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Astrophysical constraints on scalar field models

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We use stellar structure dynamics arguments to extract bounds on the relevant parameters of two scalar field models: the putative scalar field mediator of a fifth force with a Yukawa potential and the new variable mass particle models. We also analyze the impact of a constant solar inbound acceleration, such as the one reported by the Pioneer anomaly, on stellar astrophysics. We consider the polytropic gas model to estimate the effect of these models on the hydrostatic equilibrium equation and fundamental quantities such as the central temperature. The current bound on the solar luminosity is used to constrain the relevant parameters of each model.

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I. INTRODUCTION

Scalar fields play a crucial role in particle physics and cosmology. Indeed, in inflation, the potential of a scalar field, the inflaton, acts as a dynamical vacuum energy that allows for an elegant solution of the initial conditions problem [1]. This prominent role of scalar fields is also evident in models to explain the late time accelerated expansion of the Universe in vacuum energy evolving and quintessence models [2], as well as in the Chaplygin gas dark energy/dark matter unification model [3]. Scalar fields have also been proposed as dark matter candidates [4]. Furthermore, it was recently proposed that a scalar field can be also at the source of the anomalous acceleration detected by the Pioneer spacecraft [5].

Scalar fields may also have astrophysical implications as, for instance, the mediating boson of a hypothetical fifth force, which should yield the measurable effects on celestial bodies, besides the other known forces of nature. Although the origin of these fields is rather speculative, they can all be described by the Yukawa potential, written here as $V_Y(r) = Ae^{-mr}/r$, where A is the coupling strength and m is the mass of the field, which sets the range of the interaction, $\lambda_Y \equiv m^{-1}$.

Models leading to a Yukawa-type potential can be found in widely distinct areas such as brane-world models, scalar-tensor theories of gravity, and in the study of topological defects. In brane-world models, one considers our Universe as a 3-dimensional world sheet embedded in a higher dimensional bulk space [6]. Symmetry considerations about the brane and its topological properties can be implemented to constrain the evolution of matter on the brane and gravity on the brane and in the bulk.

Brane-world models are rather trendy in cosmology and allow, for instance, for a solution for the hierarchy problem, whether the typical mass scale of the bulk is comparable with the electroweak breaking scale, $M_{\mathrm{EW}} \sim 1$ TeV. As a result, a tower of Kaluza-Klein (KK) massive tensorial perturbations to the metric appears. Following the KK dimensional reduction scheme, the masses (eigenvalues) of these gravitons (eigenfunctions) are ordered. Most brane-world models consider one first light mode with cosmological range, and hence all ensuing modes have submillimeter range.

Given the relevance of scalar fields, the search for bounds on the Yukawa parameters is crucial, so to exclude unviable models and achieve some progress in the study on those that appear feasible. Most experimental tests of a ‘‘fifth’’ force have been conducted in the vacuum; in the authors opinion, a study on the way this force should affect stellar equilibrium is lacking and constitutes one of the motivations for this work.

The bounds on parameters A and $\lambda_Y \equiv m^{-1}$ include the following (see [7,8], and references therein): laboratory experiments devised to measure deviations from the inverse-square law, sensitive to the range $10^{-2}m < \lambda_Y < 1m$, and constraining A to be smaller than 10^{-4}; nucleosynthesis bounds which imply that $A < 4 \times 10^{-1}$ for $\lambda_Y < 1m$; gravimetric experiments, sensitive in the range of $10m < \lambda_Y < 10^3 m$, suggesting $A < 10^{-3}$; satellite tests probing ranges of about $10^5 m < \lambda_Y < 10^7 m$, showing that $A < 10^{-5}$; and radiometric data of the Pioneer 10/11, Galileo, and Ulysses spacecrafts suggesting the existence of a new force with parameters $A = -10^{-3}$ and $\lambda_Y = 4 \times 10^{13} m$ [9], despite contrary claims [10,11]. It is striking that, for $\lambda_Y < 10^{-3} m$ and $\lambda_Y > 10^{14} m$, A is essentially unconstrained. Considerations on higher dimensional superstring motivated cosmological solutions hint that modifications to Newtonian gravity will occur in the short range region, $\lambda_Y < 10^{-3} m$ (cf. Fig. 2). This range also emerges if one assumes the observed vacuum energy density to be related with scalar or vector/tensor excitations [12].

In high energy physics, it is widely accepted that the mass of fermions results from the Higgs mechanism, in which a scalar field coupled to the right and left components of a particle acquires a vacuum expectation value (vev) that acts as a mass term in the Lagrangian density. This behavior of the Higgs-scalar field depends on the

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presence of a potential which acquires nonvanishing minima and, therefore, cannot evolve monotonically.

Can one relax this last feature of the Higgs mechanism? This has been the main motivation behind the variable mass particle proposal [13]. In these models it is assumed that there are some yet unknown fermions whose mass results not from a coupling to the Higgs boson, but to a quintessence-type scalar field with a monotonically decreasing potential. This potential has no minima, yet the coupling of the scalar field ϕ to matter can be included in an effective potential of the following form $V_{\rm eff}(\phi) = V(\phi) + \lambda n_\psi \phi$, where n_ϕ is the number density of fermionic variable mass particle (VAMP) models and λ is their Yukawa coupling. In this way, a minimum is developed and the ensuing vev is responsible for the particle's mass. Since in a cosmological setting the density depends on the scalar factor $a(t)$, this mass will vary on a cosmological time scale.

Before proceeding, note that the present analogy is not perfect. Indeed, while the Higgs mechanism relies on a spontaneous symmetry breaking, where the vev experiences a transition from a vanishing to a finite value, the VAMP idea assumes that *no* vev exists if the matter term of the effective potential is "switched off," whereas it is always nonvanishing when the latter is considered.

