Quasilocal contribution to the gravitational self-force

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The gravitational self-force on a point particle moving in a vacuum background space-time can be expressed as an integral over the past world line of the particle, the so-called tail term. In this paper, we consider that piece of the self-force obtained by integrating over a portion of the past world line that extends a proper time $\Delta\tau$ into the past, provided that $\Delta\tau$ does not extend beyond the normal neighborhood of the particle. We express this quasilocal piece as a power series in the proper time interval $\Delta\tau$. We argue from symmetries and dimensional considerations that the $O(\Delta\tau^0)$ and $O(\Delta\tau)$ terms in this power series must vanish, and compute the first two nonvanishing terms which occur at $O(\Delta\tau^2)$ and $O(\Delta\tau^3)$. The coefficients in the expansion depend only on the particle's four velocity and on the Weyl tensor and its derivatives at the particle's location. The result may be useful as a foundation for a practical computational method for gravitational self-forces in the Kerr space-time, in which the portion of the tail integral in the distant past is computed numerically from a mode-sum decomposition.

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I. INTRODUCTION AND SUMMARY

One of the outstanding open problems of classical general relativity is the calculation of the gravitational self-force experienced by a massive particle moving in a curved background space-time. Here by particle we do not mean a point particle, but rather an extended object whose internal structure has a negligible effect on its "center-of-mass" motion. Such a particle will not follow a geodesic of the background space-time, but rather a geodesic of the total space-time, whose curvature reflects both the background and the mass/energy of the particle itself. However, if the mass of the particle is much less than the natural length scale of the background (i.e., the square root of the inverse of the curvature scale of the background), the deviation of the particle's trajectory from a background geodesic will be small over time scales that are less than the natural length scale of the background. In this case, one can treat the difference between the total space-time metric and the background space-time metric to linear order as spin-2 field generated by the particle and living on the background space-time. This field couples to the particle, and in this picture, it is understood as causing a force, the *self-force*, which causes the particle to deviate from the background geodesic.

With this description, the gravitational self-force is analogous to the self-force experienced by a electrically charged point particle coupled to a Maxwell field in curved space-time, or to the self-force experienced by a particle carrying a scalar charge coupled to a massless linear scalar field in curved space-time. In each of these cases, the interaction of a particle with its own field alters its motion. In the flat space-time limit, these forces reduce to the familiar radiation reaction forces, as they are only associated with accelerated motion and the resultant emission of radiation. In curved space-times, however, the notion of emission of radiation can, in general, be ambiguous. Furthermore, even in stationary space-times, the self-forces can have conservative terms, as well as dissipative pieces reminiscent of radiation reaction forces. Thus, in curved space-times, the notion of a radiation reaction force is replaced by the more general notion of a self-force.

Recently, there has been renewed interest in finding the gravitational self-force experienced by a massive particle traveling in a curved background space-time. The primary impetus for this interest is the imminent construction of space-based gravitational wave observatories, such as Laser Interferometer Space Antenna (LISA) [1]. One of the most interesting gravitational wave sources for these instruments will be compact solar mass objects inspiralling into black holes of 10^3 to 10^8 solar masses out to Gpc distances [2]. The extraction of the maximal amount of information from such observations, however, can only be effected if accurate theoretical waveform templates exist. The calculation of such templates requires precise knowledge of the orbital evolution of the smaller object in the gravitational background of the massive black hole, which in turn requires an accurate calculation of the gravitational self-force.

In flat space-time, the radiation reaction force can be thought of as the recoil force experienced by the particle as it emits radiation. Since the radiation carries momentum, and since the radiation field is not generally spherically symmetric, there is a nonzero net momentum transferred to the particle. This change of momentum corresponds to a force, which for an electrically charged particle in flat space-time is given by the Abraham-Lorentz-Dirac formula [3,4]. This explanation of the radiation reaction force is, of course, highly simplified. It is not straightforward to relate the force directly to the radiation—the force is local to the particle while radiation is a far-field phenomenon. Nonetheless, this simplified picture can provide intuitive insight into the phenomenon.

In curved space-times, qualitatively new types of self-interactions occur due to the failure of Huygens' principle (in its modern incarnation) to hold in most geometries.[1] More pedantically, it is due to the fact that solutions to wave equations on most curved manifolds depend not only on Cauchy data directly intersected by the past light cone, but also on Cauchy data interior to that intersection. The portion of the field that propagates in the null directions along the characteristics is called the *direct* part. The portion of the field which propagates in the timelike directions in the interior of the light cone is called the *tail*. Clearly, the tail part of a particle's field can interact with the particle, leading to a contribution to the self-force.

A general expression for the self-force on an electrically charged particle in a curved space-time was obtained in the seminal paper by DeWitt and Brehme [8].[2] More recently, similar expressions have been obtained for the gravitational self-force by Mino, Sasaki and Tanaka [10] and by Quinn and Wald [11], which we review in Sec. II below, and for the scalar self-force by Quinn [12]. These results have resolved many of the issues of principle in computing self-forces in curved space-time. See Ref. [13] for a detailed review of and simplified versions of these computations.

However, for applications to gravitational wave observations, one needs to translate the formal expressions of Refs. [10,11] into practical computational schemes for computing orbits of particles in the Kerr space-time. The expressions for the self-force involve the retarded Green's function for the wave equation, and the standard method of computing this Green's function is to use a decomposition of the field into modes. This mode decomposition method combines together the tail and singular pieces of the fields in a manner that is difficult to disentangle, and it is the tail piece of the self-field that determines the self-force. Thus, the results of Refs. [8,10–12] do not directly give a simple method of computing self-forces in black hole space-times.

Nonetheless, some progress has been made in calculating the self-force for particular particle trajectories in particular space-times. For specific geometries, and also in the weak-field approximation, it has been possible to compute the tail of the Green's function. Some classic results have been obtained for scalar or electric charges in static, radial, or circular trajectories about black holes or cosmic strings [14–24]. In Schwarzschild, and for circular or equatorial orbits in Kerr, the time-averaged nonconservative (i.e., radiation reaction) contributions to the gravitational self-force may be deduced using energy and angular-momentum balance arguments involving the flux of radiation to infinity and down the black hole [25–32]. Furthermore, in the weak-field, slow-motion limit of general relativity, one may use the post-Newtonian expansion to obtain the gravitational self-force [33–35], and the result agrees with that obtained by specializing the formal results of Refs. [10,11] to weak fields [24].

While these results are encouraging, it is important and desirable to have a framework in which arbitrary motions in black hole space-times can be computed. Recently considerable progress has been made in developing practical computational schemes for obtaining self-forces.[3] Most of these schemes are based on computing the full retarded field, which is infinite at the position of the particle on its world line, and regularizing it in some way to effect the subtraction of the direct part of the field, leaving the desired tail part. Barack and Ori have derived a mode-sum regularization scheme [36–38] that has been successfully applied in a number of cases [38–43]. The regularization parameters for this scheme have been derived from the fundamental Mino-Sasaki-Tanaka-Quinn-Wald equation of motion for Schwarzschild in Refs. [44,45] and for Kerr in Ref. [46]. Mino, Nakano and Sasaki have devel-

[1]This statement is in fact a modification of a conjecture by Hadamard [5] that the only hyperbolic differential operators whose solutions obey Huygens' principle are conformally related to the ordinary wave operator in Minkowski space with even numbers of space-time dimensions. Hadamard originally formulated this conjecture with scalar operators in mind, but it has since been trivially extended to operators for fields with higher spin. Counterexamples to Hadamard's conjecture have been found, originally in ≥ 6 (but even) dimensions by Stellmacher [6] and later in the more physically relevant case of ≥ 4 (even) dimensions by Günther [7]. Nonetheless, these counterexamples are believed (but not yet proved) to be isolated cases. This modified Hadamard conjecture has been proved in broad classes of space-times (e.g., some algebraically special space-times) and work continues in this area.

[2]The expression obtained by DeWitt and Brehme is missing a term due to a trivial calculational error; see Hobbs [9] for the correction.

[3]See the proceedings of the Capra Ranch meetings on radiation reaction at http://www.lsc-group.phys.uwm.edu/~patrick/ireland99/, http://www.aei-potsdam.mpg.de/lousto/capra/, http://cgwp.gravity.psu.edu/events/Capra5/capra5-BKP_2002-05-24-1200.shtml and http://cgwa.phys.utb.edu/Events/agendaView.php?EventID=3.

oped two regularization schemes, one of which is a mode-by-mode regularization and the second of which they dub the ''power expansion regularization'' which involves a post-Newtonian expansion of the Green's function [47–49]. Both their methods have been applied [50,51]. Another scheme is that of Lousto, who used zeta-function regularization of modes for a radially infalling scalar particle in Schwarzschild [52].

A qualitatively different method for computing self-forces in black hole space-times has been suggested by Poisson and Wiseman [53]. It is based on a direct computation of the tail field, rather than a regularization of the total retarded field. The tail field can be expressed as an integral over the past world line of the particle. The idea is to split this integral into two pieces, a piece that extends back into the past a proper time interval $\Delta\tau$, which we call the *quasilocal* piece, and the remainder of the integral. The second piece, from the more distant past, can be computed using standard multipolar decomposition of the full retarded field; no difficulties involving divergences occur here, and thus no regularization is needed. The first, quasilocal piece can be computed approximately as a power series expansion in $\Delta\tau$.

In this paper, we compute the expansion in powers of $\Delta\tau$ of the quasilocal piece of the gravitational self-force for an arbitrary vacuum space-time, to the first two nontrivial orders in $\Delta\tau$. Our result is given in Eq. (3.17) below, and may be useful as a foundation for the Poisson-Wiseman scheme. Alternatively it may be useful as a check of numerical codes that use some regularization scheme. At the core of our analysis is a local expansion of the tail piece of the Green's function for linearized perturbations. Such local expansions of Green's functions can be found in the literature on quantum field theory in curved space-time; see Refs. [54–56] for the scalar case, Ref. [55] for the electromagnetic case, and Ref. [57] for the gravitational case. We extend the expansion of Ref. [57] to one higher order, and apply the result to compute the quasilocal piece of the gravitational self-force.

The organization of this paper is as follows. In Sec. II we review the formal expression for the gravitational self-force obtained by Mino, Sasaki and Tanaka [10] and by Quinn and Wald [11], and define the quasilocal contribution to the self-force. In Appendix A, we use symmetry and dimensional arguments to deduce the possible terms that can appear in the expansion of the quasilocal contribution, thereby reducing the computation to obtaining one universal numerical coefficient at the leading nontrivial order $[O(\Delta\tau^2)]$ and four universal numerical coefficients at the next higher order $[O(\Delta\tau^3)]$. Section III computes those numerical coefficients; the final result is given in Eq. (3.17). Some of the details of the computation are relegated to Appendices B, C, D, and E. In Sec. IV we calculate the application of the general expression to some interesting cases. Finally, we make some concluding remarks in Sec. V.

Throughout this paper we use geometrized units in which $G = c = 1$, and we adopt the sign conventions of Ref. [58].

II. THE GRAVITATIONAL SELF-FORCE

A. The Mino-Sasaki-Tanaka-Quinn-Wald formula

Consider a point particle of mass μ moving on a geodesic $x^\alpha(\tau)$ of a background space-time $(M, g_{\beta\gamma})$, parametrized by proper time τ. Throughout this paper we assume that the background space-time satisfies the vacuum Einstein equation $R_{\alpha\beta} = 0$. The particle will perturb the background geometry to linear order in μ. We denote the linearized metric perturbation by $h_{\alpha\beta}$, and the more convenient trace-reversed form of this perturbation by

$$\psi_{\beta\gamma} \equiv h_{\beta\gamma} - \tfrac{1}{2} g_{\beta\gamma} h_{\mu\nu} g^{\mu\nu}. \tag{2.1}$$

We raise and lower indices with the background metric. We specialize throughout to the Lorentz gauge defined by

$$\psi^{\beta\gamma}{}_{;\gamma} = 0, \tag{2.2}$$

where the semicolon denotes a covariant derivative with respect to the background metric $g_{\alpha\beta}$. In this gauge the linearized Einstein field equations take the form of the simple wave equation

$$(\Box g_{\mu\alpha} g_{\nu\beta} + 2C_{\mu\alpha\nu\beta})\psi^{\mu\nu} = -16\pi T_{\alpha\beta}, \tag{2.3}$$

where $\Box$ and $C_{\mu\alpha\nu\beta}$ are the D'Alembertian and Weyl tensor associated with the background metric $g_{\beta\gamma}$, and $T_{\alpha\beta}$ is the linearized stress energy tensor. The wave equation (2.3) can be solved using the retarded Green's function $G_{\rm ret}^{\mu\nu\alpha'\beta'}$, which is defined by the equation

$$(\Box g_{\mu\alpha} g_{\nu\beta} + 2C_{\mu\alpha\nu\beta}) G_{\rm ret}^{\mu\nu\alpha'\beta'}(x, x') = -g_{(\alpha}{}^{\alpha'} g_{\beta)}{}^{\beta'} \delta^4(x, x'), \tag{2.4}$$

and by the fact that it has support only when x' is in the causal past of x. Here $g_\alpha{}^{\alpha'}$ is the parallel displacement bivector [8,59,60], and $\delta^4(x, x') = \delta^4(x - x')/\sqrt{-g}$ is a generalized Dirac delta function.[4] The retarded solution to Eq. (2.3) can be written in terms of the Green's function as

$$\psi_{\rm ret}^{\mu\nu}(x) = 16\pi \int d^4x' \sqrt{-g(x')} G^{\mu\nu}_{{\rm ret}\,\alpha'\beta'}(x, x') T^{\alpha'\beta'}(x'). \tag{2.5}$$

For the point particle source, the stress energy tensor is given by

[4]Note that because we have used the sign convention of Misner, Thorne and Wheeler [58], our Weyl tensor has the opposite sign to that of Mino, Sasaki and Tanaka [10]. Note also that our Green's function is defined to be one-half that of Ref. [10].

$$T_{\alpha\beta}(x) = \mu \int_{-\infty}^{\infty} \delta^4[x, x'(\tau')] u_\alpha(\tau') u_\beta(\tau') d\tau', \qquad (2.6)$$

where $u_\alpha(\tau)$ is the particle's four velocity, and inserting this into Eq. (2.5) gives

$$\psi^{\mu\nu}_{\rm ret}(x) = 16\pi\mu \int_{-\infty}^{\infty} d\tau' G^{\mu\nu}_{{\rm ret}\alpha'\beta'}[x, x'(\tau')] u^{\alpha'}(\tau') u^{\beta'}(\tau'). \qquad (2.7)$$

Now one would expect the particle to move on a geodesic of the total metric $g_{\alpha\beta} + \psi^{\rm ret}_{\alpha\beta} - g_{\alpha\beta} g^{\gamma\delta} \psi^{\rm ret}_{\gamma\delta}/2$. Such geodesic motion would be equivalent to motion for which the mass times acceleration with respect to $g_{\alpha\beta}$ is

$$f^\alpha = \mu P^{\alpha\beta\gamma\delta} \psi^{\rm ret}_{\beta\gamma;\delta}, \qquad (2.8)$$

where the tensor $P_{\alpha\beta\gamma\delta}$ is given by

$$P^{\alpha\beta\gamma\delta} \equiv -\tfrac{1}{2} u^\alpha u^\beta u^\gamma u^\delta - g^{\alpha(\beta} u^{\gamma)} u^\delta + \tfrac{1}{2} g^{\alpha\delta} u^\beta u^\gamma + \tfrac{1}{4} u^\alpha g^{\beta\gamma} u^\delta + \tfrac{1}{4} g^{\alpha\delta} g^{\beta\gamma}. \qquad (2.9)$$

