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Universality of the Hawking effect

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Addressing the question of whether the Hawking effect depends on degrees of freedom at ultrahigh (e.g., Planckian) energies/momenta, we propose three rather general conditions on these degrees of freedom under which the Hawking effect is reproduced to lowest order. As a generalization of Corley's results, we present a rather general model based on nonlinear dispersion relations satisfying these conditions together with a derivation of the Hawking effect for that model. However, we also demonstrate counter-examples, which do not appear to be unphysical or artificial, displaying strong deviations from Hawking's result. Therefore, whether real black holes emit Hawking radiation remains an open question and could give nontrivial information about Planckian physics.

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I. INTRODUCTION

The striking similarity between the laws of black hole physics and the (zeroth till third) law of thermodynamics motivated the idea to assign thermodynamic properties such as temperature and entropy to black holes [1]. Hawking's prediction [2] that black holes should emit thermal radiation with the temperature being consistent with the thermodynamic interpretation strongly supported this idea. As a consequence, the concept of black hole entropy as given by the surface (horizon) area of the black hole in Planckian units (instead of the volume, for example) is now used in many ways to estimate the total entropy of other objects—which is expected to be a measure of the number of fundamental degrees of freedom of the underlying theory (including quantum gravity).

However, in view of the (exponential) gravitational redshift near the horizon, the outgoing particles of the Hawking radiation originate from modes with extremely large (e.g., trans-Planckian) wavenumbers. As the known equations of quantum fields in curved space-times are expected to break down at such wavenumbers, the derivation of the Hawking radiation has the flaw that it applies a theory beyond its region of validity. This observation poses the question of whether the Hawking effect is independent of Planckian physics or not.

One way to address this question is to model the breakdown of the (usual) local Lorentz invariance (to be expected at the Planck scale) by a (nonlinear) deviation from the linear dispersion relation at high wavenumbers, see, e.g., [3,4]. This method is inspired by the black hole analogues which exploit the analogy between the propagation of excitations (e.g., sound waves) in laboratory-physics systems and quantum fields in curved space-times, see, e.g., [5–7].

In Secs. II, III, IV, and V we generalize and simplify the model and the results presented by Corley in [3] (see also [4]) trying to identify and to present the crucial points. Section VI is devoted to the question of which conditions and assumptions regarding Planckian physics are needed to reproduce Hawking's result—together with some counter-examples.

II. LINEAR MODEL

At first we consider a subluminal dispersion relation cf. Fig. 1, which is in some sense conceptually more clear because the in-modes generating the Hawking radiation come from outside the black hole. The horizon acts as a classical turning point where the JWKB (geometric optics) approximation breaks down allowing phenomena like particle creation. In contrast to Ref. [3], we shall not specify the shape of the dispersion relation apart from some rather general assumptions.

A. Wave Equation

The geometry as seen by the low-energy particles is described in terms of the $1 + 1$ dimensional Painlevé-Gullstrand-Lemaître [8] metric ($\hbar = c = 1$ throughout)

$$ds^2 = dt^2 - [dx - \upsilon(x)dt]^2 = [1 - \upsilon^2]dt^2 + 2\upsilon dt dx - dx^2. \tag{1}$$

The quantity $\upsilon(x)$ can be interpreted as the local velocity of the freely falling frames measured with respect to the time t corresponding to the Killing vector ∂_t of that stationary metric. In terms of the sonic black hole analogues, t is the laboratory time and υ is just the position-dependent velocity of the fluid with the (assumed to be constant) speed of sound being absorbed by a redefinition of the coordinates. Since the behavior near the horizon in arbitrary dimensions is essentially $1 + 1$ dimensional for each mode, we restrict ourselves to $1 + 1$ dimensions. Furthermore, we neglect backscattering (as induced by the angular-momentum barrier, for example).

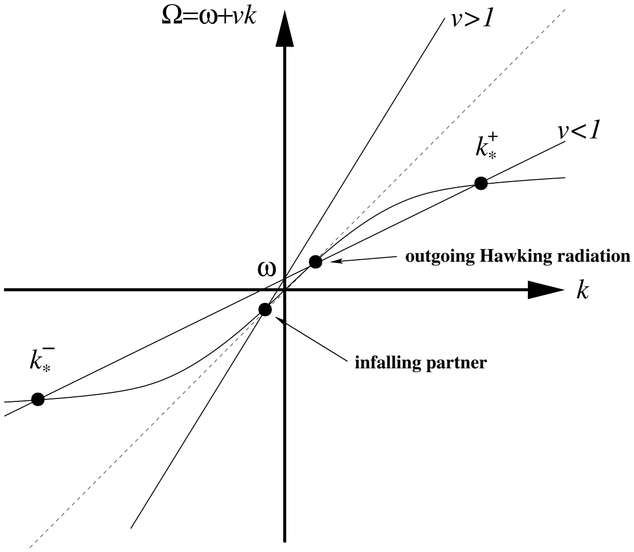

FIG. 1. Subluminal dispersion relation (not to scale). The points of intersection (black circles) with the two lines for $v > 1$ (i.e., $x < 0$) and $v < 1$ (i.e., $x > 0$) determine the solutions of the dispersion relation for a given ω. The two points corresponding to large wavenumbers $k_*^\pm$ have group velocities smaller than v, i.e., they are "swept away" and approach the horizon from above $x > 0$. Hence these solutions are the in-modes. The other solution at $x > 0$ with the group velocity exceeding v represents the outgoing Hawking radiation. The only solution beyond the horizon $x < 0$ again has a group velocity smaller than v. The corresponding wavefunction represents the infalling partner particles of the outgoing Hawking radiation, which have a negative energy as measured from infinity. During the evolution, the high-wavenumber in-modes $k_*^\pm$ (x decreases $\rightsquigarrow$ v increases) are being converted into the low-wavenumber Hawking radiation plus partner particles—where the break down of the JWKB approximation near the horizon leads to a mixing of these modes resulting in particle creation.

In order to ensure hyperbolicity, causality, and stability, we only allow second time-derivatives. Hence the generalized Klein-Fock-Gordon equation reads

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} v(x)\right]\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right]\phi = \left(\frac{\partial^2}{\partial x^2} + F\left[\frac{\partial^2}{\partial x^2}\right]\right)\phi, \tag{2}$$

with the function F representing the nontrivial dispersion relation. In general, the function F might contain an arbitrary number of derivatives—, i.e., be nonlocal (think of a lattice, for example). Note that we do not take into account absorption (i.e., F is real). The resulting dispersion relation $(\omega + vk)^2 = k^2 - F[-k^2]$ is plotted in Fig. 1.

For a stationary metric as in Eq. (1), we may separate the most general solution of the Klein-Fock-Gordon Eq. (2) into stationary modes with frequencies ω

$$\left[F\left[\frac{\partial^2}{\partial x^2}\right] + [1 - v^2]\frac{\partial^2}{\partial x^2} + 2v(i\omega - v')\frac{\partial}{\partial x} - i\omega(i\omega - v')\right]\phi_\omega = 0. \tag{3}$$

The black-hole horizon is assumed to be located at $x = 0$ and hence the Taylor expansion of the velocity around this point reads

$$v(x) = -1 + \kappa x + \mathcal{O}(\kappa^2 x^2), \tag{4}$$

with κ denoting the surface gravity.