For definitiveness, we choose a potential of the quintessence-type form, $V(\phi) = u_0 \phi^{-p}$, where p is an integer, u_0 has dimensionality M^{p+4}. The effective potential $V_{\rm eff}(\phi)$, acquires a vev given by

$$\phi_0 \equiv \langle \phi \rangle = \left(\frac{p u_0}{\lambda n_\psi} \right)^{1/(1+p)}. \qquad (1)$$

Since the number density evolves as $n_\psi(t) = n_{\psi 0} a(t)^{-3}$, while ϕ evolves as $\phi(t) = \phi_0 a^{3/(1+p)}$, where ϕ_0 is the present value of the scalar field, then its mass is given by

$$m_\phi^2 \equiv \left[\frac{\partial^2 V}{\partial \phi^2} \right]_{\phi_0} = p(p+1) u_0 \phi_0^{-(p+2)} a^{-3(2+p)/(1+p)} \qquad (2)$$

and the mass of the VAMP fermions is [13]

$$m_\psi = \lambda \phi_0 a^{3/(1+p)}. \qquad (3)$$

Considering the evolution of the energy density contributions as a function of the redshift, z, one can compute the age of the Universe

$$t = \int_0^a \frac{da'}{a'} = H_0^{-1} \int_0^{1+z^{-1}} [1 - \Omega_0 + \Omega_{M0} x^{-1} + \Omega_V x^{(2+p)/(1+p)}]^{1/2} dx, \qquad (4)$$

where $H_0 = 100\ {\rm h\, Km\, s^{-1}\, Mpc^{-1}}$, $0.65 \leq h \leq 0.75$ is the observational uncertainty on the expansion of the Universe, Ω_{M0} is the energy density of normal baryonic plus dark matter, Ω_V is the energy density due to the potential driving ϕ, and Ω_0 is the total energy density of the Universe. The limiting case where $\Omega_0 = \Omega_V = 1$, $\Omega_{M0} = 0$ yields

$$t_0 = \frac{2}{2} H_0^{-1} (1 + p^{-1}). \qquad (5)$$

Assuming that the VAMP particles, ψ, were relativistic when they decoupled from thermal equilibrium, then it follows that [13]

$$m_\psi = 12.7 \Omega_{\psi 0} h^2 r_\psi^{-1} a^{3/2}\ {\rm eV}, \qquad (6)$$

where r_ψ is the ratio of $g_{\rm eff}$, the effective number of degrees of freedom of ψ, to g_{*f}, the total effective number of relativistic degrees of freedom at freeze-out. Furthermore, in terms of the Yukawa coupling,

$$u_0 = 1.02 \times 10^{-9} \frac{\Omega_{\psi 0}^2 h^4}{\lambda r_\psi}\ {\rm eV}^5, \qquad (7)$$

and thus

$$m_\phi = 1.00 \times 10^{-6} \frac{\lambda r_\psi}{\Omega_{\psi 0}^{1/2} h} a^{-9/4}\ {\rm eV}. \qquad (8)$$

However interesting, VAMP models have not been subjected to a more concrete analysis mostly due to a significant caveat, namely, the introduction of exotic ψ fermions, the VAMP particles, and the consequent derivation of the cosmologically relevant quantities in terms of their unknown relative density, $\Omega_{\psi 0}$. As a result of this somewhat arbitrary parameter, plus the unknown coupling constant with the scalar field and the potential strength, there is little one can do in order to draw definitive conclusions from VAMP models.

The present study attempts to overcome this drawback, by asserting that, aside from the hypothetical existence of the assumed exotic particles, all fermions couple to the quintessence scalar field. In this assumption one considers that fermionic matter couples mainly to the Higgs boson, so that the VAMP mass term is a small correction to the mass acquired by the Higgs mechanism. Thus, exotic VAMP particles are defined by their lack of the Higgs coupling.

The extension of the VAMP proposal to usual fermionic matter implies in a correction to the cosmologically relevant results obtained in Refs. [13,14]. However, these corrections should be negligible, as one assumes that the "Higgs to quintessence" coupling ratio is small, that is, at cosmological scales the exotic VAMP particles dominate the VAMP sector of usual fermions.

Notice that the vev resulting from the effective potential depends crucially on the particle number density n_ψ. Hence, it is logical to expect that the effect of this variable mass term in a stellar environment should be more significant than in the vacuum. This sidesteps the model from the usual cosmological scenario with a temporal variation, to the astrophysical case with an isotropic spatial dependence, thus allowing one to attain bounds on model parameters from known stellar physics observables.

Another object of this study concerns the Pioneer anomaly. This consists in an anomalous acceleration inbound to the sun and with a constant magnitude of $a_A \simeq (8.5 \pm 1.3) \times 10^{-10}\ \mathrm{m\,s^{-2}}$, revealed by the analysis of radiometric data from the Pioneer 10/11, Galileo, and Ulysses spacecrafts. Extensive attempts to explain this phenomena as a result of poor accounting of thermal and mechanical effects and/or errors in the tracking algorithms were presented, but are now commonly accepted as unsuccessful [9].

The two Pioneer spacecraft follow approximate opposite hyperbolic trajectories away from the solar system, while Galileo and Ulysses describe closed orbits. Given this and recalling that one has three geometrically distinct designs, an ''engineering'' solution for the anomaly seems not very plausible. Hence, although not entirely proven and even poorly understood (see, e.g., Ref. [5] and references within), the Pioneer anomaly, if a real physical phenomenon, should be the manifestation of a new force. This force, in principle, acts upon the sun itself, and thus lends itself to scrutiny under the scope of this study. It turns out that one can model its effect and constrain, even though poorly, the only parameter involved, the anomalous acceleration a_A. Notice that we do not intend to explain the anomaly, which is hence modeled simply by a constant term added to the usual Newtonian force. This study, however, shows that such a constant inbound acceleration is allowed by the central solar temperature constraint up to values well above the measured Pioneer anomaly.