However, the retarded field $\psi^{\rm ret}_{\mu\nu}$ and its gradient $\psi^{\rm ret}_{\mu\nu;\lambda}$ are divergent on the particle's world line, so the naive expression (2.8) for the self-force is ill defined. Instead, the correct expression for the self-force is given by Eq. (2.8) with the retarded field replaced by the so-called tail field $\psi^{\rm tail}_{\mu\nu}$ [10,11]:

$$f^\alpha = \mu P^{\alpha\beta\gamma\delta} \langle \psi^{\rm tail}_{\beta\gamma;\delta} \rangle. \qquad (2.10)$$

Here the angular brackets $\langle \ldots \rangle$ denote the result obtained by averaging over a two sphere of some small radius r about the particle in the spatial hypersurface orthogonal to u^α, and by taking the limit $r \to 0$. The tail field $\psi^{\rm tail}_{\mu\nu}(x)$ is defined by truncating the integral (2.7) over the particle's world line to exclude the contribution from the direct part of the Green's function:

$$\psi^{\mu\nu}_{\rm tail}(x) = 16\pi\mu \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau_{\rm ret}(x)-\epsilon} d\tau' G^{\mu\nu}_{{\rm ret}\alpha'\beta'}[x, x'(\tau')] \times u^{\alpha'}(\tau') u^{\beta'}(\tau'). \qquad (2.11)$$

Here $\tau_{\rm ret}(x)$ is the value of proper time τ at the point where the world line intersects the past light cone of the point x. The remaining, direct portion of the field is

$$\psi^{\mu\nu}_{\rm direct}(x) = \psi^{\mu\nu}_{\rm ret}(x) - \psi^{\mu\nu}_{\rm tail}(x) = 16\pi\mu \lim_{\epsilon \to 0^+} \int_{\tau_{\rm ret}(x)-\epsilon}^{\tau_{\rm ret}(x)+\epsilon} d\tau' G^{\mu\nu}_{{\rm ret}\alpha'\beta'}[x, x'(\tau')] \times u^{\alpha'}(\tau') u^{\beta'}(\tau'). \qquad (2.12)$$

If we now take the gradient of the tail field (2.11) in order to substitute into the formula (2.10) for the self-force, we obtain two terms: a term generated by the action of the gradient operator on the quantity $\tau_{\rm ret}(x)$, and a term generated by the action of the gradient operator on the retarded Green's function. The first contribution gives an expression which has a direction dependent limit on the world line, but which vanishes once the average $\langle \ldots \rangle$ is taken [11,61]. The second contribution is continuous on the world line (so the averaging can be dispensed with), and yields for the self-force the expression [10,11]

$$f^\alpha(\tau) = 16\pi\mu^2 P^{\alpha\beta\gamma\delta} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\tau-\epsilon} d\tau' G^{\rm ret}_{\beta\gamma\beta'\gamma';\delta}[x, x'(\tau')] \times u^{\beta'}(\tau') u^{\gamma'}(\tau'). \qquad (2.13)$$

As an aside, we note that Detweiler and Whiting [61] have introduced an alternative splitting of the retarded field of the form

$$\psi^{\mu\nu}_{\rm ret} = \psi^{\mu\nu}_{\rm sing} + \psi^{\mu\nu}_{\rm regular}, \qquad (2.14)$$

where the singular piece $\psi^{\mu\nu}_{\rm sing}$ is a solution of the inhomogeneous wave equation (2.3) which can be computed locally, and $\psi^{\mu\nu}_{\rm regular}$ is a solution of the corresponding homogeneous wave equation such that the self-force is given by the expression (2.8) with $\psi^{\mu\nu}_{\rm ret}$ replaced by $\psi^{\mu\nu}_{\rm regular}$. This alternative formulation also gives rise to the final formula (2.13) for the self-force.

B. Equation of motion

The self-force formula (2.13) discussed above was defined only for a geodesic world line. Therefore it is necessary to supplement the self-force formula with a prescription for computing the motion of a point particle that includes the effect of the self-force to leading order in the particle's mass μ. The reason that finding such a prescription is not entirely trivial is the following [10,11]: the linearized Einstein equation admits solutions only if its source, the particle's stress energy tensor (2.6), is conserved. However, the stress energy tensor for a point particle is conserved only if the world line is a geodesic. Therefore it is not straightforward to define the self-force on a nongeodesic world line.

Quinn and Wald [11] suggested the following method of resolving this difficulty. They define a self-force for a nongeodesic world line by relaxing the Lorentz gauge condition (2.2) while retaining the form (2.3) of the linearized Einstein equation. The justification for relaxing the gauge condition is that the associated errors are quadratic in the mass μ, while the self-acceleration is linear in μ. Their equation of motion is then

$$\mu a^\alpha = f^\alpha, \qquad (2.15)$$

with f^α given by Eq. (2.13), modified as described above, and a^α is the 4-acceleration with respect to the background metric. The equation of motion (2.15) is an integro-differential equation which is nonlocal in time.

Here, we suggest an alternative equation of motion, which gives the same results as Eq. (2.15) to leading order in μ. First, for any point $\mathcal{P}$ in space-time, and for any unit future-directed timelike vector $\vec{u}$ at $\mathcal{P}$, we define the self-force vector $f^\alpha = f^\alpha(\mathcal{P}, \vec{u})$ to be self-force obtained from the prescription (2.13) for the particular geodesic which extends into the past from $\mathcal{P}$ whose tangent at $\mathcal{P}$ is $\vec{u}$. The equation of motion for the perturbed world line $x^\alpha(\tau)$ is then

$$\mu a^\alpha = f^\alpha[\vec{x}(\tau), \vec{u}(\tau)]. \tag{2.16}$$

In other words, the world line is such that its acceleration at any point is the acceleration that is obtained from the integral (2.13) for the geodesic which is tangent to the world line at that point. The justification for using the instantaneously tangential geodesic rather than the true world line is essentially the reduction of order argument discussed in Refs. [11,62]. The equation of motion (2.16) is a second order differential equation which is local in time.

C. Definition of the quasilocal piece of the self-force

In this paper we evaluate not the entire integral over the world line in the self-force formula (2.13), but instead that portion of that integral near the particle with proper times τ' in the range $\tau - \Delta\tau \le \tau' \le \tau$. Specifically, we define

$$f^\alpha_{\rm QL}(\tau, \Delta\tau) = 16\pi\mu^2 P^{\alpha\beta\gamma\delta} \lim_{\epsilon\to 0^+} \int_{\tau-\Delta\tau}^{\tau-\epsilon} d\tau' G^{\rm ret}_{\beta\gamma\beta'\gamma';\delta}[x, x'(\tau')] u^{\beta'}(\tau') u^{\gamma'}(\tau'). \tag{2.17}$$

Here the subscript "QL" denotes "quasilocal." By comparing with Eq. (2.13) we see that the entire self-force is obtained from $f^\alpha_{\rm QL}(\tau, \Delta\tau)$ in the limit where $\Delta\tau \to \infty$. We will derive an approximate power series expansion of $f^\alpha_{\rm QL}(\tau, \Delta\tau)$ that is valid in the limit $\Delta\tau \to 0$.

Within a sufficiently small neighborhood of the point $x^\alpha = x^\alpha(\tau)$, the retarded Green's function can be written in the Hadamard form

$$G^{\mu\nu\alpha'\beta'}_{\rm ret}(x, x') = \frac{\Theta[\Sigma(x, x')]}{4\pi}\{U^{\mu\nu\alpha'\beta'}(x, x')\delta[\sigma(x, x')] - V^{\mu\nu\alpha'\beta'}(x, x')\Theta[-\sigma(x, x')]\}. \tag{2.18}$$

Here $\Sigma(x, x')$ is an arbitrary function which is positive for x in the causal future of x' and negative otherwise, $\Theta[\cdot]$ is the Heaviside step function, $\sigma(x, x')$ is Synge's world function [13,63], $\delta[\cdot]$ is the ordinary Dirac delta distribution, and $U^{\mu\nu\alpha'\beta'}(x, x')$ and $V^{\mu\nu\alpha'\beta'}(x, x')$ are smooth functions. The part of the Green's function proportional to $\delta(\sigma)$ is called the direct part, and the part of the Green's function proportional to $\Theta(\sigma)$ is the tail part. For sufficiently small $\Delta\tau$, the portion of the world line between $x^\alpha(\tau - \Delta\tau)$ and $x^\alpha(\tau)$ that arises in Eq. (2.17) will lie inside the neighborhood where the Hadamard expression (2.18) is valid. Inserting this expression into the formula (2.17) for $f^\alpha_{\rm QL}$ gives

$$f^\alpha_{\rm QL}(\tau, \Delta\tau) = -4\mu^2 P^{\alpha\beta\gamma\delta} \int_{\tau-\Delta\tau}^{\tau} V_{\beta\gamma\beta'\gamma';\delta}[x, x'(\tau')] \times u^{\beta'}(\tau') u^{\gamma'}(\tau') d\tau'. \tag{2.19}$$

Note that the direct part of the Green's function does not contribute to the expression (2.19), because of the limiting process involving ϵ in Eq. (2.17). That limiting process is unnecessary for the tail contribution which gives an integrand that is finite at $\tau' = \tau$; hence there is no limiting process in the final result (2.19).

The integral (2.19) can be approximately evaluated for small $\Delta\tau$ using a covariant local expansion of $V_{\beta\gamma\beta'\gamma'}(x, x')$ in a neighborhood of $x' = x$. The result is of the form

$$f_{{\rm QL}\alpha}(\tau, \Delta\tau) = f^{(0)}_\alpha(\tau) + f^{(1)}_\alpha(\tau)\Delta\tau + f^{(2)}_\alpha(\tau)\Delta\tau^2 + f^{(3)}_\alpha(\tau)\Delta\tau^3 + O(\Delta\tau^4). \tag{2.20}$$

Because this expansion arises from a covariant Taylor series, the coefficients $f^{(j)}_\alpha(\tau)$ are purely local geometric quantities evaluated at the present position $x^\alpha(\tau)$ of the particle. In Appendix A we derive the geometric content of these coefficients using simple counting and dimensional arguments. We find that

$$f^{(0)}_\alpha = 0, \tag{2.21}$$

$$f^{(1)}_\alpha = 0, \tag{2.22}$$

$$f^{(2)}_\alpha = c_0\mu^2(\delta^\beta_\alpha + u_\alpha u^\beta) C_{\beta\gamma\delta\varepsilon} C_\sigma{}^\gamma{}_\rho{}^\varepsilon u^\delta u^\sigma u^\rho, \tag{2.23}$$

$$\begin{aligned} f^{(3)}_\alpha = \mu^2(\delta^\beta_\alpha + u_\alpha u^\beta) u^\gamma u^\delta\{&c_1 C_{\gamma\mu\delta\nu} C_\varepsilon{}^\mu{}_\sigma{}^\nu{}_{;\beta} u^\varepsilon u^\sigma \\ &+ c_2 C_{\beta\gamma\mu\delta;\nu} C_\varepsilon{}^\mu{}_\sigma{}^\nu u^\varepsilon u^\sigma + c_3[\tfrac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_\delta{}^\lambda{}_{;\beta} \\ &+ C_{\mu\varepsilon\gamma\lambda} C^\mu{}_{\sigma\delta}{}^\lambda{}_{;\beta} u^\varepsilon u^\sigma] + c_4[\tfrac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_{\delta\beta}{}^{;\lambda} \\ &+ C_{\mu\varepsilon\gamma\lambda} C^\mu{}_{\sigma\delta\beta}{}^{;\lambda} u^\varepsilon u^\sigma]\}, \end{aligned} \tag{2.24}$$

where $c_0, c_1, \ldots, c_4$ are dimensionless numerical coefficients that are as yet undetermined. The calculation of these coefficients proves to be the most difficult part of determining the expansion of $f^\alpha_{\rm QL}$, and is the topic of the next section.

III. EXPANSION OF THE SELF-FORCE

In this section we use a local covariant expansion of the retarded Green's function to compute the numerical coefficients appearing in the expansion (2.20) of the self-force. We use the formalism of bitensors developed by DeWitt and Brehme [8]; see Poisson [13] for a recent detailed review of this formalism. A fundamental role in this formalism is played by Synge's world function $\sigma(x, x')$ and its derivative $\sigma_{;\alpha}(x, x')$ (see Appendix B). The local covariant expansion of any bitensor $T(x, x')$ takes the form

$$T(x, x') = \sum_{n=0}^{\infty} \frac{1}{n!} t_n^{\alpha_1 \ldots \alpha_n}(x) \sigma_{;\alpha_1}(x, x') \ldots \sigma_{;\alpha_n}(x, x'), \quad (3.1)$$

where the coefficients $t_n^{\alpha_1 \ldots \alpha_n}$ are local tensors at x. For bookkeeping purposes we define $s^2 = |\sigma|$, then it follows from Eq. (B5) below that the nth term in Eq. (3.1) scales as s^n. We shall use s as an expansion parameter throughout our computations.

As a foundation for the expansion of the retarded Green's function, we compute in Appendix B local covariant expansions of a number of fundamental bitensors, including the second derivative $\sigma_{;\alpha\beta}$ of the world function, various covariant derivatives of the parallel displacement bivector $g_\alpha{}^{\alpha'}$, and the Van Vleck-Morette determinant Δ, to order $O(s^5)$ beyond the leading order. These expansions were originally computed to $O(s^4)$ by Christensen [59,60], and extended to $O(s^5)$ by Brown and Ottewill [54,55]. Our results agree with those of Brown and Ottewill, except for one case where we correct their result [Eq. (B33) below].

In Appendix C we compute the expansion to order $O(s^3)$ of tail portion $V_{\alpha\beta\alpha'\beta'}$ of the retarded Green's function, extending previous work of Allen, Folacci and Ottewill [57] who computed the expansion to order $O(s^2)$. The result is of the form

$$\begin{aligned} V_{\alpha\beta\alpha'\beta'} = g_{\alpha'}{}^{\gamma} g_{\beta'}{}^{\delta}[&v^0_{\alpha\beta\gamma\delta}(x) + v^0_{\alpha\beta\gamma\delta\varepsilon}(x)\sigma^{;\varepsilon} \\ &+ \tfrac{1}{2} v^0_{\alpha\beta\gamma\delta\varepsilon\zeta}(x)\sigma^{;\varepsilon}\sigma^{;\zeta} \\ &+ \tfrac{1}{6} v^0_{\alpha\beta\gamma\delta\varepsilon\zeta\eta}(x)\sigma^{;\varepsilon}\sigma^{;\zeta}\sigma^{;\eta} + v^1_{\alpha\beta\gamma\delta}(x)\sigma \\ &+ v^1_{\alpha\beta\gamma\delta\varepsilon}(x)\sigma\sigma^{;\varepsilon} + O(s^4)], \end{aligned} \quad (3.2)$$

cf. Eqs. (C9) and (C14) below. Here the various expansion coefficients $v^0_{\alpha\ldots\eta}(x)$ and $v^1_{\alpha\ldots\eta}(x)$ are given in Eqs. (C15)–(C18) and (C22) and (C23) below.