B. Assumptions

Let us summarize the assumptions that will be used for deriving the Hawking effect:

Obviously, the surface gravity of the black hole (and hence the temperature of the Hawking radiation) must be much smaller than the cut-off scale where the concept of geometry and metric breaks down (i.e., F is not negligible anymore)

$$\kappa \lll k_{\text{cutoff}}. \tag{5}$$

Furthermore, we shall assume that particle creation—necessitating a break-down of the JWKB (geometric optics) approximation—occurs in the vicinity of the horizon only. Hence we shall consider an intermediate regime: close to the horizon at $x = 0$ in units of κ

$$\kappa|x| \ll 1, \tag{6}$$

but still many cut-off lengths away from the horizon

$$|x|k_{\text{cutoff}} \ggg 1. \tag{7}$$

Based on the above assumptions, we may neglect terms of second and higher order in κ and ω since we are interested in low-frequency modes $\omega = \mathcal{O}(\kappa)$ only (Hawking radiation). Accordingly, the wave Eq. (3) simplifies to

$$\left[F\left[\frac{\partial^2}{\partial x^2}\right] + 2\kappa x\frac{\partial^2}{\partial x^2} - 2(i\omega - \kappa)\frac{\partial}{\partial x}\right]\phi_\omega = 0. \tag{8}$$

At this stage, it is advantageous to Laplace transform this equation via

$$\phi_\omega(x) = \int_C ds e^{xs}\tilde{\phi}_\omega(s). \tag{9}$$

where the contour C in the complex plane will be discussed below. Note the change of sign in the second term due to the integration by parts. The wave equation for the Laplace transformed mode $\tilde{\phi}_\omega(s)$ in terms of the complex variable s reads

$$\left[F[s^2] - 2\kappa\frac{\partial}{\partial s}s^2 - 2(i\omega - \kappa)s\right]\tilde{\phi}_\omega = 0. \tag{10}$$

In the following, we shall impose the following conditions on the dispersion relation $F[s^2]$:

(i) The dispersion relation $F[s^2]$ is assumed to be an analytic function of s^2.

(ii) Hence it possesses a Laurent/Taylor expansion

$$F[s^2] = k_{\rm cutoff}^2 \sum_{n=2}^{\infty} a_n \left(\frac{s}{k_{\rm cutoff}}\right)^{2n}, \tag{11}$$

where the (nonvanishing) coefficients a_n and the radius of convergence are supposed to be of order one—, i.e., the dispersion relation does not depend on small quantities like $\kappa/k_{\rm cutoff}$.

(iii) Furthermore, we assume a subluminal dispersion relation, i.e., in the rest frame, we have

$$\left(\frac{d\omega}{dk}\right)^2 \leq 1 \rightsquigarrow 0 \leq \omega^2 = k^2 - F[-k^2] \leq k^2 \rightsquigarrow 0 \leq F[-k^2] \leq k^2. \tag{12}$$

(iv) Finally, we assume that asymptotically $k^2 \uparrow \infty$, the dispersion relation is well separated from the line $\omega = k$, i.e., the phase velocity does not approach unity

$$\lim_{k^2\uparrow\infty} \frac{\omega^2}{k^2} < 1 \rightsquigarrow \lim_{k^2\uparrow\infty} \frac{F[-k^2]}{k^2} = F_\infty > 0. \tag{13}$$

Apart from these assumptions we do not need to specify the dispersion relation any further. For convenience, we shall choose units in which $k_{\rm cutoff} = 1$ and omit it in the following equations.

III. ANALYTICAL DERIVATION

A. Complex Plane and Asymptotics

After a separation of variables, the Laplace transformed wave Eq. (10) can be cast into the following form

$$\frac{\partial}{\partial s} \ln(s^2 \tilde{\phi}_\omega) = \frac{F[s^2] - 2(i\omega - \kappa)s}{2\kappa s^2}. \tag{14}$$

Up to an irrelevant prefactor due to the integration constant, its solution reads

$$\tilde{\phi}_\omega(s) = \frac{s^{-i\omega/\kappa}}{s} \exp\left\{\int ds \frac{F[s^2]}{2\kappa s^2}\right\}. \tag{15}$$

Since $F[s^2]/s^2$ is analytic, the Laplace transform $\tilde{\phi}_\omega(s)$ has a singularity at $s = 0$ and a branch cut from $s = 0$ to infinity—but no further singularities at finite values of s. We choose the negative real axis $\Im(s) = 0$ and $\Re(s) < 0$ for the branch cut since this choice will be most convenient for deriving Hawking radiation—for an alternative choice, see Sec. V.

As in the usual Fourier transform, we choose a contour that approaches infinity along the imaginary axis $s = ik$. In this case, the overall exponent in Eq. (9) is purely imaginary and behaves for large $|s| = |k|$ as

$$\exp\left\{xs + \int ds \frac{F[s^2]}{2\kappa s^2}\right\} \approx \exp\left\{xs - \frac{sF_\infty}{2\kappa}\right\}, \tag{16}$$

according to assumption (13). Hence the exponential function is rapidly oscillating at large $|s|$ and thus yields (again at large $|s|$) no contribution to the integral in Eq. (9). The k-integral over $\exp\{ik[x - F_\infty/(2\kappa)]\}$ gives $\delta(2\kappa x - F_\infty)$ and hence vanishes since $\kappa|x| \ll 1$. (As we shall see below, the same result can be obtained by deforming the contour of integration in the complex plane.) From a physical point of view, this result is not very surprising since—given a nontrivial dispersion relation—one would not expect momenta which are much larger than the cut-off to contribute. (This expectation is however false for $F = 0$.) The significant contributions will be found by the stationary phase method, or, after deforming the contour, the saddle-point method.

B. Saddle Point Method

In order to apply the saddle point method, let us rewrite the Laplace transformation in Eq. (9) as

$$\phi_\omega(x) = \int_C ds g(s) e^{xf(s)}, \tag{17}$$

with the two auxiliary functions

$$g(s) = \frac{s^{-i\omega/\kappa}}{s}, \tag{18}$$

and

$$f(s) = s + \frac{1}{x}\left(\int ds \frac{F[s^2]}{2\kappa s^2}\right). \tag{19}$$

The saddle points s_* of $f(s)$ are determined by

$$\left(\frac{df}{ds}\right)_{s=s_*} = 0 \rightsquigarrow 2\kappa s_*^2 x + F[s_*^2] = 0. \tag{20}$$

Many cut-off lengths away from the horizon $|x| \ggg 1$ (but still $\kappa|x| \ll 1$), we can approximate the integral in Eq. (9) by the saddle-point expansion

$$\phi_\omega(x) \approx \sqrt{\frac{2\pi}{-xf''(s_*)}} e^{xf(s_*)} g(s_*), \tag{21}$$

where a sum over multiple saddle points is implied with proper orientation. The next terms of the saddle-point expansion are suppressed by a factor of order

$$\frac{g''(s_*)}{g(s_*)xf''(s_*)} = \mathcal{O}\left(\frac{1}{xs_*}\right), \tag{22}$$

and can be neglected if x is sufficiently large to overcome the smallness of s_*, which depends on the small quantity κx via Eq. (20).