II. THE POLYTROPIC GAS STELLAR MODEL

Realistic stellar models, arising from the assumptions of hydrostatic equilibrium and Newtonian gravity, rely on four differential equations, together with appropriate definitions [15–17]. This intricate system requires heavy-duty numerical integration with complex code designs, and an analysis of the perturbations induced by a variable mass is beyond the scope of the present study. Instead, we focus on the polytropic gas model for stellar structure: this assumes an equation of the state of the form $P = K\rho^{n+1/n}$, where n is the so-called polytropic index, that defines intermediate cases between isothermic and adiabatic thermodynamical processes, and K is the polytropic constant, defined below. This assumption leads to several scaling laws for the relevant thermodynamical quantities,

$$\begin{aligned} \rho &= \rho_c \theta^n(\xi) \qquad (a), \\ T &= T_c \theta(\xi) \qquad (b), \\ P &= P_c \theta(\xi)^{n+1} \qquad (c), \end{aligned} \tag{9}$$

where ρ_c, T_c, and P_c are the values of the density, temperature, and pressure at the center of the star.

The function θ, responsible for the scaling of P, ρ, and T, is a dimensionless function of the dimensionless variable ξ, related to the physical distance to the star's center by $r = \alpha\xi$, where

$$\alpha = \left[\frac{(n+1)K}{4\pi G}\rho_c^{(1-n)/n}\right]^{1/2}, \tag{10}$$

$$K = N_n G M^{(n-1)/n} R^{(3-n)/n}, \tag{11}$$

and

$$N_n = \left[\frac{n+1}{(4\pi)^{1/n}}\xi^{(3-n)/n}\left(-\xi^2\frac{d\theta}{d\xi}\right)^{(n-1)/n)}\right]^{-1}_{\xi_1}, \tag{12}$$

so that R is the star's radius, M its mass, and ξ_1, defined by $\theta(\xi_1) \equiv 0$, corresponds to the surface of the star (actually, this definition states that all quantities tend to zero as one approaches the surface). Its unperturbed value is $\xi_1^{(0)} = 6.89685$, as given in Ref. [15].

The function $\theta(\xi)$ obeys a differential equation arising from the hydrostatic equilibrium condition

$$\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho}\right) = -G\frac{dM(r)}{dr}, \tag{13}$$

the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) = -\theta^n. \tag{14}$$

We point out that the physical radius and mass of a star appear only in Eq. (11) so that the behavior of the scaling function $\theta(\xi)$ is unaffected by their values. Hence, the stability of a star is independent of its size or mass, and different types of stars correspond to different polytropic indices n. This kind of scale-independent behavior is related to the homology symmetry of the Lane-Emden equation.

The first solar model ever considered corresponds to a polytropic star with $n = 3$ and was studied by Eddington in 1926. Although somewhat incomplete, this simplified model gives rise to relevant constraints on the physical quantities.

III. RESULTS

A. Yukawa potential induced perturbation

In this section we look at the hydrostatic equilibrium equation with a Yukawa potential:

$$dP = -\frac{GM(r)[1 + Ae^{-mr}]}{r^2}\rho(r)dr, \tag{15}$$

which, after a small algebraic manipulation, implies that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{dP}{dr}\frac{r^2}{\rho}\right) = -4\pi G\rho[1 + Ae^{-mr}] + \frac{GM(r)Ame^{-mr}}{r^2}. \quad (16)$$

The last term is a perturbation to the usual Lane-Emden equation, obtained by substituting $r = \alpha\xi$ and $\rho = \rho_c\theta^n$; one gets

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n[1 + Ae^{-m\alpha\xi}] + \frac{M(\xi)Ame^{-\alpha m\xi}}{4\pi\rho_c\alpha^2\xi^2}. \quad (17)$$

Since

$$\alpha = \left(\frac{(n+1)K}{4\pi G}\rho_c^{(1-n)/n}\right)^{1/2} \quad (18)$$

and

$$M(\xi) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{(3-n)/2n}\xi^2\frac{d\theta}{d\xi}, \quad (19)$$

the second perturbation term can be written as

$$\frac{M(\xi)Ame^{-\alpha m\xi}}{4\pi\rho_c\alpha^2\xi^2} = -\left[\frac{(n+1)K}{4\pi G}\right]^{1/2}\rho_c^{(1-n)/2n}\frac{d\theta}{d\xi}Ame^{-\alpha m\xi}. \quad (20)$$

Furthermore, one has

$$K = N_nGM^{(n-1)/n}R^{(3-n)/n}, \qquad P_c = W_nGM^2/R^4, \qquad P_c = K\rho_c^{(n+1)/n}, \quad (21)$$

where R, M are the radius and the mass of the star and N_n, W_n are numbers which depend on n, and for which one takes the tabulated values, valid for the unperturbed equation. Hence,

$$\rho_c = \left(\frac{W_n}{N_n}\right)^{(1-n)/2(1+n)}\left(\frac{M}{R^3}\right)^{(1-n)/2n}. \quad (22)$$

Substituting into Eq. (21), one obtains

$$-\left[\frac{(n+1)K}{4\pi G}\right]^{1/2}\rho_c^{(1-n)/2n}\frac{d\theta}{d\xi}Ame^{-\alpha m\xi} = -\sqrt{\frac{n+1}{4\pi}}N_n^{n/(n+1)}W_n^{(1-n)/2(n+1)}R\frac{d\theta}{d\xi}Ame^{-\alpha m\xi}. \quad (23)$$

