

**Generalized cosmological scaling solutions**Edmund J. Copeland,<sup>1</sup> Seung-Joo Lee,<sup>1</sup> James E. Lidsey,<sup>2</sup> and Shuntaro Mizuno<sup>3</sup><sup>1</sup>*Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN1 9QJ, United Kingdom*<sup>2</sup>*Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, United Kingdom*<sup>3</sup>*Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

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Motivated by the recent interest in cosmologies arising from energy density modifications to the Friedmann equation, we analyze the scaling behavior for a broad class of these cosmologies comprised of scalar fields and background barotropic fluid sources. In particular, we determine the corresponding scalar field potentials which lead to attractor scaling solutions in a wide class of braneworld and dark energy scenarios. We show how a number of recent proposals for modifying the Friedmann equation can be thought of as being dual to one another, and determine the conditions under which such dualities arise.

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**I. INTRODUCTION**

The Friedmann equation is one of the cornerstones of modern cosmology relating the expansion of the Universe to the total energy density within it. It forms the starting point for almost all investigations in cosmology. However, over the past few years, possible corrections to the Friedmann equation have been derived or proposed in a number of different contexts, generally inspired by brane-world investigations. These corrections are often of a form that involves the total energy density  $\rho$ , and are such that they tend to play a role early in the history of the Universe, fading away as we enter the late-time, post-nucleosynthesis era (although that is not always the case as we shall see). Up to now, the different models have been presented in the literature without any attempt to relate them. In this paper, by introducing a generalized form for the correction, we will provide a formalism which allows us to relate a large class of modified Friedmann cosmologies. Assuming the total energy density to be comprised of a canonical scalar field  $\phi$  with potential  $V(\phi)$ , together with some form of barotropic fluid, we will demonstrate how the existence of scaling solutions determines the form of  $V(\phi)$ , and in doing so we will establish a direct relation between the form of the potential and the functional form of the modification to the Friedmann equation. Scaling (attractor) solutions in cosmology are very important because they allow one to understand the asymptotic behavior of a particular cosmology and to determine whether such behavior is stable or not. They have also been advocated by a number of authors as a way of establishing the behavior of general scalar fields in a cosmological setting both in the context of conventional Friedmann cosmologies [1–10] and in particular classes of modified Friedmann cosmologies [11–23].

In Sec. II we present the equations of motion arising out of the modified Friedmann equations and introduce variables which allow the scaling solutions to be determined. The general conditions for scaling behavior are then estab-

lished in Sec. III and we show that these can be written as a closed form relationship between the scalar field and the functional form of the modification to the Friedmann equation. In Sec. IV we demonstrate the existence of duality symmetries between different scaling solutions, and determine the conditions which must be satisfied in terms of the modification to the Friedmann equation for such duality properties to be obtained. Section V then applies the results of the previous sections to a general class of models, evaluating the scaling potentials (and their duals), as well as the explicit evolution of the scale factor in the scaling regimes. In particular, we apply the technique to a number of recently investigated cosmologies of relevance both to braneworld and dark energy scenarios, including the Randall-Sundrum [24], Shtanov-Sahni (S-S) [25] and Cardassian [26] models. The Dvali-Gabadadze-Porrati (DGP) [27] braneworld scenario is investigated in Sec. VI and we summarize our results in Sec. VII. Throughout units are chosen such that  $\hbar = c = 1$ .

**II. EQUATIONS OF MOTION**

We consider spatially flat Friedmann-Robertson-Walker (FRW) cosmologies such that the dynamics is determined by an effective Friedmann equation of the form

$$H^2 = \frac{8\pi}{3m_4^2} \rho L^2(\rho), \quad (1)$$

where  $H \equiv \dot{a}/a$  is the Hubble parameter,  $a$  is the scale factor,  $\rho$  is the total energy density of the Universe, a dot denotes differentiation with respect to cosmic time and  $m_4$  is the four-dimensional Planck mass. Modifications to standard relativistic cosmology are parametrized by the correction function  $L(\rho)$  and this is assumed to be positive-definite without loss of generality.

We will investigate models where the Universe is sourced by a self-interacting scalar field  $\phi$  with potential  $V(\phi)$  together with a barotropic fluid with equation of state  $P_\gamma = (\gamma - 1)\rho_\gamma$ , where  $\gamma$  is the adiabatic index. The

energy density and pressure of the scalar field are given by  $\rho_\phi = \dot{\phi}^2/2 + V$  and  $P_\phi = \dot{\phi}^2/2 - V$ , respectively. As in conventional cosmologies, we assume that the energy-momenta of these matter fields is covariantly conserved and this implies that

$$\dot{\rho}_\gamma = -3\gamma H \rho_\gamma, \quad (2)$$

$$\ddot{\phi} = -3H\dot{\phi} - dV/d\phi. \quad (3)$$

Equations (1)–(3) close the system that determines the cosmic dynamics.

In standard cosmology the stability of scaling solutions is analyzed by introducing the variables [4]:

$$X_{\text{CLW}} \equiv \sqrt{\frac{4\pi}{3m_4^2}} \frac{\dot{\phi}}{H}, \quad Y_{\text{CLW}} \equiv \sqrt{\frac{8\pi}{3m_4^2}} \frac{\sqrt{V}}{H} \quad (4)$$

and rewriting the field equations as an autonomous system. Following [12,14], we define the new pair of variables:

$$X \equiv \frac{\dot{\phi}}{\sqrt{2\rho}}, \quad Y \equiv \frac{\sqrt{V}}{\sqrt{\rho}}, \quad (5)$$

that are related to those of Eq. (4) by

$$\frac{X}{X_{\text{CLW}}} = \frac{Y}{Y_{\text{CLW}}} = \epsilon L, \quad (6)$$

where  $\epsilon = \pm 1$  for expanding and contracting universes, respectively. In what follows, we consider expanding models unless otherwise stated.