Along the imaginary axis $s_* = ik_*$, the possible saddle points are given by

$$F[-k_*^2] = 2\kappa x k_*^2 = (1 - v^2)k_*^2 + \mathcal{O}(\kappa^2), \qquad (23)$$

i.e., they exactly coincide with the solutions of the dispersion relation (see Fig. 1) for $k \gg \omega$

$$k_*^2 - F[-k_*^2] = (\omega + v k_*)^2 \approx v^2 k_*^2, \qquad (24)$$

since $\omega = \mathcal{O}(\kappa)$ and $|x| \ggg 1$.

C. Contour for $x < 0$

Since the solutions of the dispersion relation and hence the saddle points depend on the sign of x, i.e., on which side of the horizon is considered, it is convenient to choose different contours in the complex plane for $x > 0$ and $x < 0$ making sure that they are deformable to each other as x goes through zero. Let us first study the case $x < 0$, i.e., the solution of the wave equation beyond the horizon cf. Figure 2.

Along the imaginary axis $s = ik$ the exponent in Eq. (21) is purely imaginary

$$\Re\{f(ik)\} = 0, \qquad (25)$$

whereas its derivative is purely real and positive

$$f'(ik) = 1 - \frac{F[-k^2]}{2\kappa x k^2} > 0, \qquad (26)$$

because $x < 0$ and $F[-k^2] \geq 0$ cf. assumption (12). As a result $|\exp\{xf(s)\}|$ decreases rapidly ($x < 0$) with increasing $\Re\{s\}$ and there are no saddle points on the imaginary axis $s = ik$. Hence we can deform the contour into the valley at $\Re\{s\} > 0$ until we hit possible saddle points at

$$\Re\{s_*[x < 0]\} > 0, \qquad (27)$$

with values

$$\Re\{f(s_*[x < 0])\} > 0. \qquad (28)$$

Since the coefficients in the Laurent/Taylor expansion of $F[s^2]$ are of order one, the real part of the saddle point satisfying $2\kappa s_*^2 x + F[s_*^2] = 0$ is mainly determined by the small quantity $\kappa x \ll 1$. Again we assume that the size of $x \ggg 1$ overcomes the smallness of $\kappa x \ll 1$ and consequently obtain a solution

$$\phi_\omega(x < 0) \approx \sqrt{\frac{2\pi}{-xf''(s_*)}} e^{xf(s_*)} g(s_*), \qquad (29)$$

which decays exponentially fast beyond the horizon.

Note that, even if we encounter no saddle points, the contour can still be deformed such that the contributions are exponentially small cf. Fig. 2. In this way the chosen contour yields basically no contribution beyond the horizon $x < 0$—which is exactly what we want for the derivation of the outgoing Hawking radiation.

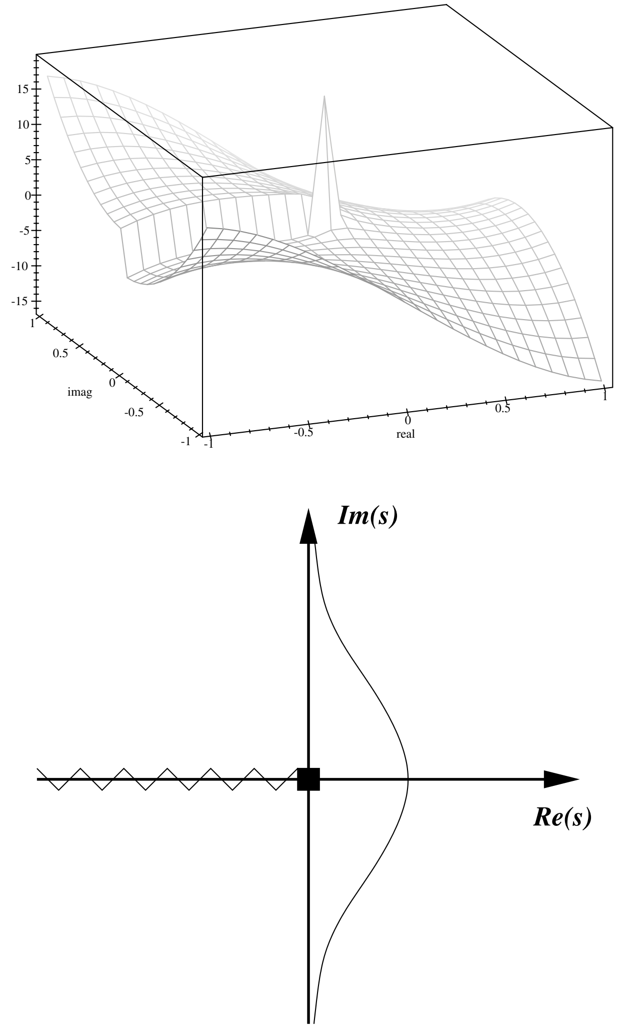

FIG. 2. Landscape plot (top) of the real part of the logarithm of the integrand in Eq. (9) for a subluminal dispersion relation and the case $x = -5$ and $\omega = \kappa = 1/30$ as well as contour in the complex plane (bottom). The black square denotes the singularity and the zig-zag line is the branch cut. The behavior of the landscape near the imaginary axis is generic, but the structure away from that axis (e.g., existence of further saddle points) depends on the particular form of the (subluminal) dispersion relation (here $F[s^2] = s^4$).

D. Contour for $x > 0$

In order to derive the solution outside the horizon $x > 0$, another contour is needed for applying the saddle point method cf. Figure 3. The exponent in Eq. (21) is still purely imaginary along the imaginary axis $s = ik$, but the slope