If one now defines the dimensionless quantities

$$C_n \equiv \left(\frac{n+1}{4\pi}\right)^{1/2}N_n^{n/(n+1)}W_n^{(1-n)/2(n+1)}, \quad (24)$$

and $\gamma \equiv mR$, the perturbed Lane-Emden equation acquires the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\frac{d\theta}{d\xi} = -\theta^n[1 + Ae^{-\alpha m\xi}] - \gamma AC_n\frac{d\theta}{d\xi}e^{-\alpha m\xi}. \quad (25)$$

One can eliminate α on the exponential term by writing $-\alpha m\xi = -\alpha mR\xi/R = -\gamma\xi/\xi_s$, where ξ_s corresponds to the value at the surface of the star. Since one must specify it prior to integration of the differential equation, one assumes that $\xi_s \simeq \xi_1$, the latter being the tabulated value for $\theta(\xi_1) = 0$. This is in good approximation, since the solar temperature is very small when compared at the surface to its central value, $T_s = 5.778 \times 10^3 K = 3.7 \times 10^{-4} T_c$. Hence, one gets

$$\frac{1}{\xi^2}\frac{d}{d\xi}\frac{d\theta}{d\xi} = -\theta^n[1 + Ae^{-\gamma\xi/\xi_1}] - \gamma AC_n\frac{d\theta}{d\xi}e^{-\gamma\xi/\xi_1}. \quad (26)$$

This notation makes clear that the perturbation vanishes for $A \rightarrow 0$. Notice that, if $m \sim R^{-1}$, then $\gamma \sim 1$. However, since m should arise from a fundamental theory, this would be the case only for stars of a particular size $m^{-1} \equiv \lambda_Y$. Since $\gamma \equiv mR = R/\lambda_Y$, it is clear that large stars ($R \gg \lambda_Y \rightarrow \gamma \gg 1$) are perturbed only within a small central region, where $\xi/\xi_1 \ll 1$.

It is also apparent that the Yukawa-type perturbation breaks the invariance under homologous transformations, since $\gamma = mR$ explicitly depends on R, the radius of the star. Hence, a bound on γ obtained from the luminosity or other observables is, for a given m, equivalent to a bound on the maximum size of a polytropic star of index n.

The boundary conditions are unaffected by the perturbation: from the definition $\rho = \rho_c\theta(\xi)$, one gets $\theta(0) = 1$; the hydrostatic equation Eq. (17), in the limit $\xi \rightarrow 0$, still imposes that $|d\theta/d\xi|_{\xi=0} = 0$.

By specifying A and γ, we may solve Eq. (17) numerically and get relative deviations from the unperturbed solution. Since a star's central temperature is given as a function of M and R and in terms of the mean molecular weight μ, the hydrogen mass H and the Boltzmann constant k by $T_c = Y_n\mu GM/R$, with

$$Y_n = \left(\frac{3}{4}\right)^{1/n}(H/k)N_n\left[-\frac{\xi}{3\frac{d\theta}{d\xi}}\right]_{\xi_1}^{1/n}, \quad (27)$$

one can compute the relative changes on T_c for different values of A and m. The results are presented in Fig. 1. The parameters were chosen so that the Yukawa interaction λ_Y ranges from $0.1R$ to $10R$: in the first case, the interaction is mainly located in the interior of the star, while in the second case it reaches outward and could be considered approximately constant within it. The Yukawa coupling was chosen so that the effect on T_c could be sizable [that is, of order $O(10^{-4})$].

This enables us to build the exclusion plot of Fig. 2, by imposing that $\Delta T_c < 4 \times 10^{-3}$, the accepted bound derived from solar luminosity constraints [15]. It is superimposed on the different bounds available [7]. One expects that a refinement of the calculus through numerical integration of the stellar dynamics differential equations and realistic assumptions for its behavior should yield more

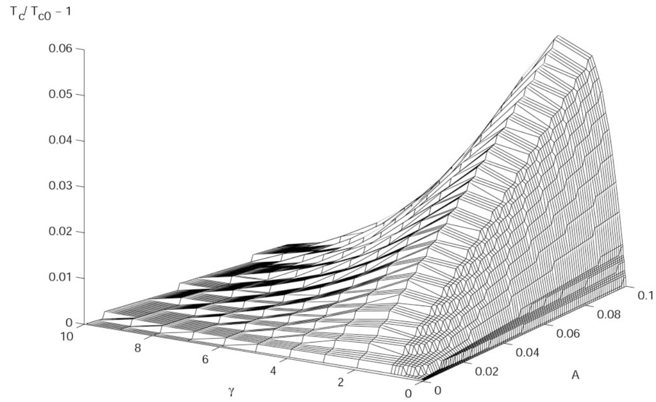

FIG. 1. Relative deviation from unperturbed central temperature $T_c/T_{c0} - 1$, for A ranging from 10^{-3} to 10^{-1}, and γ from 10^{-1} to 10.

interesting results [15,16]. Notice that the central temperature is not precisely known and it is clear that constraining its uncertainty below 10^{-4} would yield a larger exclusion region in the parameter space.

B. VAMP models

As previously discussed, we consider usual fermions with a variable mass term δm given by the coupling to the quintessence-type scalar field; its mass is then given by the usual Higgs-mechanism related term m plus this new VAMP term, a small contribution: $m = m_{\text{Higgs}} + \delta m$. We adopt, however, a "worst case" scenario, in which the electron mass is not mainly due by the Higgs mechanism plus a minor VAMP sector contribution, but fully given by the said VAMP component alone. This implies that the cosmological expectation value should be weakly perturbed, so that the electron mass does not undergo large variations.

