Induced gravity with a nonminimally coupled scalar field on the brane

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We present the four-dimensional equations on a brane with a scalar field nonminimally coupled to the induced Ricci curvature, embedded in a five-dimensional bulk with a cosmological constant. This is a natural extension to a brane-world context of scalar-tensor (Brans-Dicke) gravity. In particular we consider the cosmological evolution of a homogeneous and isotropic (FRW) brane. We identify low-energy and strong-coupling limits in which we recover effectively four-dimensional evolution. We find de Sitter brane solutions with both constant and evolving scalar field. We also consider the special case of a conformally coupled scalar field for which it is possible (when the conformal energy density exactly cancels the effect of the bulk black hole) to recover a conventional four-dimensional Friedmann equation for all energy densities.

I. INTRODUCTION

Over recent years there has been a great deal of interest in higher-dimensional models of space-time where matter fields are restricted to a lower-dimensional brane in a higher-dimensional bulk space-time: the simplest case being a 3-brane of codimension one in a five-dimensional (5D) bulk.

This raises the possibility that the four-dimensional (4D) gravity we observe is the projection of a higher-dimensional gravity. In particular Randall and Sundrum [1] discovered that conventional 4D gravity can be recovered at large scales (low energies) on a Minkowski brane-world embedded in a 5D anti-de Sitter space-time. Even if there is no 4D Einstein-Hilbert term in the classical theory then such a term should be induced by loop-corrections from matter fields [2]. Dvali, Gabadadze and Porrati [3] argued that in this case 4D gravity can then be recovered at small scales (high energies) on a Minkowski brane-world in 5D Minkowski space-time. More generally one can consider the effect of an induced gravity term as a quantum correction in any brane-world model such as the Randall-Sundrum model.

Cosmology is a natural arena in which to put to the test alternative theories of gravity. In particular the DGP model admits late-time accelerating solutions. The cosmology of induced gravity corrections to Randall-Sundrum type models have been considered by several authors [4–9].

In this paper we will consider the effect of an induced gravity term which is an arbitrary function of a scalar field on the brane. Scalar fields play an important role both in models of the early universe and late-time acceleration. They also provide a simple dynamical model for matter fields in a brane-world model. In the context of induced gravity corrections it is then natural to consider a non-

minimal coupling of the scalar field to the intrinsic (Ricci) curvature on the brane that is a function of the field. The resulting theory can be thought of as a generalization of Brans-Dicke type scalar-tensor gravity in a brane-world context.

The layout of this paper is as follows. In Sec. II we present the five- and four-dimensional terms in the action and then use the geometrical approach of Shiromizu, Maeda and Sasaki [10] to give the effective Einstein equations projected onto the brane. Although in general these equations are not closed, due to the presence of the projected 5D Weyl tensor, the symmetries of a homogeneous and isotropic brane cosmology are sufficient to determine the evolution of the projected Weyl tensor on the brane [11–13]. In Sec. III we identify two regimes in which we expect to recover effectively 4D behavior and in Sec. IV we show that this is indeed the case for cosmological (homogeneous and isotropic) branes. We discuss static (de Sitter or Minkowski) brane solutions in Sec. V and then consider the special case of a conformally coupled scalar field on the brane in Sec. VI. The rather complicated form of the modified Friedmann equation on the brane is somewhat simpler for a conformally coupled field and we show that it is even possible to recover a conventional four-dimensional Friedmann equation, at all energies, as a special case. Finally we summarise our results in Sec. VII.

II. INDUCED SCALAR-TENSOR GRAVITY ACTION

A. 5D gravity

We consider a brane, described by a 4D hypersurface (b, metric g), embedded in a 5D bulk space-time ($\mathcal{B}$, metric $g^{(5)}$), whose action is given by

$$S = \int_{\mathcal{B}} d^5 X \sqrt{-g^{(5)}} \left\{ \frac{1}{2\kappa_5^2} R[g^{(5)}] + \mathcal{L}_5 \right\}$$
$$+ \int_b d^4 X \sqrt{-g} \left\{ \frac{1}{\kappa_5^2} K + \mathcal{L}_4 \right\}, \qquad (2.1)$$

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where κ_5^2 is the 5D gravitational constant, $R[g^{(5)}]$ is the Ricci scalar in the bulk and K the extrinsic curvature of the brane in the higher-dimensional bulk, corresponding to the York-Gibbons-Hawking boundary term [14]. Thus we have 5D Einstein gravity with a 4D boundary.

We will consider the simplest case of a constant vacuum energy density on the bulk, $\mathcal{L}_5 = -U$, i.e., a cosmological constant. In this case the bulk geometry is given by an Einstein space with constant scalar curvature

$$G_{MN}[g^{(5)}] = -\kappa_5^2 U g_{MN}^{(5)}. \qquad (2.2)$$

B. 4D induced gravity

For simplicity we will assume a $\mathbb{Z}_2$-symmetry at the brane (which is also motivated by specific M-theory constructions [15,16]). In practice one can easily generalize to non-$\mathbb{Z}_2$-symmetric branes [17]. The effective Einstein equation on the brane is then [10]

$$G_{\mu\nu}[g] = -\frac{1}{2}\kappa_5^2 U g_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}, \qquad (2.3)$$

where g is the induced metric on the brane. $\Pi_{\mu\nu}$ is the quadratic energy-momentum tensor [10]

$$\Pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\sigma}\tau_\nu^\sigma + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\left(\tau_{\rho\sigma}\tau^{\rho\sigma} - \frac{1}{3}\tau^2\right), \qquad (2.4)$$

and $\tau_{\mu\nu}$ is the total energy-momentum tensor for fields on the brane defined by

$$\tau_{\mu\nu} = -2\frac{\delta\mathcal{L}_4}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_4. \qquad (2.5)$$

$E_{\mu\nu}$ is the (trace-free) projected Weyl tensor on the brane. The trace-free property determines the isotropic effective pressure of this projected Weyl tensor in terms of its effective density, but the anisotropic effective pressure due to this nonlocal term cannot in general be determined without some additional information about the 5D gravitational field.