$$f'(ik) = 1 - \frac{F[-k^2]}{2\kappa x k^2}, \qquad (30)$$

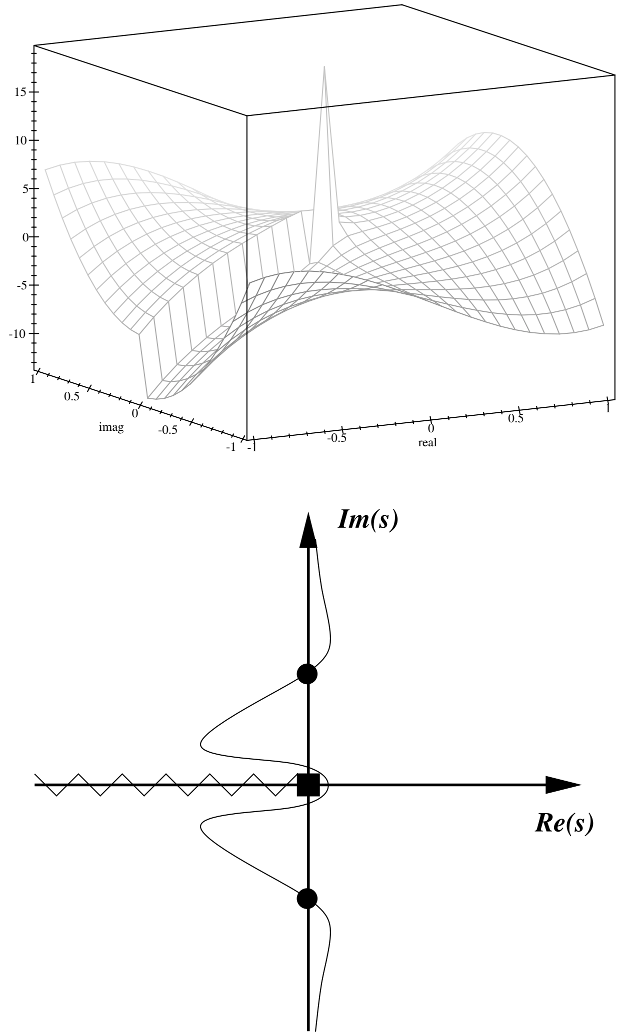

FIG. 3. Landscape plot (top) of the real part of the logarithm of the integrand in Eq. (9) for a subluminal dispersion relation and the case $x = +5$ and $\omega = \kappa = 1/30$ as well as contour in the complex plane (bottom). The black square denotes the singularity, the black dots are the saddle points, and the zig-zag line is the branch cut. The behavior of the landscape near the imaginary axis is generic, but the structure away from that axis (e.g., existence of further saddle points) depends on the particular form of the (subluminal) dispersion relation (here $F[s^2] = s^4$).

changes its sign at saddle points $s_*^\pm = \pm ik_*$. The condition (11) ensures the existence of exactly two (symmetric) saddle points along the imaginary axis—i.e., solutions of the dispersion relation with finite values of $f''(s_*)$—but the analysis can easily be generalized to the case of more than two saddle points.

For small $k \ll 1$, the first term dominates according to Eq. (11)

$$F[-k^2 \uparrow 0] \ll k^2 \rightsquigarrow 1 - \frac{F[-k^2]}{2\kappa x k^2} \approx 1, \tag{31}$$

and hence the valley is on the side $\Re\{s\} < 0$ of the imaginary axis—whereas for $k^2 > k_*^2$, the slope $f'(ik)$ is negative and thus the valley is on the other side $\Re\{s\} > 0$. Hence the contour must cross these two saddle points (cf. Figure 3) and pick up the corresponding contributions

$$\begin{aligned} \phi_\omega^\pm(x > 0) &\approx \sqrt{\frac{2\pi}{-xf''(s_*^\pm)}} e^{xf(s_*^\pm)} g(s_*^\pm) \\ &= \sqrt{\frac{2\pi}{-xf''(\pm ik_*)}} e^{xf(\pm ik_*)} g(\pm ik_*) \\ &= \sqrt{\frac{2\pi}{\mp xf''(ik_*)}} e^{\pm xf(ik_*)} g(\pm ik_*). \end{aligned} \tag{32}$$

As $f(ik_*)$ is purely imaginary, the only difference in the absolute values of the two contributions is determined by the branch cut in $g(s)$

$$\left| \frac{\phi_\omega(s_*^+, x > 0)}{\phi_\omega(s_*^-, x > 0)} \right| = \left| \frac{g(s_*^+)}{g(s_*^-)} \right| = e^{\pi\omega/\kappa}. \tag{33}$$

Ergo, the two saddle points at $s_*^\pm = \pm ik_*$ yield two rapidly ($x \ggg 1$) and oppositely oscillating contributions, whose absolute values satisfy the above relation (which will become important later on).

E. Branch Cut

Between the two saddle points $-k_* < k < k_*$, the valley lies on the same side $\Re\{s\} < 0$ of the imaginary axis as the branch cut does cf. Fig. 3. If it was not for the branch cut, the contour (originating from infinity) could be closed in this valley after crossing the two saddle points at $s_*^\pm = \pm ik_*$ such that all additional contributions (possibly further saddle points) are exponentially smaller than those of the saddle points at $s_*^\pm = \pm ik_*$. However, the branch cut demands that we integrate along it to $s = 0$ from both sides (with the proper orientation) and in circumventing the branch cut, we pick up the difference in the values of $g(s)$

$$g[\Im(s) \downarrow 0] - g[\Im(s) \uparrow 0] = 2\sinh\left(\frac{\pi\omega}{\kappa}\right) \frac{|s|^{-i\omega/\kappa}}{s}. \tag{34}$$

In this way, we obtain an additional contribution

$$\begin{aligned} \phi_\omega^{\text{rest}}(x) = 2\sinh\left(\frac{\pi\omega}{\kappa}\right) \int \frac{ds}{s} |s|^{-i\omega/\kappa} \\ \times \exp\left\{xs + \int ds \frac{F[s^2]}{2\kappa s^2}\right\}, \end{aligned} \tag{35}$$

where the integral runs from 0 along the negative real axis up to the intersection point of the contour with the branch cut. In view of $x \ggg 1$, we can omit the second term in the integrand and extend the interval to $-\infty$

$$\phi_\omega^{\text{rest}}(x > 0) \approx 2\sinh\left(\frac{\pi\omega}{\kappa}\right) \int_0^{-\infty} \frac{ds}{s} \frac{\exp\{xs\}}{|s|^{i\omega/\kappa}}. \tag{36}$$

Outside the horizon $x > 0$—the region we are interested in—we may substitute $\chi = |xs|$ and obtain

$$\phi_\omega^{\rm rest}(x>0) \approx -2\sinh\left(\frac{\pi\omega}{\kappa}\right)x^{i\omega/\kappa}\int_0^\infty \frac{d\chi}{\chi}\frac{e^{-\chi}}{\chi^{i\omega/\kappa}}, \quad (37)$$

which is just an integral representation of the Γ-function, i.e.,

$$\phi_\omega^{\rm rest}(x>0) \approx -2\sinh\left(\frac{\pi\omega}{\kappa}\right)\Gamma\left(-\frac{i\omega}{\kappa}\right)x^{i\omega/\kappa}. \quad (38)$$

Together with the contributions in Eq. (32), this completes the (approximate) solution of the wave equation outside the horizon—for the case that the solution basically vanishes beyond the horizon. Of course, we can only draw this conclusion if the two contours for $x > 0$ and $x < 0$ in Figs. 2 and 3 are deformable to each other as x crosses zero. This property is ensured by assumption (11) since the part of the complex plane covered during the deformation of the contours ($\kappa|x| \ll 1 \leadsto |s_*| \ll 1$) is well inside the radius of convergence of order one.

F. Bogoliubov Coefficients

Let us identify the various parts of the solution. The contribution generated by the branch cut $\phi_\omega^{\rm rest}(x)$ is the wavefunction of an outgoing particle with a low wavenumber $k = \mathcal{O}(\omega)$ (e.g., Hawking radiation). The saddle point contributions $\phi_\omega^\pm(x)$, on the other hand, are rapidly oscillating, since the largeness of $x \ggg 1$ is supposed to be stronger than the smallness of $s_*(\kappa x)$.