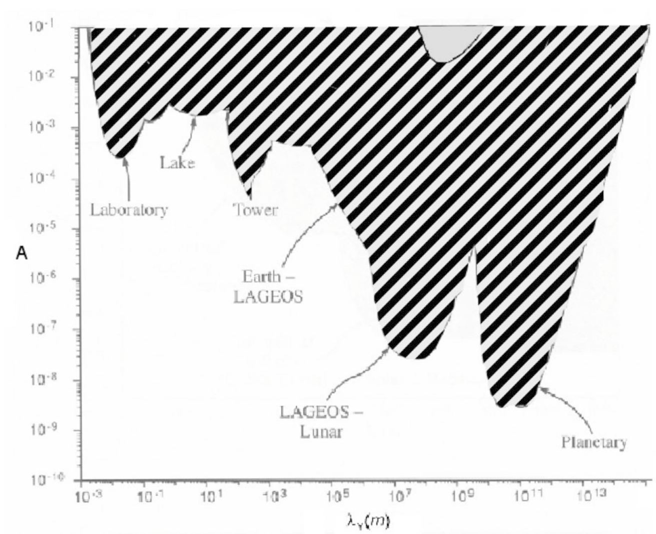

FIG. 2. Exclusion plot for the relative deviation from unperturbed central temperature T_c, for A ranging from 10^{-3} to 10^{-1}, and γ from 10^{-1} to 10 (tip at the top), superimposed on the available bounds [7].

It can be shown (see the Appendix) that this variable term leads to a geodesic deviation equation of the form

$$\ddot{x}^a = \left[\Gamma^a_{bc} + \frac{\alpha_a}{2\alpha} g_{bc}\right]\dot{x}^b \dot{x}^c + \frac{2\dot{\alpha}}{\alpha}\dot{x}^a. \tag{28}$$

Assuming isotropy, one obtains the acceleration

$$\begin{aligned}\vec{a} &= \vec{a}_{\text{Newton}} + \frac{m'}{m} g_{bc}\dot{x}^b\dot{x}^c\vec{u}_r - \frac{\dot{m}}{m}\vec{v} \\ &= \vec{a}_{\text{Newton}} + \frac{\phi'}{\phi} g_{bc}\dot{x}^b\dot{x}^c\vec{u}_r - \frac{\dot{\phi}}{\phi}\vec{v},\end{aligned} \tag{29}$$

where the prime and the dot denote derivatives with respect to r and t, respectively. Considering the Newtonian limit $g_{ab} = \text{diag}(1, -1, -1, -, 1)$ so that $g_{bc}\dot{x}^b\dot{x}^c = 1 - v^2 \simeq 1$, one finds a radial, anomalous acceleration plus a time-dependent drag force:

$$a_A = \frac{\phi'}{\phi} < 0, \qquad a_D = -\frac{\dot{\phi}}{\phi} < 0. \tag{30}$$

Notice that this radial sun bound acceleration has the qualitative features for a possible anomalous acceleration measured by the Pioneer probes [9].

The time-dependent component should vary on cosmological time scales and can thus be absorbed in the usual Higgs mass term. Hence, one considers only the perturbation to the Lane-Emden equation given by the radial force, $a_A \simeq \phi'/\phi$:

$$dP = [-GM(r) + a_A r^2]\frac{\rho dr}{r^2}, \tag{31}$$

which translates into

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho - \frac{c^2}{\phi(r)}\left[\frac{2\phi'(r)}{r}\right. \\ \left. + \phi''(r) - \frac{\phi'^2(r)}{\phi(r)}\right].\end{aligned} \tag{32}$$

Defining the dimensionless quantities

$$\begin{aligned}U &\equiv \frac{GM}{Rc^2} = 2.12 \times 10^{-6}, \\ C_n^{-1} &\equiv (n+1)N_n^{n/(n+1)}W_n^{1/(n+1)},\end{aligned} \tag{33}$$

where

$$W_n = \frac{1}{4\pi(n+1)(\frac{d\theta}{d\xi})^2_{\xi_1}}, \tag{34}$$

one obtains the perturbed Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n(\xi) - \frac{C_n}{U}\frac{1}{\phi(\xi)}\left[\phi''(\xi) + \frac{2}{\xi}\phi'(\xi) - \frac{\phi'^2(\xi)}{\phi(\xi)}\right]. \quad (35)$$

The Klein-Gordon–type equation for the scalar field, written in terms of the ξ variable is, inside the star, given by

$$\frac{1}{\alpha^2}\left[\phi''(\xi) + \frac{2}{\xi}\phi'(\xi)\right] = -pu_0\phi^{-(p+1)}(\xi) + \frac{\lambda\rho_c}{\mu}\theta^n(\xi), \quad (36)$$

where now the prime denotes derivation with respect to ξ. The parameter μ is the mean molecular weight and it is assumed that there are two electrons per molecule, that is, that the star is composed by "hydrogen" with a molecular weight μ.

Beyond the star, in the Klein-Gordon equation one has the coupling to the constant number density of fermions in the vacuum $n_\psi \equiv n_V = 3m^{-3}$,

$$\frac{1}{\alpha^2}\left[\phi''(\xi) + \frac{2}{\xi}\phi'(\xi)\right] = -pu_0\phi^{-(p+1)}(\xi) + \lambda n_V. \quad (37)$$

These equations constitute a set of coupled differential equations for $\phi(\xi)$ and $\theta(\xi)$ and, of course, the continuity of ϕ across the surface of the star must be addressed. A complete derivation can be found in Ref. [5]. In here, α and ρ_c depend on W_n, N_n and related quantities, which are evaluated after the solution $\theta(\xi)$ is known, together with M and R. Hence, one considers their unperturbed values for the sun: $\alpha = R/\xi_1^{(0)} = 1.009 \times 10^8$ m and $\rho_c = 1.622 \times 10^5\ \mathrm{kg\,m^{-3}}$. Moreover, current solar estimatives indicate that $\mu \simeq 0.62m_p$, m_p being the hydrogen atomic mass [15].