When expressed in terms of the new variables (5), the equations of motion (1)–(3) can be written in the form:

$$X' = -3X + \epsilon\lambda\sqrt{\frac{3}{2}}Y^2 + \frac{3}{2}X[2X^2 + \gamma(1 - X^2 - Y^2)], \quad (7)$$

$$Y' = -\epsilon\lambda\sqrt{\frac{3}{2}}XY + \frac{3}{2}Y[2X^2 + \gamma(1 - X^2 - Y^2)], \quad (8)$$

$$\lambda' = -\epsilon\sqrt{6}\lambda^2(\Gamma - 1)X + 3\lambda[2X^2 + \gamma(1 - X^2 - Y^2)]\rho \frac{d}{d\rho}[\ln L(\rho)], \quad (9)$$

where

$$\lambda \equiv -\frac{1}{L} \frac{m_4}{\sqrt{8\pi}} \frac{dV/d\phi}{V}, \quad (10)$$

$$\Gamma \equiv V \frac{d^2V/d\phi^2}{(dV/d\phi)^2}, \quad (11)$$

and a prime denotes differentiation with respect to the logarithm of the scale factor,  $N \equiv \ln a$ . As in Ref. [10], Eqs. (10) and (11) generalize the expressions introduced in

Refs. [2,4,9,14]. In particular,  $\lambda$  is related to the parameter  $\lambda_{\text{CLW}} \equiv -\frac{m_4}{\sqrt{8\pi}} \frac{dV/d\phi}{V}$  introduced in Ref. [4] such that

$$\frac{\lambda}{\lambda_{\text{CLW}}} = \frac{1}{L}. \quad (12)$$

We refer to  $\lambda$  as the ‘‘scaling parameter.’’

The system of equations (7)–(11) for the variables  $X$ ,  $Y$  and  $\lambda$  do not appear to be closed due to the presence of the term involving  $\Gamma$  in Eq. (9). However, it follows from Eqs. (5), (10), and (11) that we have  $\Gamma = \Gamma(\phi)$ ,  $\phi = \phi(\rho, Y)$  and  $\rho = \rho(\lambda, \phi)$  and hence that  $\Gamma = \Gamma(\lambda, Y)$ . This implies that the equations are indeed closed.

The definition of the total energy density implies that the variables (5) satisfy the constraint equation

$$X^2 + Y^2 + \frac{\rho_\gamma}{\rho} = 1 \quad (13)$$

and, since the energy density of the barotropic fluid is semi-positive-definite, any cosmological model can be represented as a trajectory in the  $(X, Y)$ -plane that is bounded within the unit circle, i.e.,  $\Omega_\phi \equiv \rho_\phi/\rho = X^2 + Y^2 \leq 1$ . Furthermore, since  $Y \geq 0$  by definition, it is sufficient to consider the evolution in the upper half of the disc.

Equations (7)–(9) exhibit an important property. For the case where  $\lambda$  is constant, they have an identical form to that of the plane-autonomous system of standard relativistic cosmology that is formulated in terms of the variables  $\{X_{\text{CLW}}, Y_{\text{CLW}}, \lambda_{\text{CLW}}\}$ . This duality immediately implies that the system (7) and (8) admits an identical set of critical points to that of the standard scenario when these solutions are expressed in terms of the variables  $\{X, Y, \lambda\}$ .

A further consequence of such a duality is that the stability of each fixed point solution can be determined directly from the stability analysis of Ref. [4]. In total, there are five critical solutions to Eqs. (7) and (8) where the variables  $\{X, Y, \lambda\} = \{X_c, Y_c, \lambda_c\}$  are constants. Three of these represent the unstable solutions ( $X_c = 1, Y_c = 0$ ), ( $X_c = -1, Y_c = 0$ ), ( $X_c = 0, Y_c = 0$ ) for all values of  $\lambda$  and  $\gamma$ . The value of  $\lambda$  determines the nature of the other two points. For  $\lambda^2 > 3\gamma$ , there exists an attractor solution

$$X_c = \sqrt{\frac{3}{2}} \frac{\gamma}{\lambda}, \quad Y_c = \sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}, \quad (14)$$

where the effective adiabatic index of the scalar field, defined by

$$\gamma_\phi \equiv \frac{2X^2}{X^2 + Y^2}, \quad (15)$$

satisfies the condition  $\gamma_\phi = \gamma$ . For this late-time attractor solution, the relative contribution of the scalar field’s energy density to the total energy density of the Universe is constant,  $\Omega_{\phi c} \equiv X^2 + Y^2 = 3\gamma/\lambda^2$ , and consequently, the energy densities of the scalar field and fluid redshift at the same rate as the Universe expands.

The fifth critical point arises if  $\lambda^2 < 6$  and is given by

$$X_c = \frac{\lambda}{\sqrt{6}}, \quad Y_c = \sqrt{1 - \frac{\lambda^2}{6}}. \quad (16)$$

This corresponds to the case where the scalar field dominates the fluid ( $\Omega_{\phi c} = 1$ ) and has an effective adiabatic index  $\gamma_\phi = \lambda^2/3$ . The solution is stable if  $\gamma_\phi < \gamma$ , i.e.,  $\lambda^2 < 3\gamma$ .

### III. GENERAL CONDITIONS FOR SCALING SOLUTIONS

Since the expressions (6) and (12) relating the standard and modified FRW cosmologies involve the correction function  $L(\rho)$ , the scalar field potential that gives rise to the fixed point attractor solutions (14) and (16) in a given generalized scenario will depend on the specific form of this function. In particular, the potential will differ from the purely exponential form that leads to scaling solutions in the conventional FRW model. In this section we establish the correspondence between the modified Friedmann equation and the scaling potential.