As one can observe in Fig. 1, the group velocity of the low-energy mode $\phi_\omega^{\rm rest}(x)$ exceeds $v(x)$, as one should expect for an outgoing particle—whereas the group velocity of the rapidly oscillating modes $\phi_\omega^\pm(x)$ is smaller than $v(x)$. Hence these are the in-modes $\phi_\pm^{\rm in}$.

Furthermore, the frequencies of the rapidly oscillating modes in the freely falling frame $\Omega_\pm = \omega \pm vk_*$ have different signs because $k_* \gg \omega$ (although $\omega > 0$ for both modes). As a result, the low-energy outgoing particle (e.g., Hawking radiation) is a mixture of positive and negative frequency (with respect to the freely falling frame) in-modes—which can be described in terms of the Bogoliubov coefficients

$$\phi_\omega^{\rm out} = \alpha_\omega \phi_+^{\rm in} + \beta_\omega \phi_-^{\rm in}. \quad (39)$$

A nonvanishing Bogoliubov β_ω-coefficient of course corresponds to the phenomenon of particle creation—i.e., the in-vacuum with respect to the freely falling frame is converted into a quantum state containing particles (Hawking radiation) by the horizon. The ratio of the Bogoliubov coefficients is determined by Eq. (33)

$$|\beta_\omega| = e^{-\pi\omega/\kappa}|\alpha_\omega|, \quad (40)$$

which is the well-known relation leading to the thermal Hawking spectrum. E.g., applying the unitarity relation of the Bogoliubov coefficients $\alpha\cdot\alpha^\dagger - \beta\cdot\beta^\dagger = \mathbf{1}$, we immediately obtain the thermal spectrum $\langle\hat{N}_\omega\rangle = |\beta_\omega|^2 \propto 1/(e^{2\pi\omega/\kappa} - 1)$.

Note that the quantum state generated by the time-evolution of the in-vacuum also contains particles with low wavenumbers beyond the horizon (see Fig. 1). As required by energy conservation and unitarity, for each outgoing particle of the Hawking radiation, there is a corresponding partner particle with negative energy (as measured from infinity) inside of the black hole—but this does not alter presented calculation cf. [3]. There are, however, correlations between the outgoing Hawking particle and its partner beyond the horizon—which generate the true thermal character (mixed state instead of pure state) of the Hawking radiation for any outside observer (thermo-field formalism [4,9]), see Sec. V below.

The branch cut in the complex plane caused by the horizon turns out to be a main ingredient for deriving the Hawking effect—it generates the contribution $\phi_\omega^{\rm rest}(x)$ as well as the ratio in Eq. (33)—which are both essential features.

IV. SUPER-LUMINAL DISPERSION

So far, we restricted our attention to a subluminal dispersion relation only. As we shall see now, the case of a superluminal dispersion as in Fig. 4 can be treated in

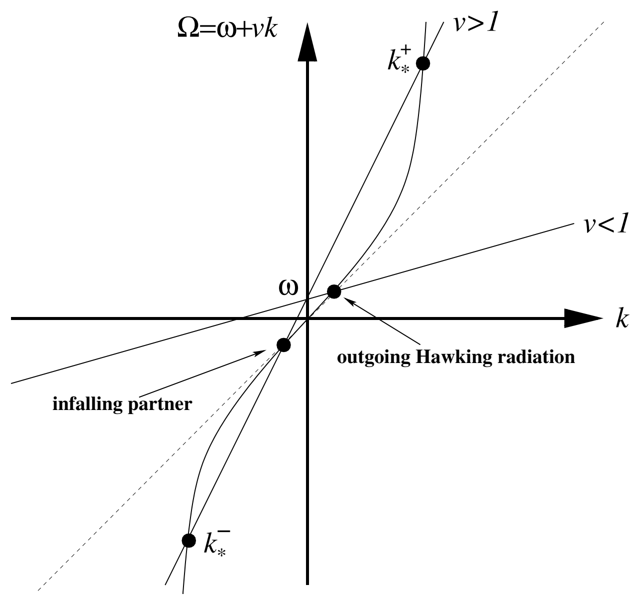

FIG. 4. Superluminal dispersion relation (not to scale). The points of intersection (black circles) with the two lines for $v > 1$ (i.e., $x < 0$) and $v < 1$ (i.e., $x > 0$) determine the solutions of the dispersion relation for a given ω. As in the subluminal case, the out-modes are the low-wavenumber solutions corresponding to the outgoing particles of the Hawking radiation ($x > 0$) and their infalling partners ($x < 0$). However, the high-wavenumber in-modes $k_*^\pm$ have group velocities exceeding v and hence are approaching the horizon from the inside $x < 0$.

basically the same way. The steps and derivations from Eq. (1) to Eq. (11) are identical, and we choose the same branch cut. Of course, for a superluminal dispersion relation we have to modify assumptions (12) and (13) accordingly in order to ensure a vanishing asymptotical contribution at the imaginary axis $s = ik$. The derivations in Secs. IIIA and IIIB apply in the same way, but now the solutions of the dispersion relation with large wavenumbers (the in-modes) are superluminal, i.e., they originate from inside the black hole, see Fig. 4.

A. Contour for $x > 0$

Let us first consider the solution outside the black hole, see Fig. 5. In contrast to the subluminal case, the function $f(s)$ in the exponent has a positive slope for all k-values

$$f'(ik) = 1 - \frac{F[-k^2]}{2\kappa x k^2} > 0, \tag{41}$$

because $F[-k^2] < 0$, and, consequently, the valley is now situated at negative real parts of s. Deforming the contour into the valley, all contributions become exponentially small—but (again) we have to circumvent the branch cut. The contribution of the branch cut yields the same result as in Sec. IIIE. Ergo, outside the black hole, we have only the outgoing Hawking particle—which is exactly what one would expect in the superluminal case.

B. Contour for $x < 0$

For $x < 0$, i.e., beyond the horizon, the slope $f'(ik)$ changes its sign at the saddle points (i.e., the solutions of the dispersion relation with large wavenumbers) and the contour has to cross the imaginary axis picking up the saddle point contributions cf. Fig. 6. For those contributions, we basically obtain the same results as in Eq. (32) and hence in Eq. (33), because the branch cut is identical.

Therefore, we reach the same conclusion as in Sec. IIIF where now the in-modes originate from inside the black hole. Hence we reproduce Hawking radiation also for a superluminal dispersion—provided that the in-modes (large wavenumbers) are initially in their ground state with respect to the freely falling frame.