The most general scalar field Lagrangian $\mathcal{L}_4$ for a scalar field, ϕ, confined on the brane can be written as

$$\mathcal{L}_4 = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) + \alpha(\phi)R[g], \qquad (2.6)$$

where ∇_μ is the covariant derivative associated with the induced metric on the brane g. Previous studies of scalar fields in induced brane-world gravity [8] are restricted to the case $\alpha = $ constant. Here we include a coupling between the scalar field ϕ and the induced gravity term on the brane, given by $\alpha(\phi)$. In this case, substituting (2.6) into (2.5), the total energy-momentum tensor on the brane becomes

$$\tau_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - g_{\mu\nu}V(\phi) - 2\alpha\mathcal{G}_{\mu\nu}[g,\phi]. \qquad (2.7)$$

This includes the "Einstein-Brans-Dicke" tensor

$$\mathcal{G}_{\mu\nu}[g,\alpha] \equiv G_{\mu\nu}[g] + \frac{1}{\alpha}(g_{\mu\nu}g^{\rho\sigma} - g_\mu^\rho g_\nu^\sigma)(\alpha'\nabla_\rho\nabla_\sigma\phi + \alpha''\nabla_\rho\phi\nabla_\sigma\phi), \qquad (2.8)$$

due to the nonminimal coupling, $\alpha(\phi)$, between the scalar field ϕ and the scalar curvature $R[g]$. In this expression the prime denotes derivative with respect to ϕ.

We can thus split the total energy-momentum tensor as follows

$$\tau_{\mu\nu} = T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\alpha)} - 2\alpha G_{\mu\nu}[g], \qquad (2.9)$$

where the canonical (minimally coupled) scalar field energy-momentum tensor is given by

$$T_{\mu\nu}^{(\phi)} \equiv \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - g_{\mu\nu}V(\phi), \qquad (2.10)$$

and the extra terms arising from the dependence of the induced gravity term upon ϕ are given by

$$T_{\mu\nu}^{(\alpha)} \equiv -2(g_{\mu\nu}g^{\rho\sigma} - g_\mu^\rho g_\nu^\sigma)(\alpha'\nabla_\rho\nabla_\sigma\phi + \alpha''\nabla_\rho\phi\nabla_\sigma\phi). \qquad (2.11)$$

Using the 5D Codacci equation one can show that the total energy-momentum tensor $\tau_{\mu\nu}$ must be conserved on the brane [10]

$$\nabla^\nu\tau_{\mu\nu} = 0. \qquad (2.12)$$

C. Scalar field wave equation

Finally, the equation of motion for the scalar field reads

$$\nabla^\mu\nabla_\mu\phi = V' - \alpha'R[g]. \qquad (2.13)$$

This is the same as the standard equation of motion for a nonminimally coupled scalar field in 4D, but it is often rewritten using the Einstein-Brans-Dicke equations to give R in terms of the trace of the energy-momentum tensor. Here we must take the trace of the effective Einstein equations on the brane (2.3) to give

$$R = 2\kappa_5^2 U - \kappa_5^4\Pi_\mu^\mu, \qquad (2.14)$$

where

$$\Pi_\mu^\mu = \frac{1}{4}\tau_{\mu\nu}\tau^{\mu\nu} - \frac{1}{12}\tau^2. \qquad (2.15)$$

Although the wave equation (2.13) is sufficient to determine the evolution of the scalar field ϕ given the induced metric on the brane, the effective Einstein equation (2.3) is not in general sufficient to determine the evolution of the induced metric given ϕ. This is due to the presence of the nonlocal term $E_{\mu\nu}$, representing the bulk gravita-

tional field. Nonetheless if we restrict our analysis to homogeneous and isotropic brane-worlds these symmetries restrict the bulk solution to either (anti-)de Sitter or Schwarzschild-(anti-)de Sitter and the equations become closed [11].

III. EQUATIONS IN LOW-ENERGY AND STRONG-COUPLING LIMITS

A. Low-energy limit

In order to obtain the effective Einstein equations (2.3) in a low-energy limit close to the Randall-Sundrum solution [1] it is helpful to define a "renormalized" energy-momentum tensor on the brane

$$\bar{\tau}_{\mu\nu} = \tau_{\mu\nu} + \sigma g_{\mu\nu}, \tag{3.1}$$

where σ is a constant brane tension. The quadratic tensor $\Pi_{\mu\nu}$ defined in Eq. (2.4) then becomes

$$\Pi_{\mu\nu} = -\frac{1}{12}\sigma^2 g_{\mu\nu} + \frac{1}{6}\sigma\bar{\tau}_{\mu\nu} + \bar{\Pi}_{\mu\nu}, \tag{3.2}$$

where $\bar{\Pi}_{\mu\nu}$ is the quadratic energy-momentum tensor (2.4) formed from $\bar{\tau}_{\mu\nu}$ instead of $\tau_{\mu\nu}$.

Substituting Eq. (3.2) for $\Pi_{\mu\nu}$ into Eq. (2.3) gives

$$G_{\mu\nu}[g] = -\Lambda_4 g_{\mu\nu} + \frac{\kappa_5^4\sigma}{6}\bar{\tau}_{\mu\nu} + \kappa_5^4\bar{\Pi}_{\mu\nu} - E_{\mu\nu}, \tag{3.3}$$

where we have defined

$$\Lambda_4 = \frac{\kappa_5^2}{2}U + \frac{\kappa_5^4\sigma^2}{12}. \tag{3.4}$$

For $U < 0$ we can choose $\sigma = \sqrt{-6U/\kappa_5^2}$ so that $\Lambda_4 = 0$, but in principle we can work with any value of σ and hence Λ_4.

The energy-momentum tensor on the right-hand-side of Eq. (3.3) includes a contribution from the Einstein tensor, so ultimately we can rewrite the induced gravity equations on the brane as

$$2\Phi_{\text{lo}}G_{\mu\nu}[g, \Phi_{\text{lo}}] = -\frac{6\Lambda_4}{\kappa_5^4\sigma}g_{\mu\nu} + \bar{T}_{\mu\nu}^{(\phi)}$$
$$+ \frac{6}{\kappa_5^4\sigma}(\kappa_5^4\bar{\Pi}_{\mu\nu} - E_{\mu\nu}), \tag{3.5}$$

where $G_{\mu\nu}[g, \Phi_{\text{lo}}]$ is the Einstein-Brans-Dicke tensor (2.8) for the effective Brans-Dicke field

$$\Phi_{\text{lo}}(\phi) \equiv \frac{3}{\kappa_5^4\sigma}\left[1 + \frac{\kappa_5^4\sigma}{3}\alpha(\phi)\right]. \tag{3.6}$$

Thus at low energies, if we can neglect the quadratic tensor $\bar{\Pi}$, and in a conformally flat bulk ($E_{\mu\nu} = 0$), we will recover the usual Brans-Dicke equations for a nonminimally coupled scalar field in four-dimensions. Moreover, for $\alpha = $ constant we recover Einstein gravity with a mini-

mally coupled scalar field on the brane and an effective gravitational coupling $\kappa_4^2 = (2\Phi_{\text{lo}})^{-1} = $ constant.