It can be shown by direct substitution that both sets of critical points (14) and (16) represent solutions to the field equations (7)–(9) of the form  $X' = Y' = \lambda' = 0$  if the relation

$$\Gamma = 1 + \rho \frac{d}{d\rho} [\ln L(\rho)] \quad (17)$$

is satisfied. Since  $\rho = V/Y_c^2$  for these solutions, Eq. (17) may be written in the form

$$\rho \frac{d^2 \rho / d\phi^2}{(d\rho/d\phi)^2} - 1 - \rho \frac{d}{d\rho} [\ln L] = 0, \quad (18)$$

and multiplying Eq. (18) by  $(d\rho/d\phi)/\rho$  then implies that

$$\frac{d(\ln(d\rho/d\phi))}{d\phi} - \frac{d(\ln\rho)}{d\phi} - \frac{d(\ln L)}{d\phi} = 0. \quad (19)$$

Equation (19) may be integrated twice to yield a necessary and sufficient condition on the scalar field potential if the solution is to represent a scaling solution for a given choice of correction function  $L(\rho)$ . We find the important result:

$$\int \frac{d\rho}{\rho L} = -\frac{\sqrt{8\pi}\lambda}{m_4} \phi, \quad (20)$$

where one of the integration constants has been set to zero without loss of generality by performing a linear shift in the value of the scalar field and the constant of proportionality on the right-hand side follows by requiring consistency with Eq. (10).

It is also of interest to determine the evolution of the scale factor for a given class of scaling solutions. Since  $X_c$  is a nonzero constant for these solutions, Eq. (5) implies

that the scalar field is a monotonically varying function of proper time ( $\dot{\phi} \neq 0$ ). It is natural, therefore, to view the value of the field as the dynamical variable of the system and to express all time-dependent parameters in terms of this variable.

In general, the scalar field Eq. (3) can be expressed in the form

$$\dot{\rho}_\phi = -3H\dot{\phi}^2 \quad (21)$$

or, equivalently, as

$$\frac{d\rho_\phi}{d\phi} = -3H\dot{\phi}. \quad (22)$$

It then follows from the definition of the Hubble parameter that

$$3H^2 = -\frac{1}{a} \frac{da}{d\phi} \frac{d\rho_\phi}{d\phi} \quad (23)$$

and substituting Eq. (23) into Eq. (1) implies that the Friedmann equation can be expressed in the form

$$\frac{da}{d\phi} \frac{d\rho}{d\phi} = -\frac{8\pi}{\Omega_{\phi c} m_4^2} a \rho L^2(\rho). \quad (24)$$

Introducing a new variable

$$b(\phi) = \exp\left[\Omega_{\phi c} \int^\rho d\rho \frac{1}{\rho L^2(\rho)}\right] \quad (25)$$

simplifies Eq. (24) to

$$\frac{da}{d\phi} \frac{db}{d\phi} = -\frac{8\pi}{m_4^2} ab \quad (26)$$

and the scale factor is then determined up to a single quadrature:

$$a(\phi) = \exp\left[-\frac{8\pi}{m_4^2} \int^\phi d\phi b(\phi) \left(\frac{db}{d\phi}\right)^{-1}\right]. \quad (27)$$

Thus, for a given cosmological scenario characterized by a correction function  $L(\rho)$ , the potential (and equivalently the total energy density) yielding the scaling solution is determined by integrating Eq. (20). Integration of Eq. (25) then yields the dependence of  $b(\phi)$  on the scalar field and the evolution of the scale factor follows after integration of Eq. (27). Finally, the time-dependence of the scale factor can in principle be deduced by integrating Eq. (22),

$$t = -\frac{\sqrt{24\pi}}{\Omega_{\phi c} m_4} \int^\phi d\phi L(\phi) \rho^{1/2}(\phi) \left(\frac{d\rho}{d\phi}\right)^{-1}, \quad (28)$$

and inverting the result.

In the following section, we employ the above formalism to establish a link between different classes of scaling solutions that arise for various choices of the modification to the Friedmann equation.

#### IV. DUALITY BETWEEN SCALING SOLUTIONS

A duality between different scaling solutions can be established by noting that Eq. (26) is invariant under the simultaneous interchange

$$a(\phi) \rightarrow b^p(\phi), \quad b(\phi) \rightarrow a^{1/p}(\phi), \quad (29)$$

where  $p$  is an arbitrary constant. This symmetry implies that a given scaling solution may be employed as a seed to generate a new scaling cosmology for a different Friedmann equation and associated scalar field potential. To be specific, let us consider the scaling solution parametrized by the functions  $\{a_+(\phi), b_+(\phi), \rho_+(\phi)\}$  that arises for a specific choice of correction function  $L_+(\rho)$ . We now denote the ‘‘dual’’ scaling solution as  $\{a_-(\phi), b_-(\phi), \rho_-(\phi)\}$  and assume an *ansatz* of the form

$$b_-(\phi) = [a_+(\phi)]^{1/p}. \quad (30)$$

The new scale factor is then determined from Eq. (27):

$$a_-(\phi) = \exp\left[-\frac{8\pi p}{m_4^2} \int d\phi a_+(\phi) \left(\frac{da_+}{d\phi}\right)^{-1}\right]. \quad (31)$$

However, since the function  $a_+(\phi)$  is itself a solution to the Friedmann equation (26), Eq. (31) simplifies after integration to

$$a_-(\phi) = [b_+(\phi)]^p \quad (32)$$

modulo an arbitrary (constant) prefactor.