We now turn to evaluation of the integrand in the expression (2.19) for the quasilocal piece of the self-force. First, we note that the four velocity is parallel transported along the world line, so we can make the replacement

$$V^{\beta\gamma}{}_{\beta'\gamma';\delta} u^{\beta'} u^{\gamma'} = [V^{\beta\gamma}{}_{\beta'\gamma';\delta} g^{\beta'}{}_{\mu} g^{\gamma'}{}_{\nu}] u^\mu u^\nu. \quad (3.3)$$

We can rewrite the first factor on the right-hand side as

$$\begin{aligned} V^{\beta\gamma}{}_{\beta'\gamma';\delta} g^{\beta'}{}_{\mu} g^{\gamma'}{}_{\nu} = &\bar{V}^{\beta\gamma}{}_{\mu\nu;\delta} + Q_{\mu\varepsilon\delta} \bar{V}^{\beta\gamma\varepsilon}{}_{\nu} \\ &+ Q_{\nu\varepsilon\delta} \bar{V}^{\beta\gamma}{}_{\mu}{}^{\varepsilon} \end{aligned} \quad (3.4)$$

where we have defined $\bar{V}_{\alpha\beta\gamma\delta} = g_\gamma{}^{\alpha'} g_\delta{}^{\beta'} V_{\alpha\beta\alpha'\beta'}$ and we have used the definition (B18) of the tensor $Q_{\alpha\beta\gamma}$. Using the expansions (3.2), (B15), and (B25) we now obtain

$$\begin{aligned} V_{\beta\gamma\beta'\gamma';\delta} g^{\beta'}{}_{\mu} g^{\gamma'}{}_{\nu} = &\mathcal{V}_{\beta\gamma\mu\nu\delta} + \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho}\sigma_{;\rho} \\ &+ \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho\eta}\sigma_{;\rho}\sigma_{;\eta} + O(s^3), \end{aligned} \quad (3.5)$$

where

$$\mathcal{V}_{\beta\gamma\mu\nu\delta} = v^0_{\beta\gamma\mu\nu;\delta} + v^0_{\beta\gamma\mu\nu\delta}, \quad (3.6)$$

$$\begin{aligned} \mathcal{V}_{\beta\gamma\mu\nu\delta\rho} = &v^0_{\beta\gamma\mu\nu\rho;\delta} + v^0_{\beta\gamma\mu\nu\rho\delta} + v^1_{\beta\gamma\mu\nu} g_{\rho\delta} \\ &- v^0_{\beta\gamma\lambda\nu} C^\lambda{}_{\mu\delta\rho}, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \mathcal{V}_{\beta\gamma\mu\nu\delta\rho\tau} = &-\tfrac{1}{3} v^0_{\beta\gamma\mu\nu\lambda} C^\lambda{}_{\rho\delta\tau} + \tfrac{1}{2} v^0_{\beta\gamma\mu\nu\rho\tau;\delta} \\ &+ \tfrac{1}{2} v^1_{\beta\gamma\mu\nu;\delta} g_{\rho\tau} + \tfrac{1}{2} v^0_{\beta\gamma\mu\nu\rho\tau\delta} \\ &+ \tfrac{3}{2} v^1_{\beta\gamma\mu\nu(\delta} g_{\rho\tau)} - v^0_{\beta\gamma\lambda\nu\rho} C^\lambda{}_{\mu\delta\tau} \\ &+ \tfrac{1}{3} v^0_{\beta\gamma\lambda\nu} C^\lambda{}_{\mu\delta\rho;\tau}. \end{aligned} \quad (3.8)$$

It is understood that the right-hand sides of Eqs. (3.6), (3.7), and (3.8) are to be symmetrized on the index pair $(\mu\nu)$, and on the index pair $(\rho\tau)$ if present. We also note that $\sigma^{;\alpha}(x, x')$ is proportional to the tangent to the geodesic joining $x = x^\alpha(\tau)$ to $x' = x^{\alpha'}(\tau')$, i.e., the four velocity u^α. It follows from the normalization condition (B5) that

$$\sigma^{;\alpha}(x, x') = (\tau - \tau') u^\alpha, \quad (3.9)$$

where u^α is the four velocity at $x^\alpha(\tau)$. Finally, substituting the formulas (3.3), (3.5), and (3.9) into the expression (2.19) gives

$$\begin{aligned} f^\alpha_{\rm QL}(\tau, \Delta\tau) = -4\mu^2 P^{\alpha\beta\gamma\delta} u^\mu u^\nu \int_{\tau-\Delta\tau}^{\tau} d\tau' \{&\mathcal{V}_{\beta\gamma\mu\nu\delta} \\ &+ \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho} u_\rho (\tau - \tau') \\ &+ \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho\tau} u_\rho u_\tau (\tau - \tau')^2 \\ &+ O[(\tau - \tau')^3]\}. \end{aligned} \quad (3.10)$$

Evaluating the integral over τ' gives an expansion of $f^\alpha_{\rm QL}$ of the form (2.20), where the coefficients are

$$f^{(0)\alpha} = 0, \quad (3.11)$$

$$f^{(1)\alpha} = -4\mu^2 P^{\alpha\beta\gamma\delta} u^\mu u^\nu \mathcal{V}_{\beta\gamma\mu\nu\delta}, \quad (3.12)$$

$$f^{(2)\alpha} = -2\mu^2 P^{\alpha\beta\gamma\delta} u^\mu u^\nu \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho} u_\rho, \quad (3.13)$$

$$f^{(3)\alpha} = -\tfrac{4}{3}\mu^2 P^{\alpha\beta\gamma\delta} u^\mu u^\nu \mathcal{V}_{\beta\gamma\mu\nu\delta}{}^{\rho\tau} u_\rho u_\tau. \quad (3.14)$$

The various coefficients $\mathcal{V}_{\alpha\ldots\eta}$ are obtained from the formulas (3.6), (3.7), and (3.8) and are tabulated in Appendix D. We now substitute those expressions into Eqs. (3.11), (3.12), (3.13), and (3.14) and use the definition (2.9) of the projection tensor $P^{\alpha\beta\gamma\delta}$. After a considerable amount of algebra we obtain coefficients $f^{(j)}_\alpha$ of the form (2.21), (2.22), and (2.23), as expected. The numerical values of the coefficients are

$$c_0 = -1 \quad (3.15)$$

and

$$c_1 = \frac{1}{6}, \qquad c_2 = -\frac{3}{20}, \qquad c_3 = \frac{1}{3}, \qquad c_4 = -\frac{19}{60}. \tag{3.16}$$

Our final expression for the quasilocal piece of the self-force is therefore

$$\begin{aligned} f_{\mathrm{QL}\alpha}(\tau, \Delta\tau) = &-\mu^2(\delta_\alpha^\beta + u_\alpha u^\beta) C_{\beta\gamma\delta\varepsilon} C_\sigma{}^\gamma{}_\rho{}^\varepsilon u^\delta u^\sigma u^\rho \Delta\tau^2 + \mu^2(\delta_\alpha^\beta + u_\alpha u^\beta) u^\gamma u^\delta \{\tfrac{1}{6} C_{\gamma\mu\delta\nu} C_\varepsilon{}^\mu{}_\sigma{}^\nu{}_{;\beta} u^\varepsilon u^\sigma \\ &- \tfrac{3}{20} C_{\beta\gamma\mu\delta;\nu} C_\varepsilon{}^\mu{}_\sigma{}^\nu u^\varepsilon u^\sigma + \tfrac{1}{3}[\tfrac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_\delta{}^\lambda{}_{;\beta} + C_{\mu\varepsilon\gamma\lambda} C^\mu{}_{\sigma\delta}{}^\lambda{}_{;\beta} u^\varepsilon u^\sigma] - \tfrac{19}{60}[\tfrac{1}{2} C_{\mu\nu\gamma\lambda} C^{\mu\nu}{}_{\delta\beta}{}^{;\lambda} \\ &+ C_{\mu\varepsilon\gamma\lambda} C^\mu{}_{\sigma\delta\beta}{}^{;\lambda} u^\varepsilon u^\sigma]\}\Delta\tau^3 + O(\Delta\tau^4). \end{aligned} \tag{3.17}$$

IV. SOME SPECIAL CASES

Our expression (3.17) for the quasilocal contribution to the self-force is quite general, applying to a massive particle with any four velocity in any vacuum background space-time. There are, however, cases which are of more intrinsic interest than others. In particular, cases in which the background is a black hole are of interest for developing templates for LISA. We now examine the form that f_{QL}^α takes for several particle four velocities in a Schwarzschild background and for two simple four velocities in a Kerr background.

We begin with the Schwarzschild background. In standard Schwarzschild coordinates, the line element is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.1}$$

where m is the mass of the Schwarzschild black hole. Consider a general particle four velocity. Because of the spherical symmetry of the background, we can always arrange that at any instant, the particle as well as the spatial projection of the particle's four velocity be in the equatorial plane. Doing so, we have that $\theta = \pi/2$ and $u^\theta = 0$ for a general particle four velocity. This leaves three nonvanishing components for u^α, however, the normalization condition $u^\alpha u_\alpha = -1$ can be used to eliminate one (we have chosen to eliminate u^t). Thus, in this background, the velocity of the particle is completely specified in general by a choice of $u^r = dr/d\tau = \dot{r}$ and $u^\phi = d\phi/d\tau = \dot{\phi}$.

Putting this general four velocity and the Schwarzschild metric into (3.17) yields

$$\begin{aligned} f_{\mathrm{QL}}^t = &\mu^2 \frac{m^2}{r^7(r-2m)} \sqrt{r[r\dot{r}^2 + (r-2m)(\dot{\phi}^2 r^2 + 1)]} \\ &\times \bigg[9r^3\dot{\phi}^2(2\dot{\phi}^2 r^2 + 1)\Delta\tau^2 \\ &+ \frac{3}{10}\dot{r}(64\dot{\phi}^2 r^2 + 150\dot{\phi}^4 r^4 + 1)\Delta\tau^3 + O(\Delta\tau^4)\bigg] \end{aligned} \tag{4.2}$$

$$\begin{aligned} f_{\mathrm{QL}}^r = \mu^2 \frac{m^2}{r^8} \bigg[&-9r^4(2\dot{\phi}^2 r^2 + 1)\dot{\phi}^2 \dot{r}\Delta\tau^2 \\ &+ \frac{3}{20}(62\dot{\phi}^2 r^2 m + 80m\dot{\phi}^4 r^4 + 4m - 40\dot{\phi}^4 r^5 \\ &- 2r\dot{r}^2 - 300r^5\dot{r}^2\dot{\phi}^4 - 2r - 128r^3\dot{r}^2\dot{\phi}^2 \\ &- 31\dot{\phi}^2 r^3)\Delta\tau^3 + O(\Delta\tau^4)\bigg] \end{aligned} \tag{4.3}$$

$$f_{\mathrm{QL}}^\theta = 0 \tag{4.4}$$

$$\begin{aligned} f_{\mathrm{QL}}^\phi = &-\mu^2 \frac{m^2}{r^7} \dot{\phi} \bigg[9r(\dot{\phi}^2 r^2 + 1)(2\dot{\phi}^2 r^2 + 1)\Delta\tau^2 + \frac{3}{20} \\ &\times (388\dot{\phi}^2 r^2 + 300\dot{\phi}^4 r^4 + 99)\dot{r}\Delta\tau^3 + O(\Delta\tau^4)\bigg]. \end{aligned} \tag{4.5}$$

One expects, of course, the vanishing of the θ component by symmetry arguments. However, we notice several other features. For purely radial motion ($\dot{\phi} = 0$), the quasilocal part of the self-force vanishes to $O(\Delta\tau^3)$, and the f_{QL}^ϕ component vanishes to $O(\Delta\tau^4)$—indeed, by symmetry arguments it must vanish to all orders. Thus, the quasilocal part of the self-force is directed toward the black hole to order $O(\Delta\tau^4)$ in this case. For purely tangential motion ($\dot{r} = 0$), f_{QL}^r vanishes to $O(\Delta\tau^2)$. Further, there is no $\Delta\tau^3$ contribution to f_{QL}^t or f_{QL}^ϕ in this case. On the other hand, f_{QL}^r does not vanish even if for a static particle, where $\dot{\phi} = \dot{r} = 0$.

A particularly interesting case is that of a particle following a circular geodesic around the black hole. In this case,

$$\dot{r} = 0, \qquad \dot{\phi} = \frac{1}{r}\sqrt{\frac{m}{r-3m}}, \tag{4.6}$$

which gives a quasilocal self-force contribution of

$$f_{\mathrm{QL}}^t = 9\mu^2 \frac{m^3}{r^6} \sqrt{\frac{r}{r-3m}} \frac{(r-m)}{(r-3m)^2} \Delta\tau^2 + O(\Delta\tau^4) \tag{4.7}$$

$$f^r_{\rm QL} = \frac{3}{20}\mu^2\frac{m^2}{r^8}(35m^2 - 19mr - 2r^2)\frac{(r-2m)}{(r-3m)^2}\Delta\tau^3 + O(\Delta\tau^4) \tag{4.8}$$

$$f^\theta_{\rm QL} = 0 \tag{4.9}$$

$$f^\phi_{\rm QL} = -9\mu^2\frac{m^2}{r^7}(r-2m)\sqrt{\frac{m}{r-3m}}\frac{(r-m)}{(r-3m)^2}\Delta\tau^2 + O(\Delta\tau^4). \tag{4.10}$$

Interestingly, we can assign physical meanings to two of these components. Assuming that the self-force causes an adiabatic deviation from the background geodesic, we can define an energy for the particle of $E = u_t$ and an angular momentum of $L = u_\phi$. Thus, $dE/d\tau = du^t/d\tau = f^t/\mu$ and $dL/d\tau = du^\phi/d\tau = f^\phi/\mu$. In other words, $f^t_{\rm QL}/\mu$ and $f^\phi_{\rm QL}/\mu$ can, respectively, be interpreted as the energy and angular momentum radiated by the particle due to the quasilocal part of the self-force.

Of course, we expect astrophysical black holes to provide a Kerr background space-time, so this background is more astrophysically relevant than Schwarzschild. Unfortunately, as is often the case, calculations in the Kerr background lead to longer and less manageable expressions. Using symbolic algebra programs, it is straightforward to calculate, for example, for the case of general equatorial motion in Kerr, and we have done so using MAPLE. The expressions, however, are so unwieldy that we choose not to display them here. Rather, we illustrate with the two simplest motions in a Kerr background, stationary with respect to an observer at rest at infinity and corotating with the black hole. We restrict the particle to be in the equatorial plane in both cases for convenience.

We start with the metric in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - 2a\sin^2\theta\left(\frac{r^2+a^2-\Delta}{\Sigma}\right)dtd\phi \\ &+ \left[\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{4.11}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{4.12}$$

$$\Delta = r^2 + a^2 - 2mr. \tag{4.13}$$

As usual, a is the spin parameter for the Kerr black hole and m is its mass. We will only be considering the special case where the particle is located at $\theta = \pi/2$, and only particles with $u^r = u^\theta = 0$. Again, we have the normalization condition $u^\alpha u_\alpha = -1$ which we use to further eliminate u^t. We therefore have only one specifiable four velocity component, $\dot\phi$.

