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Axi-dilaton gravity in $D \geq 4$ dimensional space-times with torsion

H. Cebeci*

Department of Physics, Middle East Technical University, 06531 Ankara, Turkey

T. Dereli†

Department of Physics, Koç University, 03450 Sarıyer, İstanbul, Turkey

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We study models of axi-dilaton gravity in space-time geometries with torsion. We discuss conformal rescaling rules in both Riemannian and non-Riemannian formulations. We give static, spherically symmetric solutions and examine their singularity behavior.

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I. INTRODUCTION

Gravitational interactions are formulated on a space-time manifold M equipped with a metric tensor field g and a metric compatible connection ∇ defined on the bundle of orthonormal frames. Most commonly, interactions coupled with gravity are studied in a geometry where the connection ∇ is constrained to be the unique torsion-free Levi-Cività connection. In this context, massive test particles are postulated to follow timelike geodesics associated with space-time metric and torsion-free connection. On the other hand a metric compatible connection with torsion provides new independent degrees of freedom. It has been shown that the scalar field interactions coupled with gravity can yield connections with nonzero torsion [1]. In that case, a space-time history of particles may be determined by the autoparallels of a connection with torsion [2–4]. We know that the independent variation of any action with respect to connection determines space-time torsion. In particular, the bosonic part of effective superstring interactions can produce a torsion that is proportional to the gradient of the dilaton (scalar) field. Hence, it would be of interest to formulate such types of interactions in frames where torsion exists.

It is an exciting conjecture that all superstring models belong to an 11 dimensional M theory that accommodates their apparent dualities. M theory as a classical theory can be considered in a low-energy limit where only the low-lying excitation modes contribute to an effective field theory. As such it would be the same as $D = 11$ dimensional supergravity theory. A subsequent Kaluza-Klein reduction to $D = 10$ dimensions would bring it to a string model whose gravitational sector consists of space-time metric tensor g, dilaton scalar ϕ, and the axion potential $(p + 1)$-form A that would minimally couple to p branes. We call such an effective gravitational field theory an axi-dilaton gravity in D dimensions. Axi-dilaton gravity theory can be studied in the Einstein frame. However, by working out the theory in the Brans-Dicke frame [5], one can see the difference between formulation of theory with a torsion-free connection and formulation with a connection with torsion. In the latter case, we vary the action treating the metric and the connection as independent variables. We have shown that the corresponding field equations in both cases with or without torsion are equivalent provided a shift in the Brans-Dicke coupling parameter ω is allowed. We further assume a direct coupling of the kth power of the dilaton scalar with the axionic kinetic term. The conformal scaling properties are examined in both geometries. In Sec. III we investigate a class of static, spherically symmetric solutions which depend on the coupling parameters ω and k in dimensions $D \geq 4$. In particular, we point out a new class of conformal black hole solutions obtained for the scale invariant parameter values.

II. AXI-DILATON GRAVITY IN D DIMENSIONS

We start with an action

$$I[g, \phi, A] = \int_M \mathcal{L}, \tag{1}$$

where the Lagrangian density D-form $\mathcal{L}$ is given in the Brans-Dicke frame in a geometry based on the Riemannian formulation, by imposing as a constraint that the connection is Levi-Cività:

$$\mathcal{L} = \frac{\phi}{2} R^{ab} \wedge *(e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi - \frac{\phi^k}{2} H \wedge *H. \tag{2}$$

Here the basic gravitational variables are the coframe 1-forms e^a in terms of which the space-time metric $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = \mathrm{diag}(- + + + + + \cdots)$. The Hodge $*$ map is defined so that the oriented volume form $*1 = e^0 \wedge e^1 \wedge \cdots \wedge e^{D-1}$. Levi-Cività connection 1-forms ${}^{(0)}\omega^a{}_b$ are obtained from the first Cartan structure equations

$$de^a + {}^{(0)}\omega^a{}_b \wedge e^b = 0, \tag{3}$$

*Present Address: Feza Gürsey Institute, 34680 Çengelköy, İstanbul, Turkey.
Email address: hcebeci@metu.edu.tr
†Email address: tdereli@ku.edu.tr

where the metric compatibility requires ${}^{(0)}\omega_{ab} = -{}^{(0)}\omega_{ba}$ and corresponding curvature 2-forms are obtained from the second Cartan structure equations

$$ {}^{(0)}R^{ab} = d^{(0)}\omega^{ab} + {}^{(0)}\omega^{a}{}_{c} \wedge {}^{(0)}\omega^{cb}. \tag{4} $$