We may now determine the condition that the dual correction function  $L_-(\rho)$  must satisfy for the solution (32) to also represent a scaling solution that satisfies Eq. (20). If we assume *a priori* that the two solutions  $a_{\pm}(\phi)$  represent scaling solutions characterized by  $\lambda_{\pm}$ , respectively, Eq. (20) implies that

$$\frac{1}{\lambda_+ \rho_+ L_+} \frac{d\rho_+}{d\phi} = \frac{1}{\lambda_- \rho_- L_-} \frac{d\rho_-}{d\phi}. \quad (33)$$

It then follows, after substitution of Eq. (24) into the right-hand side of Eq. (33), that

$$\frac{1}{\rho_+ L_+} \frac{d\rho_+}{d\phi} = -\frac{8\pi\lambda_+}{\Omega_{\phi c} m_4^2 \lambda_-} \frac{L_- a_-}{(da_-/d\phi)} \quad (34)$$

and Eq. (26) then implies that

$$\frac{1}{\rho_+ L_+} \frac{d\rho_+}{d\phi} = \frac{\lambda_+}{\lambda_- \Omega_{\phi c}} \frac{L_-}{b_-} \frac{db_-}{d\phi}. \quad (35)$$

On the other hand, substituting the *ansatz* (30) into Eq. (35) and employing Eq. (24) for the positive-branch solution yields the condition

$$\frac{1}{\rho_+^2 L_+^2} \left(\frac{d\rho_+}{d\phi}\right)^2 = -\frac{8\pi}{\Omega_{\phi c}^2 m_4^2} \frac{\lambda_+}{\lambda_- p} L_- L_+. \quad (36)$$

Consistency with Eq. (20) therefore implies that a necessary and sufficient condition for the dual cosmology

$\{a_-(\phi), L_-(\phi)\}$  to represent a scaling solution is that *the correction functions arising in the respective Friedmann equations must be proportional to the inverse of each other when both are expressed as functions of the scalar field:*

$$L_+(\phi)L_-(\phi) = -p\lambda_+\lambda_-\Omega_{\phi c}^2. \quad (37)$$

It is interesting that the standard relativistic cosmology ( $L = 1$ ) represents the self-dual model when  $p\Omega_{\phi c}^2\lambda_+\lambda_- = -1$ .

Finally, we find after substituting Eq. (37) into Eq. (33) and employing Eq. (25) that the energy density of the dual scaling solution is given by

$$\frac{1}{\rho_-} \frac{d\rho_-}{d\phi} = -p\Omega_{\phi c}\lambda_-^2 \frac{1}{b_+} \frac{db_+}{d\phi}. \quad (38)$$

The dual potential then follows immediately from Eqs. (5) and (25):

$$V_-(\phi) = Y_c^2 \exp\left[-p\lambda_-^2 \Omega_{\phi c}^2 \int d\rho_+ \frac{1}{\rho_+ L_+^2}\right]. \quad (39)$$

In the following section we employ the techniques developed above to determine scaling solutions (and their duals) in a number of different cosmological settings.

#### V. UNIFICATION OF MODIFIED FRIEDMANN COSMOLOGIES

##### A. Generalized class of scaling cosmologies

A wide class of scenarios that have been considered recently predict deviations from the standard cosmology of the form

$$L(\rho) = \sqrt{1 + A\rho^\nu}, \quad (40)$$

where  $\nu$  is an arbitrary dimensionless constant and  $A$  is an arbitrary constant with dimension  $m^{-4\nu}$ . In this case, the form of the potential leading to scaling (fixed point) solutions is determined by integrating Eq. (20). It is found that

$$V_+(\phi) = Y_c^2 A^{-1/\nu} \text{cosech}^{2/\nu}\left(-\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi\right), \quad (41)$$

if  $A > 0$  and

$$V_-(\phi) = Y_c^2 |A|^{-1/\nu} \text{sech}^{2/\nu}\left(-\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi\right), \quad (42)$$

if  $A < 0$ .

Given the form of the scalar potential (41), the parametric solution for the case  $A > 0$  is determined by integrating Eqs. (25) and (27), respectively:

$$\begin{aligned}\rho_+(\phi) &= A^{-1/\nu} \text{cosech}^{2/\nu} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \\ b_+(\phi) &= A^{-\Omega_{\phi c}/\nu} \text{sech}^{2\Omega_{\phi c}/\nu} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \\ a_+(\phi) &= \sinh^{2/(\lambda^2\Omega_{\phi c}\nu)} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right).\end{aligned}\quad (43)$$

The time dependence of the solution follows by substituting Eq. (43) into Eq. (40) to yield the Friedmann correction function:

$$L_+(\phi) = \text{cotanh} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \quad (44)$$

and then evaluating the integrand in Eq. (28):

$$t = -\frac{\sqrt{3A^{1/\nu}}}{\lambda\Omega_{\phi c}} \int d\phi \sinh^{1/\nu} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right). \quad (45)$$

The integral (45) can be performed analytically for various choices of  $\nu$ , whereas the late-time behavior can be analyzed for arbitrary  $\nu$ . In particular, we find from Eq. (45) that the late-time limit corresponds to large  $\phi$ , and it therefore follows from Eq. (44) that  $L \rightarrow 1$  as  $t \rightarrow \infty$ . This in turn is the limit corresponding to the case of an exponential potential.

