Staticity theorem for a higher dimensional generalized Einstein-Maxwell system

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I derive formulas for variations of mass, angular momentum, and canonical energy in Einstein $(n-2)$-gauge form field theory by means of the Arnowitt-Deser-Misner formalism. Considering the initial data for the manifold with an interior boundary which has the topology of $(n-2)$ sphere, I obtained the generalized first law of black hole thermodynamics. Supposing that a black hole event horizon comprises a bifurcation Killing horizon with a bifurcate surface, I find that the solution is static in the exterior world, when the Killing timelike vector field is normal to the horizon and has vanishing *electric* or *magnetic* fields on static slices.

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I. INTRODUCTION

Nowadays, there has been a significant resurgence of interest in gravity and black holes in more than four dimensions. It stems from the attempts at building a consistent quantum gravity theory in the realm of M/string theory as well as in the range of TeV gravity, where the large or infinite dimensions are taken into account. Especially the mathematical aspects of classification of n-dimensional black holes have recently attracted more attention. As far as the problem of classification of nonsingular black hole solutions in four dimensions was concerned, Israel [1], Müller zum Hagen *et al.* [2], and Robinson [3] presented first proofs. The most complete results were provided in Refs. [4–8]. The classification of both static vacuum black hole solutions as well as the Einstein-Maxwell black holes was finished in Refs. [9,10].

The problem of the uniqueness black hole theorem for stationary axisymmetric spacetime turned out to be more complicated. It was elaborated in Ref. [11], but the complete proof was provided by Mazur [12] and Bunting [13] (for a review of the uniqueness of black hole solutions story, see [14] and references therein).

Attempts at building a consistent quantum gravity theory triggered the research concerning the mathematical aspects of the low-energy string theory black holes. The uniqueness of the black hole solutions in dilaton gravity was proved in Refs. [15,16], while the uniqueness of the static dilaton $\mathrm{U}(1)^2$ black holes being the solution of $N = 4, d = 4$ supergravity was provided in Ref. [17]. The extension of the uniqueness proof to the case of static dilaton black holes with $\mathrm{U}(1)^N$ gauge fields was established in Ref. [18].

On the other hand, the n-dimensional black hole uniqueness theorem, in both vacuum and charged cases, was given in Refs. [19–22]. The case of a nonlinear self-gravitating σ model in higher dimensions was treated in Ref. [23]. The complete classification of n-dimensional charged black holes having both degenerate and nondegenerate components of an event horizon was provided in Ref. [24]. In Ref. [25] it was pointed out that a black hole being the source of both magnetic and electric components of 2-form $F_{\mu\nu}$ was a striking coincidence. Hence, in order to treat this problem in n-dimensional gravity, one should consider both *electric* and *magnetic* components of $(n-2)$-gauge form $F_{\mu_1 \cdots \mu_{n-2}}$. In Ref. [26] the proof of the uniqueness of a static higher dimensional *electrically* and *magnetically* charged black hole containing an asymptotically flat hypersurface with compact interior and nondegenerate components of the event horizon was given.

Proving the uniqueness theorem for stationary n-dimensional black holes is much more complicated. It turned out that generalization of Kerr metric to arbitrary n dimensions proposed by Myers and Perry [25] is not unique. The counterexample showing that a five-dimensional rotating black hole ring solution with the same angular momentum and mass but the horizon of which was homeomorphic to $S^2 \times S^1$ was presented in Ref. [27] (see also Ref. [28]). In Ref. [29] it was shown that the Myers-Perry solution is the unique black hole in five dimensions in the class of spherical topology and three commuting Killing vectors [29], while in Ref. [30] the problem of a stationary nonlinear self-gravitating σ model in five-dimensional spacetime was considered. It was proved that, when we assume that the horizon had the topology of S^3, the Myers-Perry vacuum Kerr solution is the only maximally extended, stationary, axisymmetric flat solution having the regular rotating event horizon with constant mapping.

The uniqueness theorem for black holes is closely related to the problem of staticity for nonrotating black holes and circularity for rotating ones. For the first time, the problem of staticity was tackled by Lichnerowicz [31]. The next extension to the vacuum spacetime was attributed to Hawking [32], while the extension taking into account electromagnetic fields was provided by Carter [33]. But only recently was the complete proof of the staticity theorem [34,35] by means of the Arnowitt-Deser-Misner (ADM) formalism given. In the case of the low-energy string theory, the problem of staticity was studied in Refs. [36,37].