We begin with the case where $\dot\phi = 0$. This is the case of a particle which is momentarily stationary with respect to an observer at spatial infinity. In that case, the quasilocal part of the self-force becomes

$$f^t_{\rm QL} = 18\mu^2\frac{a^2m^3}{r^{10}}\frac{(r^2-2mr+2a^2)}{(r^2-2mr+a^2)^3}\times\frac{[r(r-2m)(r^2-2mr+a^2)^2]^{3/2}}{(r-2m)^4}\Delta\tau^2 + O(\Delta\tau^4), \tag{4.14}$$

$$f^r_{\rm QL} = -\frac{3}{10}\mu^2\frac{m^2}{r^{11}}\frac{(r^2-2mr+a^2)}{(r-2m)^2}(10a^4 - 16mra^2 + 8a^2r^2 - 4r^3m + r^4 + 4m^2r^2)\Delta\tau^3 + O(\Delta\tau^4), \tag{4.15}$$

$$f^\theta = 0, \tag{4.16}$$

$$f^\phi_{\rm QL} = -9\mu^2\frac{am^2}{r^{10}}\frac{(r^2-2mr+2a^2)}{(r^2-2mr+a^2)^3}\times\frac{[r(r-2m)(r^2-2mr+a^2)^2]^{3/2}}{(r-2m)^3}\Delta\tau^2 + O(\Delta\tau^4). \tag{4.17}$$

It is interesting to note that there is a self-force at order $\Delta\tau^2$ in this case, unlike the case of a static particle in Schwarzschild.

One might argue that this is not unexpected, since in Kerr the particle is "moving" with respect to the rotating background. A fairer comparison, therefore, might be with a particle that is corotating in the Kerr background, i.e., one for which $u_\phi = 0$. However, in this case we find the quasilocal part of the self-force is

$$\begin{aligned} f^r_{\rm QL} = &-\frac{3}{10}\mu^2\frac{m^2}{r^{11}}\frac{(r^2-2mr+a^2)}{(r^3+2ma^2+a^2r)^2}(22a^4m^2r^2 - 27a^2r^5m - 71a^4r^3m - 44a^6rm + r^8 + 10r^6a^2 \\ &+ 27a^4r^4 + 28a^6r^2 + 10a^8)\Delta\tau^3 + O(\Delta\tau^4) \end{aligned} \tag{4.18}$$

$$f^\phi_{\rm QL} = -9\mu^2\frac{m^2a}{r^9}\frac{(r^2+a^2)(r^4+3a^2r^2-2a^2mr+2a^4)}{(r^2-2mr+a^2)(r^3+a^2r+2ma^2)^4}\times[r(r^3+a^2r+2ma^2)(r^2-2mr+a^2)]^{3/2}\times\Delta\tau^2 + O(\Delta\tau^4). \tag{4.19}$$

Thus, in both analogues of the static particle in a Schwarzschild background, the rotation of the Kerr background induces a radiation reaction force at order $\Delta\tau^3$.

We end this section with a warning. While the expressions derived here might be useful for comparisons, they do not, in general, have any intrinsic meaning (the case of the circular geodesic in Schwarzschild being somewhat of an exception). One obvious reason for this is that we have found only a part of the self-force which reflects a very limited part of the particle's world line. However, there is a more subtle limitation as well. The self-force is a gauge dependent quantity, depending on the perturbation gauge chosen. Our expressions, which are composed of quantities that are gauge invariants of the background, are nonetheless tied to the Lorentz perturbation gauge. Thus, these expressions may, in general, only be legitimately compared to other expressions for the self-force in the Lorentz gauge.

V. CONCLUSION

In this paper, we have discussed a novel approach to calculating the self-force experienced by a massive particle moving on a geodesic in a curved background. In this approach, proposed by Poisson and Wiseman, one does not regularize the retarded Green's functions nor the retarded field. Rather, one explicitly calculates the tail part of the retarded Green's function in the normal neighborhood of its current position. From this, one can calculate the quasilocal part of the self-force, which arises from some finite portion of the particle's world line within this neighborhood, of duration $\Delta\tau$. The rest of the self-force can then be obtained using the full retarded Green's function (without need of regularization), since it is well behaved (and gives the correct contribution) for the portion of the particle's world line beyond $\Delta\tau$ to the past.

We have also carried out the first step in this procedure, the calculation of the first two nonvanishing terms of an expansion of the quasilocal part of the self-force in $\Delta\tau$. Our expression has some remarkable properties. First, it is quite general, in that it does not rely on prior specification of the particle motion nor on prior specification of the background geometry (we have restricted our attention to vacuum backgrounds in this paper, but this was for convenience, and exactly the same procedure can be used for calculating the quasilocal part of the self-force in backgrounds with matter). Second, we express the quasilocal part of the self-force in terms of quantities that are purely local to the particle. Thus, our expression does not require a detailed understanding of the past history of the particle—it is only a function of the current position and velocity of the particle. This can hardly be surprising—within a normal neighborhood, there is a unique geodesic specified by any four vector at a given point, which the particle is assumed to be travelling along.

Much work remains in order to determine even if the Poisson-Wiseman prescription is feasible, let alone to calculate a complete self-force using it. While the approach here is quite general, it is likely that the retarded Green's function from which the rest of the self-force is calculated will be most easily obtained in most scenarios of interest by a mode-sum expansion. It remains to be seen whether it is technically feasible to calculate the quasilocal part of the self-force to sufficient order that it has sufficient precision at a distance from the particle at which the mode-sum converges well. Work is currently under way to begin addressing such questions [64].

Some hope arises from the work of Anderson and Hu [65], who have shown how to calculate the tail part of the retarded Green's function for a scalar particle in Schwarzschild using the Hadamard-WKB approximation. This approach, which should work for spin-1 (electromagnetic) and spin-2 (gravitational) fields on a Schwarzschild background, allows one to calculate to a much higher order in $\Delta\tau$ for a fixed amount of effort. Furthermore, it might be extendible to Kerr backgrounds. In any case, we echo their sentiment, that this approach warrants further investigation.

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APPENDIX A: GEOMETRIC CONTENT OF THE EXPANSION COEFFICIENTS

In this appendix, we use symmetry and dimensional arguments to deduce the forms of the coefficients $f_\alpha^{(0)}$, $f_\alpha^{(1)}$, $f_\alpha^{(2)}$ and $f_\alpha^{(3)}$ appearing in the expansion (2.20) of the quasilocal piece of the self-force, up to some unknown numerical coefficients. We start by noting that the self-force is proportional to the square μ^2 of the particle's mass, from Eq. (2.19). Factoring out this factor of μ^2, we can write the expansion (2.20) as

$$f_{\mathrm{QL}\alpha}(\tau, \Delta\tau) = \mu^2[\hat{f}_\alpha^{(0)} + \hat{f}_\alpha^{(1)}\Delta\tau + \hat{f}_\alpha^{(2)}\Delta\tau^2 + \hat{f}_\alpha^{(3)}\Delta\tau^3 + O(\Delta\tau^4)]. \tag{A1}$$

Here the coefficients $\hat{f}_\alpha^{(j)} = f_\alpha^{(j)}/\mu^2$ satisfy the following key properties:

(i) They are independent of the particle mass μ.

(ii) They must be constructed as polynomial expressions in the following tensors at $x^\alpha(\tau)$: the metric $g_{\alpha\beta}$, the four velocity u^α, and the Weyl tensor and its various symmetrized derivatives $C_{\alpha\beta\gamma\delta}$,

$C_{\alpha\beta\gamma\delta;\varepsilon}$, $C_{\alpha\beta\gamma\delta;(\varepsilon\rho)}$, etc. The Ricci tensor does not appear since we are assuming a vacuum background, and the four acceleration of the curve a^α does not appear since we are assuming a geodesic curve at zeroth order. The Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$ cannot appear, since the expression (2.19) for the self-force is invariant under the parity transformation $\epsilon_{\alpha\beta\gamma\delta} \to -\epsilon_{\alpha\beta\gamma\delta}$.

(iii) Since the force is dimensionless (in geometric units in which $G = c = 1$), and $\Delta\tau$ has dimensions of length, each coefficient $\hat{f}^{(j)}_\alpha$ has dimension $(\text{length})^{-2-j}$.

(iv) Each coefficient $\hat{f}^{(j)}_\alpha$ must be orthogonal to the four velocity u^α, since the total force (2.19) has this property.

We now apply these properties to deduce the most general allowed forms of the various coefficients, which are given in Eqs. (2.21), (2.22), (2.23), and (2.24) above.

1. The zeroth order coefficient $\hat{f}^{(0)}_\alpha$

The coefficient $\hat{f}^{(0)}_\alpha$ has dimension $(\text{length})^{-2}$, and hence must be linear in $C_{\alpha\beta\gamma\delta}$ which also has dimension $(\text{length})^{-2}$. None of the derivatives of the Weyl tensor can appear. However, there is no nonvanishing vector that can be constructed out of $C_{\alpha\beta\gamma\delta}$, $g_{\alpha\beta}$ and u^α that is orthogonal to u^α. One needs to contract at least three of the indices on the Weyl tensor with the metric or the four velocity. However the Weyl tensor is traceless on all pairs of indices, so one cannot use the metric to contract any pair of indices. Also the antisymmetry properties $C_{(\alpha\beta)\gamma\delta} = C_{\alpha\beta(\gamma\delta)} = 0$ of the Weyl tensor mean that one cannot contract with three factors of four velocity. Hence the coefficient $\hat{f}^{(0)}_\alpha$ must vanish cf. Eq. (2.21) above. A version of this argument was first given by Ori, and was used to deduce the fact that the formula for the gravitational self-force could contain only the tail term and could not contain any local terms, unlike the scalar and electromagnetic self-force expressions [66].

An alternative, simpler version of the argument can be obtained by considering the independent, electric and magnetic components of the Weyl tensor. Introduce an orthonormal basis $e^\alpha_{\hat{0}} = u^\alpha$ and $e^\alpha_{\hat{j}}$, $1 \le j \le 3$. Then we define

$$\mathcal{E}_{\hat{i}\hat{j}} = C_{\hat{0}\hat{i}\hat{0}\hat{j}} \equiv e^\alpha_{\hat{0}} e^\beta_{\hat{i}} e^\gamma_{\hat{0}} e^\delta_{\hat{j}} C_{\alpha\beta\gamma\delta} \tag{A2}$$

and

$$\mathcal{B}_{\hat{i}\hat{j}} = -\tfrac{1}{2}\epsilon_{\hat{i}\hat{k}\hat{l}} C_{\hat{k}\hat{l}\hat{0}\hat{j}} \equiv -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta} C^{\gamma\delta}{}_{\varepsilon\rho} u^\alpha e^\beta_{\hat{i}} u^\varepsilon e^\rho_{\hat{j}}. \tag{A3}$$

In vacuum these are symmetric, traceless tensors, and the Weyl tensor can be expressed in terms of these tensors via the formulas

$$C_{\hat{0}\hat{i}\hat{j}\hat{k}} = -\epsilon_{\hat{l}\hat{j}\hat{k}} \mathcal{B}_{\hat{l}\hat{i}} \tag{A4}$$

and

$$C_{\hat{i}\hat{j}\hat{k}\hat{l}} = -\epsilon_{\hat{i}\hat{j}\hat{p}} \epsilon_{\hat{k}\hat{l}\hat{q}} \mathcal{E}_{\hat{p}\hat{q}} \tag{A5}$$

together with Eq. (A2). The coefficient $\hat{f}^{(0)}_{\hat{i}}$ must be a three vector that is linear in $\mathcal{E}_{\hat{i}\hat{j}}$, or linear in $\mathcal{B}_{\hat{i}\hat{j}}$ with one factor of $\epsilon_{\hat{i}\hat{j}\hat{k}}$ by parity arguments, and there is no such vector since $\mathcal{B}_{[\hat{i}\hat{j}]} = 0$.

2. The first order coefficient $\hat{f}^{(1)}_\alpha$

The coefficient $\hat{f}^{(1)}_\alpha$ has dimension $(\text{length})^{-3}$ and so must be linear in $C_{\alpha\beta\gamma\delta;\varepsilon}$. Again, it is impossible to form a nonvanishing four vector that is orthogonal to u^α by contracting $C_{\alpha\beta\gamma\delta;\varepsilon}$ with the metric and/or four velocity. The additional derivative index ε cannot be contracted with any of the indices on the Weyl tensor, since by the Bianchi identity the Weyl tensor is divergence free on all its indices in vacuum:

$$C_{\alpha\beta\gamma\delta}{}^{;\delta} = 0. \tag{A6}$$

The ε index can be contracted with the four velocity to form $C_{\alpha\beta\gamma\delta;\varepsilon} u^\varepsilon$, but then one is faced with the same problem as above of contracting three of the four remaining free indices to obtain a vector. It follows that $\hat{f}^{(1)}_\alpha$ must vanish cf. Eq. (2.22) above.

One can also phrase this argument in terms of the electric and magnetic components of the Weyl tensor as before. The components of $C_{\alpha\beta\gamma\delta;\varepsilon}$ can be represented as the "time derivatives" and "spatial derivatives" of $\mathcal{E}_{\hat{i}\hat{j}}$ and $\mathcal{B}_{\hat{i}\hat{j}}$:

$$\dot{\mathcal{E}}_{\hat{i}\hat{j}} \equiv C_{\alpha\beta\gamma\delta;\varepsilon} u^\alpha e^\beta_{\hat{i}} u^\gamma e^\delta_{\hat{j}} u^\varepsilon \tag{A7}$$

$$\mathcal{E}_{\hat{i}\hat{j},\hat{k}} \equiv C_{\alpha\beta\gamma\delta;\varepsilon} u^\alpha e^\beta_{\hat{i}} u^\gamma e^\delta_{\hat{j}} e^\varepsilon_{\hat{k}} \tag{A8}$$

$$\dot{\mathcal{B}}_{\hat{i}\hat{j}} \equiv -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta} C^{\gamma\delta}{}_{\varepsilon\rho;\sigma} u^\alpha e^\beta_{\hat{i}} u^\varepsilon e^\rho_{\hat{j}} u^\sigma. \tag{A9}$$

$$\mathcal{B}_{\hat{i}\hat{j},\hat{k}} \equiv -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta} C^{\gamma\delta}{}_{\varepsilon\rho;\sigma} u^\alpha e^\beta_{\hat{i}} u^\varepsilon e^\rho_{\hat{j}} e^\sigma_{\hat{k}}. \tag{A10}$$

These quantities are not all independent but obey constraints that follow from the Bianchi identity:

$$\mathcal{E}_{\hat{i}\hat{j},\hat{j}} = 0, \tag{A11}$$

$$\mathcal{B}_{\hat{i}\hat{j},\hat{j}} = 0, \tag{A12}$$

$$\dot{\mathcal{E}}_{\hat{i}\hat{j}} = -\epsilon_{\hat{i}\hat{k}\hat{l}} \mathcal{B}_{\hat{j}\hat{k},\hat{l}}, \tag{A13}$$

$$\dot{\mathcal{B}}_{\hat{i}\hat{j}} = \epsilon_{\hat{i}\hat{k}\hat{l}} \mathcal{E}_{\hat{j}\hat{k},\hat{l}}. \tag{A14}$$

Since the time derivatives can be obtained from the spatial derivatives, we can without loss of generality restrict attention to the spatial derivatives. Thus, the coefficient $\hat{f}^{(1)}_{\hat{i}}$ must depend linearly on $\mathcal{E}_{\hat{i}\hat{j},\hat{k}}$ and/or $\mathcal{B}_{\hat{i}\hat{j},\hat{k}}$, and can depend in addition only on $\delta_{\hat{i}\hat{j}}$ and on $\epsilon_{\hat{i}\hat{j}\hat{k}}$. Consider first the derivative $\mathcal{E}_{\hat{i}\hat{j},\hat{k}}$. This quantity cannot be contracted with $\epsilon_{\hat{i}\hat{j}\hat{k}}$ by parity arguments, and it is easy to see that one cannot obtain a nonvanishing spatial vector. The only candidate vectors are the divergence $\mathcal{E}_{\hat{i}\hat{j},\hat{k}}\delta^{\hat{j}\hat{k}}$ which vanishes by Eq. (A11), and the contraction $\mathcal{E}_{\hat{i}\hat{j},\hat{k}}\delta^{\hat{i}\hat{j}}$, which vanishes by the traceless property of $\mathcal{E}_{\hat{i}\hat{j}}$. Next, consider the derivative $\mathcal{B}_{\hat{i}\hat{j},\hat{k}}$. By parity, this quantity must be accompanied by one factor of $\epsilon_{\hat{i}\hat{j}\hat{k}}$ (or an odd number of factors of it). Any such tensor will have an even number of indices, and consequently it is impossible to obtain by contraction a vector.