The corresponding scaling solution for  $A < 0$  driven by the potential (42) is deduced by applying the duality transformation (29) to the solution (43) for a particular value of the constant  $p$ , where the scaling parameters are chosen to be equal,  $\lambda_+ = \lambda_- = \lambda$ . For the case where  $p = -1/(\lambda^2\Omega_{\phi c}^2)$ , the duality transformation (37) implies that the dual correction function is given by

$$L_-(\phi) = \tanh \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right), \quad (46)$$

while integrating Eq. (39) with the form for  $\rho_+(\phi)$  given in Eq. (43) implies that the dual potential has precisely the form of Eq. (42). We may conclude, therefore, that the dual correction function is given by Eq. (40) with  $A < 0$ . In this sense, a model with  $A > 0$  and a specific value of  $\nu$  is twinned with the model where the value of  $\nu$  is the same but the sign of  $A$  is changed. In general, the dual scale factor is deduced from Eqs. (32) and (43):

$$a_-(\phi) = \cosh^{2/(\lambda^2\Omega_{\phi c}\nu)} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \quad (47)$$

and the time-dependence follows from Eq. (28):

$$t = -\frac{\sqrt{3|A|^{1/\nu}}}{\lambda\Omega_{\phi c}} \int d\phi \cosh^{1/\nu} \left( -\frac{\lambda\nu}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right). \quad (48)$$

It is of course trivial to show that for this choice of  $p$ , the standard cosmology solution corresponding to the case

$L = 1$  reproduces the well known exponential potential for the scalar field [28].

In the following subsections, we consider some of the specific models that belong to the class of corrections given by Eq. (40).

### B. Randall-Sundrum type II braneworld cosmology

The case  $A = 1/2\sigma$  and  $\nu = 1$  corresponds to the Randall-Sundrum type II (R-S II) braneworld scenario [24,29–31], where a codimension one brane with positive tension  $\sigma$  is embedded in five-dimensional anti-de Sitter (AdS<sub>5</sub>) space:

$$L(\rho) = \sqrt{1 + \frac{\rho}{2\sigma}}. \quad (49)$$

For this case, the scaling potential yielding the fixed point solution is given by

$$V(\phi) = 2\sigma Y_c^2 \text{cosech}^2 \left( -\frac{\lambda}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \quad (50)$$

and the time-dependences of the scalar field and scale factor are deduced by evaluating the integral (45) for  $\nu = 1$  and substituting the result into Eq. (43):

$$\cosh \left( -\frac{\lambda}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) = \left( \frac{2\pi}{3Am_4^2} \right)^{1/2} \lambda^2 \Omega_{\phi c} t \quad (51)$$

$$a(t) = \left[ \frac{2\pi\Omega_{\phi c}^2 \lambda^4}{3Am_4^2} t^2 - 1 \right]^{1/(\lambda^2\Omega_{\phi c})}. \quad (52)$$

These results for the scaling solutions associated with the R-S II model confirm those previously obtained using a different method in Ref. [32]. An important feature of our approach is that it shows the solution is a fixed point attractor solution. Another important feature that emerges is that in the limit where the quadratic energy density term dominates, i.e., when  $\sqrt{8\pi}\lambda\phi \ll 2m_4$ , the potential (50) asymptotes to  $V \propto \phi^{-2}$ , consistent with earlier analyses [33,34]. Similarly, once the energy density has decreased so that  $L \sim 1$ , the value of the scalar field becomes large, and  $V \sim \exp[-\lambda(\sqrt{8\pi}/m_4)\phi]$ , in agreement with general relativistic results [3,4].

In Figs. 1 and 2, we confirm numerically how the above potential leads to the expected attractor solutions for a model with a barotropic fluid of radiation ( $\gamma = 4/3$ ).

### C. Shtanov-Sahni braneworld cosmology

The case  $A = -1/(2|\sigma|)$  and  $\nu = 1$  represents a class of braneworld inspired cosmologies due to Shtanov and Sahni [25,35,36]. In this scenario, a codimension one brane with negative tension  $\sigma$  is embedded in a five-dimensional conformally flat Einstein space, where the signature of the fifth dimension is timelike. In this model, the deviation from the conventional Friedmann cosmology is character-

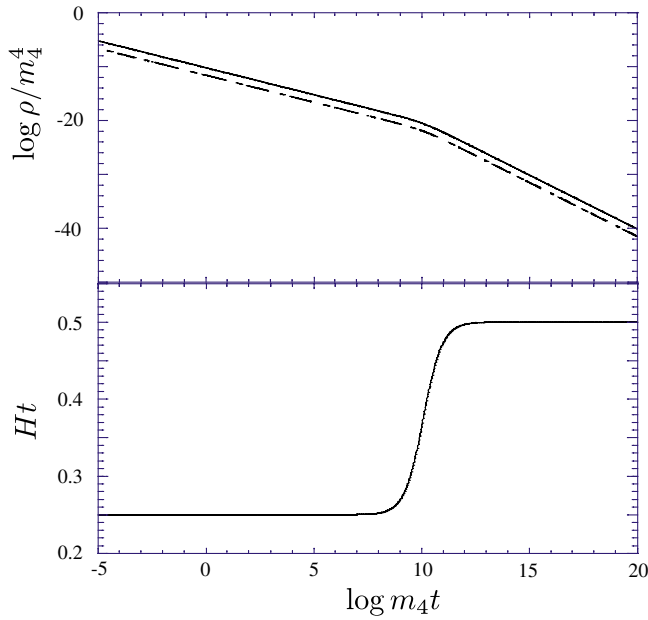


FIG. 1 (color online). Scaling solution for R-S II brane cosmology including a radiation fluid and a scalar field with potential given by Eq. (50). In order to obtain the scaling solution, we choose  $\lambda = 10$  and for simplicity specify the energy scale of the brane tension to be  $\sigma/m_4^4 = 10^{-20}$ . In the upper figure, the time evolution of the energy density for both radiation  $\rho_r$  (solid curve) and the scalar field  $\rho_\phi$  (dashed curve) are shown. Note that around  $\log m_4 t \sim 10$  there is a rapid change as the quadratic correction becomes negligible and the standard cosmological evolution is recovered. However, the energy density of the scalar field mimics that of the radiation fluid throughout the entire evolution, i.e., there is scaling behavior. In the lower figure, we show the time dependence of the cosmic expansion law. At early times, before  $\log m_4 t \sim 10$ , the scale factor grows as  $a \propto t^{1/4}$ , and represents a solution for a radiation-dominated universe in a  $\rho^2$  dominated cosmology. After this time, the conventional expansion rate  $a \propto t^{1/2}$  arises.

ized by

$$L(\rho) = \sqrt{1 - \frac{\rho}{2|\sigma|}}. \quad (53)$$

This model has recently been invoked to develop a nonsingular oscillating universe, where the turning points in both the contracting and expanding phases are induced by the quadratic correction [37].