The effective potential for ϕ can be written as

$$V_{\text{lo}}(\phi) = V(\phi) - \frac{\sigma}{2} + \frac{3U}{\kappa_5^2\sigma}, \tag{3.7}$$

and the effective Brans-Dicke parameter is

$$\omega_{\text{lo}} \equiv \frac{\Phi_{\text{lo}}}{2[\Phi_{\text{lo}}'(\phi)]^2} = \frac{3}{2\kappa_5^4\sigma\alpha'^2}\left[1 + \frac{\kappa_5^4\sigma}{3}\alpha(\phi)\right]. \tag{3.8}$$

B. Strong-coupling limit

There is an alternative limiting case to consider where the 5D curvature is negligible, or the induced coupling α is large. In this case we expect the conventional 4D Lagrangian $\mathcal{L}_4$ given in Eq. (2.6) to dominate in the action (2.1). In this case we have the standard 4D Einstein-Brans-Dicke equation

$$2\alpha G_{\mu\nu}[g, \alpha] = T_{\mu\nu}^{(\phi)}, \tag{3.9}$$

with effective Brans-Dicke field

$$\Phi_{\text{hi}}(\phi) \equiv \alpha(\phi), \tag{3.10}$$

effective potential

$$V_{\text{hi}}(\phi) = V(\phi), \tag{3.11}$$

and dimensionless Brans-Dicke parameter

$$\omega_{\text{hi}}(\phi) = \frac{\alpha}{2\alpha'^2}. \tag{3.12}$$

Note that this coincides with the limiting form of Eq. (3.8) in the strong-coupling limit, i.e., for $\kappa_5^4\sigma\alpha \gg 1$.

IV. DYNAMICS OF A HOMOGENEOUS AND ISOTROPIC BRANE

In the present section we will consider the cosmological evolution of a Friedmann-Robertson-Walker (FRW) brane with a nonminimally coupled scalar field. In the special case of an isotropic brane geometry the projected Weyl tensor $E_{\mu\nu}$ necessarily has a vanishing anisotropic stress and the projected field Eqs. (2.3) and (2.13) form a closed set of evolution equations for scalar field and metric on the brane. Indeed, it can be shown that for an expanding FRW brane the unique bulk space-time (in Einstein gravity in vacuum, as we assume here) is 5D Schwarzschild-anti de Sitter space-time [11,18].

The trace-free property of the projected Weyl tensor implies that it acts like a "dark radiation" [12,13] and hence

$$\dot{E}_0^0 + 4HE_0^0 = 0, \tag{4.1}$$

where a dot denotes derivatives with respect to proper cosmic time and H is the Hubble rate. Thus $E_{\mu\nu}$ evolves

like a radiation fluid with $E_0^0 = C/a^4$, where C is an integration constant.

After some lengthy but straightforward calculations, the modified Friedmann equation on the brane can be obtained from Eq. (2.3) as

$$3\left(H^2 + \frac{K}{a^2}\right) = \frac{\kappa_5^2 U}{2} + \frac{\kappa_5^4}{12}\left[\rho - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right]^2 + \frac{C}{a^4},$$

(4.2)

where $K = \pm 1, 0$ depending on the geometry of the spatial three-dimensional sections on the brane. The modified Friedmann equation can be rewritten as

$$H^2 + \frac{K}{a^2} = \frac{1}{6\alpha}\left\{\rho + \frac{3}{\kappa_5^4 \alpha}\left[1 \right.\right.$$
$$\left.\left. \pm \sqrt{1 + \frac{2}{3}\kappa_5^4 \alpha\left(\rho - \kappa_5^2 \alpha U - 2\alpha\frac{C}{a^4}\right)}\right]\right\}, \quad (4.3)$$

which shows the existence of two branches of solution for H^2 as a function of ρ. The modified Raychaudhuri equation is

$$\left\{1 + \frac{\kappa_5^4}{3}\alpha\left[\rho - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right]\right\}\left(\dot{H} - \frac{K}{a^2}\right)$$
$$= -\frac{\kappa_5^4}{12}(\rho + P)\left[\rho - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right] - \frac{2}{3}\frac{C}{a^4}. \quad (4.4)$$

Thus the modified Einstein equations can be written in exactly the same form as obtained for constant α [7]. The effect of the nonminimal coupling of the ϕ field is hidden in the definition of the effective energy density, ρ, of the scalar field which includes nonminimal terms. In the limit $\alpha \rightarrow 0$ we recover the modified Einstein equations of the Randall-Sundrum model [13] with a minimally coupled scalar field on the negative branch (lower sign in Eq. (4.3)).

Following the notation introduced in Eq. (2.9) we will write

$$\rho = \rho^{(\phi)} + \rho^{(\alpha)}, \quad (4.5)$$

$$P = P^{(\phi)} + P^{(\alpha)}. \quad (4.6)$$

The effective energy density and pressure of the scalar field has been split into a part associated with the canonical scalar field energy-momentum tensor, given from Eq. (2.10) as

$$\rho^{(\phi)} \equiv -T_0^{0(\phi)} = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$
$$P^{(\phi)} \equiv T_i^{i(\phi)} = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (4.7)$$

and a part due to the nonminimal coupling, given from Eq. (2.11) as

$$\rho^{(\alpha)} \equiv -T_0^{0(\alpha)} = -6\alpha' H\dot{\phi},$$
$$P^{(\alpha)} \equiv T_i^{i(\alpha)} = 2(\alpha'\ddot{\phi} + 2H\alpha'\dot{\phi} + \alpha''\dot{\phi}^2), \quad (4.8)$$

where $i = 1, \ldots, 3$ labels the spatial coordinates on the brane.

