the gauge potentials  $A_i^+ = u^A A_{Ai}$ ,  $A_i^- = -\bar{w}^A A_{Ai}$  and  $A_{\dot{A}}^i$ , respectively. We also introduce the additional derivatives

$$\begin{aligned} D^{++} &= u^A \frac{\partial}{\partial \bar{w}^A}, \quad D^{--} = -\bar{w}^A \frac{\partial}{\partial u^A}, \\ D^0 &= \left( u^A \frac{\partial}{\partial u^A} - \bar{w}^A \frac{\partial}{\partial \bar{w}^A} \right). \end{aligned} \quad (28)$$

Notice that  $D_0$  is a charge operator, assigning  $+1$  charge to  $u^A$  and  $-1$  charge to  $\bar{w}^A$ . The superscripts in (26) and (28) indicate the value of this charge for each of the derivatives.

The constraints of the theory can now be displayed as

$$\{\mathcal{D}_i^+, \mathcal{D}_j^+\} = 0, \quad (29)$$

$$[\mathcal{D}^{++}, \mathcal{D}_i^+] = 0, \quad (30)$$

$$[\mathcal{D}^{--}, \mathcal{D}_i^+] = \mathcal{D}_i^-, \quad (31)$$

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = -D_0, \quad (32)$$

$$\begin{aligned} \{\mathcal{D}_i^+, \mathcal{D}_j^-\} + \{\mathcal{D}_i^-, \mathcal{D}_j^+\} &= 0, \quad \{\mathcal{D}_i^-, \mathcal{D}_j^-\} = 0, \\ [\mathcal{D}^{++}, \mathcal{D}_i^-] &= -\mathcal{D}_i^+, \quad [\mathcal{D}^{--}, \mathcal{D}_i^-] = 0, \end{aligned} \quad (33)$$

$$[\mathcal{D}^{++}, \mathcal{D}_{\dot{A}}^i] = 0, \quad [\mathcal{D}^{--}, \mathcal{D}_{\dot{A}}^i] = 0, \quad (34)$$

$$\{\mathcal{D}_i^+, \mathcal{D}_{\dot{A}}^j\} = \delta_i^j u^A \mathcal{D}_{A\dot{A}}, \quad \{\mathcal{D}_i^-, \mathcal{D}_{\dot{A}}^j\} = -\delta_i^j \bar{w}^A \mathcal{D}_{A\dot{A}}, \quad (35)$$

$$\{\mathcal{D}_{\dot{A}}^i, \mathcal{D}_{\dot{B}}^j\} + \{\mathcal{D}_{\dot{B}}^i, \mathcal{D}_{\dot{A}}^j\} = 0. \quad (36)$$

Even though we have written the gauged versions  $\mathcal{D}^{\pm\pm}$ , the gauge potentials  $A^{++}$ ,  $A^{--}$  are zero at this stage; these constraints are thus equivalent to the previous constraints (27). Further, even though we introduced  $u^A$ ,  $\bar{w}^A$ , the constraints do not depend on all components of these spinors. The constraints are homogeneous (of different degrees) and so, one of the components, say  $\alpha$  (and  $\bar{\alpha}$ ) can be factored out.

So far, the introduction of the additional variables and the potentials is really a meaningless redundancy since their potentials are zero. However, we now notice that, because of the constraints (29),  $A_i^+$  is of the form  $-D_i^+ g g^{-1}$ , for some matrix  $g$  (which is generally not unitary). The matrix  $g$  is in general a function of  $x^\mu$ ,  $\theta^{Ai}$ ,  $\bar{\theta}_{\dot{A}}^i$  and the new coordinates  $u^A$ ,  $\bar{w}^A$ . (If it did not depend on  $u^A$ ,  $\bar{w}^A$ ,  $W_{ij}$  would be zero.) This property of  $A_i^+$