The S-S braneworld is dual to that of the R-S II scenario in the sense discussed above. The scaling potential follows directly from Eq. (42):

$$V(\phi) = 2|\sigma|Y_c^2 \operatorname{sech}^2\left(-\frac{\lambda}{2} \frac{\sqrt{8\pi}}{m_4} \phi\right), \quad (54)$$

whereas the time-dependences of the scalar field and scale factor follow from Eqs. (48) and (47), respectively, after substituting for  $A = -1/(2|\sigma|)$  and  $\nu = 1$ :

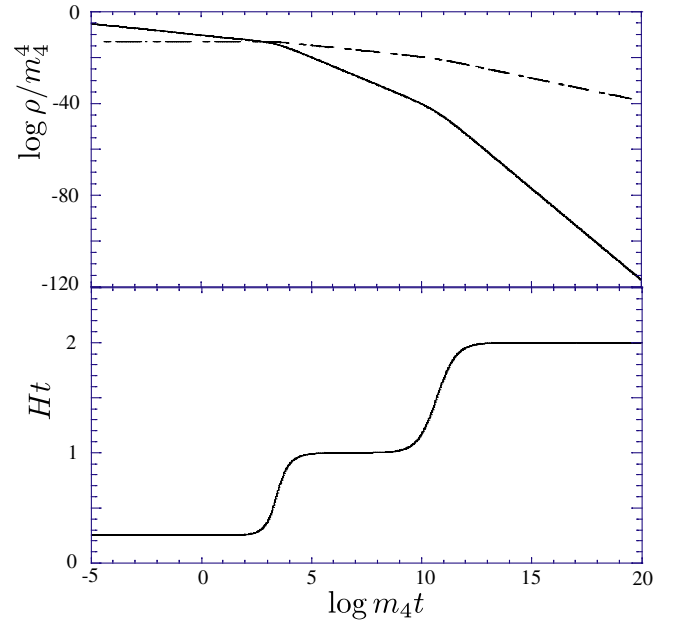


FIG. 2 (color online). As in Fig. 1, although now we specify  $\lambda = 1$ . This leads to a scalar field dominated universe. In the lower figure, the time dependence of the cosmic expansion law is shown. As expected in a universe dominated by a scalar field, the scale factor grows as  $a \propto t$  at early times, corresponding to the  $\rho^2$  dominated phase, and as  $a \propto t^2$  at late times when the linear  $\rho$  term is important.

$$\sinh\left(-\frac{\lambda}{2} \frac{\sqrt{8\pi}}{m_4} \phi\right) = \left(\frac{2\pi}{3|A|m_4^2}\right)^{1/2} \lambda^2 \Omega_{\phi c} t \quad (55)$$

$$a(t) = \left[1 + \frac{2\pi\Omega_{\phi c}^2 \lambda^4}{3|A|m_4^2} t^2\right]^{1/(\lambda^2 \Omega_{\phi c})}. \quad (56)$$

Such a scaling solution is phenomenologically interesting since it represents a nonsingular bouncing cosmology. The universe collapses from infinity ( $t \rightarrow -\infty$ ) to a finite size at  $t = 0$  and then bounces into an expanding phase. The scalar field rolls up the potential during the collapse, reaches the maximum of the potential at  $\phi = 0$  at the instant of the bounce, and then rolls monotonically down the other side during the expansion era.

#### D. Cardassian cosmology

In the above classes of models, the modifications to the Friedmann equation become significant at high energy scales (early times). On the other hand, recent cosmic microwave background (CMB) and large-scale structure observations indicate that the Universe is entering a stage of accelerated expansion at the present epoch and a number of phenomenological models have been developed in an attempt to provide a geometrical interpretation of these observations. In Cardassian cosmology [26,38], for example, the modification term in the Friedmann equation

is given by Eq. (40) with  $A > 0$  and  $\nu \equiv n$  [26,38]:

$$L(\rho) = \sqrt{1 + A\rho^n} \quad (57)$$

and the present-day cosmological acceleration can be explained even when the energy density is comprised of only ordinary matter sources if  $n < -1/3$ . The characteristic feature of this model, therefore, is that the modification term becomes significant at late times.

Although no scalar field is present in the scenario considered in Refs. [26,38], it is instructive to show that the equivalent background cosmology can be obtained from a model comprised of a barotropic fluid and a self-interacting scalar field. We see immediately from Eq. (41) that the corresponding potential which provides the fixed point attractor solution is given by

$$V(\phi) = Y_c^2 A^{-1/n} \text{cosech}^{2/n} \left( -\frac{\lambda n}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right) \quad (58)$$

and in Fig. 3 we demonstrate numerically how the above potential leads to the expected attractor solution. Although such a homogeneous solution is indistinguishable from the