The equation of motion (2.13) for the scalar field, ϕ, in the FRW geometry is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \alpha' R[g], \quad (4.9)$$

where the intrinsic Ricci scalar for a FRW brane is

$$R[g] = 6\left(\dot{H} + 2H^2 + \frac{K}{a^2}\right). \quad (4.10)$$

In conventional 4D scalar-tensor gravity the Ricci scalar is often eliminated from the scalar field equation of motion in favor of the trace of the energy-momentum tensor, using the contracted Einstein-Brans-Dicke equation for the Einstein tensor. In our brane-world scenario the contracted effective Einstein Eq. (2.3) yields a more complicated expression for the Ricci scalar.

The nonminimal coupling of the scalar field to the Ricci curvature on the brane through $\alpha(\phi)$ leads to the non-conservation of the scalar field effective energy density

$$\dot{\rho} + 3H(\rho + P) = 6\alpha'\dot{\phi}\left(H^2 + \frac{K}{a^2}\right). \quad (4.11)$$

This equation can be deduced from the definition of ρ and P [see Eqs. (4.5), (4.6), (4.7), and (4.8)] and the equation of motion for ϕ (4.9). We see that ρ and P are conserved whenever α is constant, i.e., when ϕ is a minimally coupled scalar field. For this particular case, ρ and P reduce to $\rho^{(\phi)}$ and $P^{(\phi)}$ [see Eq. (4.7)], respectively.

In general, although the scalar field effective energy density ρ is not conserved, it is always possible to construct a total energy density from the total energy-momentum tensor $\tau_{\mu\nu}$, defined in Eq. (2.5),

$$\rho^{\text{tot}} = \rho^{(\phi)} + \rho^{(\alpha)} - 6\alpha\left(H^2 + \frac{K}{a^2}\right), \quad (4.12)$$

which is locally conserved on the brane, in accordance with Eq. (2.12).

A. Low-energy regime

In order to analyze the different possible regimes for the effective Friedmann equation on the brane, we introduce an (arbitrary) constant brane tension σ, as in Eq. (3.1) so that

$$\bar{\rho} = \rho - \sigma, \qquad \bar{P} = P + \sigma. \quad (4.13)$$

If we then expand the quadratic term on the right-hand side of the modified Friedmann equation (4.2), we obtain

$$3\left(H^2 + \frac{K}{a^2}\right) = \Lambda_4 + \frac{\kappa_5^4 \sigma}{6}\left[\bar{\rho} - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right]$$
$$+ \frac{\kappa_5^4}{12}\left[\bar{\rho} - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right]^2 + \frac{C}{a^4}, \quad (4.14)$$

where Λ_4 is given by Eq. (3.4).

We can identify a low-energy regime corresponding to

$$\left|\bar{\rho} - 6\alpha\left(H^2 + \frac{K}{a^2}\right)\right| \ll \sigma, \qquad (4.15)$$

where we recover from (4.14) an effective 4D Friedmann equation

$$3\left(1 + \frac{\kappa_5^4 \sigma \alpha}{3}\right)\left(H^2 + \frac{K}{a^2}\right) \simeq \Lambda_4 + \frac{\kappa_5^4 \sigma}{6}\bar{\rho} + \frac{C}{a^4}. \quad (4.16)$$

with the effective gravitational coupling given by Eq. (3.6).

If we choose $\sigma = \sqrt{-6U}/\kappa_5$, i.e., set $\Lambda_4 = 0$, and consider an anti-de Sitter bulk ($C = 0$) then the constraint Eq. (4.16) allows us to express the low-energy condition (4.15) as

$$\bar{\rho} \ll \sigma\left(1 + \frac{\kappa_5^4 \sigma \alpha}{3}\right). \qquad (4.17)$$

B. Strong-coupling regime

In order to identify the strong-coupling regime we rewrite the modified Friedmann (4.2) equation as

$$\left[1 - \frac{\rho}{6\alpha(H^2 + \frac{K}{a^2})}\right]^2 = \frac{1}{\kappa_5^4 \alpha^2 (H^2 + \frac{K}{a^2})}\left[1 - \frac{\kappa_5^2 U}{6(H^2 + \frac{K}{a^2})}\right.$$
$$\left. - \frac{C}{3a^4(H^2 + \frac{K}{a^2})}\right]. \qquad (4.18)$$

We identify a strong-coupling regime where

$$\alpha^2 \gg \frac{1}{\kappa_5^4(H^2 + \frac{K}{a^2})}\left[1 - \frac{\kappa_5^2 U}{6(H^2 + \frac{K}{a^2})} - \frac{C}{3a^4(H^2 + \frac{K}{a^2})}\right], \qquad (4.19)$$

in which case we recover from Eq. (4.18) an effective 4D Friedmann equation

$$6\alpha\left(H^2 + \frac{K}{a^2}\right) \simeq \rho, \qquad (4.20)$$

with the effective gravitational coupling given by Eq. (3.10).

Consistency of the last two equations implies that strong-coupling also requires a lower bound on the energy density

$$\rho \gg \frac{6}{\kappa_5^4 \alpha}\left|1 - \frac{\kappa_4^2 \alpha U}{\rho} - \frac{2\alpha C}{a^4 \rho}\right|. \qquad (4.21)$$

Note that the strong-coupling form for the Friedmann equation, (4.20), can also be obtained from the low-energy regime, Eq. (4.16), for $\kappa_5^4 \sigma \alpha \gg 1$ and $\Lambda_4 = C = 0$.

C. Intermediate energy and weak coupling regime

Having shown that one recovers two effectively 4D regimes in the limits of low-energy or strong-coupling, it is interesting to consider whether or not one can recover an essentially 5D regime where $H^2 \propto \rho^2$ as is found in Randall-Sundrum cosmology (where $\alpha = 0$) at high energies [13,19].