ϕ is the dilaton 0-form and H is a $(p+2)$-form field that is derived from the axion potential $(p+1)$-form A so that $H = dA$. ω and k are real coupling parameters. Coframe e^a variations of this action lead to the Einstein field equations

$$ \frac{1}{2}\phi^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{\omega}{2\phi}\tau_c[\phi] - \frac{\phi^k}{2}\tau_c[H] - {}^{(0)}D(\iota_c(*d\phi)), \tag{5} $$

where dilaton and axion stress-energy $(D-1)$-forms are given, respectively, by

$$ \tau_c[\phi] = \iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi) \tag{6} $$

and

$$ \tau_c[H] = \iota_c H \wedge *H - (-1)^p H \wedge \iota_c(*H). \tag{7} $$

ϕ variation of (2) yields

$$ \frac{1}{2}{}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = -\omega d\left(\frac{*d\phi}{\phi}\right) - \frac{\omega}{2\phi^2} d\phi \wedge *d\phi + k\frac{\phi^{k-1}}{2} H \wedge *H. \tag{8} $$

We trace (5) by considering its exterior multiplication by e^c and multiply (8) by $(D-2)\phi$. The resulting two equations are then subtracted side by side to obtain the dilaton field equation

$$ \left(\omega + \frac{D-1}{D-2}\right) d * d\phi = \frac{\phi^k}{2}\alpha H \wedge *H, \tag{9} $$

where $\alpha = [2p - (D-4)/(D-2)] + k$. Finally, independent axion potential A variations lead to

$$ d(\phi^k * H) = 0 \tag{10} $$

together with $dH = 0$.

Next we consider the following action in which connection 1-forms are varied independently of the metric of space-time:

$$ \mathcal{L} = \frac{\phi}{2} R^{ab} \wedge *(e_a \wedge e_b) - \frac{c}{2\phi} d\phi \wedge *d\phi - \frac{\phi^k}{2} H \wedge *H. \tag{11} $$

Coframe variations of this action give the Einstein field equations

$$ \frac{1}{2}\phi R^{ab} \wedge (e_a \wedge e_b \wedge e_c) = -\frac{c}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^k \tau_c[H], \tag{12} $$

where $\tau_c[\phi]$ and $\tau_c[H]$ are as given by (6) and (7), respectively. Scalar field variations of the action give

$$ \frac{1}{2} R^{ab} \wedge (e_a \wedge e_b) = -cd\left(\frac{*d\phi}{\phi}\right) - \frac{c}{2\phi^2} d\phi \wedge *d\phi + k\frac{\phi^{k-1}}{2} H \wedge *H. \tag{13} $$

When we trace (12) and compare it with (13) multiplied by $(D-2)\phi$, we obtain the dilaton field equation

$$ cd * d\phi = \frac{\alpha}{2}\phi^k H \wedge *H. \tag{14} $$

Independent connection variations of (11) lead to

$$ D\left[\frac{\phi}{2} * (e^a \wedge e^b)\right] = 0 \tag{15} $$

from which we can readily solve for the torsion 2-forms:

$$ T^a = e^a \wedge \frac{d\phi}{(D-2)\phi}. \tag{16} $$

We can decompose the connection 1-forms in a unique way according to

$$ \omega^a{}_b = {}^0\omega^a{}_b + K^a{}_b, \tag{17} $$

where the contortion 1-forms $K^a{}_b$ satisfy

$$ K^a{}_b \wedge e^b = T^a. \tag{18} $$

Substitution of (16) into (18) gives

$$ K^a{}_b = \frac{1}{(D-2)\phi}(e^a \iota_b d\phi - e_b \iota^a d\phi). \tag{19} $$