suggests that we can make a gauge transformation using  $g$  and eliminate it. When this is done, the potentials  $A^{\pm\pm}$  are no longer zero; rather  $A^{++} = g^{-1} D^{++} g$ . In this new gauge  $A_i^+ = 0$ , the constraints (29) to (33) become

$$D_i^+ A^{++} = 0, \quad (37)$$

$$A_i^- = -D_i^+ A^{--}, \quad (38)$$

$$D^{++} A^{--} - D^{--} A^{++} + [A^{++}, A^{--}] = 0, \quad (39)$$

$$D_i^+ A_j^- + D_j^+ A_i^- = 0,$$

$$D_i^- A_j^- + D_j^- A_i^- + \{A_i^-, A_j^-\} = 0, \quad (40)$$

$$D^{++} A_i^- - D_i^- A^{++} + [A^{++}, A_i^-] = 0,$$

$$D^{--} A_i^- - D_i^- A^{--} + [A^{--}, A_i^-] = 0.$$

In addition to these, we still have the constraints (34) and (35) as well as (36) or  $F_{\dot{A}\dot{B}}^{ij} + (\dot{A} \leftrightarrow \dot{B}) = 0$  in (27).

These equations show how we can obtain a solution to the theory. We can start with  $A^{++}$  as the given quantity. It must be chosen such that it satisfies an analyticity condition (37).<sup>2</sup> Equation (39) then defines  $A^{--}$ . Given  $A^{--}$  we can use (38) to obtain  $A_i^- = -D_i^+ A^{--}$ . This will give us both  $A_i^+$  (which is zero) and  $A_i^-$ ; one can even transform back to the original gauge, if it is convenient. To show that this is indeed a solution, we must also check the constraints (40) using  $A_i^- = -D_i^+ A^{--}$ . This can be done in a straightforward way.

We have thus solved half of the constraints; we must now consider the dotted-sector and the mixed constraints (34). Having obtained  $A_i^\pm$ , we can, in principle, transform them back to the original gauge and take conjugates to get  $A_{\dot{A}}^i$ . This will take care of the constraints  $F_{\dot{A}\dot{B}}^{ij} + (\dot{A} \leftrightarrow \dot{B}) = 0$ . The constraints (35) can be taken as the definition of  $A_{A\dot{A}}$ . The only difficulty is with the constraints (34). This constraint reads

$$D^{\pm\pm} A_{\dot{A}}^i - D_{\dot{A}}^i A^{\pm\pm} + [A^{\pm\pm}, A_{\dot{A}}^i] = 0. \quad (41)$$

We do not have a way to deal with this in generality, but we notice that a particular solution may be obtained by setting  $D_{\dot{A}}^i A^{++}$  or  $D_{\dot{A}}^i A^{--}$  to zero. This imposes a chirality condition on  $A^{++}$  [and via (39) on  $A^{--}$ ]. Thus for our special solution we have

$$D_{\dot{A}}^i A^{++} = 0. \quad (42)$$

What we have shown is that if we find an  $A^{++}$  obeying the analyticity condition (37) and the chirality condition (42), then we can find a solution to the constraints of the  $\mathcal{N} = 4$  theory. The only nontrivial condition is Eq. (39).

<sup>2</sup>This term came from the definition of analyticity in harmonic superspace [16].

We now turn to the solution of (37), (42), and (39). The solution to the chirality condition (42) is well known:  $A^{++}$  must depend on  $\bar{\theta}_i^{\dot{A}}$  only through the combination  $y^\mu = x^\mu - i\bar{\theta}_i^{\dot{A}}\theta^{Ai}(\sigma^\mu)_{A\dot{A}}$ .<sup>3</sup> We will look for solutions of the form  $A^{++} = A_p \exp(ip \cdot y)$ ; the analyticity condition (37) then tells us that

$$\left[ u^A \frac{\partial A_p}{\partial \theta^{Ai}} - 2u^A(\sigma \cdot p)_{A\dot{A}} \bar{\theta}_i^{\dot{A}} A_p \right] = 0. \quad (43)$$