The high-energy regime in the Randall-Sundrum model corresponds to

$$\rho \gg \sigma_{RS}, \qquad (4.22)$$

where $\sigma_{RS} = \sqrt{6|U|}/\kappa_5$ corresponds to the brane tension required for a static Minkowski brane. In the induced gravity model we must add the additional condition

$$\rho \gg \left|6\alpha\left(H^2 + \frac{K}{a^2}\right)\right|. \qquad (4.23)$$

Thus we require both high-energy and weak coupling. In this case, the modified Friedmann Eq. (4.2) reads

$$3\left(H^2 + \frac{K}{a^2}\right) \simeq \frac{\kappa_5^4}{12}\rho^2 + \frac{C}{a^4}. \qquad (4.24)$$

Substituting Eq. (4.24) into the inequality (4.23) requires

$$\rho_- \ll \rho \ll \rho_+, \qquad (4.25)$$

where

$$\rho_{\pm} = \left|\frac{3}{\kappa_5^4 \alpha}\left(1 \pm \sqrt{1 - \frac{4\kappa_5^4 \alpha^2 C}{3a^4}}\right)\right|. \qquad (4.26)$$

For this intermediate regime to exist requires both

$$\frac{\kappa_5^4 \alpha^2 C}{a^4} \ll 1, \qquad (4.27)$$

and

$$\kappa_5^4 |\alpha| \rho \ll 1. \qquad (4.28)$$

Finally combining (4.22) and (4.28) we obtain the consistency condition

$$\sigma_{RS} \ll \rho \ll \frac{1}{\kappa_5^4 |\alpha|}, \qquad (4.29)$$

which only exists for sufficiently weak coupling

$$|\alpha| \ll \frac{1}{\kappa_5^4 \sigma_{RS}}. \qquad (4.30)$$

V. DE SITTER AND MINKOWSKI BRANES

In this section, we describe some maximally symmetric branes that can be obtained in the framework given in

Sec. IV. In particular, we will obtain inflationary branes with de Sitter geometry or purely Minkowski space-times on the brane. We consider that the bulk is given by a 5D maximally symmetric space-time and therefore the projected Weyl tensor on the brane is zero. For simplicity we will use the spatially flat coordinate chart on the brane so that $K = 0$ and the Ricci scalar $R = 12H^2 = $ constant.

From the Friedmann Eq. (4.2) we see that we require $\rho - 6\alpha H^2 = $ constant. In addition, the last condition and the continuity Eq. (4.11) implies that $P = -\rho$ for $H \neq 0$. Note however that unlike 4D general relativity, we do not necessarily require $\rho = $ constant.

Eqs. (4.5) and (4.6) for the density and pressure of the nonminimally coupled scalar field give

$$\rho + P = (1 + \alpha'')\dot{\phi}^2 + 2\alpha'(\ddot{\phi} - H\dot{\phi}). \quad (5.1)$$

The scalar field Eq. (4.9) and the condition $P = -\rho$ then gives the first-order constraint

$$(1 + 2\alpha'')\dot{\phi}^2 + 2\alpha'(12\alpha'H^2 - 4H\dot{\phi} - V') = 0. \quad (5.2)$$

If the scalar field does not evolve in time ($\dot{\phi} = 0$) and $\alpha' \neq 0$, then we require $V' = 12H^2\alpha'$, i.e., the potential gradient is balanced by the nonminimal coupling to the scalar curvature. The scalar field has to be at an extremum of the potential ($V' = 0$) if ϕ is constant in time and $H = 0$.

For a Minkowski brane $H = 0$, the Raychaudhuri Eq. (4.4) requires either $\rho + P = 0$ or $\rho = 0$. For $\rho = 0$ we must have $U = 0$ from the Friedmann Eq. (4.2), but we may in principle have $P \neq 0$. Equation (5.1) together with the equation of motion (4.9) then yields for $\rho = 0$

$$P = (1 + 2\alpha'')\dot{\phi}^2 - 2\alpha'V'. \quad (5.3)$$

However, in the following we will restrict our discussion to de Sitter or Minkowski branes with $P = -\rho$.

A. De Sitter branes with $\dot{\phi} = 0$

In the following, we will describe the fixed points of the theory, i.e., values of the scalar field, $\phi = \phi_c$ such that $\dot{\phi} = \ddot{\phi} = 0$ and $\dot{H} = 0$, where $H = H_c$ corresponds to the Hubble parameter for $\phi = \phi_c$. For $\dot{\phi} = \ddot{\phi} = 0$ we necessarily have $\rho = -P = V_c$, where $V_c = V(\phi)$.

Using Friedmann Eq. (4.2) and the equation of motion of the scalar field (4.9), we obtain

$$H_c^2 = \frac{1}{6\alpha_c}\left\{V_c + \frac{3}{\kappa_5^4\alpha_c}\left[1 \right.\right.$$

$$\left.\left. \pm\sqrt{1 - \frac{2}{3}\kappa_5^4\alpha_c(\kappa_5^2\alpha_c U - V_c)}\right]\right\}, \quad (5.4)$$

$$V_c' = 12H_c^2\alpha_c', \quad (5.5)$$

where V_c', α_c and α_c' correspond to $V'(\phi_c)$, $\alpha(\phi_c)$ and $\alpha'(\phi_c)$, respectively. Note that we require $2\kappa_5^4\alpha_c(\kappa_5^2\alpha_c U - V_c) < 3$ for H^2 to be real.

To obtain a Minkowski brane with $H_c = 0$ requires [see Eq. (4.2)] the usual Randall-Sundrum fine-tuning between the 5D cosmological constant and the potential

$$V^2(\phi_c) = -\frac{6U}{\kappa_5^2} \geq 0. \quad (5.6)$$

In addition ϕ_c must coincide with an extremum of the potential, $V_c' = 0$. The Minkowski brane is only obtained for the branch corresponding to the upper choice of sign in Eq. (5.4) for $\kappa_5^4\alpha_c V_c + 3 \leq 0$ or lower sign for $\kappa_5^4\alpha_c V_c + 3 \geq 0$. Only for $\alpha_c = -3/\kappa_5^4 V_c$ do we obtain $H_c = 0$ for both branches.