Curvature 2-forms R^{ab} can be similarly decomposed as

$$ R^{ab} = {}^{(0)}R^{ab} + {}^{(0)}DK^{ab} + K^a{}_c \wedge K^{cb}, \tag{20} $$

where

$$ {}^{(0)}DK^{ab} = dK^{ab} + {}^{(0)}\omega^b{}_c \wedge K^{ac} + {}^{(0)}\omega^a{}_c \wedge K^{cb}. \tag{21} $$

Then we calculate

$$ R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{2}{\phi}{}^{(0)}D(\iota_c(*d\phi)) - \frac{2(D-1)}{(D-2)\phi^2} d\phi \wedge \iota_c(*d\phi) - \frac{D-1}{(D-2)\phi^2}\iota_c(d\phi \wedge *d\phi) \tag{22} $$

and

$$ R^{ab} \wedge *(e_a \wedge e_b) = {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{2(D-1)}{(D-2)} \times \left(*\frac{d\phi}{\phi}\right) - \frac{D-1}{(D-2)\phi^2} d\phi \wedge *d\phi. \tag{23} $$

If we insert (23) into (11), action density reduces to

$$ \mathcal{L} = \frac{1}{2}\phi^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \left(c - \frac{D-1}{D-2}\right)\frac{1}{2\phi} d\phi \wedge *d\phi - \frac{\phi^k}{2} H \wedge *H \tag{24} $$

up to a closed form. Substituting (22) into the Einstein field equations (12), we obtain

$$\frac{1}{2}\phi^{(0)}R^{ab}\wedge *(e_a\wedge e_b\wedge e_c) = -\left(c-\frac{D-1}{D-2}\right)\frac{1}{2\phi}\tau_c[\phi] - \frac{\phi^k}{2}\tau_c[H] - {}^0D(\iota_c(*d\phi)). \quad (25)$$

Similarly, substituting (23) into the dilaton field equation (13), we obtain

$$\frac{1}{2}{}^{(0)}R^{ab}\wedge *(e_a\wedge e_b) = \frac{(c-\frac{D-1}{D-2})}{2\phi^2}d\phi\wedge *d\phi - \left(c-\frac{D-1}{D-2}\right)\times\frac{1}{\phi}d(*d\phi) + k\frac{\phi^{k-1}}{2}H\wedge *H. \quad (26)$$

We have thus shown that provided the coupling constants are identified as

$$\omega = c - \frac{D-1}{(D-2)}, \quad (27)$$

the coupled field equations (25) and (26) are equivalent to the field equations (5) and (8).

Let us now consider conformal rescalings of the metric induced by the coframe rescalings

$$e^a \rightarrow e^{\sigma(x)}e^a. \quad (28)$$

These imply the transformation

$${}^{(0)}\omega_{ab} \rightarrow {}^{(0)}\omega_{ab} - e_b\iota_a d\sigma + \iota_b d\sigma e_a \quad (29)$$

of the Levi-Cività connection 1-forms. If we also postulate the following rescaling of the Brans-Dicke scalar field

$$\phi \rightarrow e^{-(D-2)\sigma}\phi, \quad (30)$$

then a straightforward calculation shows that the action (2) is scale invariant for $\omega = -(D-1)/(D-2)$ and $k = -(2p+4-D)/(D-2)$, or for $c=0$ and $\alpha = 0$. In terms of the geometry described by the action (11), the above rescaling rules imply the transformation

$$K_{ab} \rightarrow K_{ab} + \iota_a d\sigma e_b - \iota_b d\sigma e_a \quad (31)$$

so that the connection with torsion does not scale:

$$\omega_{ab} \rightarrow \omega_{ab}. \quad (32)$$

Hence

$$R_{ab} \rightarrow R_{ab} \quad (33)$$

and

$$T^a \rightarrow e^{\sigma}(T^a + d\sigma\wedge e^a). \quad (34)$$

We can reformulate our axi-dilaton gravity in the so-called Einstein frame by adopting the coframes