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In this paper, we shall study the problem of the staticity theorem in the Einstein $(n-2)$-gauge form $F_{(n-2)}$ theory. Section II will be devoted to the canonical formalism of the underlying theory. In Sec. III we tackle the problem of canonical energy and angular momentum and derive the first law of thermodynamics for black holes with $(n-2)$-gauge form $F_{(n-2)}$ fields. Our derivation of the first law of black hole thermodynamics relies on the assumption that the event horizon is a Killing bifurcation $(n-2)$-dimensional sphere. Then we find the conditions for staticity for nonrotating black holes in n dimensions.

In what follows, the Greek indices will range from 0 to n. They denote tensors on an n-dimensional manifold, while the Latin ones run from 1 to n and denote tensors on a spacelike hypersurface Σ. The adequate covariant derivatives are signed, respectively, as ∇_α and ∇_i.

II. HIGHER DIMENSIONAL GENERALIZED EINSTEIN-MAXWELL SYSTEM

In this section, we shall examine the generalized Maxwell $(n-2)$-gauge form $F_{\mu_1\ldots\mu_{n-2}}$ in n-dimensional spacetime described by the following action:

$$I = \int d^n x \sqrt{-g}[{}^{(n)}R - F^2_{(n-2)}], \tag{1}$$

where $g_{\mu\nu}$ is an n-dimensional metric tensor, and $F_{(n-2)} = dA_{(n-3)}$ is the $(n-2)$-gauge form field. The canonical formalism divides the metric into spatial and temporal parts, as follows:

$$ds^2 = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \tag{2}$$

where general covariance implies the great arbitrariness in the choice of lapse and shift functions $N^\mu(N, N^a)$.

A point in the phase space for the underlying theory is related to the specification of the fields $(h_{ab}, \pi_{ab}, A_{j_1\ldots j_{n-3}}, E_{j_1\ldots j_{n-3}})$ on $(n-1)$-dimensional hypersurface Σ. The field momenta are found in the usual way by varying the Lagrangian with respect to $\nabla_0 h_{ab}, \nabla_0 A_{j_1\ldots j_{n-3}}$, where ∇_0 denotes the derivative with respect to the time coordinate. Thus, the momentum canonically conjugates to a Riemannian metric

$$\pi^{ab} = \sqrt{h}(K^{ab} - h^{ab}K), \tag{3}$$

where K_{ab} is the extrinsic curvature of the hypersurface Σ. Similarly, the momentum canonically conjugates to $(n-2)$-gauge form field $F_{\mu_1\ldots\mu_{n-2}}$ defined as

$$\pi^{(F)}_{j_1\ldots j_{n-3}} = \frac{\delta \mathcal{L}}{\delta(\nabla_0 A^{j_1\ldots j_{n-3}})} = 2(n-2)E_{j_1\ldots j_{n-3}}, \tag{4}$$

while the electric field $E_{j_1\ldots j_{n-3}}$ implies

$$E_{j_1\ldots j_{n-3}} = \sqrt{h}F_{\alpha j_1\ldots j_{n-3}} n^\alpha, \tag{5}$$

where n^μ is the unit normal timelike vector to the hypersurface Σ. The Hamiltonian defined by the Legendre transform may be written as follows:

$$\begin{aligned}\mathcal{H} &= \pi^{ab}\nabla_0 h_{ab} + \pi^{j_1\ldots j_{n-3}}_{(F)}\nabla_0 A_{j_1\ldots j_{n-3}} - \mathcal{L}(R, F_{(n-2)}) \\ &= N^\mu C_\mu + \tilde{\mathcal{A}}_{0 j_2\ldots j_{n-3}}\tilde{B}^{j_2\ldots j_{n-3}} + \mathcal{H}_{\text{div}},\end{aligned} \tag{6}$$

where for brevity of notation I have denoted by $\tilde{\mathcal{A}}_{j_1\ldots j_{n-3}} = (n-3)!\,A_{j_1\ldots j_{n-3}}$. On the other hand, the total derivative part of the Hamiltonian $\mathcal{H}_{\text{div}}$ is given by

$$\begin{aligned}\mathcal{H}_{\text{div}} &= 2(n-3)(n-2)\nabla_{j_1}(E^{j_1\ldots j_{n-3}}\tilde{\mathcal{A}}_{0 j_2\ldots j_{n-3}}) \\ &\quad + 2\sqrt{h}\nabla_i\left(\frac{N_j \pi^{ij}}{\sqrt{h}}\right).\end{aligned} \tag{7}$$