Since the first term does not have a factor of  $\bar{\theta}_i^{\dot{A}}$ , we get a nonzero solution only if

$$u^A \frac{\partial A_p}{\partial \theta^{Ai}} = 0, \quad u^A(\sigma \cdot p)_{A\dot{A}} = 0. \quad (44)$$

The first equation tells us that  $A_p$  must depend on  $\theta^{Ai}$  only through  $\xi^i = u_A \theta^{Ai}$ , so that we can write

$$A_p = t^a \left( a_-^a + \xi^i a_i^a + \frac{1}{2} \xi^i \xi^j a_{ij}^a + \frac{1}{3!} \xi^i \xi^j \xi^k \epsilon_{ijkl} \bar{a}^{al} + \xi^1 \xi^2 \xi^3 \xi^4 a_+^a \right), \quad (45)$$

where the coefficients  $a_\pm^a, a_i^a, a_{ij}^a, \bar{a}^{al}$  are arbitrary functions of  $p$ .<sup>4</sup> Notice that we have essentially recovered the superfield of (18) from the (special) solution of the constraints of the  $\mathcal{N} = 4$  theory, except for the appearance of  $y^\mu$ , instead of  $x^\mu$ , in the plane wave part  $\exp(ip \cdot y)$ . (It is immaterial that  $y^\mu$  appears rather than  $x^\mu$  since we will be integrating over  $x^\mu$  anyway.)

The second condition in (44) shows that  $p_\mu$  must be a null vector. Thus the solution must be on shell, as we knew it would be from the general statement that the constraints put the  $\mathcal{N} = 4$  theory on shell.  $u^a$  is an eigenvector of  $\sigma \cdot p$  with zero eigenvalue. Since  $p_\mu$  is real, we can write  $p_{A\dot{A}} = u_A \bar{u}_{\dot{A}}$ .

For a general solution to (37) and (42) we can do a superposition by integrating over the null momenta. But recall that  $u^A$  was part of the space, so we do not have the full freedom of integration. All of our constraints really depend only on the  $\mathbf{CP}^1$  subspace whose local coordinate we have taken as  $z = \beta/\alpha$ ; there is freedom to divide out by an appropriate number of  $\alpha, \bar{\alpha}$  because there is a balance of charges. So what can be freely integrated over is just the  $\alpha$  part. We choose this measure to be consistent with Lorentz invariance; this brings us to the combination  $A_{\bar{z}}$  given in (16) in terms of an integral over  $\tilde{A} = A_p \exp(ip \cdot y)$ .

<sup>3</sup>One might consider  $y^\mu = x^\mu - i\bar{\theta}_i^{\dot{A}}\theta^{Ai}(\sigma^\mu)_{A\dot{A}} + F$  with an arbitrary function  $F = F(\theta^{Ai}, u^A, \bar{\omega}^A)$  such as  $\theta^{Ai}(\sigma^\mu)_{A\dot{A}}\bar{u}_{\dot{A}}$ , but there is no combination satisfying the condition of being a singlet for the supersymmetry indices  $i$  and having vanishing  $D^0$  charge.

<sup>4</sup>There may exist  $(\bar{w}u)$  dependence, but this would not have an important role in the dynamics with a fixed choice of  $K^{AA}$ .

The final equation to be solved, namely, (39), is now straightforward. First of all, we express it in local coordinates. While  $u^A$  and  $(u^A)^*$  define the usual complex coordinates in terms of which we can get the local coordinates  $z, \bar{z}$ , we had to introduce a vector  $K$  to obtain Lorentz-invariant contractions and to define the derivatives  $D^{\pm\pm}$ . If we choose  $K_{\dot{A}}^A = \delta_{\dot{A}}^A$ , this will correspond to the usual description of  $\mathbf{CP}^1$  where we use  $u^A$  and  $\bar{u}^A$ . (This corresponds to the use of a particular frame to define the derivatives  $D^{\pm\pm}$ , but our final results will be Lorentz-invariant.) We now define a set of local gauge potentials  $A_{\bar{z}}, \mathcal{A}_{\bar{z}}$  by

$$A^{++} = \frac{\alpha}{\bar{\alpha}}(1 + z\bar{z})A_{\bar{z}}, \quad A^{--} = \frac{\bar{\alpha}}{\alpha}(1 + z\bar{z})\mathcal{A}_{\bar{z}}. \quad (46)$$