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0}\right)^{1/(D-2)}e^a, \quad (35)$$

where ϕ_0 is a constant. The new coframes $\tilde{e}^a$ become orthonormal with respect to space-time metric

$$\tilde{g} = \left(\frac{\phi}{\phi_0}\right)^{2/(D-2)}g. \quad (36)$$

In terms of this metric the associated Hodge dual is denoted by $\tilde{*}$. For an arbitrary frame independent p-form Ω,

$$*\Omega = \left(\frac{\phi}{\phi_0}\right)^{(2p-D)/(D-2)}\tilde{*}\Omega. \quad (37)$$

In the reformulation of action (2) in terms of $\tilde{g}$, new connection fields $\tilde{\omega}^{ab}$ can be written in terms of ${}^{(0)}\omega^{ab}$ as

$$\tilde{\omega}^{ab} = \Gamma^{ab} + {}^{(0)}\omega^{ab}, \quad (38)$$

where

$$\Gamma^{ab} = \frac{1}{(D-2)\phi}(e^a\iota^b d\phi - e^b\iota^a d\phi). \quad (39)$$

The corresponding curvature 2-forms become

$$\tilde{R}^{ab} = {}^{(0)}R^{ab} + {}^{(0)}D\Gamma^{ab} + \Gamma^{ac}\wedge\Gamma_c{}^b. \quad (40)$$

In terms of $\tilde{g}$, (2) becomes

$$\mathcal{L} = \frac{1}{2}\phi_0\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b) - \frac{c}{2}\phi_0\frac{1}{\phi^2}d\phi\wedge\tilde{*}d\phi - \frac{\phi^{\alpha}}{2}\phi_0^{(k-\alpha)}H\wedge\tilde{*}H, \quad (41)$$

up to a closed form. Introducing a massless scalar field $\Phi = \ln|\frac{\phi}{\phi_0}|$, (41) reads

$$\mathcal{L} = \frac{1}{2}\phi_0\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b) - \frac{c}{2}\phi_0 d\Phi\wedge\tilde{*}d\Phi - \frac{1}{2}(\phi_0)^k\exp(\alpha\Phi)H\wedge\tilde{*}H. \quad (42)$$

Einstein field equations obtained by coframe variations of (42) are

$$\frac{1}{2}\phi_0\tilde{R}^{ab}\wedge\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b\wedge\tilde{e}_c) = -\frac{c}{2}\phi_0\tilde{\tau}_c[\Phi] - \frac{1}{2}\times(\phi_0)^k e^{\alpha\Phi}\tilde{\tau}_c[H], \quad (43)$$

where

$$\tilde{\tau}_c[\Phi] = \tilde{\iota}_c d\Phi\wedge\tilde{*}d\Phi + d\Phi\wedge\tilde{\iota}_c(\tilde{*}d\Phi) \quad (44)$$

and

$$\tilde{\tau}_c[H] = \tilde{\iota}_c H\wedge\tilde{*}H - (-1)^p H\wedge\tilde{\iota}_c(\tilde{*}H). \quad (45)$$

On the other hand variations with respect to connection 1-forms $\tilde{\omega}^{ab}$ yield

$$D(\tilde{\omega})(\tilde{*}(\tilde{e}_a\wedge\tilde{e}_b)) = 0, \quad (46)$$

from which we obtain $\tilde{T}^a = 0$. Finally, we give the scalar field equation

$$c\phi_0 d(\tilde{*}d\Phi) = \frac{1}{2}(\phi_0)^k \alpha e^{\alpha\Phi} H \wedge \tilde{*}H, \quad (47)$$

and the axion field equation

$$d(e^{\alpha\Phi}\tilde{*}H) = 0. \quad (48)$$

Interestingly, by another conformal rescaling of the coframes in the Einstein frame, we can obtain the so-called string frame action. Applying the transformation