For simplicity, we deal only with the case of $p = 1$, as in Ref. [13]. Instead of the potential strength u_0, we work with the potential energy density $\Omega_V < 1$ of the scalar field. Before presenting the obtained numerical solutions, we develop the expression for $\Omega_V = V(\phi_c)/\rho_{\mathrm{crit}}$, with $\rho_{\mathrm{crit}} \simeq 1.88 \times 10^{-29}\ \mathrm{h^2\,g\,cm^{-3}}$; in what follows we chose $h = 0.71$. Therefore

$$V(\phi_c) = u_0\left(\sqrt{\frac{u_0}{\lambda n_V}}\right)^{-1} + \lambda n_V\sqrt{\frac{u_0}{\lambda n_V}} = 2\sqrt{u_0\lambda n_V}, \quad (38)$$

which implies

$$u_0 = \frac{\Omega_V^2\rho_{\mathrm{crit}}^2}{4\lambda n_V}. \quad (39)$$

We now rescale the scalar field so to work with a dimensionless quantity $\Phi \equiv \phi/\phi_c^*$, where ϕ_c^* is the cosmological vev obtained by assuming as reference values $\lambda = \Omega_V = 1$,

$$\phi_c^* = \frac{\rho_{\mathrm{crit}}}{2n_V}. \quad (40)$$

Hence, the cosmological vev for general λ and u_0 is given by

$$\phi_c = \frac{\Omega_V\rho_{\mathrm{crit}}}{2\lambda n_V} = \frac{\Omega_V}{\lambda}\phi_c^*, \quad (41)$$

so that $\Phi_c = \phi_c/\phi_c^* = \Omega_V/\lambda$. Thus, the Klein-Gordon equation has the following form:

(i) Inside the star

$$\Phi''(\xi) + \frac{2}{\xi}\Phi'(\xi) = -\frac{2\alpha^2 n_V^2\Omega_V^2}{\rho_{\mathrm{crit}}\lambda}\Phi^{-2}(\xi) + \frac{2\alpha^2\lambda n_V}{\mu}\frac{\rho_c}{\rho_{\mathrm{crit}}}\theta^3(\xi). \quad (42)$$

(ii) In the vacuum

$$\Phi''(\xi) + \frac{2}{\xi}\Phi'(\xi) = -\frac{2\alpha^2 n_V^2\Omega_V^2}{\rho_{\mathrm{crit}}\lambda}\Phi^{-2}(\xi) + \frac{2\alpha^2\lambda n_V}{\rho_{\mathrm{crit}}}n_V. \quad (43)$$

The perturbed Lane-Emden equation assumes the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n(\xi) - \frac{C_n}{U}\frac{1}{\Phi(\xi)}\left[\Phi''(\xi) + \frac{2}{\xi}\Phi'(\xi) - \frac{\Phi'^2(\xi)}{\Phi(\xi)}\right]. \quad (44)$$

Since the perturbation on $\theta(\xi)$ is shown to be small, one can take the unperturbed function $\theta(\xi) \simeq \theta_0(\xi)$ when solving Eqs. (43) and (44), and then introduce the obtained solution for the scalar field $\Phi(\xi)$ in Eq. (45). First, one assumes that the scalar field is given by its cosmological vev perturbed by a small "astrophysical," $\Phi_a(\xi)$, contribution, $\Phi(\xi) = \Omega_V/\lambda + \Phi_a(\xi)$. Hence, the Klein-Gordon equation becomes

$$\Phi_a''(\xi) + \frac{2}{\xi}\Phi_a'(\xi) \approx \frac{2\alpha^2\lambda n_V}{\rho_{\mathrm{crit}}}\frac{\rho_c}{\mu}\theta^3(\xi) - \frac{2\alpha^2\lambda^2 n_V^2}{\rho_{\mathrm{crit}}}\left[1 - \frac{2\lambda\Phi_a(\xi)}{\Omega_V}\right], \quad (45)$$

inside the star, and

$$\Phi_a''(\xi) + \frac{2}{\xi}\Phi_a'(\xi) \approx \frac{2\alpha^2\lambda n_V^2}{\rho_{\mathrm{crit}}} - \frac{2\alpha^2\lambda^2 n_V^2}{\rho_{\mathrm{crit}}}\left[1 - \frac{2\lambda\Phi_a(\xi)}{\Omega_V}\right], \quad (46)$$

in the outer region.

Substituting by the sun values $\alpha = R/\xi_1^{(0)} = 1.009 \times 10^8$ m, $\rho_c = 1.622 \times 10^5\ \mathrm{kg\,m^{-3}}$, $\mu \approx 0.62m_p$, $n_\psi = 3\ \mathrm{m^{-3}}$, one gets

$$\Phi_a''(\xi) + \frac{2}{\xi}\Phi_a'(\xi) \approx 6.8\lambda\left[4.79 \times 10^{32}\theta^3(\xi) - \lambda\left(1 - \frac{2\lambda\Phi_a(\xi)}{\Omega_V}\right)\right], \quad (47)$$

inside the star, and

$$\Phi_a''(\xi) + \frac{2}{\xi}\Phi_a'(\xi) \approx 6.8\lambda\left[1 - \lambda\left(1 - \frac{2\lambda\Phi_a(\xi)}{\Omega_V}\right)\right], \quad (48)$$

in the outer region.

Numerical integration of these equations enables the computation of the central temperature's relative deviation; as boundary conditions for $\Phi(\xi)$ it is imposed that both the field and its derivative vanish beyond the solar system (about $10^5 AU$). One can see by inspection that the solution $\Phi_a(\xi)$ is practically the same, regardless of the value for Ω_V, as one always assumes $\Phi_a(\xi) \ll \Omega_V/\lambda$. However, it is highly sensitive to λ.