The substitution of these into Eq. (39) transforms it into

$$\partial_{\bar{z}}\mathcal{A}_{\bar{z}} - \partial_z A_{\bar{z}} + [A_{\bar{z}}, \mathcal{A}_{\bar{z}}] = 0. \quad (47)$$

Basically, this takes us to the Eq. (7). What we have shown is that solving (39) is equivalent to solving (47). Given a solution of (47), we can obtain a solution to (39) by using (46).

Equation (47) can be solved for  $\mathcal{A}_{\bar{z}}$  in terms of  $A_{\bar{z}}$ ; the latter is arbitrary except for the analyticity condition (37). Notice that if we substitute (46) into (37), the prefactor  $(\alpha/\bar{\alpha})(1 + z\bar{z})$  drops out; we can also factor out  $\alpha$  from  $u^A$  (which is equivalent to writing  $u^1 = 1, u^2 = z$ ). We thus obtain the same conditions (44) for  $A_{\bar{z}}$  with  $(1, z)$  in place of  $u^A$ . (This is in accordance with our earlier comment on dividing out  $\alpha, \bar{\alpha}$ .) It can be solved for  $A_{\bar{z}}$  as before, giving a function of the  $z$ 's and the momentum  $p$ ; the value of  $\alpha, \bar{\alpha}$  given by the momentum can then be used to go back to the full  $u^A$ . We see that  $A_{\bar{z}}$  is given by a superposition of fields of the form  $A_p e^{ip \cdot y}$  with  $A_p$  given by (45). We now take it to be given by

$$A_{\bar{z}} = \pi \int d(\bar{\alpha}\alpha) \frac{\bar{\alpha}}{2\alpha} A_p e^{ip \cdot y} = \pi \int d(\bar{\alpha}\alpha) \frac{\bar{\alpha}}{2\alpha} \tilde{A}. \quad (48)$$

This is essentially (16), but we have obtained it as a solution of the constraints. [It should be emphasized that since the coefficients  $a_-^a, a_i^a$ , etc., can also be functions of  $p$ , there is no loss of generality in taking the particular form (48). In other words, it is simply the choice for which the coefficients can be interpreted as the properly normalized annihilation amplitudes.]

The key issue is thus the solution of (47) for  $\mathcal{A}_{\bar{z}}$ . But rather than discussing  $\mathcal{A}_{\bar{z}}$  in its own right, we shall now turn to the  $S$  matrix. Since we are looking for tree-level amplitudes at this point, what we need, in the spirit of (21), is a classical action. The basic equation of motion for us is (39) or (47). We need an action, which for any given  $A_{\bar{z}}$ , gives the Eq. (47) for  $\mathcal{A}_{\bar{z}}$ . This action, not surprisingly, is a variant of the WZW action in the holomorphically extended superspace  $(x^\mu, \theta^{Ai}, z, \bar{z})$  and is

$$S = -k \int d\mathbf{X} S_{\text{WZW}}(U) + \frac{k}{\pi} \int d\mathbf{X} d^2z \text{Tr}(A_{\bar{z}} \partial_z U U^{-1}). \quad (49)$$