$$\hat{e}^{\,a} = \exp\left(\frac{2\Phi}{D-2}\right)\tilde{e}^a, \quad (49)$$

where $\hat{e}^a$ are assumed to satisfy the torsion-free structure equations

$$d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b = 0, \quad (50)$$

the action density (42) becomes

$$\mathcal{L} = e^{-2\Phi}\left\{\frac{1}{2}\phi_0 \hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b) - \frac{1}{2}\phi_0 \hat{k} d\Phi \wedge \hat{*}d\Phi\right\} - \frac{\phi_0{}^k}{2}\exp(\alpha_0\Phi) H \wedge \hat{*}H, \quad (51)$$

up to a closed form where coupling parameters are redefined as

$$\alpha_0 = (2p + 4 - D)\frac{3}{D-2} + k \quad (52)$$

and

$$\hat{k} = c - \frac{4(D-1)}{(D-2)}. \quad (53)$$

Action density (51) is called the string frame action in D dimensions. We would like to remark that it is possible to start directly from (51) and make independent coframe $\hat{e}^a$ and connection $\hat{\omega}^{ab}$ variations. Independent connection variations yield

$$D(\hat{\omega})(e^{-2\Phi}\hat{*}(\hat{e}_a \wedge \hat{e}_b)) = 0 \quad (54)$$

from which we can obtain torsion 2-forms $\hat{T}^a = \frac{2}{D-2} d\Phi \wedge \hat{e}^a$ [6]. Coframe variations on the other hand yield

$$\frac{1}{2}\phi_0 e^{-2\Phi}\hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b \wedge \hat{e}_c) = -\frac{1}{2}\phi_0 \hat{k} e^{-2\Phi}\hat{\tau}_c[\Phi] - \frac{1}{2}(\phi_0)^k e^{\alpha_0\Phi}\hat{\tau}_c[H], \quad (55)$$

where

$$\hat{\tau}_c[\Phi] = \hat{\iota}_c d\Phi \wedge \hat{*}d\Phi + d\Phi \wedge \hat{\iota}_c(\hat{*}d\Phi) \quad (56)$$

and

$$\hat{\tau}_c[H] = \hat{\iota}_c H \wedge \hat{*}H - (-1)^p H \wedge \hat{\iota}_c(\hat{*}H). \quad (57)$$

The scalar field Φ variation of (51) gives

$$\phi_0 e^{-2\Phi}\hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b) = \phi_0 \hat{k} e^{-2\Phi} d\Phi \wedge \hat{*}d\Phi + \hat{k}\phi_0 d(e^{-2\Phi}\hat{*}d\Phi) - \frac{1}{2} \times (\phi_0)^k \alpha_0 e^{\alpha_0\Phi} H \wedge \hat{*}H. \quad (58)$$

We consider the exterior multiplication of (55) by $\hat{e}^c$ and then multiply the equation by $\frac{2}{2-D}$. If we subtract the resulting equation from (58) and use (52), we obtain the scalar field equation

$$\phi_0 \hat{k} d(e^{-2\Phi}\hat{*}d\Phi) = \frac{1}{2}(\phi_0)^k \alpha e^{\alpha_0\Phi} H \wedge \hat{*}H. \quad (59)$$

Finally, the gauge field A variation yields

$$d(e^{\alpha_0\Phi}\hat{*}H) = 0. \quad (60)$$

The field equations without torsion in the string frame can be determined exactly in the same way we explained above.

III. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

In this section we investigate a class of static, spherically symmetric solutions of the axi-dilaton field equations (10), (12), and (14) in the Brans-Dicke frame with $p = D - 4$. Such solutions were studied previously in the Einstein and string frames [7–10] in Riemannian geometries. We emphasize again that classical solutions of the coupled field equations given in the Brans-Dicke, Einstein, and string frames, whether we consider a space-time geometry with or without torsion, are all conformally equivalent to each other. However, the scale invariant case can be most conveniently studied in the Brans-Dicke frame [11]. In terms of spherical polar coordinates ($t, r, \theta_i, i = 1, 2, 3, \ldots, D-2$), we take the metric

$$g = -f^2(r) dt \otimes dt + h^2(r) dr \otimes dr + R^2(r) d\Omega_{D-2}. \quad (61)$$