3. The second order coefficient $\hat{f}^{(2)}_\alpha$

The coefficient $\hat{f}^{(2)}_\alpha$ has dimension (length)$^{-4}$, and so must be either quadratic in $C_{\alpha\beta\gamma\delta}$ or linear in $C_{\alpha\beta\gamma\delta;(\varepsilon\sigma)}$. Consider first the second derivative term $C_{\alpha\beta\gamma\delta;(\varepsilon\sigma)}$. Let us first analyze the term $C_{\alpha\beta\gamma\delta;\varepsilon\sigma}$ without the symmetrization. The six indices on this tensor must be reduced to one index by contractions with the metric and/or the four velocity. As before, the problem is getting rid of at least three of the indices α, β, γ and δ. The only new type of contraction that is available is to contract one of these indices with the second derivative index σ [contractions with the first derivative index ε vanish by Eq. (A6)]. However, such a contraction can be reexpressed as a product of two Weyl tensors by commuting the indices. The same argument also applies to the symmetrized derivative $C_{\alpha\beta\gamma\delta;(\varepsilon\sigma)}$, since it is a linear combination of unsymmetrized derivatives. Thus, it is sufficient to consider expressions that are quadratic in the Weyl tensor.[5]

Consider therefore expressions that are quadratic in the Weyl tensor, or linear in $C_{\alpha\beta\gamma\delta}C_{\varepsilon\sigma\rho\lambda}$. We can classify such expressions in terms of the number of contractions between indices on the first Weyl tensor and indices on the second Weyl tensor. The cases of zero, four and one contractions are easy to dispense with. For example, in the case of one contraction, there is no vector orthogonal to u^α that can be obtained by contracting $C_{\alpha\beta\gamma\delta}C_{\varepsilon\sigma\rho}{}^{\delta}$ with $g_{\alpha\beta}$ and u^α (without further contractions between the two Weyl tensors). In the case of three contractions, there are two different tensors that one can construct, namely

$$V_{\alpha\beta} = C_{\alpha\gamma\delta\varepsilon}C_{\beta}{}^{\gamma\delta\varepsilon} \tag{A15}$$

and

$$W_{\alpha\beta} = C_{\alpha\gamma\delta\varepsilon}C_{\beta}{}^{\delta\gamma\varepsilon}. \tag{A16}$$

However it follows from $C_{\alpha[\beta\gamma\delta]} = 0$ that $V_{\alpha\beta} = 2W_{\alpha\beta}$, so there is only one independent tensor obtainable from three contractions. Furthermore using the formulas (A2), (A4), and (A5) one can show that $W_{\alpha\beta}$ is proportional to the metric:

$$W_{\alpha\beta} = \tfrac{1}{4}W_{\gamma}{}^{\gamma}g_{\alpha\beta}. \tag{A17}$$

Hence there is no nonvanishing vector that can be formed using three contractions.

Consider next the case of two contractions. There are three different four index tensors that can be obtained with two contractions, namely

$$X_{\alpha\beta\varepsilon\sigma} = C_{\alpha\beta\gamma\delta}C_{\varepsilon\sigma}{}^{\gamma\delta}, \tag{A18}$$

$$Y_{\alpha\gamma\varepsilon\rho} = C_{\alpha\beta\gamma\delta}C_{\varepsilon}{}^{\beta}{}_{\rho}{}^{\delta}, \tag{A19}$$

and

$$Z_{\alpha\gamma\varepsilon\sigma} = C_{\alpha\beta\gamma\delta}C_{\varepsilon\sigma}{}^{\beta\delta}. \tag{A20}$$

The tensor $X_{\alpha\beta\varepsilon\sigma}$ does not yield any candidate expressions, since one needs to contract it with three factors of the four velocity, and it is antisymmetric on the index pairs $(\alpha\beta)$ and $(\varepsilon\sigma)$. Similarly the tensor $Z_{\alpha\gamma\varepsilon\sigma}$ is antisymmetric on the index pairs $(\alpha\gamma)$ and $(\varepsilon\sigma)$, and so does not yield any candidate expressions. The tensor $Y_{\alpha\gamma\varepsilon\rho}$ can be used to construct the nonvanishing four vector $Y_{\alpha\gamma\varepsilon\rho}u^\gamma u^\varepsilon u^\rho$. Projecting this vector orthogonal to the four velocity yields the possible term

$$(\delta^\beta_\alpha + u_\alpha u^\beta)C_{\beta\gamma\delta\varepsilon}C_{\sigma}{}^{\gamma}{}_{\rho}{}^{\varepsilon}u^\delta u^\sigma u^\rho. \tag{A21}$$

This quantity is nonvanishing in general and so can appear in the expression for $\hat{f}^{(2)}_\alpha$ cf. Eq. (2.23) above. It is the only term that arises at this order in $\Delta\tau$. Using the formulas (A2) and (A4), we can express this term in terms of the electric and magnetic components of the Weyl tensor as

$$-\epsilon_{\hat{i}\hat{j}\hat{k}}\mathcal{E}_{\hat{j}\hat{l}}\mathcal{B}_{\hat{l}\hat{k}}. \tag{A22}$$

This completes the derivation of the most general allowed form of $\hat{f}^{(2)}_\alpha$.

The analysis of expressions that are quadratic in the Weyl tensor can be rephrased more simply in terms of the tensors $\mathcal{E}_{\hat{i}\hat{j}}$ and $\mathcal{B}_{\hat{i}\hat{j}}$. One needs a spatial vector that is bilinear in $\mathcal{E}_{\hat{i}\hat{j}}$ and/or $\mathcal{B}_{\hat{i}\hat{j}}\epsilon_{\hat{p}\hat{q}\hat{r}}$, since by parity each factor of $\mathcal{B}_{\hat{i}\hat{j}}$ must be accompanied by a factor of $\epsilon_{\hat{p}\hat{q}\hat{r}}$. It is easy to see that the only nonvanishing candidate expression is the product (A22).

[5] In fact, the only nontrivial candidate expression one can construct from the symmetrized second derivative vanishes. This expression is

$$C_{\alpha\beta\gamma\delta;(\varepsilon\sigma)}g^{\sigma\delta}u^\beta u^\gamma u^\varepsilon = -\tfrac{1}{2}[C_{\varepsilon\sigma\alpha}{}^{\rho}C_{\rho\beta\gamma}{}^{\sigma} + C_{\varepsilon\sigma\beta}{}^{\rho}C_{\alpha\rho\gamma}{}^{\sigma} + C_{\varepsilon\sigma\gamma\rho}C_{\alpha\beta}{}^{\rho\sigma}]g^{\sigma\delta}u^\beta u^\gamma u^\varepsilon,$$

where we have used Eq. (A6). Here the last term in the square brackets vanishes since the first factor is symmetric in the index pair $(\rho\sigma)$ by virtue of being contracted with $u^\varepsilon u^\gamma$, and the second factor is antisymmetric in $(\rho\sigma)$. The remaining two terms in the square brackets cancel against each other.

4. The third order coefficient $\hat{f}_\alpha^{(3)}$

The coefficient $\hat{f}_\alpha^{(3)}$ has dimension $(\text{length})^{-5}$, and so must be either bilinear in $C_{\alpha\beta\gamma\delta}$ and $C_{\alpha\beta\gamma\delta;\varepsilon}$, or else linear in the symmetrized third derivative $C_{\alpha\beta\gamma\delta;(\varepsilon\sigma\rho)}$. The argument used in the first paragraph of Sec. A 3 above also applies here and shows that any expression constructed from the third derivative can be expressed as a product of the Weyl tensor and a first derivative of the Weyl tensor. Therefore it is sufficient to consider such products.

For analyzing these products, a fully covariant analysis would be very complex, so we use the simpler formalism of the electric and magnetic components. We need to construct a vector that is a contraction of a product of one of the tensors

$$\mathcal{E}_{\hat{i}\hat{j}}, \qquad \mathcal{B}_{\hat{i}\hat{j}}\epsilon_{\hat{p}\hat{q}\hat{r}} \tag{A23}$$

together with one of the tensors

$$\mathcal{E}_{\hat{i}\hat{j},\hat{k}}, \qquad \mathcal{B}_{\hat{i}\hat{j},\hat{k}}\epsilon_{\hat{p}\hat{q}\hat{r}}. \tag{A24}$$

As before we can neglect the time derivatives because of Eqs. (A13) and (A14). It follows from the identities (A11)–(A14) and the fact that all of the tensors (A23) and (A24) are symmetric and tracefree on the index pair $(\hat{i}\hat{j})$ that there are only four possible nonvanishing vectors that one can construct, namely[6]

$$\mathcal{E}_{\hat{j}\hat{k}}\mathcal{E}_{\hat{j}\hat{k},\hat{i}}, \qquad \mathcal{E}_{\hat{j}\hat{k}}\mathcal{E}_{\hat{i}\hat{j},\hat{k}}, \qquad \mathcal{B}_{\hat{j}\hat{k}}\mathcal{B}_{\hat{j}\hat{k},\hat{i}}, \quad \text{and} \quad \mathcal{B}_{\hat{j}\hat{k}}\mathcal{B}_{\hat{i}\hat{j},\hat{k}}. \tag{A25}$$

The corresponding covariant expressions can be obtained from Eqs. (A2) and (A3) and are, respectively,

$$(\delta_\alpha^\beta + u_\alpha u^\beta)C_{\gamma\mu\delta\nu}C_\varepsilon{}^\mu{}_\sigma{}^\nu{}_{;\beta}u^\gamma u^\delta u^\varepsilon u^\sigma, \tag{A26}$$

$$C_{\alpha\gamma\mu\delta;\nu}C_\varepsilon{}^\mu{}_\sigma{}^\nu u^\gamma u^\delta u^\varepsilon u^\sigma, \tag{A27}$$

$$(\delta_\alpha^\beta + u_\alpha u^\beta)u^\gamma u^\delta[\tfrac{1}{2}C_{\mu\nu\gamma\lambda}C^{\mu\nu}{}_\delta{}^\lambda{}_{;\beta} + C_{\mu\varepsilon\gamma\lambda}C^\mu{}_{\sigma\delta}{}^\lambda{}_{;\beta}u^\varepsilon u^\sigma], \tag{A28}$$

and

$$u^\gamma u^\delta[\tfrac{1}{2}C_{\mu\nu\gamma\lambda}C^{\mu\nu}{}_{\delta\alpha}{}^{;\lambda} + C_{\mu\varepsilon\gamma\lambda}C^\mu{}_{\sigma\delta\alpha}{}^{;\lambda}u^\varepsilon u^\sigma], \tag{A29}$$

cf. Eq. (2.24) above.

[6]One can also construct from the time derivatives the possible terms $\epsilon_{\hat{i}\hat{j}\hat{k}}\dot{\mathcal{B}}_{\hat{j}\hat{l}}\mathcal{E}_{\hat{l}\hat{k}}$ and $\epsilon_{\hat{i}\hat{j}\hat{k}}\mathcal{B}_{\hat{j}\hat{l}}\dot{\mathcal{E}}_{\hat{l}\hat{k}}$. However these terms can be expressed in terms of the four quantities (A25) using the formulas (A13) and (A14):

$$\epsilon_{\hat{i}\hat{j}\hat{k}}\dot{\mathcal{B}}_{\hat{j}\hat{l}}\mathcal{E}_{\hat{l}\hat{k}} = \mathcal{E}_{\hat{j}\hat{k}}\mathcal{E}_{\hat{j}\hat{k},\hat{i}} - \mathcal{E}_{\hat{j}\hat{k}}\mathcal{E}_{\hat{i}\hat{j},\hat{k}},$$
$$\epsilon_{\hat{i}\hat{j}\hat{k}}\mathcal{B}_{\hat{j}\hat{l}}\dot{\mathcal{E}}_{\hat{l}\hat{k}} = \mathcal{B}_{\hat{j}\hat{k}}\mathcal{B}_{\hat{j}\hat{k},\hat{i}} - \mathcal{B}_{\hat{j}\hat{k}}\mathcal{B}_{\hat{i}\hat{j},\hat{k}}.$$

APPENDIX B: LOCAL EXPANSIONS OF SOME FUNDAMENTAL BITENSORS

In this appendix we compute local covariant expansions of a number of fundamental bitensors, including the second derivative $\sigma_{;\alpha\beta}$ of the world function, various covariant derivatives of the parallel displacement bivector $g_\alpha{}^{\alpha'}$, and the Van Vleck-Morette determinant Δ, to order $O(s^5)$ beyond the leading order. These expansions were originally computed to $O(s^4)$ by Christensen [59,60], and extended to $O(s^5)$ by Brown and Ottewill [54,55]. Our results agree with those of Brown and Ottewill, except for one case where we correct their result [Eq. (B33) below]. All of the formulas in this appendix are valid in any number of space-time dimensions, except in specific cases, noted below, which make use of the identity (B4) which is specific to four dimensions. For a detailed and pedagogic review of the computational methods used here see Ref. [13].

We first note a number of useful identities that follow from the Bianchi identity and from properties of the Riemann tensor:

$$R_{\alpha\beta\gamma\delta}R_\varepsilon{}^{\delta\gamma\beta} = \tfrac{1}{2}R_{\alpha\beta\gamma\delta}R_\varepsilon{}^{\beta\gamma\delta} \tag{B1}$$

$$R_{\alpha\beta\gamma\delta}R_\varepsilon{}^{\delta\gamma\beta}{}_{;\mu} = \tfrac{1}{2}R_{\alpha\beta\gamma\delta}R_\varepsilon{}^{\beta\gamma\delta}{}_{;\mu} \tag{B2}$$

$$R_\alpha{}^{\mu\nu\lambda}R_{\beta\mu\gamma\nu;\lambda} = -\tfrac{1}{2}R_\alpha{}^{\mu\nu\lambda}R_{\beta\mu\nu\lambda;\gamma} \tag{B3}$$

$$C_{\alpha\mu\nu\lambda}C_\beta{}^{\mu\nu\lambda} = \tfrac{1}{4}g_{\alpha\beta}C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}. \tag{B4}$$

Next, we derive the coincidence limits $x' \to x$ of symmetrized derivatives of the world function σ. Following Ref. [8], these can be obtained by repeatedly differentiating the identity

$$\sigma = \tfrac{1}{2}g^{\alpha\beta}\sigma_{;\alpha}\sigma_{;\beta} \tag{B5}$$

and by taking the coincidence limit. The results are

$$\lim_{x'\to x}\sigma_{;\alpha\beta} = g_{\alpha\beta}, \tag{B6}$$

$$\lim_{x'\to x}\sigma_{;\alpha\beta\gamma} = 0, \tag{B7}$$

$$\lim_{x'\to x}\sigma_{;\alpha\beta\gamma\delta} = -\tfrac{1}{3}[R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}], \tag{B8}$$

$$\lim_{x'\to x}\sigma_{;\alpha\beta\gamma\delta\varepsilon} = -\tfrac{1}{4}[R_{\alpha\gamma\beta\delta;\varepsilon} + R_{\alpha\delta\beta\gamma;\varepsilon} + R_{\alpha\delta\beta\varepsilon;\gamma} + R_{\alpha\varepsilon\beta\delta;\gamma} + R_{\alpha\varepsilon\beta\gamma;\delta} + R_{\alpha\gamma\beta\varepsilon;\delta}], \tag{B9}$$

$$\lim_{x'\to x}\sigma_{;\alpha\beta(\gamma\delta\varepsilon\mu)} = -\tfrac{12}{5}R_{\alpha\gamma\beta\delta;\varepsilon\mu} - \tfrac{8}{15}R_{\alpha\gamma\delta\chi}R_{\beta\varepsilon\mu}{}^\chi, \tag{B10}$$

and

$$\lim_{x'\to x}\sigma_{;\alpha\beta(\gamma\delta\varepsilon\mu\nu)} = -\tfrac{10}{3}R_{\alpha\gamma\beta\delta;\varepsilon\mu\nu} - \tfrac{10}{3}R_{\alpha\gamma\delta\chi}R_{\beta\varepsilon\mu}{}^{\chi}{}_{;\nu}. \tag{B11}$$

In Eqs. (B10) and (B11), the right-hand sides are to be symmetrized over the index pair $(\alpha\beta)$ and over as many of the indices $(\gamma\delta\varepsilon\mu\nu)$ as are present.