Here  $k$  is a normalization constant which can be thought of as the level number of the WZW action and  $d\mathbf{X} = d^4x d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4$ . The equation of motion can be obtained by varying the matrix field  $U$  and is identical to (47) with  $\mathcal{A}_z = -\partial_z U U^{-1}$ . Further, if we write  $A_{\bar{z}} = M^{\dagger-1} \partial_{\bar{z}} M^{\dagger}$ , for some matrix  $M^{\dagger}$ , the solution to (47) is evidently  $U = M^{\dagger-1}$ . We can now use the Polyakov-Wiegman identity to write

$$\begin{aligned} & -S_{\text{WZW}}(U) + \frac{1}{\pi} \int d^2z \text{Tr}(A_{\bar{z}} \partial_z U U^{-1}) \\ &= -S_{\text{WZW}}(M^{\dagger} U) + S_{\text{WZW}}(M^{\dagger}) = S_{\text{WZW}}(M^{\dagger}) \\ &= S_{\text{WZW}}(A_{\bar{z}}). \end{aligned} \quad (50)$$

This tells us that the action (49), which leads to the required equation of motion (47), when evaluated on solutions of that equation is given by the WZW action of (1) (with the additional integration with the measure  $d\mathbf{X}$ ). All we have to do at this point is to substitute the form of  $A_{\bar{z}}$  given by (48) to obtain the  $S$  matrix amplitudes, following the general formula (22). Evidently, we have recovered the formula (19).<sup>5</sup> Notice also that our final formula (19) involves only Lorentz-invariant scalar products; thus, the choice of the vector  $K$  is irrelevant. (We can recover the coupling constant by the standard scaling  $\tilde{A} \rightarrow g\tilde{A}$ . But the overall normalization of the action  $k$  is not given by the equations of motion. This is always the case classically. Thus there is one constant in the amplitudes which is not determined by our argument. This is basically Planck's constant.)

We also note that if we introduce two more variables  $\zeta_{\hat{A}}$  and write combinations like  $\tilde{D}^{i+} = \zeta^{\hat{A}} D_{\hat{A}}^i$ , then we can obtain similar results for the opposite ‘‘handedness,’’ with almost all positive helicity gluons  $\mathcal{A}(+ + \dots +) = 0$ ,  $\mathcal{A}(- + + \dots +) = 0$ , and  $\mathcal{A}(- - + + \dots +)$  by exchanging the undotted sector with dotted sector. Furthermore, if we were to introduce both  $u_A$  and  $\zeta_{\hat{A}}$ , then we are naturally led to a  $\mathbf{CP}^1 \times \mathbf{CP}^1$  structure. This has occurred before in connection with the  $\mathcal{N} = 4$  theory, for example, the paper of Rosly and Schwartz in [16] as well as [6,11]. It would be interesting to utilize this structure as well as the Chern-Simons theory description to eliminate the condition (42).

<sup>5</sup>There are additional  $\tilde{\theta}_i^{\hat{A}} \theta^{A_i}$ -terms from  $\exp(ip \cdot y)$ , since  $y^\mu = x^\mu - i\tilde{\theta}_i^{\hat{A}} \theta^{A_i} (\sigma^\mu)_{\hat{A}\hat{A}}$ , but the tree amplitudes are the same as those of (18) since  $\tilde{\mathcal{A}}(- - \dots -) \sim (\tilde{\theta})^8 |_{\tilde{\theta}_i^{\hat{A}}=0} = 0$ ,  $\mathcal{A}(+ - \dots -) \sim (\tilde{\theta})^4 |_{\tilde{\theta}_i^{\hat{A}}=0} = 0$ .

#### IV. THE NON-MHV AMPLITUDES

So far our analysis is restricted to the MHV amplitudes. In fact, we see that once we make the simplifying assumption of  $D_{\hat{A}}^i A^{++} = 0$ , we are restricted to the MHV amplitudes. The proper way to proceed would thus be to relax this condition and see how the solution to the constraints would change. However, this is rather difficult; our derivation is limited to the MHV amplitudes. As mentioned in the introduction, a suggestion was made in [8] that one could simplify the calculation of the non-MHV amplitudes by using MHV vertices and then connecting them via propagators, analogous to Wick contractions in standard perturbative field theory. While we do not have an independent justification or derivation of this result, we note that there is an elegant way to incorporate it in our formalism.