The gauge field $\tilde{\mathcal{A}}_{0 j_2\ldots j_{n-3}}$ has no kinetic terms associated with it. Therefore, one can consider it as a Lagrange multiplier corresponding to the generalized *Gauss law* of the form as follows:

$$0 = \tilde{B}^{j_2\ldots j_{n-3}} = 2(n-3)(n-2)\nabla_{j_1}(E^{j_1\ldots j_{n-3}}). \tag{8}$$

In this paper, we shall consider the asymptotically flat initial data, i.e., in an asymptotic region of hypersurface Σ which is diffeomorphic to $\mathbf{R}^{n-1} - B$, where B is compact, one has the following conditions to be satisfied:

$$h_{ab} \approx \delta_{ab} + \mathcal{O}\left(\frac{1}{r}\right), \tag{9}$$

$$\pi_{ab} \approx \mathcal{O}\left(\frac{1}{r^2}\right), \tag{10}$$

$$A_{j_1\ldots j_{n-3}} \approx \mathcal{O}\left(\frac{1}{r}\right), \tag{11}$$

$$E_{j_1\ldots j_{n-3}} \approx \mathcal{O}\left(\frac{1}{r}\right). \tag{12}$$

At infinity we also assume the standard behavior of the lapse and shift functions, i.e., $N \approx 1 + \mathcal{O}(\frac{1}{r})$ and $N^a \approx \mathcal{O}(\frac{1}{r})$. On the hypersurface Σ, the initial data are restricted to the constraint manifold on which at each point $x \in \Sigma$ the following quantities vanish:

$$\begin{aligned}0 &= C_0 \\ &= \sqrt{h}\left[-{}^{(n-1)}R + \frac{1}{h}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right] \\ &\quad + \frac{(n-2)}{\sqrt{h}}E_{j_1\ldots j_{n-3}}E^{j_1\ldots j_{n-3}} + \sqrt{h}F_{j_1\ldots j_{n-2}}F^{j_1\ldots j_{n-2}},\end{aligned} \tag{13}$$

$$0 = C_a = 2(n-2)F_{a j_1\ldots j_{n-3}}E^{j_1\ldots j_{n-3}} - 2\sqrt{h}\nabla^i\left(\frac{\pi_{ia}}{\sqrt{h}}\right),$$

$$0 = \tilde{B}^{j_2\ldots j_{n-3}} = 2(n-3)(n-2)\nabla_{j_1}(E^{j_1\ldots j_{n-3}}),$$

where ∇_j is the derivative operator on Σ, while ${}^{(n-1)}R$ denotes the scalar curvature with respect to the metric h_{ab} on the considered hypersurface. The equations of motion for this theory can be formally derived from the *volume integral contribution* $\mathcal{H}_V$ to the Hamiltonian $\mathcal{H}$ and subject to the *pure constraint* form as follows:

$$\mathcal{H}_V = \int_\Sigma d\Sigma(N^\mu C_\mu + N^\mu \tilde{\mathcal{A}}_{\mu j_2 \dots j_{n-3}} \tilde{B}^{j_2 \dots j_{n-3}}). \quad (14)$$