Next, consider any smooth bitensor $T(x, x')$ (we suppress tensor indices on T). We can expand this bitensor about x as a covariant Taylor expansion

$$T(x, x') = \sum_{n=0}^{\infty}\frac{1}{n!}t_n^{\alpha_1\ldots\alpha_n}(x)\sigma_{;\alpha_1}(x, x')\ldots\sigma_{;\alpha_n}(x, x'), \tag{B12}$$

where the coefficients $t_n^{\alpha_1\ldots\alpha_n}$ are local tensors at x. By repeatedly differentiating Eq. (B12) and taking the coincidence limit we find for $n = 0$ that

$$t_0(x) = \lim_{x'\to x}T(x, x'), \tag{B13}$$

together with the recursion relation for $n \geq 1$

$$t_{n\alpha_1\ldots\alpha_n}(x) = \lim_{x'\to x}T_{;(\alpha_1\ldots\alpha_n)}(x, x') - \sum_{r=0}^{n-1}\binom{n}{r}t_{r(\alpha_1\ldots\alpha_r;\alpha_{r+1}\ldots\alpha_n)}(x). \tag{B14}$$

We now apply the recursion relation (B14) to the bitensor $\sigma_{;\alpha\beta}(x, x')$, using the formulas (B6)–(B11). The result is the expansion

$$\begin{aligned}\sigma_{;\alpha\beta} = {} & g_{\alpha\beta} - \tfrac{1}{3}R_{\alpha\gamma\beta\delta}\sigma^{;\gamma}\sigma^{;\delta} + \tfrac{1}{12}R_{\alpha\gamma\beta\delta;\varepsilon}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon} \\ & - [\tfrac{1}{60}R_{\alpha\gamma\beta\delta;\varepsilon\mu} + \tfrac{1}{45}R_{\alpha\gamma\delta\rho}R_{\beta\varepsilon\mu}{}^{\rho}]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu} \\ & + [\tfrac{1}{360}R_{\alpha\gamma\beta\delta;\varepsilon\mu\nu} + \tfrac{1}{120}R_{\alpha\gamma\delta\rho;\nu}R_{\beta\varepsilon\mu}{}^{\rho} \\ & + \tfrac{1}{120}R_{\alpha\gamma\delta\rho}R_{\beta\varepsilon\mu}{}^{\rho}{}_{;\nu}]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu}\sigma^{;\nu} + O(s^6).\end{aligned} \tag{B15}$$

It follows from Eq. (B15) that in vacuum

$$\Box\sigma = n + O(s^4), \tag{B16}$$

where n is the number of space-time dimensions, and that

$$\Box(\sigma^{;\alpha}) = O(s^3). \tag{B17}$$

A similar computation can be carried out for the tensor $Q_{\alpha\beta\gamma}$ defined by

$$Q_{\alpha\beta\gamma} \equiv g_{\alpha}{}^{\alpha'}g_{\alpha'\beta;\gamma}. \tag{B18}$$

By starting with the identity [8]

$$g^{\beta\gamma}g_{\alpha'\alpha;\beta}\sigma_{;\gamma} = 0, \tag{B19}$$

repeatedly differentiating and taking the coincidence limit one obtains

$$\lim_{x'\to x}g_{\alpha'\alpha;\beta} = 0, \tag{B20}$$

$$\lim_{x'\to x}g_{\alpha'\alpha;\beta\gamma}g^{\alpha'}{}_{\rho} = -\tfrac{1}{2}R_{\alpha\rho\beta\gamma}, \tag{B21}$$

$$\lim_{x'\to x}g_{\alpha'\alpha;\beta(\gamma\delta)}g^{\alpha'}{}_{\rho} = -\tfrac{2}{3}R_{\alpha\rho\beta\gamma;\delta}, \tag{B22}$$

$$\lim_{x'\to x}g_{\alpha'\alpha;\beta(\gamma\delta\varepsilon)}g^{\alpha'}{}_{\rho} = -\tfrac{3}{4}R_{\alpha\rho\beta\gamma;\delta\varepsilon} - \tfrac{1}{4}R_{\alpha\rho\gamma\varphi}R_{\beta\delta\varepsilon}{}^{\varphi}, \tag{B23}$$

and

$$\begin{aligned}\lim_{x'\to x}g_{\alpha'\alpha;\beta(\gamma\delta\varepsilon\mu)}g^{\alpha'}{}_{\rho} = {} & -\tfrac{4}{5}R_{\alpha\rho\beta\gamma;\delta\varepsilon\mu} - \tfrac{8}{15}R_{\alpha\rho\gamma\varphi;\mu}R_{\beta\delta\varepsilon}{}^{\varphi} \\ & - \tfrac{3}{5}R_{\alpha\rho\gamma\varphi}R_{\beta\delta\varepsilon}{}^{\varphi}{}_{;\mu}.\end{aligned} \tag{B24}$$

In Eqs. (B20)–(B24), the right-hand sides are understood to be symmetrized over as many of the indices $(\gamma\delta\varepsilon\mu)$ as are present. Combining the coincidence limits (B20)–(B24) with the formulas (B12)–(B14) for Taylor expansions together with the definition (B18) one obtains

$$\begin{aligned}Q_{\alpha\beta\gamma} = {} & \tfrac{1}{2}R_{\alpha\beta\gamma\delta}\sigma^{;\delta} - \tfrac{1}{6}R_{\alpha\beta\gamma\delta;\varepsilon}\sigma^{;\delta}\sigma^{;\varepsilon} + \tfrac{1}{24}[R_{\alpha\beta\gamma\delta;\varepsilon\mu} \\ & + R_{\alpha\beta\delta\rho}R_{\gamma\varepsilon\mu}{}^{\rho}]\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu} - [\tfrac{1}{120}R_{\alpha\beta\gamma\delta;\varepsilon\mu\nu} \\ & + \tfrac{7}{360}R_{\alpha\beta\delta\rho;\nu}R_{\gamma\varepsilon\mu}{}^{\rho} \\ & + \tfrac{1}{60}R_{\alpha\beta\delta\rho}R_{\gamma\varepsilon\mu}{}^{\rho}{}_{;\nu}]\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu}\sigma^{;\nu} + O(s^5).\end{aligned} \tag{B25}$$

Next, we derive an expansion for the quantity $g_{\alpha}{}^{\alpha'}g_{\alpha'\beta;\gamma}{}^{;\gamma}$ by using the identity [8]

$$g_{\alpha}{}^{\alpha'}g_{\alpha'\beta;\gamma}{}^{;\gamma} = Q_{\alpha\beta\gamma}{}^{;\gamma} - Q^{\delta}{}_{\alpha\gamma}Q_{\delta\beta}{}^{\gamma}, \tag{B26}$$

together with the expansion (B25). The result is

$$\begin{aligned}g_{\alpha}{}^{\alpha'}g_{\alpha'\beta;\gamma}{}^{;\gamma} = {} & -\tfrac{1}{4}C_{\alpha\rho\gamma\varphi}C_{\beta}{}^{\rho}{}_{\delta}{}^{\varphi}\sigma^{;\gamma}\sigma^{;\delta} \\ & + [\tfrac{1}{20}C_{\alpha\rho\gamma\varphi}C_{\beta}{}^{\rho}{}_{\delta}{}^{\varphi}{}_{;\varepsilon} + \tfrac{7}{60}C_{\alpha\rho\gamma\varphi;\varepsilon}C_{\beta}{}^{\rho}{}_{\delta}{}^{\varphi} \\ & + \tfrac{2}{45}C_{\alpha\beta\rho\gamma;\varphi}C_{\delta}{}^{\rho}{}_{\varepsilon}{}^{\varphi}]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon} + O(s^4),\end{aligned} \tag{B27}$$

where we have specialized to the vacuum case $R_{\alpha\beta} = 0$.

Next, following Ref. [8] we define the tensor

$$D_{\alpha\beta} = -g_{\alpha}{}^{\alpha'}\sigma_{;\alpha'\beta} \tag{B28}$$

which is related to the Van Vleck-Morette determinant [67] Δ by

$$\Delta = \det D_{\alpha}{}^{\beta}. \tag{B29}$$

Using the identity [8]

$$D_{\alpha\beta} = Q_{\alpha\gamma\beta}\sigma^{;\gamma} + \sigma_{;\alpha\beta} \tag{B30}$$

together with the expansions (B15) and (B25) we obtain the expansion

$$D_{\alpha\beta} = g_{\alpha\beta} + \tfrac{1}{6}R_{\alpha\gamma\beta\delta}\sigma^{;\gamma}\sigma^{;\delta} - \tfrac{1}{12}R_{\alpha\gamma\beta\delta;\varepsilon}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon} + [\tfrac{1}{40}R_{\alpha\gamma\beta\delta;\varepsilon\mu} + \tfrac{7}{360}R_{\alpha\gamma\delta\rho}R_{\beta\varepsilon\mu}{}^{\rho}]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu} - [\tfrac{1}{180}R_{\alpha\gamma\beta\delta;\varepsilon\mu\nu} + \tfrac{1}{90}R_{\alpha\gamma\delta\rho;\nu}R_{\beta\varepsilon\mu}{}^{\rho} + \tfrac{1}{120}R_{\alpha\gamma\delta\rho}R_{\beta\varepsilon\mu}{}^{\rho}{}_{;\nu}]\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon}\sigma^{;\mu}\sigma^{;\nu} + O(s^6). \tag{B31}$$

The determinant of $D_\alpha{}^\beta$ can be calculated via (E4). Taking the square root of the determinant gives

$$\Delta^{1/2} = 1 + \tfrac{1}{12}R_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta} - \tfrac{1}{24}R_{\alpha\beta;\gamma}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + [\tfrac{1}{80}R_{\alpha\beta;\gamma\delta} + \tfrac{1}{360}R_{\rho\alpha\beta\varphi}R^{\rho}{}_{\gamma\delta}{}^{\varphi} + \tfrac{1}{288}R_{\alpha\beta}R_{\gamma\delta}]\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma}\sigma^{;\delta} - [\tfrac{1}{360}R_{\alpha\beta;\gamma\delta\varepsilon} + \tfrac{1}{288}R_{\alpha\beta}R_{\gamma\delta;\varepsilon} + \tfrac{1}{360}R_{\rho\alpha\beta\varphi}R^{\rho}{}_{\gamma\delta}{}^{\varphi}{}_{;\varepsilon}]\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma}\sigma^{;\delta}\sigma^{;\varepsilon} + O(s^6). \tag{B32}$$

Acting on this expression with the wave operator and using the identities (B1)–(B3) together with Eq. (B15) we obtain

$$\Box\Delta^{1/2} = \tfrac{1}{6}R + [\tfrac{1}{40}\Box R_{\alpha\beta} - \tfrac{1}{120}R_{;\alpha\beta} + \tfrac{1}{72}RR_{\alpha\beta} - \tfrac{1}{30}R_{\alpha\gamma}R^{\gamma}{}_{\beta} + \tfrac{1}{60}R_{\alpha\gamma\beta\delta}R^{\gamma\delta} + \tfrac{1}{60}R_{\alpha\gamma\delta\varepsilon}R_{\beta}{}^{\gamma\delta\varepsilon}]\sigma^{;\alpha}\sigma^{;\beta} + [\tfrac{1}{45}R_{\alpha\mu}R_{\beta}{}^{\mu}{}_{;\gamma} - \tfrac{1}{180}R_{\mu\nu}R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}{}_{;\gamma} - \tfrac{1}{120}R_{\alpha\beta}{}^{;\mu}{}_{;\mu\gamma} - \tfrac{1}{180}R_{\mu\nu;\alpha}R_{\beta}{}^{\mu}{}_{\gamma}{}^{\nu} - \tfrac{1}{90}R_{\alpha\mu\nu\lambda}R_{\beta}{}^{\mu\nu\lambda}{}_{;\gamma} - \tfrac{1}{144}RR_{\alpha\beta;\gamma} + \tfrac{1}{360}R_{;\alpha\beta\gamma}]\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + O(s^4). \tag{B33}$$

Specializing to the vacuum case $R_{\alpha\beta} = 0$ yields

$$\Box\Delta^{1/2} = \tfrac{1}{60}C_{\alpha\mu\nu\lambda}C_{\beta}{}^{\mu\nu\lambda}\sigma^{;\alpha}\sigma^{;\beta} - \tfrac{1}{90}C_{\alpha\mu\nu\lambda}C_{\beta}{}^{\mu\nu\lambda}{}_{;\gamma}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + O(s^4). \tag{B34}$$

In the special case of four space-time dimensions this can be further simplified using the identity (B4) to give

$$\Box\Delta^{1/2} = \tfrac{1}{240}C_{\varepsilon\mu\nu\lambda}C^{\varepsilon\mu\nu\lambda}g_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta} - \tfrac{1}{360}C_{\mu\nu\lambda\rho;\gamma}C^{\mu\nu\lambda\rho}g_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta}\sigma^{;\gamma} + O(s^4). \tag{B35}$$

APPENDIX C: COVARIANT EXPANSION OF THE RETARDED GREEN'S FUNCTION

In this appendix we derive the local covariant expansion of the retarded Green's function $G_{\rm ret}^{\mu\nu\alpha'\beta'}(x,x')$ which is defined by the differential equation

$$(\Box g_{\mu\alpha}g_{\nu\beta} + 2C_{\mu\alpha\nu\beta})G_{\rm ret}^{\mu\nu\alpha'\beta'}(x,x') = -[g_{(\alpha}{}^{\alpha'}g_{\beta)}{}^{\beta'} + \kappa g_{\alpha\beta}g^{\alpha'\beta'}]\delta^4(x,x'). \tag{C1}$$

Here we have introduced a real parameter κ to facilitate comparison with the work of Allen, Folacci and Ottewill (AFO) [57], who analyzed the case $\kappa = -1/2$. For this paper we are interested in the case $\kappa = 0$ cf. Eq. (2.4) above. Note that these two Green's functions, the cases $\kappa = 0$ and $\kappa = -1/2$, are related to each other by a trace reversal on the index pair $(\alpha'\beta')$. Throughout this appendix we specialize to four space-time dimensions.