The Wick contraction operator for two gluons is given by

$$\hat{W} = \exp \left[ - \int_{x,y} D(x,y) \frac{\delta}{\delta a_{\bar{z}}^a(x)} \frac{\delta}{\delta a_{\bar{z}}^a(y)} \right], \quad (51)$$

with the propagator  $D(x,y)$  which is the inverse of  $p^2$ . Consider the functional for the  $S$  matrix defined by

$$\mathcal{F} = \hat{W} \exp(i\Gamma[\tilde{A}]), \quad (52)$$

where  $\Gamma[\tilde{A}]$  is given in (19). Consider the application of this to two vertices, resulting again in a tree diagram. First of all, to include propagators, we need the off shell continuation of the amplitudes, at least for the gluon which is replaced by the propagator. This will be assumed to be done as in [8]. The prescription is the following: If  $p_\mu$  is the off shell momentum, the corresponding spinor momentum in the MHV vertex will be taken as  $u_A = p_{A\hat{A}} \xi^{\hat{A}}$ , where  $\xi^{\hat{A}}$  is a fixed spinor, taken to be the same for all off shell lines in a diagram. Secondly, the individual MHV amplitudes have a color structure of the form  $\text{Tr}(t^{a_1} \dots t^{a_n})$ . Since  $U(1)$ 's decouple from the theory, we may extend the range of the indices  $a_1, a_2$ , etc., to include a  $U(1)$  direction as well. We will take the  $t^a$ 's to be normalized so that we have the completeness relation  $(t^a)_{ij} \times (t^a)_{kl} = \delta_{jk} \delta_{il}$ . Then the contractions preserve the color structure  $\text{Tr}(t^{a_1} \dots t^{a_n})$  with cyclic ordering of the external lines from the individual vertices.

Using (52), we may calculate the subamplitude  $\mathcal{A}(1_- 2_- 3_- 4_+ 5_+ 6_+)$  for the scattering of six gluons. We find

$$\begin{aligned}
\mathcal{A}(1_2 3_4 5_6) &= \mathcal{A}(4_+ 5_+ 6_+ 1_- k_-) D_{kl}^{(1)} \mathcal{A}(l_+ 2_- 3_-) + \mathcal{A}(3_- 4_+ 5_+ 6_+ k_-) D_{kl}^{(2)} \mathcal{A}(l_+ 1_- 2_-) \\
&+ \mathcal{A}(6_+ 1_- k_-) D_{kl}^{(3)} \mathcal{A}(l_+ 2_- 3_- 4_+ 5_+) + \mathcal{A}(3_- 4_+ k_-) D_{kl}^{(4)} \mathcal{A}(l_+ 5_+ 6_+ 1_- 2_-) \\
&+ \mathcal{A}(3_- 4_+ 5_+ k_-) D_{kl}^{(5)} \mathcal{A}(l_+ 6_+ 1_- 2_-) + \mathcal{A}(5_+ 6_+ 1_- k_-) D_{kl}^{(6)} \mathcal{A}(l_+ 2_- 3_- 4_+), \quad (53)
\end{aligned}$$

where the  $D$ 's are given by

$$\begin{aligned}
D_{kl}^{(1)} &= (p_2 + p_3)^{-2}, & D_{kl}^{(2)} &= (p_1 + p_2)^{-2}, \\
D_{kl}^{(3)} &= (p_6 + p_1)^{-2}, & D_{kl}^{(4)} &= (p_3 + p_4)^{-2}, \\
D_{kl}^{(5)} &= (p_3 + p_4 + p_5)^{-2}, & D_{kl}^{(6)} &= (p_2 + p_3 + p_4)^{-2}.
\end{aligned} \quad (54)$$

This result agrees with the general prescription given in [8]. The general formula (52) can also generate loop diagrams. It is not entirely clear to us at this point whether they are identical to the one-loop amplitudes of the  $\mathcal{N} = 4$  Yang-Mills theory, although the resulting amplitudes are very similar to the recent suggestion in [17].

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