One can verify that the change caused by arbitrary infinitesimal variations $(\delta h_{ab}, \delta\pi^{ab}, \delta\tilde{\mathcal{A}}^{j_1\cdots j_{n-3}}, \delta E_{j_1\dots j_{n-3}})$ of compact support, after integration by parts, leads us to the expression

$$\delta\mathcal{H}_V = \int_\Sigma d\Sigma(\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^{j_1\cdots j_{n-3}}\delta\tilde{\mathcal{A}}^{j_1\cdots j_{n-3}} + S^{j_1\cdots j_{n-3}}\delta E_{j_1\dots j_{n-3}}), \quad (15)$$

where $\mathcal{P}^{ab}$, $\mathcal{Q}_{ab}$, $\mathcal{R}^{j_1\cdots j_{n-3}}$, $S^{j_1\cdots j_{n-3}}$ are written as

$$\mathcal{P}^{ab} = \sqrt{h}Na^{ab} + \sqrt{h}(h^{ab}\nabla^j\nabla_j N - \nabla^a\nabla^b N) - \mathcal{L}_{N^i}\pi^{ab}, \quad (16)$$

$$\mathcal{Q}_{ab} = \frac{N}{\sqrt{h}}(2\pi_{ab} - \pi h_{ab}) + 2\nabla_a N_b, \quad (17)$$

$$\mathcal{R}^{j_1\cdots j_{n-3}} = -2(n-2)[\nabla_a(F^{aj_1\cdots j_{n-3}}) + \mathcal{L}_{N^i}E^{j_1\cdots j_{n-3}}], \quad (18)$$

$$S^{j_1\cdots j_{n-3}} = \frac{2(n-2)}{\sqrt{h}}NE^{j_1\cdots j_{n-3}} + 2(n-2)\mathcal{L}_{N^i}\tilde{\mathcal{A}}^{j_1\cdots j_{n-3}} + 2(n-2)(n-3)\nabla^{j_1}(N\tilde{\mathcal{A}}_0^{j_2\cdots j_{n-3}}), \quad (19)$$

while a^{ab} takes the form

$$a^{ab} = \frac{1}{2}F_{j_1\dots j_{n-2}}F^{j_1\cdots j_{n-2}} - (n-2)F^{aj_2\cdots j_{n-2}}F^b_{j_2\dots j_{n-2}} - \frac{(n-2)}{2\sqrt{h}}h^{ab}E_{j_1\dots j_{n-3}}E^{j_1\cdots j_{n-3}} + \frac{(n-2)(n-3)}{\sqrt{h}}E^{aj_2\cdots j_{n-3}}E^b_{j_2\dots j_{n-3}} + \frac{1}{h}\left[2\pi^a_j\pi^{bj} - \pi h^{ab} - \frac{1}{2}h^{ab}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right] + {}^{(n-1)}R^{ab} - \frac{1}{2}h^{ab(n-1)}R^{ab}. \quad (20)$$

In the above relations, $\mathcal{L}_{N^i}$ denotes the Lie derivative calculated on the hypersurface Σ. The Lie derivative of $E^{j_1\cdots j_{n-3}}$ is understood as the Lie derivative of the adequate tensor density.

On using the Hamiltonian principle and evaluating variations of the compact support of Σ, we finally reach the evolution equations which can be written as follows:

$$\dot{\pi}^{ab} = -\mathcal{P}^{ab}, \quad (21)$$

$$\dot{h}_{ab} = \mathcal{Q}_{ab}, \quad (22)$$

$$\dot{E}^{j_1\cdots j_{n-3}} = -\mathcal{R}^{j_1\cdots j_{n-3}}, \quad (23)$$

$$\dot{\tilde{\mathcal{A}}}_{j_1\dots j_{n-3}} = S_{j_1\dots j_{n-3}}. \quad (24)$$

As was mentioned in Ref. [34], expression (14) depicts rather the volume integral contribution to the Hamiltonian. The nonvanishing surface terms arise when we take into account integration by parts. In order to get rid of these surface contribution terms, one can add the surface terms given by

$$\mathcal{H} = \mathcal{H}_v + \int_{S^\infty} dS_{j_1}[2(n-2)N^a\tilde{\mathcal{A}}_{aj_2\dots j_{n-3}}E^{j_1\cdots j_{n-3}} + 2(n-2)(n-3)N\tilde{\mathcal{A}}_{0j_2\dots j_{n-3}}E^{j_1\cdots j_{n-3}}] + \int_{S^\infty} dS^a\left[N(\nabla^b h_{ab} - \nabla_a h^b_b) + \frac{2N^b\pi_{ab}}{\sqrt{h}}\right]. \quad (25)$$