A convenient choice of the coframe 1-forms is

$$e^0 = f(r)dt, \qquad e^{D-1} = h(r)dr, \quad (62)$$

$$e^1 = R(r)d\theta_1,\, e^2 = R(r)\sin\theta_1 d\theta_2, \ldots,$$
$$e^{D-2} = R(r)\sin\theta_1 \cdots \sin\theta_{D-3} d\theta_{D-2}.$$

The axion field $(D-2)$-form

$$H = g(r) e^1 \wedge e^2 \wedge e^3 \wedge \cdots \wedge e^{D-2} \quad (63)$$

and the dilaton scalar

$$\phi = \phi(r). \quad (64)$$

Case: $c \neq 0$, $k \neq -\frac{D-4}{D-2}$.

Asymptotically flat solutions are given by the metric functions [11]

$$R(r) = r\left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{\alpha_3},$$
$$f(r) = \left[1 - \left(\frac{C_2}{r}\right)^{D-3}\right]^{\alpha_4}\left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{\alpha_5}, \qquad (65)$$
$$h(r) = \left[1 - \left(\frac{C_2}{r}\right)^{D-3}\right]^{\alpha_2}\left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{\alpha_1},$$

together with

$$\phi(r) = \left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{2\gamma/\alpha} \qquad (66)$$

and

$$g(r) = \frac{Q}{R^{D-2}}, \qquad (67)$$

where the exponents are related by

$$\alpha_1 = \gamma\left(\frac{1}{(D-3)} - \frac{2}{(D-2)\alpha}\right) - \frac{1}{2}, \qquad \alpha_2 = -\frac{1}{2},$$
$$\alpha_3 = \left(\frac{1}{(D-3)} - \frac{2}{(D-2)\alpha}\right)\gamma,$$
$$\alpha_4 = \frac{1}{2}, \qquad \alpha_5 = \frac{1}{2} - \left(1 + \frac{2}{(D-2)\alpha}\right)\gamma.$$

Here, we introduced a new parameter

$$\gamma = \frac{(D-2)\alpha^2}{4c(D-3) + (D-2)\alpha^2} \qquad (68)$$

and set

$$c = \omega + \frac{D-1}{D-2}. \qquad (69)$$

The integration constants C_1 and C_2 are both taken positive and should satisfy

$$Q^2 = \frac{4c(C_1C_2)^{D-3}(D-3)^2}{\alpha^2}. \qquad (70)$$

The following identification of the physical constants can be made:

$$2M \equiv \lim_{r\to\infty} r^{D-3}(1 - f^2)$$
$$= (C_2)^{D-3} + \left(1 - \frac{4\gamma}{(D-2)\alpha} - 2\gamma\right)(C_1)^{D-3} \qquad (71)$$

is the Arnowitt-Deser-Misner (ADM) mass;

$$\Sigma \equiv \lim_{r\to\infty} \frac{\phi'}{\phi} r^{D-2} = 2(D-3)(C_1)^{D-3}\frac{\gamma}{\alpha} \qquad (72)$$

is the scalar charge, and

$$Q \equiv \lim_{r\to\infty} g r^{D-2} = Q \qquad (73)$$

is the magnetic charge.

Depending on the values of the coupling constants, M can take either negative or positive signs. We will take the parameters γ and α to be positive and since

$$1 - \frac{4\gamma}{(D-2)\alpha} - 2\gamma = \frac{4c(D-3) - 4\alpha - (D-2)\alpha^2}{4c(D-3) + (D-2)\alpha^2}, \qquad (74)$$

it turns out that for

$$\alpha^2(D-2) + 4\alpha \leq 4c(D-3) \qquad (75)$$

the mass M becomes strictly positive. By eliminating $(C_2)^{D-3}$ from the mass equation (71) and substituting into (70) above, we obtain an algebraic relation satisfied by $(C_1)^{D-3}$, namely,

$$Q^2 = \frac{4c(C_1)^{D-3}[2M - (1 - \frac{4\gamma}{(D-2)\alpha} - 2\gamma)(C_1)^{D-3}](D-3)^2}{\alpha^2}. \qquad (76)$$