We use the standard method explained by Hadamard [5], DeWitt and Brehme [8], and AFO [57] in the scalar, vector and tensor cases, respectively, and we extend the expansions of AFO to one higher order. We assume for the Feynman Green's function the expression

$$G_{\rm F}^{\mu\nu\alpha'\beta'}(x,x') = \frac{1}{4\pi^2}\left\{\frac{U^{\mu\nu\alpha'\beta'}(x,x')}{\sigma + i\epsilon} + V^{\mu\nu\alpha'\beta'}(x,x')\ln[\sigma + i\epsilon] + W^{\mu\nu\alpha'\beta'}(x,x')\right\} \tag{C2}$$

for some bitensors $U^{\mu\nu\alpha'\beta'}$, $V^{\mu\nu\alpha'\beta'}$ and $W^{\mu\nu\alpha'\beta'}$, where we have introduced a regularization parameter ϵ to give the appropriate singularity structure. The expression (2.18) for the retarded Green's function can be obtained by taking the negative of the imaginary part of the Feynman Green's function (C2), and by multiplying by the function $\Theta[\Sigma(x,x')]$ defined in Sec. II C [8].

Substituting the real part of the Green's function (C2) into the homogeneous version of the differential equation (C1), and equating to zero the coefficients of $1/\sigma^2$, $\ln\sigma$, and the remainder gives the three equations

$$U^{\mu\nu\alpha'\beta';\gamma}\sigma_{;\gamma} - \tfrac{1}{2}U^{\mu\nu\alpha'\beta'}(\ln\Delta)^{;\gamma}\sigma_{;\gamma} = 0, \tag{C3}$$

$$\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}V_{\mu\nu\alpha'\beta'} = 0, \tag{C4}$$

and

$$\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}U_{\mu\nu\alpha'\beta'} + 2V_{\alpha\beta\alpha'\beta'} + 2V_{\alpha\beta\alpha'\beta'}{}^{;\gamma}\sigma_{;\gamma} - V_{\alpha\beta\alpha'\beta'}(\ln\Delta)^{;\gamma}\sigma_{;\gamma} + \sigma\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}W_{\mu\nu\alpha'\beta'} = 0. \tag{C5}$$

Here we have defined the differential operator

$$\mathfrak{D}_{\alpha\beta}{}^{\mu\nu} = \Box\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + 2C_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}, \tag{C6}$$

and we have made use of the identity [8]

$$\Box\sigma = 4 - (\ln\Delta)^{;\gamma}\sigma_{;\gamma}, \tag{C7}$$

where Δ is the Van Vleck-Morette determinant (B29).

The solution to the differential equation (C3) for $U^{\alpha\beta\alpha'\beta'}$ which is appropriate for the source term on the right-hand side of Eq. (C1) is

$$U_{\alpha\beta\alpha'\beta'} = \Delta^{1/2}[g_{\alpha'(\alpha}g_{\beta)\beta'} + \kappa g_{\alpha\beta}g_{\alpha'\beta'}], \tag{C8}$$

where we have used the identity (B19).

Next, we assume formal power series expansions for $V_{\alpha\beta\alpha'\beta'}$ and $W_{\alpha\beta\alpha'\beta'}$ of the form

$$V_{\alpha\beta\alpha'\beta'}(x,x') = \sum_{n=0}^{\infty} V^n_{\alpha\beta\alpha'\beta'}(x,x')\sigma^n \tag{C9}$$

and

$$W_{\alpha\beta\alpha'\beta'}(x,x') = \sum_{n=0}^{\infty} W^n_{\alpha\beta\alpha'\beta'}(x,x')\sigma^n. \tag{C10}$$

Note that these expansions do not define unique representations of the bitensors $V_{\alpha\beta\alpha'\beta'}$ and $W_{\alpha\beta\alpha'\beta'}$, since the coefficients $V^n_{\alpha\beta\alpha'\beta'}$ and $W^n_{\alpha\beta\alpha'\beta'}$ can be arbitrary functions of x and x'. However, one can obtain a unique set of coefficients from the following prescription [8]. Pick a bisolution $W^0_{\alpha\beta\alpha'\beta'}$ of the homogeneous wave equation $\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}W^0_{\mu\nu\alpha'\beta'} = 0$. Then, substitute the expansions (C9) and (C10) into Eqs. (C4) and (C5), simplify using the identities (B5) and (C7), and equate to zero the coefficients of powers of σ. The result is the following recursive set of ordinary differential equations along the geodesic joining x and x' that allow one to solve for the coefficients:

$$\begin{aligned} V^0_{\alpha\beta\alpha'\beta'} &+ [V^0_{\alpha\beta\alpha'\beta';\mu} - \tfrac{1}{2}V^0_{\alpha\beta\alpha'\beta'}(\ln\Delta)_{;\mu}]\sigma^{;\mu} \\ &= -\tfrac{1}{2}\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}U_{\mu\nu\alpha'\beta'}, \end{aligned} \tag{C11}$$

$$\begin{aligned} V^n_{\alpha\beta\alpha'\beta'} &+ \frac{1}{n+1}\left[V^n_{\alpha\beta\alpha'\beta';\mu} - \frac{1}{2}V^n_{\alpha\beta\alpha'\beta'}(\ln\Delta)_{;\mu}\right]\sigma^{;\mu} \\ &= -\frac{1}{2n(n+1)}\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}V^{n-1}_{\mu\nu\alpha'\beta'} \end{aligned} \tag{C12}$$

and

$$\begin{aligned} W^n_{\alpha\beta\alpha'\beta'} &+ \frac{1}{n+1}\left[W^n_{\alpha\beta\alpha'\beta';\mu} - \frac{1}{2}W^n_{\alpha\beta\alpha'\beta'}(\ln\Delta)_{;\mu}\right]\sigma^{;\mu} \\ &= -\frac{1}{2n(n+1)}\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}W^{n-1}_{\mu\nu\alpha'\beta'} \\ &\quad -\frac{1}{n+1}V^n_{\alpha\beta\alpha'\beta'} + \frac{1}{2n^2(n+1)}\mathfrak{D}_{\alpha\beta}{}^{\mu\nu}V^{n-1}_{\mu\nu\alpha'\beta'}. \end{aligned} \tag{C13}$$

Equations (C12) and (C13) apply for $n \geq 1$. The power series (C9) and (C10) with these coefficients converge in a neighborhood of the diagonal $x = x'$ for analytic metrics [5].

In this paper we are only interested in the coefficients $V^n_{\alpha\beta\alpha'\beta'}$, which can be obtained from Eqs. (C11) and (C12). We expand each of these coefficients as covariant Taylor series of the form (B12)

$$\begin{aligned} V^n_{\alpha\beta\alpha'\beta'} = g_{\alpha'}{}^{\gamma}g_{\beta'}{}^{\delta}[&v^n_{\alpha\beta\gamma\delta}(x) + v^n_{\alpha\beta\gamma\delta\varepsilon}(x)\sigma^{;\varepsilon} \\ &+ \tfrac{1}{2}v^n_{\alpha\beta\gamma\delta\varepsilon\zeta}(x)\sigma^{;\varepsilon}\sigma^{;\zeta} \\ &+ \tfrac{1}{6}v^n_{\alpha\beta\gamma\delta\varepsilon\zeta\eta}(x)\sigma^{;\varepsilon}\sigma^{;\zeta}\sigma^{;\eta} + \ldots], \end{aligned} \tag{C14}$$

where the coefficients $v^n_{\alpha\ldots\eta}$ are local tensors at x. We now specialize to the case $n = 0$, and substitute the expansion (C14) for $V^0_{\alpha\beta\alpha'\beta'}$ and the expression (C8) for $U_{\alpha\beta\alpha'\beta'}$ into the differential equation (C11). We simplify using the identity (B19) and the definition (C6), and expand the various terms as power series in $\sigma_{;\mu}$ using the expansions (B15), (B25), (B27), (B32), and (B35). Equating the coefficients of the various powers of $\sigma_{;\mu}$ then gives a series of equations that can be solved for the coefficients $v^0_{\alpha\ldots\eta}$. The results are

$$v^0_{\alpha\beta\gamma\delta} = -C_{\alpha\gamma\beta\delta}, \tag{C15}$$

$$v^0_{\alpha\beta\gamma\delta\varepsilon} = \tfrac{1}{2}C_{\alpha\gamma\beta\delta;\varepsilon}, \tag{C16}$$

$$\begin{aligned} v^0_{\alpha\beta\gamma\delta\varepsilon\zeta} = &-\tfrac{1}{3}C_{\alpha\gamma\beta\delta;\varepsilon\zeta} - \tfrac{1}{6}C_{\alpha\gamma}{}^{\mu}{}_{\varepsilon}C_{\beta\delta\mu\zeta} \\ &+ \tfrac{1}{6}g_{\alpha\gamma}C_{\beta}{}^{\mu\nu}{}_{\varepsilon}C_{\delta\mu\nu\zeta} - \tfrac{1}{180}\Pi_{\alpha\beta\gamma\delta}C^{\mu\nu\rho}{}_{\varepsilon}C_{\mu\nu\rho\zeta}, \end{aligned} \tag{C17}$$

and

$$\begin{aligned} v^0_{\alpha\beta\gamma\delta\varepsilon\zeta\eta} = &\tfrac{1}{4}C_{\alpha\gamma\beta\delta;\varepsilon\zeta\eta} + \tfrac{1}{4}(C_{\alpha\gamma}{}^{\mu}{}_{\varepsilon}C_{\beta\delta\mu\zeta})_{;\eta} \\ &- \tfrac{1}{5}g_{\alpha\gamma}(C_{\beta}{}^{\mu\nu}{}_{\varepsilon}C_{\delta\mu\nu\zeta})_{;\eta} \\ &- \tfrac{1}{10}g_{\alpha\gamma}C_{\beta}{}^{\mu\nu}{}_{\varepsilon}C_{\delta\mu\nu\zeta;\eta} \\ &+ \tfrac{1}{15}g_{\alpha\gamma}C_{\beta\delta\mu\varepsilon;\nu}C^{\mu}{}_{\zeta}{}^{\nu}{}_{\eta} \\ &+ \tfrac{1}{240}\Pi_{\alpha\beta\gamma\delta}g_{\varepsilon\zeta}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma;\eta}, \end{aligned} \tag{C18}$$

where

$$\Pi_{\alpha\beta\gamma\delta} = \tfrac{1}{2}g_{\alpha\gamma}g_{\beta\delta} + \tfrac{1}{2}g_{\alpha\delta}g_{\beta\gamma} + \kappa g_{\alpha\beta}g_{\gamma\delta}. \tag{C19}$$

In Eqs. (C15)–(C18), the right-hand sides are understood to be symmetrized on the index pair $(\alpha\beta)$, on the index pair $(\gamma\delta)$, and on as many of the index triplet $(\varepsilon\zeta\eta)$ as are present. When $\kappa = -1/2$, the formulas (C15)–(C17) agree with Eqs. (A20)–(A22) of AFO specialized to the vacuum case.

We will also need the first two of the coefficients in the expansion of $V^1_{\alpha\beta\alpha'\beta'}$, which we obtain from Eq. (C12) with $n = 1$. The hard part of the computation is evaluating the source term on the right-hand side of this equation. Using the definition (C14) and the expansions (B15), (B25), and (B27), we obtain

$$\begin{aligned} g_{\gamma}{}^{\alpha'}g_{\delta}{}^{\beta'}\Box V^0_{\alpha\beta\alpha'\beta'} = &\Box v^0_{\alpha\beta\gamma\delta} + 2v^0_{\alpha\beta\gamma\delta\varepsilon} + v^0_{\alpha\beta\gamma\delta\varepsilon} \\ &+ [\Box v^0_{\alpha\beta\gamma\delta\varepsilon} + 2v^0_{\alpha\beta\gamma\delta\varepsilon\zeta} + v^0_{\alpha\beta\gamma\delta\varepsilon\zeta} \\ &- C_{\alpha\eta\beta\delta;\zeta}C_{\gamma}{}^{\eta\zeta}{}_{\varepsilon}]\sigma^{;\varepsilon} + O(s^2). \end{aligned} \tag{C20}$$

Now inserting suitably symmetrized versions of the formulas (C15)–(C18) for the expansion coefficients $v^0_{\alpha\ldots\eta}$ and using the identity (B4) gives

$$\begin{aligned} g_\gamma^{\alpha'} g_\delta^{\beta'} \Box V^0_{\alpha\beta\alpha'\beta'} = & -\tfrac{1}{3}\Box C_{\alpha\gamma\beta\delta} - \tfrac{1}{6}C_{\alpha\gamma\eta\zeta}C_{\beta\delta}{}^{\eta\zeta} + \tfrac{1}{24}g_{\alpha\gamma}g_{\beta\delta}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \tfrac{1}{180}\Pi_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\ & + [\tfrac{1}{12}\Box(C_{\alpha\gamma\beta\delta;\varepsilon}) + \tfrac{1}{360}\Pi_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma;\varepsilon}C^{\mu\nu\rho\sigma} - \tfrac{1}{24}g_{\alpha\gamma}g_{\beta\delta}C_{\mu\nu\rho\sigma;\varepsilon}C^{\mu\nu\rho\sigma} + \tfrac{1}{6}C_{\alpha\gamma}{}^{\mu\nu}C_{\beta\delta\mu\nu;\varepsilon} \\ & - \tfrac{1}{6}C_{\alpha\gamma}{}^{\mu\nu}C_{\beta\delta\mu\varepsilon;\nu} + \tfrac{5}{6}C_{\alpha\eta\beta\delta;\zeta}C_\varepsilon{}^\zeta{}_\gamma{}^\eta + \tfrac{1}{6}C_{\gamma\eta\beta\delta;\zeta}C_\varepsilon{}^\zeta{}_\alpha{}^\eta + \tfrac{1}{45}g_{\beta\delta}C_{\alpha\gamma\eta\zeta;\rho}C^{\eta\zeta\rho}{}_\varepsilon + \tfrac{1}{15}g_{\beta\delta}C_\alpha{}^{\eta\zeta\rho}C_{\gamma\eta\varepsilon\rho;\zeta} \\ & + \tfrac{1}{45}g_{\beta\delta}C_{\alpha\gamma\eta\zeta;\rho}C^{\rho\zeta\eta}{}_\varepsilon + \tfrac{1}{10}g_{\beta\delta}C_\gamma{}^{\eta\zeta\rho}C_{\alpha\eta\varepsilon\rho;\zeta}]\sigma^{;\varepsilon} + O(s^2), \end{aligned} \tag{C21}$$

where the right-hand side is understood to be symmetrized on the index pairs $(\alpha\beta)$ and $(\gamma\delta)$. Next, we expand both sides of Eq. (C12) with $n = 1$ as a power series in $\sigma_{;\mu}$ to $O(s)$, using the formulas (C15), (C16), and (C21) to evaluate the right-hand side, and the expansion (C14) with $n = 1$ on the left-hand side. Equating the coefficients of the various powers of $\sigma_{;\mu}$ then gives a series of equations that can be solved for the coefficients $v^1_{\alpha\ldots\eta}$. The results are

$$v^1_{\alpha\beta\gamma\delta} = \tfrac{1}{12}\Box C_{\alpha\gamma\beta\delta} + \tfrac{1}{2}C_\alpha{}^\mu{}_\beta{}^\nu C_{\gamma\mu\delta\nu} + \tfrac{1}{24}C_{\alpha\gamma}{}^{\mu\nu}C_{\beta\delta\mu\nu} - \tfrac{1}{24}g_{\alpha\gamma}C_\beta{}^{\mu\nu\rho}C_{\delta\mu\nu\rho} + \tfrac{1}{720}\Pi_{\alpha\beta\gamma\delta}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} \tag{C22}$$