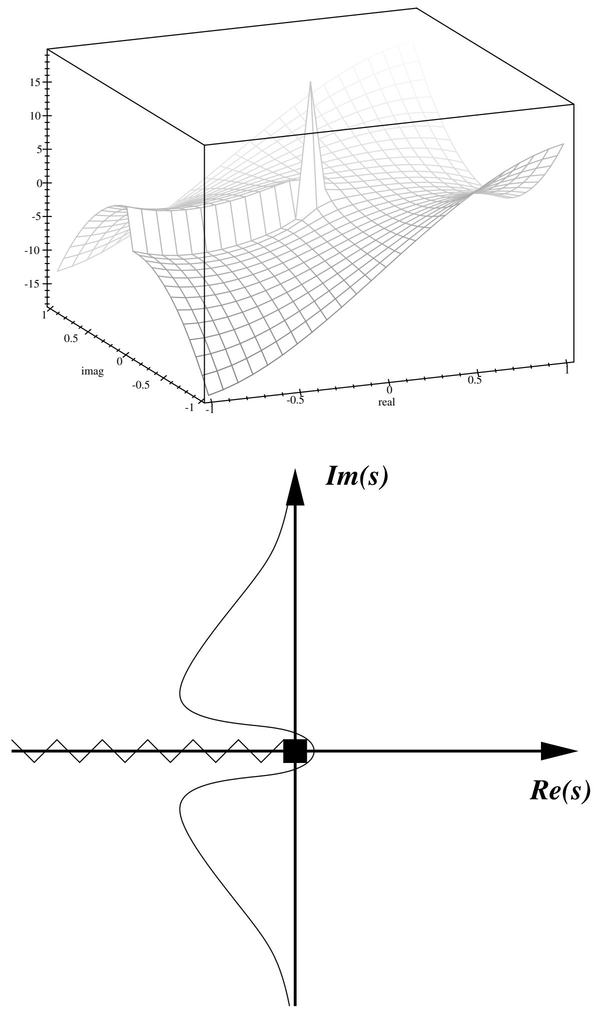

FIG. 5. Landscape plot (top) of the real part of the logarithm of the integrand in Eq. (9) for a superluminal dispersion relation and the case $x = +5$ and $\omega = \kappa = 1/30$ as well as contour in the complex plane (bottom). The black square denotes the singularity and the zig-zag line is the branch cut. The behavior of the landscape near the imaginary axis is generic, but the structure away from that axis (e.g., existence of further saddle points) depends on the particular form of the (superluminal) dispersion relation (here $F[s^2] = -s^4$).

V. ENTANGLEMENT

So far, we restricted our attention to the decomposition of the outgoing Hawking radiation in terms of the in-modes cf. Eq. (39). However, a full description of the evolution of the quantum state requires a complete set of out-modes, i.e., the outgoing Hawking particles as well as their infalling partners. Fortunately, it turns out that the Bogoliubov coefficients of the infalling partners can be inferred in complete analogy to the previous Sections if we choose the branch cut in the opposite way, i.e., along the positive real axis $\Im(s) = 0$ and $\Re(s) > 0$. Circumventing this branch cut then reproduces the wavefunction of the infalling partner particles as in Sec. IIIE (but with $x \to -x$), and, consistently, this contribution only occurs for $x < 0$, i.e., beyond the horizon. Again, for both cases (sub- and superluminal), one obtains basically the same relation

$$\phi_\omega^{\text{partner}} = \beta_\omega^{\text{partner}} \phi_+^{\text{in}} + \alpha_\omega^{\text{partner}} \phi_-^{\text{in}}. \tag{42}$$

Note that we have interchanged the role of the creation and annihilation operators and hence α^{partner} and β^{partner} here because the energy of the infalling partner particles is negative as measured from infinity and their pseudonorm is negative for positive ω. The opposite direction of the branch cut implies the inverse relation compared to Eq. (33) and together with the above interchange, we

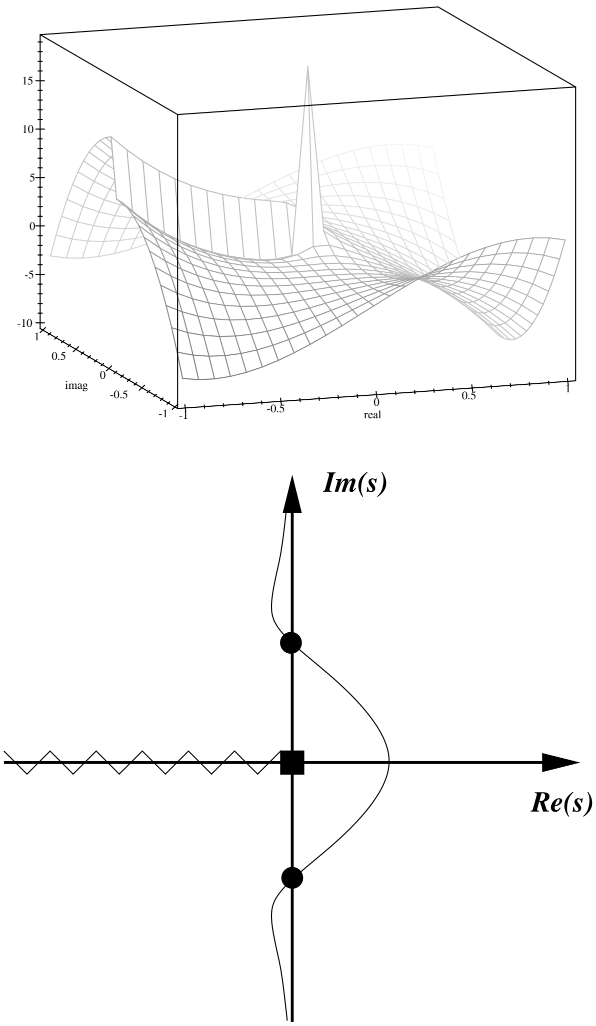

FIG. 6. Landscape plot (top) of the real part of the logarithm of the integrand in Eq. (9) for a superluminal dispersion relation and the case $x = -5$ and $\omega = \kappa = 1/30$ as well as contour in the complex plane (bottom). The black square denotes the singularity, the black dots are the saddle points, and the zig-zag line is the branch cut. The behavior of the landscape near the imaginary axis is generic, but the structure away from that axis (e.g., existence of further saddle points) depends on the particular form of the (superluminal) dispersion relation (here $F[s^2] = -s^4$).

derive the same ratio as in Eq. (40)

$$|\beta_\omega^{\text{partner}}| = e^{-\pi\omega/\kappa}|\alpha_\omega^{\text{partner}}|. \tag{43}$$

The knowledge of the complete set of out-modes (Hawking radiation $\phi_\omega^{\text{Hawking}}$ plus their infalling partners $\phi_\omega^{\text{partner}}$) facilitates the decomposition of the in-modes in terms of the out-modes

$$\phi_-^{\text{in}} = \alpha_\omega^{\text{inv}}\phi_\omega^{\text{partner}} + \beta_\omega^{\text{inv}}\phi_\omega^{\text{Hawking}}. \tag{44}$$

In view of the relations (40) and (43) as well as unitarity $\alpha \cdot \alpha^\dagger - \beta \cdot \beta^\dagger = \mathbf{1}$, the inverse Bogoliubov coefficients satisfy an analogous condition

$$|\beta_\omega^{\text{inv}}| = e^{-\pi\omega/\kappa}|\alpha_\omega^{\text{inv}}|. \tag{45}$$