Then both C_1 and C_2 being real, we obtain the following BPS inequality between the mass and the magnetic charge:

$$M \geq \frac{\alpha}{2c^{1/2}(D-3)}\sqrt{\frac{4c(D-3) - 4\alpha - (D-2)\alpha^2}{4c(D-3) + (D-2)\alpha^2}}|Q|. \qquad (77)$$

At this point, let us assume that $C_2 > C_1$. Then the curvature scalar of the Levi-Cività connection

$$^{(0)}\mathcal{R} = \frac{1}{r^{2(D-2)}}\left\{\left(\frac{D-4}{D-2} - \frac{(D-1)\alpha}{c(D-2)}\right) \times Q^2\left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{\{[2(k-1)\gamma/\alpha]-2(D-2)\alpha_3\}} - \omega\left(\frac{2\gamma}{\alpha}(C_1)^{D-3}(D-3)\right)^2\left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{-2-2\alpha_1} \times \left[1 - \left(\frac{C_2}{r}\right)^{D-3}\right]\right\} \qquad (78)$$

is finite at $r_+ = C_2$. That is, for $Q \neq 0$, the metric functions admit an outer horizon at $r_+ = C_2$. The calculation of the corresponding quadratic curvature invariant on the other hand yields

$$*(R_{ab} \wedge *R^{ab}) \sim \left[1 - \left(\frac{C_1}{r}\right)^{D-3}\right]^{-4-4\alpha_1} r^{-4(D-2)}, \qquad (79)$$

which shows that $r = 0$ is an essential singularity. We should also discuss the nature of the singularity of solutions on the inner surface $r_- = C_1$. If the following conditions

$$-2 - 2\alpha_1 = -\frac{2(k-1)\gamma}{(D-3)\alpha} - 1 \geq 0 \qquad (80)$$

and

$$\frac{2(k-1)\gamma}{\alpha} - 2(D-2)\alpha_3 = -\frac{2(k-1)\gamma}{\alpha} \geq 0 \qquad (81)$$

are met, the curvature scalar would be finite at $r_- = C_1$. For this to hold, it is sufficient to examine the positivity of (80). The positivity of (81) will follow. It is clear from our definitions that condition (80) does not hold for $k \geq 1$. We checked after a tedious calculation that it does not hold for $k < 1$ either. Therefore $r_- = C_1$ is a singular surface for any value of k . Nevertheless, we can conclude that our solutions describe the exterior of black holes when $r > C_2 > C_1$.

It is also interesting to see what happens if the geometry of space-time is equipped with a connection with torsion. Then the corresponding curvature scalar

$$\begin{aligned}\mathcal{R} = \frac{1}{r^{2(D-2)}}\Bigg\{&\left(\frac{D-4}{D-3}\right)\\ &\times Q^2\left[1-\left(\frac{C_1}{r}\right)^{D-3}\right]^{\{[2(k-1)\gamma/\alpha]-2(D-2)\alpha_3\}}\\ &- c\left(\frac{2\gamma}{\alpha}(C_1)^{D-3}(D-3)\right)^2\left[1-\left(\frac{C_1}{r}\right)^{D-3}\right]^{-2-2\alpha_1}\\ &\times\left[1-\left(\frac{C_2}{r}\right)^{D-3}\right]\Bigg\} \end{aligned} \tag{82}$$

is again finite at $r_+ = C_2$ while $r_- = C_1$ is singular. $r = 0$ is still an essential singularity. Hence, the nature of the outer horizon and the inner singularities are not affected by torsion.

Case: $c = 0$, $k = -\frac{D-4}{D-2}$.