and

$$\begin{aligned} v^1_{\alpha\beta\gamma\delta\varepsilon} = & -\tfrac{1}{24}\Box(C_{\alpha\gamma\beta\delta;\varepsilon}) - \tfrac{1}{720}\Pi_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma;\varepsilon}C^{\mu\nu\rho\sigma} + \tfrac{1}{80}g_{\alpha\gamma}g_{\beta\delta}C_{\mu\nu\rho\sigma;\varepsilon}C^{\mu\nu\rho\sigma} - \tfrac{1}{18}C_{\alpha\gamma}{}^{\mu\nu}C_{\beta\delta\mu\nu;\varepsilon} \\ & - \tfrac{1}{3}C_\alpha{}^\mu{}_\beta{}^\nu C_{\mu\gamma\nu\delta;\varepsilon} - \tfrac{1}{6}C_\alpha{}^\mu{}_\beta{}^\nu{}_{;\varepsilon}C_{\mu\gamma\nu\delta} - \tfrac{1}{12}C_\varepsilon{}^\eta{}_\alpha{}^\zeta C_{\zeta\gamma\beta\delta;\eta} - \tfrac{1}{4}C_{\varepsilon\eta\gamma}{}^\zeta C_{\alpha\zeta\beta\delta}{}^{;\eta} + \tfrac{1}{36}C_{\alpha\gamma\zeta\eta}C_{\beta\delta}{}^\zeta{}_\varepsilon{}^{;\eta} \\ & - \tfrac{1}{270}g_{\beta\delta}C_{\alpha\gamma\eta\zeta;\rho}C^{\eta\zeta\rho}{}_\varepsilon - \tfrac{1}{90}g_{\beta\delta}C_\alpha{}^{\eta\zeta\rho}C_{\gamma\eta\varepsilon\rho;\zeta} - \tfrac{1}{270}g_{\beta\delta}C_{\alpha\gamma\eta\zeta;\rho}C^{\rho\zeta\eta}{}_\varepsilon - \tfrac{1}{60}g_{\beta\delta}C_\gamma{}^{\eta\zeta\rho}C_{\alpha\eta\varepsilon\rho;\zeta} \\ & + \tfrac{1}{180}g_{\beta\delta}C_\alpha{}^{\mu\nu\rho}C_{\gamma\mu\nu\rho;\varepsilon}, \end{aligned} \tag{C23}$$

where again there is implicit symmetrization on the index pairs $(\alpha\beta)$ and $(\gamma\delta)$. When $\kappa = -1/2$, Eq. (C22) agrees with Eq. (A23) of AFO specialized to the vacuum case.[7]

APPENDIX D: EXPANSION COEFFICIENTS

In this appendix we list the expressions for the coefficients $\mathcal{V}_{\alpha\ldots\eta}$ which appear in the expansion (3.5). These expressions are obtained by substituting Eqs. (C15)–(C18) and (C22) and (C23), specialized to $\kappa = 0$, into Eqs. (3.6), (3.7), and (3.8). The results are

$$\mathcal{V}_{\alpha\beta\gamma\delta\varepsilon} = -\tfrac{1}{2}C_{\alpha\gamma\beta\delta;\varepsilon}, \tag{D1}$$

$$\begin{aligned} \mathcal{V}_{\alpha\beta\gamma\delta\varepsilon}{}^\sigma = & \tfrac{1}{6}C_{\alpha\gamma\beta\delta;\varepsilon}{}^{;\sigma}, + \tfrac{2}{3}C_\varepsilon{}^\sigma{}_\alpha{}^\rho C_{\rho\gamma\beta\delta} - \tfrac{1}{3}C_\varepsilon{}^\sigma{}_\gamma{}^\rho C_{\alpha\rho\beta\delta} - \tfrac{1}{6}C_{\alpha\gamma}{}^\rho{}_\varepsilon C_{\beta\gamma\rho}{}^\sigma + \tfrac{1}{12}g_{\alpha\gamma}C_\beta{}^{\rho\tau}{}_\varepsilon C_{\delta\rho\tau}{}^\sigma \\ & + \tfrac{1}{12}g_{\alpha\gamma}C_\beta{}^{\rho\tau\sigma}C_{\delta\rho\tau\varepsilon} + \delta^\sigma_\varepsilon[\tfrac{1}{12}\Box C_{\alpha\gamma\beta\delta} + \tfrac{1}{2}C_\alpha{}^\rho{}_\beta{}^\tau C_{\gamma\rho\delta\tau} + \tfrac{1}{24}C_{\alpha\gamma}{}^{\rho\tau}C_{\beta\delta\rho\tau} - \tfrac{1}{96}g_{\alpha\gamma}g_{\beta\delta}C_{\rho\tau\mu\nu}C^{\rho\tau\mu\nu}], \end{aligned} \tag{D2}$$

and

$$\begin{aligned} \mathcal{V}_{\alpha\beta(\gamma\delta|\varepsilon|\sigma\rho)} = & -\tfrac{1}{24}C_{\alpha\gamma\beta\delta;\varepsilon\sigma\rho} + \tfrac{1}{720}g_{\alpha\gamma}g_{\beta\delta}g_{\varepsilon\sigma}C_{\tau\lambda\mu\nu}C^{\tau\lambda\mu\nu}{}_{;\rho} + \tfrac{1}{12}C_{\alpha\gamma\tau\delta}C_{\beta\sigma}{}^\tau{}_{\varepsilon;\rho} - \tfrac{1}{6}C_{\alpha\tau\varepsilon\gamma;\delta}C^\tau{}_{\sigma\beta\rho} + \tfrac{1}{6}C_{\alpha\tau\beta\gamma}C_{\varepsilon\delta\sigma}{}^\tau{}_{;\rho} \\ & + \tfrac{1}{12}C_{\alpha\gamma\tau\varepsilon}C_{\beta\delta}{}^\tau{}_{\sigma;\rho} - \tfrac{5}{12}C_{\alpha\tau\varepsilon\gamma}C^\tau{}_{\delta\beta\sigma;\rho} + \tfrac{1}{12}C_{\alpha\tau\beta\gamma;\delta}C_{\varepsilon\sigma\rho}{}^\tau + \tfrac{1}{24}C_{\alpha\gamma\beta\delta;\tau}C_{\varepsilon\sigma\rho}{}^\tau - \tfrac{1}{12}C_{\alpha\gamma\tau\delta}C_{\beta\sigma}{}^\tau{}_{\rho;\varepsilon} \\ & - g_{\beta\gamma}[\tfrac{1}{90}C_{\delta\alpha\tau\varepsilon;\lambda}C_\sigma{}^\tau{}_\rho{}^\lambda + \tfrac{1}{30}C_{\delta\tau\sigma\lambda}C_\alpha{}^\tau{}_\varepsilon{}^\lambda{}_{;\rho} + \tfrac{1}{20}C_{\delta\tau\varepsilon\lambda;\sigma}C_\alpha{}^\tau{}_\rho{}^\lambda + \tfrac{1}{90}C_{\delta\alpha\tau\sigma;\lambda}C_\varepsilon{}^\tau{}_\rho{}^\lambda + \tfrac{1}{90}C_{\delta\alpha\tau\sigma;\lambda}C_\rho{}^\tau{}_\varepsilon{}^\lambda \\ & + \tfrac{1}{30}C_{\delta\tau\varepsilon\lambda}C_\alpha{}^\tau{}_\sigma{}^\lambda{}_{;\rho} + \tfrac{1}{20}C_{\delta\tau\sigma\lambda;\rho}C_\alpha{}^\tau{}_\varepsilon{}^\lambda - \tfrac{1}{20}C_{\delta\tau\sigma\lambda}C_\alpha{}^\tau{}_\rho{}^\lambda{}_{;\varepsilon} - \tfrac{1}{30}C_{\delta\tau\sigma\lambda;\varepsilon}C_\alpha{}^\tau{}_\rho{}^\lambda] \\ & + g_{\varepsilon\rho}[\tfrac{1}{90}g_{\alpha\gamma}g_{\beta\delta}C_{\tau\lambda\mu\nu;\sigma}C^{\tau\lambda\mu\nu} - \tfrac{1}{24}\Box(C_{\alpha\gamma\beta\delta;\sigma}) - \tfrac{1}{6}C_{\alpha\tau\beta\lambda;\gamma}C^\tau{}_\delta{}^\lambda{}_\sigma - \tfrac{1}{18}C_{\alpha\gamma\tau\lambda;\delta}C_{\beta\sigma}{}^{\tau\lambda} - \tfrac{1}{4}C_{\gamma\tau\delta\lambda}C_\alpha{}^\lambda{}_{\beta\sigma}{}^{;\tau} \\ & - \tfrac{1}{12}C_{\gamma\tau\alpha\lambda}C^\lambda{}_{\delta\beta\sigma}{}^{;\tau} + \tfrac{1}{36}C_{\alpha\gamma\tau\delta;\lambda}C_{\beta\sigma}{}^{\tau\lambda} + \tfrac{1}{270}g_{\beta\gamma}C_{\delta\alpha\tau\lambda;\mu}C^{\lambda\tau}{}_\sigma{}^\mu - \tfrac{1}{90}g_{\beta\gamma}C_{\delta\tau\sigma\lambda;\mu}C_\alpha{}^{\tau\mu\lambda} \\ & - \tfrac{1}{60}g_{\beta\gamma}C_{\delta\tau\lambda\mu}C_\alpha{}^\tau{}_\sigma{}^{\mu;\lambda} + \tfrac{1}{270}g_{\beta\gamma}C_{\gamma\alpha\tau\lambda;\mu}C_\sigma{}^{\tau\lambda\mu} + \tfrac{1}{180}g_{\beta\gamma}C_\alpha{}^{\tau\lambda\mu}C_{\delta\tau\lambda\mu;\sigma} - \tfrac{1}{3}C_{\alpha\tau\beta\lambda}C_\gamma{}^\tau{}_\delta{}^\lambda{}_{;\sigma}] \\ & + g_{\sigma\rho}[-\tfrac{1}{240}g_{\alpha\gamma}g_{\beta\delta}C_{\tau\lambda\mu\nu}C^{\tau\lambda\mu\nu}{}_{;\varepsilon} + \tfrac{1}{48}\Box(C_{\alpha\gamma\beta\delta;\varepsilon}) + \tfrac{1}{24}C_{\alpha\tau\beta\gamma;\lambda}C_\varepsilon{}^\lambda{}_\delta{}^\tau + \tfrac{1}{8}C_{\varepsilon\tau\alpha\lambda}C^\lambda{}_{\gamma\beta\delta}{}^{;\tau} + \tfrac{1}{72}C_{\alpha\gamma\tau\varepsilon;\lambda}C_{\beta\delta}{}^{\tau\lambda} \\ & + \tfrac{1}{6}C_{\alpha\tau\beta\lambda;\varepsilon}C_\gamma{}^\tau{}_\delta{}^\lambda + \tfrac{1}{72}C_{\alpha\gamma\tau\lambda;\varepsilon}C_{\beta\delta}{}^{\tau\lambda} + \tfrac{1}{12}C_{\alpha\tau\beta\lambda}C_\gamma{}^\tau{}_\delta{}^\lambda{}_{;\varepsilon} + \tfrac{1}{540}g_{\beta\gamma}C_{\delta\alpha\tau\lambda;\mu}C^{\lambda\tau}{}_\varepsilon{}^\mu - \tfrac{1}{180}g_{\beta\gamma}C_{\delta\tau\varepsilon\lambda;\mu}C_\alpha{}^{\tau\mu\lambda} \\ & - \tfrac{1}{120}g_{\beta\gamma}C_{\delta\tau\lambda\mu}C_\alpha{}^\tau{}_\varepsilon{}^{\mu;\lambda} + \tfrac{1}{540}g_{\beta\gamma}C_{\delta\alpha\tau\lambda;\mu}C_\varepsilon{}^{\tau\lambda\mu} + \tfrac{1}{360}g_{\beta\gamma}C_{\alpha\tau\lambda\mu}C_\delta{}^{\tau\lambda\mu}{}_{;\varepsilon}]. \end{aligned} \tag{D3}$$

[7]Note that the expressions (C15)–(C18) and (C22) and (C23) are all traceless on the index pair $(\gamma\delta)$, aside from the terms involving the tensor $\Pi_{\alpha\beta\gamma\delta}$. This means that performing a trace reversal on the index pair $(\gamma\delta)$ is equivalent to changing the value of κ from 0 to $-1/2$, in agreement with the discussion after Eq. (C1) above.

The right-hand sides of Eqs. (D1) and (D2) are understood to be symmetrized over the index pairs $(\alpha\beta)$ and $(\gamma\delta)$. In Eq. (D3), as the notation on the left-hand side indicates, we have only computed the piece of $\mathcal{V}_{\alpha\beta\gamma\delta\varepsilon\sigma\rho}$ which is totally symmetric on the indices $(\gamma\delta\sigma\rho)$. This is because only this piece of $\mathcal{V}_{\alpha\beta\gamma\delta\varepsilon\sigma\beta}$ is needed for computing the expansion coefficient $f^{(2)}_{\alpha}$, from Eq. (3.14) above. The right-hand side of Eq. (D3) is understood to be symmetrized on the index pair $(\alpha\beta)$ and on the indices $(\gamma\delta\sigma\rho)$.

APPENDIX E: SOME USEFUL DETERMINANT IDENTITIES

One occasionally encounters determinants of rank two tensors in relativity. The normal expressions for evaluating determinants do not lend themselves naturally to expression using standard tensor notation. However, it is possible to write the determinant of any square matrix in terms of powers of the matrix and their traces, both of which are easily expressed in tensor notation. We list the three simplest here for a matrix A of various sizes

Dimension	Identity	
2×2	$\mathrm{Det}A = [(\mathrm{Tr}A)^2 - \mathrm{Tr}(A^2)]/2$	(E1)
3×3	$\mathrm{Det}A = [(\mathrm{Tr}A)^3 - 3(\mathrm{Tr}A)\mathrm{Tr}(A^2) + 2\mathrm{Tr}(A^3)]/6$	(E2)
4×4	$\mathrm{Det}A = \{(\mathrm{Tr}A)^4 - 6(\mathrm{Tr}A)^2\mathrm{Tr}(A^2) + 8(\mathrm{Tr}A)\mathrm{Tr}(A^3) + 3[\mathrm{Tr}(A^2)]^2 - 6\mathrm{Tr}(A^4)\}/24.$	(E3)

Similar identities can be obtained for any dimension through a straightforward recursive application of Newton's identities.

To make the usefulness of these identities more apparent, we express the determinant in the 4×4 case in tensor notation using (E3):

$$\mathrm{D\,et}A^{\alpha}{}_{\beta} = [(A^{\alpha}{}_{\alpha})^4 - 6(A^{\alpha}{}_{\alpha})^2 A^{\beta}{}_{\gamma}A^{\gamma}{}_{\beta} + 8A^{\alpha}{}_{\alpha}A^{\beta}{}_{\gamma}A^{\gamma}{}_{\delta}A^{\delta}{}_{\beta} + 3(A^{\alpha}{}_{\beta}A^{\beta}{}_{\gamma})^2 - 6A^{\alpha}{}_{\beta}A^{\beta}{}_{\gamma}A^{\gamma}{}_{\delta}A^{\delta}{}_{\alpha}]/24. \tag{E4}$$

While this expression is not a computationally efficient way of calculating the determinant, it clearly has the advantage of reducing the determinant operation to more familiar tensor operations. We include it here because we have not been successful in locating it in the literature.

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