Also, one must verify the validity of the condition $\Phi_a \ll \Omega_V/\lambda$ for chosen Ω_V and λ values. For this, note that $\Phi_a(\xi)$ evolves as λ^{-1} and, as stated above, it is fairly independent of Ω_V to a very good approximation. Hence, choosing a smaller value for Ω_V amounts to reducing λ, both by lowering the field $\Phi_a(\xi)$ and increasing its upper limit, Ω_V/λ.

By the same token, each value of Ω_V corresponds to a maximum allowed value for the coupling, $\lambda_{\max}(\Omega_V)$. One then uses these values to numerically obtain to first order solutions for $\theta(\xi)$ and $\phi(\xi)$, for say $\Omega_V = 0.1$, 0.4 and 0.7, as presented in Figs. 3 and 4. This enables one to extract the variation of the central temperature, T_c. The maximum allowed values for λ (depending on the chosen Ω_V) are the following: for $\Omega_V = 0.1$, $\lambda_{\max} = 1.24 \times 10^{-14}$; for $\Omega_V = 0.4$, $\lambda \leq 2.45 \times 10^{-14}$; and for $\Omega_V = 0.7$, $\lambda \leq 3.3 \times 10^{-14}$. The limiting case $\Omega_V = 1$ yields $\lambda \leq 3.93 \times 10^{-14}$.

None of the presented curves exceed the maximum allowed variation for T_c of 0.4%: the maximum of $\delta T_c = 2.82 \times 10^{-8}$ occurs for $\Omega_V = 0.7$, $\lambda = 2.82 \times 10^{-14}$. Hence, the luminosity constraint is always respected and the bound one must respect is $\lambda < 10^{-14}$.

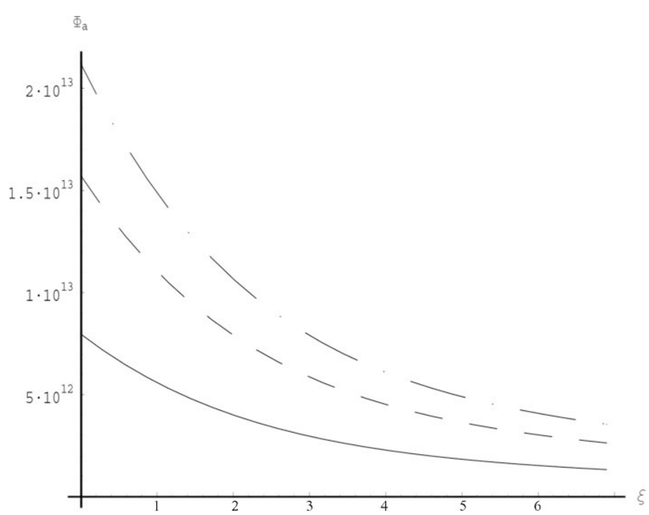

FIG. 3. $\Phi_a(\xi)$ field profile, for the ($\Omega_V = 0.1$, $\lambda = 1.24 \times 10^{-14}$) (solid line), ($\Omega_V = 0.4$, $\lambda = 2.45 \times 10^{-15}$) (dashed line), and ($\Omega_V = 0.7$ and $\lambda = 3.3 \times 10^{-14}$) (dash-dotted line) cases.

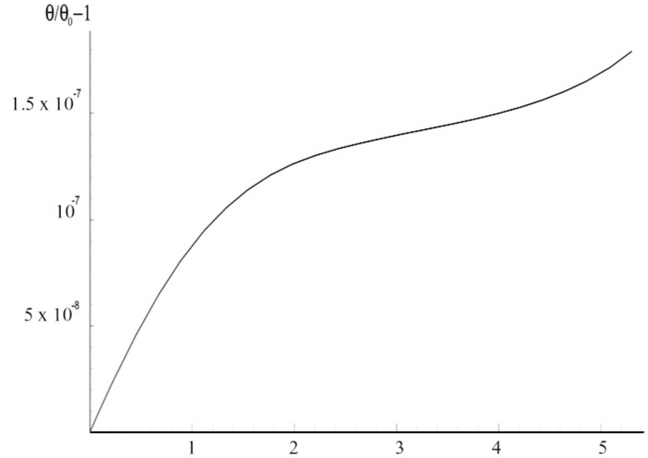

FIG. 4. Perturbed solutions for $\theta(\xi) \sim T(\xi)$, for the $\Omega_V = 0.1$, $\lambda = 1.06 \times 10^{-14}$ case; other solutions overlap.

C. The Pioneer anomaly

As outlined before, we now aim to establish the effect of anomalous, constant inbound acceleration a_A superimposed on the usual Newtonian acceleration. No particular model is considered to explain its origin, which would, in principle, introduce further corrections (see, e.g., the Yukawa negative coupling model [18] and a "exotic" scalar field model [5]). We plot the results for variable a_A, and identify the reported Pioneer anomaly with the case $a_A = a_P \sim 8.5 \times 10^{-10}\ \mathrm{m\,s^{-2}}$.

Following the method discussed above, we look at the hydrostatic equilibrium equation for a constant perturbation:

$$dP = [-GM(r) + a_A r^2]\frac{\rho dr}{r^2}, \quad (49)$$

and therefore

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho + \frac{2a_A}{r}. \quad (50)$$

The last term is a perturbation to the usual Lane-Emden equation, which is given by

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n(\xi) + \frac{a_A}{2\pi G\rho_c\alpha\xi}. \quad (51)$$