Consequently, the in-vacuum defined via $\hat{a}_{\text{in}}|0_{\text{in}}\rangle = 0$ will be annihilated by a linear combination of the operators corresponding to the out-modes

$$[\hat{a}_\omega^{\text{partner}} + e^{-\pi\omega/\kappa}(\hat{a}_\omega^{\text{Hawking}})^\dagger]|0_{\text{in}}\rangle = 0, \tag{46}$$

where an irrelevant phase has been absorbed by the redefinition of $\hat{a}_\omega^{\text{partner}}$. This well-known relation (see, e.g., [10]) induces the entanglement between the particles of the Hawking radiation $\hat{a}_\omega^{\text{Hawking}}$ and their infalling partners $\hat{a}_\omega^{\text{partner}}$

$$|0\rangle_{\text{in}}^\omega \propto \exp\{e^{-\pi\omega/\kappa}(\hat{a}_\omega^{\text{partner}}\hat{a}_\omega^{\text{Hawking}})^\dagger\}|0\rangle_{\text{out}}^\omega, \tag{47}$$

which in turn generates the thermal density matrix after averaging over the unobservable infalling partners.

VI. UNIVERSALITY

The derivation presented in the previous Sections demonstrates that the Hawking effect does (to lowest order) not depend on the details of the dispersion relation at high wavenumbers—given the model assumptions discussed above. Let us try to identify more general conditions under which the Hawking should remain unchanged by the details of the physics at large wavenumbers. For convenience, we shall assume that the cut-off scale coincides with the Planck scale and use the terms sub-Planckian for effects according to the known laws of physics (e.g., linear dispersion) and trans-Planckian for new physics (e.g., nonlinear dispersion).

First of all, we assume that the JWKB (geometric optics or eikonal) approximation breaks down (thereby allowing for the phenomenon of particle creation) in the vicinity of the horizon only, where the gravitational red-shift induces a transition of trans-Planckian into sub-Planckian modes. An example where this assumption does not apply will be discussed in Sec. VIB.

Given that assumption, the crucial point is the quantum state of the modes when they leave the Planckian regime. If the modes leave the Planckian regime (''are born'') in their ground state with respect to freely falling observers near the horizon, then one obtains Hawking radiation cf. [10,11]. Let us review the standard argument leading to that conclusion. In terms of the Regge-Wheeler tortoise coordinate r_*, the 1 + 1 dimensional Schwarzschild metric can be cast into the conformally flat form

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt^2 - dr_*^2) \simeq \exp\left\{\frac{r_*}{2M}\right\}(dt^2 - dr_*^2), \tag{48}$$

where the $\simeq$ applies near the horizon. The trajectory of a freely falling observer is given by (A and B are integration constants)

$$r_*(t \uparrow \infty) \simeq -t - A \exp\left\{-\frac{t}{2M}\right\} + B, \tag{49}$$

and its proper time $d\tau^2 = ds^2[t, r_*(t)]$ accordingly reads

$$\tau \sim \exp\left\{-\frac{t}{2M}\right\}. \tag{50}$$

Hence the freely falling observers would define their ground state via the positive frequency solutions

$$F_\omega^{\text{in}}(U) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega U}, \tag{51}$$

with respect to the Kruskal coordinate

$$U = -4Me^{-u/(4M)} = -4Me^{-[t-r_*]/(4M)}. \tag{52}$$

The doubly exponential behavior of these modes—when expressed in terms of the coordinates t, r of an outside observer—lead to the thermal particle content.

The remaining issue is, of course, to determine in which cases the modes do indeed leave the Planckian regime in their ground state with respect to freely falling observers (near the horizon). As a very natural example, one could ensure this property by means of the following three assumptions:

(a) **Freely falling frame** If we assume that the usual local Lorentz invariance is broken at the Planck scale via the introduction of preferred frames (where preferred frames are the frames in which Planckian physics displays maximal symmetry under time-inversion, for example) then the freely falling frame should be preferred (instead of the rest frame of the black hole, for example).

(b) **Ground state** The Planckian excitations are assumed to start off in their ground state (with respect to the freely falling frame, see point above) subject to possible constraints such as conservation laws etc.

(c) **Adiabatic evolution** Finally the evolution of the modes is supposed to be adiabatic—, i.e., the Planckian dynamics is supposed to be much faster than all external (sub-Planckian) variations (e.g., experienced by a traveling wavepacket). This condition demands the absence of level crossing and long time-scales in Planckian physics.

E.g., for the sonic black hole analogues (”dumb holes”) such as a fluid flowing trough a Laval nozzle (accelerated from subsonic to supersonic speed), the freely falling frame corresponds to the local rest frame of the flowing fluid, whereas the rest frame of the walls of the nozzle is analogous to the global rest frame of black hole. Of course, one can easily imagine situations where at least one of the above assumptions fails. E.g., for a superluminal dispersion relation, the modes with large wavenumbers originate from inside the black hole, i.e., ultimately from the singularity (of from a turbulent regime), and it is not obvious why they should be in their ground state. Further examples for the failure of the above assumptions, where the reference frame for Planckian physics is not the local freely falling frame but the global rest frame of black hole; or where the adabaticity breaks down, are the subject of the next Sections.

A. Miles Instability

As an example, in which the aforementioned set of assumptions fails and which does not reproduce Hawking radiation, let us consider the following fluid model: Apart from a deviation from the usual dispersion relation as described by a k-dependent phase velocity $v_{\text{ph}}^2(k)$, we suppose a coupling to a reservoir of Planckian degrees of freedom in the rest frame of the black hole (i.e., *not* the freely falling frame) manifesting itself as an effective dissipation term in the dispersion relation

$$(\omega + v_{\text{fl}}k)^2 = k^2 v_{\text{ph}}^2(k) - 2i\omega\gamma(k). \tag{53}$$

The k-dependence of the damping term $\gamma(k)$ ensures that it is completely negligible at sub-Planckian wavenumbers.

In flat space-time (fluid at rest $v_{\text{fl}} = 0$), the damping term just implies a decay of the Planckian modes (as one would expect). For a black hole, the Planckian modes giving rise to Hawking radiation, however, behave in a different way. As one can easily perceive from Figs. 1 and 4, for solutions of the dispersion relation with large (Planckian) wavenumbers, we have

$$|v_{\text{ph}}^2(k) - v_{\text{fl}}^2| \ll 1. \tag{54}$$

Expanding the relevant solution of Eq. (53) for ω in powers of this small quantity, we obtain

$$\omega_+ \approx \frac{k^2}{2} \frac{v_{\text{ph}}^2(k) - v_{\text{fl}}^2}{v_{\text{fl}}k + i\gamma(k)}. \tag{55}$$

Assuming a real wavenumber, the imaginary part of the frequency changes its sign if the fluid velocity exceeds the phase velocity

$$\Im(\omega) > 0, \tag{56}$$

which indicates an instability. This phenomenon is basically the Miles instability—which is responsible for the generation of water waves by wind, for example [12].