In the following, we see under which conditions the fixed points correspond to stable solutions. The potential $V(\phi)$ and the coupling $\alpha(\phi)$ can be approximated near ϕ_c by

$$V(\phi) \simeq V_c + V_c'(\phi - \phi_c) + \frac{1}{2}V_c''(\phi - \phi_c)^2, \quad (5.7)$$

$$\alpha(\phi) \simeq \alpha_c + \alpha_c'(\phi - \phi_c) + \frac{1}{2}\alpha_c''(\phi - \phi_c)^2, \quad (5.8)$$

where V_c'' and α_c'' are $V''(\phi_c)$ and $\alpha''(\phi_c)$, respectively. The equation of motion (4.9) for a small perturbation $\delta\phi = \phi - \phi_c$, to first-order in $\delta\phi$ becomes

$$\delta\ddot{\phi} + 3H_c\delta\dot{\phi} + (V_c'' - 12H_c^2\alpha_c'')\delta\phi = \alpha_c'\delta R, \quad (5.9)$$

where $\delta R = R(\phi_c + \delta\phi) - 12H_c^2$. If δR is negligible, we have that a fixed point is stable when $V_c'' - 12H_c^2\alpha_c'' > 0$. However, in general the perturbed Ricci scalar (4.10) can be calculated using the Friedmann and Raychaudhuri Eqs. (4.2) and (4.4), which gives

$$\delta R = -\frac{\kappa_5^4(V_c - 6\alpha_c H_c^2)}{1 + (1/3)\kappa_5^4\alpha_c(V_c - 6\alpha_c H_c^2)}[\alpha_c'\delta\ddot{\phi}$$

$$+ 3\alpha_c'H_c\delta\dot{\phi} - 4\alpha_c'H_c^2\delta\phi]. \quad (5.10)$$

Substituting this into the equation of motion of $\delta\phi$, Eq. (5.9), we obtain (for $\kappa_5^4(\alpha_c + 3\alpha_c'^2)(V_c - 6\alpha_c H_c^2) \neq -3$) the canonical equation of motion for a field perturbation in a de Sitter background

$$\delta\ddot{\phi} + 3H_c\delta\dot{\phi} + m_{\text{eff}}^2\delta\phi = 0, \quad (5.11)$$

where the effective mass

$$m_{\text{eff}}^2 \equiv \frac{V_c'' - 12H_c^2\alpha_c'' - 4\beta H_c^2}{1 + \beta} > 0, \quad (5.12)$$

with

$$\beta = \frac{\kappa_5^4\alpha_c'^2(V_c - 6\alpha_c H_c^2)}{1 + (\alpha_c/3)\kappa_5^4(V_c - 6\alpha_c H_c^2)}. \quad (5.13)$$

The general solution to Eq. (5.11) is given by

$$\delta\phi = C_+\exp(\lambda_+ t) + C_-\exp(\lambda_- t), \quad (5.14)$$

where

$$\lambda_{\pm} = \frac{-3H_c \pm \sqrt{9H_c^2 - 4m_{\text{eff}}^2}}{2}. \qquad (5.15)$$

Thus the stability condition for the fixed point is simply $m_{\text{eff}}^2 > 0$. For a constant (minimal) coupling ($\alpha_c' = \alpha_c'' = 0$) this stability condition takes the usual form $V_c'' > 0$. But nonminimal coupling can stabilize the fixed point even for $V_c'' < 0$. For example, for a Minkowski brane with $H_c = 0$, the stability condition $m_{\text{eff}}^2 > 0$ reduces to ϕ_c being a maximum (or minimum) of the potential for $1 + \beta$ negative (or positive).

a. Quadratic model As an illustration, we apply the previous analysis to the following simple quadratic model for the nonminimal coupling and potential:

$$\alpha(\phi) = \alpha_0 + \frac{1}{2}\alpha_2\phi^2, \qquad (5.16)$$

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2, \qquad (5.17)$$

where α_0, α_2, V_0 and m^2 are constants.

Imposing the condition (5.5), yields two possible fixed points for the model:

(i) $\phi_c = 0$. The square of the Hubble parameter is given by Eq. (5.4) with $V_c = V_0$ and $\alpha_c = \alpha_0$. The parameter $\beta = 0$ and the effective mass $m_{\text{eff}}^2 = m^2 - 12\alpha_2 H_c^2$. The fixed point is stable as long as

$$m^2 > \frac{2\alpha_2}{\alpha_0}\left\{V_0 + \frac{3}{\kappa_5^4\alpha_0}\left[1 \right.\right.$$
$$\left.\left. \pm \sqrt{1 - \frac{2}{3}\kappa_5^4\alpha_0(\kappa_5^2\alpha_0 U - V_0)}\right]\right\}. \qquad (5.18)$$

(ii) $\phi_c \neq 0$. This fixed point is obtained for $\alpha_2 \neq 0$ when the Hubble constant satisfies $H_c^2 = m^2/12\alpha_2$. The square of the Hubble parameter is given by Eq. (5.4) with α_c given by $\alpha_0 + (1/2)\alpha_2\phi_c^2$. From Eq. (4.2) we obtain

$$\phi_c^2 = \frac{2\alpha_0}{\alpha_2} - \frac{4V_0}{m^2} \pm \frac{4}{\kappa_5^2 m^2}\sqrt{\frac{3m^2}{\alpha_2} + 6\kappa_5^2 U}.$$

Thus this fixed point only exists for $m^2 > -2\alpha_2\kappa_5^2 U$. From Eq. (5.12) we find $m_{\text{eff}}^2 = -4\beta H_c^2/(1 + \beta)$ and hence this fixed point is stable as long as $-1 \leq \beta < 0$.

B. De Sitter branes with $\dot{\phi} \neq 0$

We cannot find the general solution of the constraint Eq. (5.2) for $\dot{\phi} \neq 0$ without specifying the form of $V(\phi)$ and $\alpha(\phi)$. Adopting the simple quadratic model introduced in Eqs. (5.16) and (5.17) it can be shown that there are

solutions to the scalar field equation of motion (4.9) with $\dot{\phi} \neq 0$, which satisfy Eq. (5.2), when the mass of the scalar field satisfies

$$\frac{m^2}{H^2} = \frac{2\alpha_2(1 + 6\alpha_2)(3 + 16\alpha_2)}{(1 + 4\alpha_2)^2}. \qquad (5.19)$$

In this case we have a solution to the equation of motion (4.9) where the scalar field evolves exponentially with respect to cosmic time

$$\phi = \phi_0 \exp(\mu H t), \qquad (5.20)$$

where the dimensionless parameter μ is given by

$$\mu = \frac{2\alpha_2}{1 + 4\alpha_2}. \qquad (5.21)$$

For $-1/4 < \alpha_2 < 0$ this describes the decay of the scalar field to the fixed point with $\phi = 0$, but for $\alpha_2 > 0$ or $< -1/4$ the $\phi = 0$ fixed point is clearly unstable.