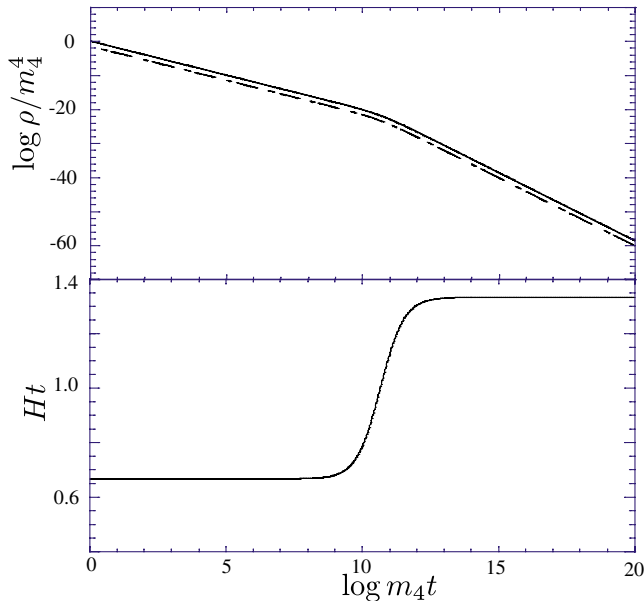


FIG. 3 (color online). Scaling solution for a Cardassian-type cosmology including a matter fluid source and a scalar field with a potential given by Eq. (58). We specify  $n = -0.5$  and choose  $\lambda = 10$  in order to obtain the scaling solution. For simplicity, we set the energy scale where modifications to the standard scenario become significant to be  $A/m_4^2 = 10^{-10}$ . In the upper figure, the time evolution of the energy density of matter  $\rho_m$  (solid curve) and that of the scalar field  $\rho_\phi$  are shown. When  $\log m_4 t \sim 10$ , the time-dependence of the energy density changes as the late-time modification term becomes significant. In the lower figure, we show the time dependence of the cosmic expansion law. For  $\log m_4 t \sim 10$ , we see that  $a \propto t^{2/3}$ , corresponding to the conventional matter dominated universe, whereas for  $\log m_4 t > 10$ , the correction term leads to an accelerating universe,  $a \propto t^{4/3}$ .

purely perfect fluid background, it is possible that the presence of a scalar field may modify the evolution of perturbations and the clustering properties of matter. In principle, this could result in potentially observational signatures [39] and a scaling solution of this type provides a framework for quantitatively investigating the evolution of perturbations in these models.

Finally, before concluding this section, we illustrate the duality transformation that relates the Cardassian cosmologies with the S-S braneworld. Denoting the former with a subscript “+” and the latter by “−”, we may substitute the form for  $b_+(\phi)$  in the Cardassian scenario, as given by Eq. (43), into Eq. (38) to deduce that

$$V_-(\phi) \propto \text{sech}^{-2p\Omega_{\phi c}^2 \lambda_-^2/n} \left( -\frac{\lambda_+ n}{2} \frac{\sqrt{8\pi}}{m_4} \phi \right). \quad (59)$$

This reduces to the S-S scaling potential Eq. (42) (with  $\nu = 1$  and  $\lambda = \lambda_-$ ) when  $p\Omega_{\phi c}^2 \lambda_-^2 = -n$  and  $\lambda_+ = \lambda_-/n$ . Moreover, in this case, it can be verified that the dual correction function satisfying Eq. (37) reduces to Eq. (46) with  $\nu = 1$  and  $\lambda = \lambda_-$ .

We now proceed in the following section to determine the scaling solutions in a further braneworld scenario, where the corrections to the Friedmann equation become significant at late times.

## VI. DVALI-GABADADZE-PORRATI BRANEWORLD COSMOLOGY

The Dvali-Gabadadze-Porrati braneworld scenario [27,40] corresponds to a 3-brane embedded in flat five-dimensional Minkowski spacetime, where a Ricci scalar term is included in the brane action. All energy-momentum is confined to the brane since the bulk is empty. The modified Friedmann equation for the DGP model is given by [40]

$$H^2 \pm \frac{H}{r_0} = \frac{8\pi}{3m_4^2} \rho, \quad (60)$$

where  $r_0 \equiv m_4^2/(2m_5^3)$  and  $m_5$  is the five-dimensional Planck scale. In the DGP model, gravity behaves as four-dimensional Einstein gravity at short scales, whereas it propagates into the bulk at large scales. This induces corrections to the standard Friedmann equation at low energies and the parameter  $r_0$  determines the scale at which these corrections become important. There are two inequivalent ways of embedding the brane in the bulk and this is reflected in the different choices of sign in Eq. (60). In this section, we refer to these as the (+) and (−) branches, respectively. In subsequent expressions, where different signs may be taken, the upper case corresponds to the (+) branch.

It proves convenient to express the Friedmann equation (60) in the equivalent form

$$H = \frac{1}{2r_0} [\mp 1 + \sqrt{1 + B\rho}], \quad (61)$$

where

$$B \equiv \frac{32\pi r_0^2}{3m_4^2}. \quad (62)$$

For the  $(-)$  branch, the late-time attractor is de Sitter (exponential) expansion for any decreasing energy density [40]. For the  $(+)$  branch, on the other hand, expanding Eq. (61) as a Taylor series to lowest-order implies that  $H \approx 8\pi r_0 \rho / (3m_4^2)$ . Modulo a rescaling of the four-dimensional Planck mass, this corresponds formally to the high-energy limit ( $\rho \gg 2\sigma$ ) of the R-S II braneworld (49). Consequently, the early-time analysis of the latter model performed in Ref. [41] is directly applicable to the late-time behavior of this branch of the DGP model. In particular, we may conclude immediately that the potential driving the scaling solution in this limit is the inverse power-law potential  $V \propto \phi^{-2}$ .