By the direct calculations, it can be seen that not only for asymptotically flat perturbations of a compact support of the hypersurface Σ but also for N^μ and $\tilde{\mathcal{A}}_{j_1\dots j_{n-3}}$ satisfying the asymptotic conditions at infinity, we get

$$\delta\mathcal{H} = \int_\Sigma d\Sigma(\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^{j_1\cdots j_{n-3}}\delta\tilde{\mathcal{A}}^{j_1\cdots j_{n-3}} + S^{j_1\cdots j_{n-3}}\delta E_{j_1\dots j_{n-3}}). \quad (26)$$

III. FIRST LAW OF BLACK HOLE MECHANICS

We can define the canonical energy as the Hamiltonian function corresponding to the case when N^μ is an asymptotical translation at infinity. Thus, one has that $N \to 1$, $N^a \to 0$. We multiply the Hamiltonian function by 1/2 and reach the expression

$$\mathcal{E} = \alpha M + (n-2)(n-3)\int_{S^\infty} dS_{j_1}N\tilde{\mathcal{A}}_{0j_2\dots j_{n-3}}E^{j_1\cdots j_{n-3}}, \quad (27)$$

where M is the ADM mass defined as follows:

$$\alpha M = \frac{1}{2}\int_{S^\infty} dS^a[N(\nabla^b h_{ab} - \nabla_a h^j_j)], \quad (28)$$

and $\alpha = \frac{n-3}{n-2}$. The remaining term is highly gauge dependent because of the arbitrary choice of $\tilde{\mathcal{A}}_{0j_2\dots j_{n-3}}$. It yields

$$\mathcal{E}_F = (n-2)(n-3)\int_{S^\infty} dS_{j_1}N\tilde{\mathcal{A}}_{0j_2\dots j_{n-3}}E^{j_1\cdots j_{n-3}}. \quad (29)$$

We shall call $\mathcal{E}_F$ *canonical energy* of $(n-2)$-gauge form fields. We define also the canonical angular momentum $J_{(i)}$ on the constraint submanifold of the phase space as the Hamiltonian $\mathcal{H}$ multiplied by the factor 1/2, when $N \to 0$ and the shift vector tends to the appropriate Killing vector fields responsible for rotation in the adequate directions. Thus, it reduces to

$$J_{(i)}^{(\infty)} = -\frac{1}{2}\int_{S^\infty} dS_a[2\phi_{(i)}^b \pi_b^a + 2(n-2)(n-3)\phi_{(i)}^m \tilde{\mathcal{A}}_{mj_2\cdots j_{n-3}} E^{aj_2\cdots j_{n-3}}]. \quad (30)$$

If one considers the case of hypersurface Σ having an asymptotic region and smooth interior boundary S and takes into account the linear combinations of the translation and rotations at infinity, then we reach the following expression:

$$2\left(\delta\mathcal{E} - \sum_{i=1}^{n-1}\Omega_{(i)} J^{(i)(\infty)}\right) = \int_\Sigma d\Sigma(\mathcal{P}^{ab}\delta h_{ab} + \mathcal{Q}_{ab}\delta\pi^{ab} + \mathcal{R}^{j_1\cdots j_{n-3}}\delta\tilde{\mathcal{A}}^{j_1\cdots j_{n-3}} + S^{j_1\cdots j_{n-3}}\delta E_{j_1\cdots j_{n-3}}) + \delta(\text{surface terms}). \quad (31)$$

As in Ref. [34], one can take an asymptotically flat hypersurface Σ which intersects the sphere S of a stationary n-dimensional black hole. We assume also that $(n-2)$-sphere S is a bifurcation Killing horizon and set $N^\mu = \chi^\mu = t^\mu + \sum_{i=1}^{n-1}\Omega_{(i)}\phi^{\mu(i)}$, where $\Omega_{(i)}$ describe angular velocities of the direction established by $\phi^{\mu(i)}$. We also choose $\tilde{\mathcal{A}}_{0j_2\cdots j_{n-3}}$ so that $\dot{A}_{j_1\cdots j_{n-3}} = \dot{E}_{j_1\cdots j_{n-3}} = 0$. Using Eqs. (21)–(24) one can draw a conclusion that the integral over Σ vanishes while only one surface term survives because of the fact that on sphere S we have $N^\mu = 0$. The nonzero term is equal to $2\pi\kappa\delta A$, where κ is the surface gravity constant on S, while A is the area of the $(n-2)$-dimensional sphere S. Thus we reach the following:

Theorem.—Let $(h_{ij}, \pi^{ij}, \tilde{\mathcal{A}}_{j_1\cdots j_{n-3}}, E^{j_1\cdots j_{n-3}})$ be smooth asymptotically flat initial data for a stationary black hole with $(n-2)$-gauge form field on a hypersurface Σ with $(n-2)$-dimensional bifurcation sphere S. If $(\delta h_{ij}, \delta\pi^{ij}, \delta\tilde{\mathcal{A}}_{j_1\cdots j_{n-3}}, \delta E^{j_1\cdots j_{n-3}})$ are arbitrary smooth asymptotically flat solutions of the linearized constraints on a hypersurface Σ, then the following is fulfilled:

$$\alpha\delta M + \delta\mathcal{E}_F - \sum_{i=1}^{n-1}\Omega_{(i)} J^{(i)(\infty)} = \kappa\delta A. \quad (32)$$

Taking into account (32) one can see that any stationary black hole with a bifurcate Killing horizon is an extremum of mass M at fixed *canonical energy* of $(n-2)$-gauge form fields, canonical momentum, and horizon area. We get the extension of the first law of black hole mechanics which is true for arbitrary asymptotically flat perturbations of a stationary n-dimensional black hole (in four dimensions, a similar result was obtained by Sudarsky and Wald [34] in Einstein Yang-Mills theory and in the case of Einstein-Maxwell axion-dilaton black holes in Ref. [36]), contrary to the first law of black holes mechanics derived in Ref. [38] valid for perturbations to a nearby stationary black hole.

IV. STATICITY CONDITIONS

Now we proceed to find the staticity theorem for nonrotating n-dimensional black holes with $(n-2)$-gauge field $F_{\mu_1\cdots\mu_{n-2}}$. To begin with, let us suppose that a stationary black hole is regular on and outside a Killing horizon of a Killing vector field of the form

$$\chi^\mu = t^\mu + \sum_{i=1}^{n-1}\Omega_{(i)}\phi^{\mu(i)} \quad (33)$$

is normal. The mass of a black hole implies [25] the following:

$$M = -\frac{1}{\alpha}\int_S \epsilon_{j_1\cdots j_{n-2}ab}\nabla^a t^b. \quad (34)$$

Furthermore, we define the angular momentum of black hole associated with a rotational Killing vector $\phi_{(i)}$ expressed as a covariant surface integral

$$I_{(i)\mathrm{BH}} = \frac{1}{2}\int_H \epsilon_{j_1\cdots j_{n-2}ab}\nabla^a\phi_{(i)}^b. \quad (35)$$

The same procedure as in Ref. [38] leads us to the mass formula

$$M = \frac{2}{\alpha}\int_\Sigma d\Sigma\left(T_{\mu\nu} + \frac{g_{\mu\nu}T}{2-n}\right)t^\mu n^\nu + \frac{2}{\alpha}\kappa A + \frac{2}{\alpha}\sum_{i=1}^{n-1}\Omega_{(i)} I_{\mathrm{BH}}^{(i)}. \quad (36)$$

Rewriting the latter expression (36) in terms of the considered matter energy momentum tensor yields

$$M - \frac{2}{\alpha}\kappa A - \frac{2}{\alpha}\sum_{i=1}^{n-1}\Omega_{(i)} I_{\mathrm{BH}}^{(i)} = \int_\Sigma d\Sigma\left[\frac{(n-2)}{\sqrt{h}} t^m F_{mj_1\cdots j_{n-3}} E^{j_1\cdots j_{n-3}} + \frac{\lambda}{h}E_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}} + \lambda\left(\frac{n-3}{n-2}\right)F_{j_1\cdots j_{n-2}}F^{j_1\cdots j_{n-2}}\right], \quad (37)$$

where we defined $\lambda = -n_\beta t^\beta$. Taking account of constraint equations and changing the surface integral into a volume one, we can deduce that $J_{(i)}^{(\infty)}$ has the form

$$J_{(i)}^{(\infty)} = -\frac{1}{2}\int_\Sigma d\Sigma[\pi^{ab}\mathcal{L}_{N^i}h_{ab} + 2(n-2)\mathcal{L}_{N^i}\tilde{\mathcal{A}}_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}] + J_{(i)H}, \quad (38)$$

where we define $J_{(i)H}$ by the following expression:

$$J_{(i)H} = -\frac{1}{2}\int_S dS_a[2\phi^b_{(i)}\pi^a_b + 2(n-2) \times (n-3)\phi^m_{(i)}\tilde{\mathcal{A}}_{mj_2\cdots j_{n-3}}E^{aj_2\cdots j_{n-3}}]. \quad (39)$$