This limiting behavior where $\phi \to 0$ is consistent with linear perturbations (5.14) studied in the previous subsection about the $\phi = 0$ fixed point for the particular case (5.19). The Hubble parameter, which remains constant for all ϕ, is thus given by the corresponding solution to Eq. (5.4) for $\phi = 0$:

$$H^2 = \frac{1}{6\alpha_0}\left\{V_0 + \frac{3}{\kappa_5^4\alpha_0}\left[1 \right.\right.$$
$$\left.\left. \pm \sqrt{1 - \frac{2}{3}\kappa_5^4\alpha_0(\kappa_5^2\alpha_0 U - V_0)}\right]\right\}, \qquad (5.22)$$

which is real so long as $2\kappa_5^4\alpha_0(\kappa_5^2\alpha_0 U - V_0) < 3$.

The effective energy density ρ and the pressure P of the evolving scalar field, ϕ, can be expressed as

$$\rho = -P = V_0 + 3\alpha_2 H^2\phi^2, \qquad (5.23)$$

which will be time-dependent for $\alpha_2 \neq 0$. On the other hand, it can be checked that,

$$\rho - 6\alpha H^2 = V_0 - 6\alpha_0 H^2 = \text{constant.} \qquad (5.24)$$

VI. CONFORMALLY COUPLED SCALAR FIELD ON THE BRANE

An interesting model to consider is the case of a conformally coupled scalar field on the brane, with conformal coupling

$$\alpha = \alpha_0 - \frac{1}{12}\phi^2, \qquad (6.1)$$

where α_0 is a positive constant, and a vanishing potential $V = 0$.

It is known in 4D General Relativity that the trace of the effective energy-momentum tensor of a conformally coupled scalar field is zero. Therefore, the behavior of a spatially homogeneous conformally coupled field can be

effectively described as a radiation fluid [20]. We will show that this remains true in brane-world models.

We split the energy-momentum tensor (2.7) as follows

$$\tau_{\mu\nu} = \hat{T}_{\mu\nu} - 2\alpha_0 G_{\mu\nu}, \qquad (6.2)$$

where, from Eq. (2.9) we have

$$\hat{T}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(\alpha)}_{\mu\nu} + \frac{1}{6}\phi^2 G_{\mu\nu}. \qquad (6.3)$$

The scalar field equation (2.13) then ensures that $\hat{T}_{\mu\nu}$ is traceless for the conformal coupling given by Eq. (6.1). We recover the usual 4D result because we only use the scalar field equation on the brane (2.13) and this is formally the same as in 4D General Relativity case. This result remains true if we include a quartic self-interaction potential for the scalar $V = \lambda\phi^4$, but for simplicity we will consider here the case of a non-self-interacting field ($\lambda = 0$).

In the following, we will consider cosmological solutions where the brane is homogeneous and isotropic. For convenience, we define a dimensionless scalar field $\chi = \phi/a$. Now, the scalar field equation of motion (4.9) can be rewritten as

$$\frac{d^2\chi}{d\eta^2} + K\chi = 0, \qquad (6.4)$$

where $\eta = \int dt/a$ corresponds to the conformal time on the brane and K is the spatial curvature. In addition, we have that the effective energy density $\hat{\rho}$ and pressure $\hat{P}$, described by $\hat{T}_{\mu\nu}$, are given by

$$\hat{\rho} = 3\hat{P} = \frac{1}{2a^4}\left[\left(\frac{d\chi}{d\eta}\right)^2 + K\chi^2\right]. \qquad (6.5)$$

Using Eq. (6.5) and the first integral of the scalar field equation of motion (6.4), we obtain $\hat{\rho} = B/a^4$, where B is an integration constant.

If in addition, we consider a nonvanishing, but constant potential $V = V_0$ on the brane, we have that $\hat{T}_{\mu\nu}$ is no longer trace-free. On the other hand, the evolution of the scalar field χ is unchanged and given by Eq. (6.4), while $\hat{\rho}$ and $\hat{P}$ are shifted such that

$$\hat{\rho} = \frac{B}{a^4} + V_0, \qquad (6.6)$$

$$\hat{P} = \frac{B}{3a^4} - V_0. \qquad (6.7)$$

The cosmological evolution of the brane is given by Eqs. (4.2) and (4.4), where now ρ, P and α are substituted by $\hat{\rho}$, $\hat{P}$ and α_0, respectively. The Hubble parameter (4.3) thus reads

$$H^2 = -\frac{K}{a^2} + \frac{1}{6\alpha_0}\left\{\frac{B}{a^4} + V_0 + \frac{3}{\kappa_5^4\alpha_0}\left[1\right.\right.$$
$$\left.\left. \pm \sqrt{1 + \frac{2}{3}\kappa_5^4\alpha_0\left(-\kappa_5^2\alpha_0 U + V_0 + \frac{B - 2\alpha_0 C)}{a^4}\right)}\right]\right\}. \qquad (6.8)$$

This Friedmann equation is the same as that found in [7] for a radiation filled brane-world universe with a nonvanishing brane tension.

We note that it is possible to recover the conventional evolution for a 4D cosmology filled with radiation and an effective vacuum energy density with a fine-tuning of the parameters of the solution. This is possible, when the energy density of the conformally coupled scalar field on the brane exactly cancels the effect of the projected Weyl tensor on the brane, i.e.,

$$B = 2\alpha_0 C. \qquad (6.9)$$

Then the Friedmann Eq. (6.8) becomes

$$3\left(H^2 + \frac{K}{a^2}\right) = \frac{B}{2\alpha_0 a^4} + \Lambda_\pm, \qquad (6.10)$$

where the effective cosmological constant is given by

$$\Lambda_\pm = \frac{1}{2\alpha_0}\left\{V_0 + \frac{3}{\kappa_5^4}\left[1 \pm \sqrt{1 - \frac{2}{3}\kappa_5^4\alpha_0(\kappa_5^2\alpha_0 U - V_0)}\right]\right\}. \qquad (6.11)$$

A vanishing cosmological constant on the brane requires the usual Randall-Sundrum fine-tuning $V_0^2 = -6U/\kappa_5^2$. We then obtain $\Lambda_\pm = 0$ for $\pm(\kappa_5^4\alpha_0 V_0) \le 0$.