A direct comparison between Eqs. (1) and (61) implies that the Friedmann correction function is given by

$$L = \frac{1}{\sqrt{B\rho}} [\mp 1 + \sqrt{1 + B\rho}]. \quad (63)$$

In order to derive the scaling solutions, we define a new variable,  $\theta$ :

$$\rho \equiv \frac{1}{B} \sinh^2 \theta. \quad (64)$$

Substituting Eq. (63) into Eq. (20) then implies that the solution represents a scaling solution if

$$-\frac{\sqrt{8\pi\lambda}}{m_4} \phi = 2 \int d\theta \frac{\cosh \theta}{\cosh \theta \mp 1} \quad (65)$$

and the integral (65) may be evaluated to yield the form of the potential:

$$\frac{\sqrt{2\pi\lambda}}{m_4} \phi = \frac{\sqrt{B\rho}}{\sqrt{1 + B\rho} \mp 1} - \sinh^{-1} \sqrt{B\rho}. \quad (66)$$

The corresponding time dependence of the scaling solution can also be determined. In terms of the variable (64), the Friedmann equation (61) simplifies to

$$H(\theta) = \frac{1}{r_0} \sinh^2 \frac{\theta}{2} \quad (67)$$

for the  $(+)$  branch, and

$$H(\theta) = \frac{1}{r_0} \cosh^2 \frac{\theta}{2} \quad (68)$$

for the  $(-)$  branch. Recalling that  $\dot{\phi}^2/\rho = 2X_c^2$  and  $\rho_\phi = \Omega_{\phi c} \rho$  for the scaling solution, it follows that the scalar field equation (21) transforms to

$$t = -\frac{\Omega_{\phi c}}{3X_c^2} \int d\theta \frac{\cotanh \theta}{H(\theta)} \quad (69)$$

after substitution of Eq. (64). Substituting Eqs. (67) and (68) for the  $(+)$  and  $(-)$  branches, respectively, and evaluating the integral (69) then implies that

$$\frac{3X_c^2}{\Omega_{\phi c} r_0} t = \cotanh^{-1} \sqrt{1 + B\rho} + \frac{1}{\sqrt{1 + B\rho} \mp 1}. \quad (70)$$

## VII. SUMMARY

In this paper we have brought together a number of recent approaches to cosmology which involve modifications of the Friedmann equation. By introducing the general function  $L(\rho)$  as the way of parametrizing the modification, we have been able to establish the conditions under which the new system enters scaling solutions. Considering the case where the energy density is comprised of a scalar field and background barotropic fluid, we have obtained the general relationship that would have to be satisfied between the evolving scalar field and  $L(\rho)$ . In particular, we have obtained the corresponding potential  $V(\phi)$  which leads to scaling solutions and, for a rather general class of functions of  $L(\rho)$ , we have shown that there exist dual solutions which also exhibit similar scaling behavior. This has allowed us to relate solutions which would otherwise appear quite distinct, including those involving collapsing and expanding cosmologies.

Moreover, the duality can directly relate singular and nonsingular cosmologies. To illustrate this property, consider a particular scaling solution  $a_+(\phi)$  that is singular in the sense that the scale factor vanishes at  $a_+(0) = 0$ . (The value of the scalar field can be chosen to be  $\phi = 0$  without loss of generality). Suppose, however, that the logarithmic derivative of the scale factor with respect to the field is nonzero at this point,  $d \ln a_+ / d\phi|_0 \neq 0$ , and furthermore, that the scale factor is a monotonic function with a finite first derivative for all physical (nonzero) values of the field. These properties are satisfied, for example, in the R-S II and Cardassian models.

The qualitative behavior of the dual solution,  $a_-(\phi)$ , is then determined from Eq. (31). If we define a new parameter  $\epsilon(\phi) \equiv [d \ln a / d\phi]^{-1}$ , the value of the scale factor is simply given by the area under the curve  $\epsilon_+(\phi)$ , where the field evolves from zero to some value  $\phi$ . (We are assuming implicitly that  $\dot{\phi} > 0$  and  $p < 0$  again without loss of generality). However, due to the exponential nature of Eq. (31) the initial value of the dual scale factor is *nonzero* and this results in a nonsingular background.

On the other hand, the time reversal of the seed solution  $a_+(\phi)$  would result in a collapsing dual model where the limits in the integral (31) are taken from  $\phi$  to zero. Consequently, the dual solution can be analytically continued through  $\phi = 0$  into a contracting phase. In this



sense, therefore, any singular (expanding) scaling solution satisfying the above (very weak) conditions can generate a nonsingular bouncing cosmology, where the latter is associated with a combination of the seed solution and its time reversal. For fixed values of  $\{p, \lambda_{\pm}\}$ , the collapsing phase of the bouncing solution will be unstable if the expanding phase is stable, and vice versa. In principle, however, different seed solutions may be employed to generate distinct and stable collapsing and expanding branches that can be smoothly joined at the bounce.

This opens up the possibility that such dualities will allow us to relate singular cosmologies to nonsingular bouncing cosmologies, a topic presently of considerable interest in cosmology.

Finally, we have argued that the type of correction given by Eq. (40) arises in a number of particle physics motivated models. Further examples arise in the limit where  $A\rho^{\nu} \gg 1$ . In particular, the case  $\nu = -1/3$  corresponds to the high-energy limit of the Gauss-Bonnet braneworld [18,42]. In this model, the R-S II scenario is generalized to include a Gauss-Bonnet combination of curvature invariants in the five-dimensional bulk action. More generally, effective Friedmann equations of the form  $H^2 \propto \rho^{\nu}$ , where  $\nu$  is arbitrary [43], can arise in models based on Hořava-

Witten theory compactified on a Calabi-Yau three-fold [44]. Generalized scaling solutions driven by corrections of this form were recently investigated for a variety of scalar field models [23].

Future directions involving the use of the duality properties of these models would include an extension of our analysis to negative potentials, thereby allowing us to link these classes of solutions with those arising in the cyclic/ekpyrotic scenario [45]. On the other hand, as we have seen, the more general corrections proposed in [27,40] that arise due to modifications of gravity on large scales can lead to an explanation of the present cosmic acceleration without introducing dark energy [46]. It would be interesting to investigate the impact that the duality transformations we have described have on such a scenario.

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