Using the fact that Killing vector fields $\phi^{(i)}_\mu$ are equal to their tangential projection $\phi^{(i)}_m$, one can readily find that $J^{(\infty)}_{(i)} = J^{(i)}_H$. The first term in relation (39) is equal to $I^{(i)}_{\rm BH}$. Then from (39) it follows immediately the result

$$\sum_{i=1}^{n-1}\Omega_{(i)}(I^{(i)}_{\rm BH} - J^{(i)(\infty)}) = (n-2)\int_\Sigma d\Sigma[E^{j_1\cdots j_{n-3}}\mathcal{L}_{N^i}\tilde{\mathcal{A}}_{j_1\cdots j_{n-3}} - t^m F_{mj_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}]. \quad (40)$$

By virtue of the above equation and the constraint relation (24), we find the expression of the form

$$M - \frac{2}{\alpha}\kappa A + \frac{2}{\alpha}\mathcal{E}_F - \frac{2}{\alpha}\sum_{i=1}^{n-1}\Omega_{(i)}J^{(i)(\infty)} = \int_\Sigma d\Sigma\Big[2\lambda F_{j_1\cdots j_{n-2}}F^{j_1\cdots j_{n-2}} - 2\frac{\lambda(n-2)}{h}E_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}\Big]. \quad (41)$$

From this stage on, we shall restrict our attention to the case of the maximal hypersurface, i.e., for which $\pi^a_a = 0$. Having this in mind, we consider the initial data induced on hypersurface Σ and choose the lapse and shift function coinciding with Killing vector fields in the spacetime under consideration. It may be verified that by contracting Eq. (21) we get

$$\nabla_m\nabla^m N = \rho N, \quad (42)$$

where ρ is given by

$$\rho = \Big(\frac{n-3}{n-2}\Big)F_{j_1\cdots j_{n-2}}F^{j_1\cdots j_{n-2}} + \frac{n}{2h(n-2)}\pi_{ij}\pi^{ij} - \frac{1}{2h}[(n-1)(n-5) + (3-n)]E_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}. \quad (43)$$

We remark that ρ will be non-negative for $n \geq 4$. Consistently with this remark, the maximum principle can be applied to relation (42) provided that solutions of it can be uniquely determined by their boundary value at S and their asymptotic value at infinity.

To proceed further, we use as the lapse function λ with the boundary conditions $\lambda|_S = 0, \lambda|_\infty = 1$. Integrating Eq. (42) we obtain a *black hole mass formula* as

$$M - \frac{2}{\alpha}\kappa A = \frac{2}{\alpha}\int_\Sigma d\Sigma\lambda\rho. \quad (44)$$

Using the scaling transformation, we can transform a solution of Einstein $(n-2)$-form gauge theory into a new one with the following changes:

$$M \to \beta^{n-3}M, \quad (45)$$

$$\mathcal{E}_F \to \beta^{n-3}\mathcal{E}_F, \quad (46)$$

$$\Omega_{(i)} \to \beta^{-1}\Omega_{(i)}, \quad (47)$$

$$J^{(i)(\infty)} \to \beta^{n-2}J^{(i)(\infty)}, \quad (48)$$

$$\kappa \to \beta^{-1}\kappa, \quad (49)$$

$$A \to \beta^{n-2}A, \quad (50)$$

where β is a constant. Inserting the linearized perturbation connected with the above scaling transformation into Eq. (32), one is finally left with the second mass formula of the form

$$\alpha M - 2\kappa A - 2\sum_{i=1}^{n-1}\Omega_{(i)}J^{(i)(\infty)} + \mathcal{E}_F = 0. \quad (51)$$