A class of asymptotically flat solutions to conformally scale invariant theory has the following form:

$$\begin{aligned} R(r) &= r\left[1-\left(\frac{E_1}{r}\right)^{D-3}\right]^{-\beta/(D-2)},\\ f(r) &= \left[1-\left(\frac{E_1}{r}\right)^{D-3}\right]^{(1/2)-[\beta/(D-2)]}\left[1-\left(\frac{E_2}{r}\right)^{D-3}\right]^{1/2},\\ h(r) &= \left[1-\left(\frac{E_1}{r}\right)^{D-3}\right]^{-(1/2)-[\beta/(D-2)]}\left[1-\left(\frac{E_2}{r}\right)^{D-3}\right]^{-1/2},\\ \phi(r) &= \left[1-\left(\frac{E_1}{r}\right)^{D-3}\right]^{\beta}, \qquad g(r) = \frac{Q}{R^{D-2}}, \end{aligned} \tag{83}$$

where E_1 and E_2 are constants that satisfy

$$(E_2E_1)^{D-3} = \frac{Q^2}{(D-2)(D-3)}. \tag{84}$$

β is a free parameter proportional to the scalar charge. The special case of parameter values $Q = 0$ and $E_2 = 0$ brings (83) to the Einstein-conformal scalar field solution of Bekenstein [12]. Bekenstein proposed a black hole interpretation of this solution based on the study of conformal world lines [13]. The scalar particles are postulated to follow geodesic world lines in Brans-Dicke theory. On the other hand, if space-time geometry is equipped with a connection with torsion, the history of particles would be an autoparallel of a connection with torsion [3]. It has been shown that the conformal world lines are nothing but the autoparallel curves in the non-Riemannian reformulation of the Brans-Dicke theory [2]. In this case, the scalar curvature of the connection with torsion is calculated as

$$\mathcal{R}_c = \frac{D-4}{D-2}Q^2\left[1-\left(\frac{E_1}{r}\right)^{D-3}\right]^{2\beta/(D-2)}\frac{1}{r^{2(D-2)}}. \tag{85}$$

We again specify the integration constants E_2 and E_1 to be positive and consider the ADM mass

$$2M_c = (E_2)^{D-3} + \left(1-\frac{2\beta}{(D-2)}\right)(E_1)^{D-3}. \tag{86}$$

This is strictly positive for $\beta \leq \frac{(D-2)}{2}$. Using

$$(E_2E_1)^{D-3} = \frac{Q^2}{(D-2)(D-3)} \tag{87}$$

and Eq. (86) above, we obtain the following BPS bound between the mass M_c and magnetic charge Q:

$$M_c \geq \sqrt{\frac{1}{(D-2)(D-3)}(1-\frac{2\beta}{(D-2)})}|Q|. \tag{88}$$

Let us examine the singularity of solutions in this case. The curvature scalar (85) is regular at $r_+ = E_2$. Contrary to the previous case, it is also nonsingular at $r_- = E_1$ for $\beta \geq 0$. We still have an essential point singularity at $r = 0$. Therefore, we conclude that our conformally scale invariant solutions describe black holes with two regular event horizons provided $0 \leq \beta \leq \frac{(D-2)}{2}$.

IV. CONCLUSION

In this paper we have studied axi-dilaton gravity theories in $D \geq 4$ dimensional space-times. We have shown by making use of the conformal rescaling properties of the space-time geometry, the equivalence of the variational field equations obtained in the Brans-Dicke, Einstein, and string frames, with or without torsion.

We have investigated a class of asymptotically flat, static, spherically symmetric solutions in the Brans-Dicke frame. The black hole configurations found in the case of nonscale invariant axi-dilaton gravity generalize the well-known $D = 4$ Janis-Newman-Winicour solutions of the Einstein-Maxwell-massless scalar field equations [14]. The fact that we are working in the Brans-Dicke frame is essential to our discussion of the solutions of the scale invariant axi-dilaton gravity in D dimensions. The solutions found in this case generalize the conformal black hole solutions of Bekenstein [12,13] of $D = 4$ Einstein-conformal scalar field theory.

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