If k and γ are of order one (in Planckian units), the imaginary part of ω is of the same order as the real part

$$\Im(\omega) = \mathcal{O}[\Re(\omega)], \tag{57}$$

and since $\omega = \mathcal{O}(\kappa)$ corresponds to the inverse size of the black hole, there can be enough time for the instability to develop and to excite the modes. Note that positive frequency (trans-Planckian) modes with $v_{\text{ph}}^2(k) > v_{\text{fl}}^2$ are damped but negative frequency modes with $v_{\text{ph}}^2(k) < v_{\text{fl}}^2$ are amplified. Hence this effect destroys the balance in

Eq. (40) which generates the thermal spectrum of the Hawking radiation.

However, the above analysis based on classical solutions of the dispersion relation cannot be applied directly, i.e., without respecting the fluctuation-dissipation theorem, for example, to the quantum fluctuations that generate the Hawking radiation. In order to turn our attention to the quantum theory, let us consider the following Lagrangian density corresponding to a superluminal dispersion $v_{\rm ph}^2 = 1 + k^2$

$$\mathcal{L} = \frac{1}{2}[(\dot{\phi} + \mathbf{v} \cdot \nabla\phi)^2 - (\nabla\phi)^2 - (\nabla^2\phi)^2]. \quad (58)$$

For a stationary metric, i.e., $v(x)$, a conserved energy density with respect to global rest frame of black hole can be derived by means of the Noether theorem

$$\mathcal{E} = \frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + (\nabla^2\phi)^2 - (\mathbf{v} \cdot \nabla\phi)^2]. \quad (59)$$

Evidently the energy density is not positive definite for $\mathbf{v}^2 > v_{\rm ph}^2$, i.e., beyond the horizon (superluminal dispersion). The local energy density with respect to the local freely falling frame is of course positive definite. After a normal mode expansion into wavepackets, the total Hamiltonian is split up into nearly independent positive and negative energy modes (with respect to global rest frame of black hole). Obviously, the negative energy modes can be strongly excited by a comparably weak interaction with further Planckian degrees of freedom at the global rest frame of the black hole. In this way, these modes would not be in their ground state—even with respect to freely falling observers—and, consequently, one would not reproduce Hawking radiation.

For example, let us consider fermionic fields where the quantum states of all trans-Planckian modes are maximally excited, i.e., $|1\rangle$ instead of the ground state $|0\rangle$. In that situation, the usual relation $\beta_\omega = e^{-\pi\omega/\kappa}\alpha_\omega$ implies

$$\langle 1|\hat{N}_\omega|1\rangle = |\alpha_\omega|^2 \propto \frac{1}{1 + e^{-2\pi\omega/\kappa}}, \quad (60)$$

i.e., *not* a thermal spectrum. Note that, for fermions, the unitarity relation is $\alpha \cdot \alpha^\dagger + \beta \cdot \beta^\dagger = \mathbf{1}$ instead of $\alpha \cdot \alpha^\dagger - \beta \cdot \beta^\dagger = \mathbf{1}$ leading to the Fermi-Dirac spectrum $1/(e^{2\pi\omega/\kappa} + 1)$ for the usual Hawking radiation. If the occupation number (i.e., $|0\rangle$ or $|1\rangle$) depends on the history of the mode—e.g., the frequency ω—then one would obtain another (in general nonthermal) spectrum. With an appropriate mixture of $|0\rangle$ and $|1\rangle$, one could even obtain a state with no outgoing particle content (Hawking radiation) at all (Boulware vacuum).

In summary, an interaction with a reservoir at the Planck scale with respect to the rest frame of the black hole can invalidate Hawking's derivation. In that argument, the rest frame of the black hole is a crucial point—a damping term with respect to the freely falling frame $2i(\omega + v_{\rm fl}k)\gamma(k)$ would not induce a positive imaginary part of ω. A similar phenomenon occurs in the so-called " black hole laser" where wavepackets bounce back and forth between the inner and outer horizons—which also generates deviations from the Hawking effect [13]. This quasireflection mechanism also singles out the rest frame of the black hole (location of the two horizons) as a preferred frame for the Planckian modes. In contrast to the Miles instability, this phenomenon displays more similarities to the Pierce instability [14].

B. Breakdown of Adiabaticity

In the previous subsection VIA, the assumption (a) of Sec. VI and hence also (b) failed. Let us now give an example for the breakdown of the adiabaticity condition (c), which is closely related to the assumption that geometric optics is valid everywhere except in the vicinity of the horizon.

One version of the adiabatic theorem states that if the dynamics of all internal degrees of freedom is much faster than any external time dependence, then a system being initially in its ground state basically remains in the (time-dependent instantaneous) ground state. (Of course, for this theorem to apply we have to assume that quantum theory is still valid at the Planck scale.)

As a counter-example, where the system does not stay in its ground state, consider the dispersion relation

$$\omega^2 = \sin^2 k + m^2, \quad (61)$$

which has minima at $k \in \pi\mathbb{N}$ (Planckian units). If we assume a weakly time-dependent metric far away from the black hole $ds^2 = a^2(t)[dt^2 - dx^2]$, the wave equation reads after a normal mode expansion

$$\ddot{\phi}_k + [\sin^2 k + a^2(t)m^2]\phi_k = 0, \quad (62)$$

since the mass term breaks the conformal invariance. Therefore, it very easy to create Planckian ($k \approx \pi\mathbb{N}$) particles (e.g., via parametric resonance) by means of comparably small and slow (sub-Planckian) variations of $a(t)$ with with a characteristic scale corresponding to m (instead of the Planck mass).

As a result of this breakdown of the geometric optics approximation far away from the horizon, the Planckian modes falling towards the black hole are not in their ground state—and hence one will again obtain deviations from the Hawking effect. Note that a similar effect (occupation of Planckian modes) can occur during inflation if we assume a dispersion relation like the above.

VII. CONCLUSIONS

A. Summary

The Hawking effect is not *a priori* independent of the laws of physics at the Planck scale, but it can be made so by imposing the three assumptions **a) Freely falling frame**,

b) Ground state, and **c) Adiabatic evolution**, explained in more detail in Sec. VI. As one example, we generalized the analytical method of Ref. [3] to arbitrary dispersion relations subject to some rather general assumptions.

However, we have also demonstrated counter-examples, which do not appear to be unphysical or artificial, displaying deviations from Hawking's result. Therefore, whether real black holes emit Hawking radiation or not remains an open question and gives nontrivial information about Planckian physics.

B. Outlook

Another example, where sub-Planckian phenomena have their origin in trans-Planckian modes, is the generation of inhomogeneities during the cosmic epoch of inflation (according to our present standard model of cosmology) from quantum fluctuations of the inflaton field. In this case, the investigation of the universality, or, conversely, the dependence of this mechanism on Planckian physics including higher-order corrections has an additional aspect, because observations of the cosmic microwave background, for example, might yield signatures of Planckian physics, see, e.g., [15,16]. E.g., a dispersion relation with minima for large k-values as in Sec. VIB potentially allows particle creation leading to a change in the spectrum, see [15].

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