Then, using (41), (44), and (51) one solves them for $\mathcal{E}_F$ and $\sum_{i=1}^{n-1}\Omega_{(i)}J^{(i)(\infty)}$. The results become

$$\mathcal{E}_F = \int_\Sigma d\Sigma\Big[4\lambda\Big(\frac{n-3}{n-2}\Big)F_{j_1\cdots j_{n-2}}F^{j_1\cdots j_{n-2}} + 4\lambda\frac{n-3}{h}E_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}\Big] \quad (52)$$

while the formula for the angular momenta can be written as

$$\sum_{i=1}^{n-1}\Omega_{(i)}J^{(i)(\infty)} = \int_\Sigma d\Sigma\Big\{3\lambda\Big(\frac{n-3}{n-2}\Big)F_{j_1\cdots j_{n-2}}F^{j_1\cdots j_{n-2}} + \frac{\lambda n}{2h(n-2)}\pi_{ij}\pi^{ij} + \frac{\lambda}{h}\Big[\frac{3(3-n)-(n-1)(n-5)}{2(2-n)}\Big] \times E_{j_1\cdots j_{n-3}}E^{j_1\cdots j_{n-3}}\Big\}. \quad (53)$$

In the case of four-dimensional spacetime, the coefficient for $E^2_{(n-3)}$ in Eq. (53) is equal to zero. Thus, we have the same result for $n = 4$ as was obtained in Ref. [35].

In Ref. [39] it was pointed out that the exterior region of a black hole can be foliated by maximal hypersurfaces with boundary S, asymptotically orthogonal to the timelike Killing vector field t_μ, when the strong energy condition for every timelike vector is satisfied. As one can check, this is the case in the considered theory. In light of what has been shown above, we can establish the following:

Theorem.—Let us consider an asymptotically flat solution to Einstein $(n-2)$-gauge theory possessing a stationary Killing vector field and describing a stationary black hole comprising a bifurcate Killing horizon. Suppose, moreover, that $\sum_{i=1}^{n-1} \Omega_{(i)} J^{(i)(\infty)} = 0$, then the solution is static and has vanishing electric $E^{j_1 \cdots j_{n-3}}$ or magnetic $F_{j_1 \ldots j_{n-2}}$ fields on static hypersurfaces.

One can readily verify the above by applying Eq. (53) to the maximal hypersurfaces. It will be noticed that on the considered hypersurfaces Σ_t one has the condition $\pi_{ij} = 0$. Let N denote the lapse function for the maximal hypersurface and n^μ depict the unit normal to this hypersurface. We choose $N^\mu = N n^\mu$ as the evolution vector field for these slices. This is sufficient to establish that

$$\mathcal{L}_{N^\mu} \pi^{ij} = \dot{\pi}^{ij} = 0. \tag{54}$$

From Eqs. (17) and (22), since $\pi^{ab} = 0$ and $N^a = 0$, we obtain that $\mathcal{L}_{N^\mu} h_{ab} = \dot{h}_{ab} = 0$. We shall first consider the case when $E^{j_1 \cdots j_{n-3}} = 0$. Consequently, it yields the result as follows:

$$\mathcal{L}_{N^\mu} E^{j_1 \cdots j_{n-3}} = \dot{E}^{j_1 \cdots j_{n-3}} = 0. \tag{55}$$

It can be verified that, considering Eqs. (19) and (24) and choosing $A_{0 j_2 \ldots j_{n-3}} = 0$, one gets that $\dot{\tilde{\mathcal{A}}}_{j_1 \ldots j_{n-3}} = 0$. By virtue of this, we can conclude that the solution is static.

Now we take into account the case when $F_{j_1 \ldots j_{n-2}} = 0$. To begin with let us consider relation (18) from which, because of the fact that $N^a = 0$, one has that $\mathcal{L}_{N^\mu} E^{j_1 \cdots j_{n-3}} = 0$. Thus, we see that $\dot{E}^{j_1 \cdots j_{n-3}} = 0$. Now consider Eqs. (19) and (24) and choose $A_{0 j_2 \cdots j_{n-3}} = 0$ as well as $E^{j_1 \cdots j_{n-3}} = 0$. Then one can draw a conclusion that $\dot{\tilde{\mathcal{A}}}_{j_1 \cdots j_{n-3}} = 0$ and the solution is static.

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