Going back to the projected Einstein equations (2.3) we can see that it is possible to recover the standard 4D Einstein equations

$$G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + \frac{1}{2\alpha_0}T^{\text{c.c.}}_{\mu\nu}, \qquad (6.12)$$

for the special case of a conformally coupled field with (trace-free) energy-momentum tensor if it exactly matches the projected Weyl tensor

$$T^{\text{c.c.}}_{\mu\nu} = -2\alpha_0 E_{\mu\nu}. \qquad (6.13)$$

In this case the conformally coupled field and the projected Weyl tensor exactly cancel out in the total effective energy-momentum tensor (6.2) on the brane, $\tau_{\mu\nu} = (2\alpha_0\Lambda_4 - V_0)g_{\mu\nu}$, and hence $\Pi_{\mu\nu} \propto g_{\mu\nu}$ in Eq. (2.4). The conformally coupled energy-momentum tensor, $T^{\text{c.c.}}_{\mu\nu}$ (or equivalently the projected Weyl tensor, $E_{\mu\nu}$) then only appears *linearly* in the induced Einstein equations (2.3).

VII. DISCUSSION

In this paper we have studied the field equations for a scalar field living on a 4D brane embedded in 5D vacuum space-time, including the effect of a nonminimal coupling of the field to the 4D scalar curvature on the brane. This is a natural generalization of previous studies of the dynamics of minimally coupled scalar fields on the brane, just as Brans-Dicke scalar-tensor models are a natural generalization of minimally coupled fields in 4D general relativity. Such a nonminimal coupling would be expected to arise as a quantum correction for any self-gravitating field, but in the present paper we have just considered the classical dynamics of an effective theory with nonminimal coupling.

In a 4D scalar-tensor gravity theory with a nonminimally coupled scalar field it is always possible to perform a conformal transformation [21] $g_{\mu\nu} \rightarrow \Omega^2(\phi)g_{\mu\nu}$ to recast the theory as Einstein gravity plus a minimally coupled field, in what is known as the Einstein frame [22]. This is no longer possible in a brane-world context with a nonminimally coupled scalar field on the brane, as the bulk gravity already defines a "5D Einstein frame" [23]. The nonminimal coupling of the scalar on the brane results in the Einstein-Brans-Dicke tensor (2.8) appearing as a source term in the total energy-momentum tensor on the brane. If one attempts to simplify this by a conformal transformation to the "4D Einstein frame" on the brane, then this simplifies the total energy-momentum source term on the brane, but results in more complicated effective gravitational field equations in the bulk. There seems to be no easy way to avoid the rather messy gravitational field equations for a nonminimally coupled scalar field on the brane.

We identify two different regimes in which the evolution reduces to the usual 4D form. At low energies (relative to the brane tension σ) the projected 5D Einstein equations reduce to an effective 4D gravity theory (3.5), which is a generalization of the Randall-Sundrum model [1]. The nonminimal coupling $\alpha(\phi)$ leads to a correction to the effective gravitational constant on the brane (3.6). On the other hand, if the nonminimal coupling term is large so that the effects of the bulk gravity is negligible, we recover an effective 4D scalar-tensor gravity theory (3.9) where $\alpha(\phi)$ describes the gravitational coupling (3.10).

We give the form of the modified Friedmann equation for homogeneous and isotropic cosmologies with a nonminimally coupled scalar field. For a FRW brane moving in 5D anti-de Sitter space-time it is then possible to give expressions for the 4D low-energy and strong-coupling regimes in terms of the energy density. Only for sufficiently weak coupling (4.30) is it possible to recover an intermediate "5D" regime where the Hubble expansion is linearly proportional to the scalar field energy density on the brane [19].

We have given the projected field equations on the brane following the approach of Shiromizu, Maeda and Sasaki [10] where the nonlocal effect of bulk gravity is described by the projection of the 5D Weyl tensor. The most general 5D vacuum solution respecting the symmetries of a homogeneous and isotropic (FRW) brane is 5D Schwarzschild-anti-de Sitter where the projected Weyl tensor acts like a radiation fluid.

An interesting special case is that of a conformally coupled scalar field on the brane. As in 4D gravity, one can use the scalar field equation of motion to define a trace-free energy-momentum tensor (6.3) for a conformally coupled field on the brane. In general this obeys the same modified Friedmann equation as found previously [5,7] for a radiation fluid on a brane with fixed induced gravity coupling α_0. But for particular values of the conformal field's energy density it is possible for it to exactly cancel out the nonlocal effect from the projected Weyl tensor and we recover a standard 4D Friedmann equation for a conformal field.

We also identify de Sitter brane solutions with constant H. We find solutions with a constant scalar field displaced from the minimum of the potential, where the potential gradient is balanced by the gradient of the nonminimal coupling term. But for some scalar field Lagrangians it is also possible to find de Sitter solutions with constant 4D Ricci scalar, but nonconstant scalar field.

It is natural to consider extending previous analyses of slow-roll inflation due to a self-interacting scalar field on the brane [24] to include the effect of a nonminimal coupling for the scalar field to the induced Ricci curvature on the brane. Several authors have considered the spectrum of scalar metric perturbations produced by quantum fluctuations of an inflaton field on the brane in the presence of a constant induced gravity correction [8,9]. Indeed we have recently shown that the 4D consistency relation for the tensor-scalar ratio from inflation remains true with a constant induced gravity correction. It would be interesting to see whether this remains true for a scalar field with nonminimal coupling $\alpha(\phi)$. However our ability to relate the scalar metric perturbations produced during inflation to observables at late times may be limited due to the nonconservation of the scalar field energy density ρ in Eq. (4.7). Only the total effective energy density $\rho^{\rm tot}$ in Eq. (4.12) is locally conserved and so we require strictly adiabatic perturbations in this total effective energy density in order for the scalar curvature perturbation to remain constant in the large scale limit [25]. We leave this interesting question for future work.

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