The factor in the perturbation term can be written as

$$2\pi G\rho_c\alpha = [(n+1)GK\pi]^{1/2}\rho_c^{(n+1)/2n}. \quad (52)$$

Following the same steps as before, and defining the dimensionless quantities

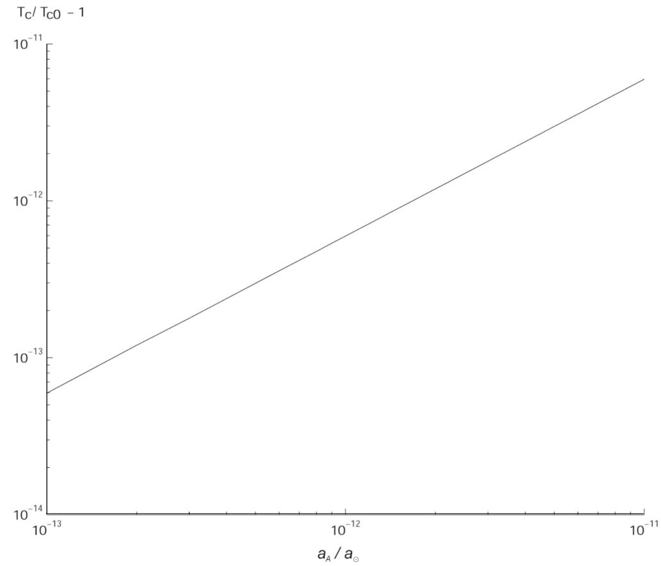

FIG. 5. Relative deviation from the unperturbed central temperature, for a_A ranging from $10^{-13}a_\odot$ to $10^{-11}a_\odot$.

$$C_n^{-1} \equiv \sqrt{(n+1)\pi W_n}, \quad \beta \equiv \frac{a_A R^2}{GM} \equiv \frac{a_A}{a_\odot} = 3.65 \times 10^{-3} a_A, \tag{53}$$

one obtains the perturbed Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi}\right] = -\theta^n(\xi) + \beta C_n \frac{1}{\xi}. \tag{54}$$

As previously, the boundary conditions for this modified Lane-Emden equation are unaffected by the perturbation: from the definition $\rho = \rho_c \theta(\xi)$, one gets $\theta(0) = 1$; the hydrostatic equation (50), in the limit $\xi \longrightarrow 0$, still imposes $|d\theta/d\xi|_{\xi=0} = 0$. In the present case one has only one model parameter, β, which can be constrained by the same luminosity bounds as before. Solutions for this equation with β varying from $10^{-13}a_\odot$ to $10^{-11}a_\odot$ (the reported value is of magnitude $a_P \sim 3 \times 10^{-12} a_\odot$) enable one to compute the relative central temperature deviation as a function of β, as presented in Fig. 5.

From Fig. 5, one concludes that the relative deviation of the central temperature scales linearly with a_A, as $\delta T_c \sim a_A/a_\odot$. Thus, the bound $\delta T_c < 4 \times 10^{-3}$ is satisfied for values of this constant anomalous acceleration up to $a_{\max} \sim 10^{-4} a_\odot$. The reported value is then well within the allowed region and has a negligible impact on the astrophysics of the sun.

IV. CONCLUSIONS

In this work we have studied solutions of the perturbed Lane-Emden equation for three different cases, related to relevant scalar field models. We obtain bounds on the parameter space of each model from solar luminosity constraints.

The exclusion plot obtained for a Yukawa perturbation produces no new exclusion region in the parameter space A-λ_Y. This results from the low accuracy to which the central temperature T_c is known, when compared to the sensibility of dedicated experiments [7]. Therefore, it is fair to expect that improvements in the knowledge of the sun's central temperature could yield a new way of exploring the available range of parameters.

For the VAMP case we have shown that the Yukawa coupling of the VAMP sector is constrained to be $\lambda < 10^{-14}$, and that the solar luminosity constraint is always respected. The numerical analysis reveals that Ω_V and λ should satisfy the relation $\lambda/\Omega_V < 10^{-13}$.

It has also been shown that the scalar field acquires its "cosmological" value just outside the star, leading to no differential shifts of the particle masses in the vacuum; thus, there is no observable variation of fermionic masses and hence no violation of the weak equivalence principle.

Finally, we found that an anomalous, constant acceleration such as the one reported on the Pioneer 10/11 spacecraft is allowed within the sun for values up to $10^{-4}a_\odot$, thus clearly stating that the observed value $a_P \sim 10^{-12} a_\odot$ has negligible impact on the central temperature and other stellar parameters.

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APPENDIX

In order to encompass models with a variable mass, we consider the generalized Lagrangian density

$$L = \sqrt{\alpha(x)}\sqrt{g_{ab}\dot{x}^a\dot{x}^b}. \tag{A1}$$

The function α is, in the homogeneous and time-independent case, identified with the square of the rest mass. We now deduce the Euler-Lagrange equation for the timelike geodesics. Notice that there is no right side terms because τ is an affine parameter:

$$
\begin{aligned}
0 &= \frac{\partial L^2}{\partial x^c} - \frac{d}{d\tau}\frac{\partial L^2}{\partial \dot{x}^c} = \alpha_{,c} g_{ab}\dot{x}^a\dot{x}^b + \alpha g_{ab,c}\dot{x}^a\dot{x}^b - \frac{d}{d\tau}(\alpha^2 g_{ac}\dot{x}^a) \\
&= (\alpha_{,c} g_{ab} + \alpha g_{ab,c})\dot{x}^a\dot{x}^b - 2\dot{\alpha} g_{ac}\dot{x}^a - 2\alpha g_{ac,b}\dot{x}^a\dot{x}^b - 2\alpha g_{ac}\dot{x}^a \\
&= \left(\frac{\alpha_{,c}}{\alpha} g_{ab} + g_{ab,c}\right)\dot{x}^a\dot{x}^b - \frac{2\dot{\alpha}}{\alpha} g_{ac}\dot{x}^a - 2g_{ac,b}\dot{x}^a\dot{x}^b - 2g_{ac}\ddot{x}^a \\
&= \left(\frac{\alpha_{,c}}{\alpha} g_{ab} + g_{ab,c} - 2g_{ac,b}\right)\dot{x}^a\dot{x}^b - \frac{2\dot{\alpha}}{\alpha} g_{ac}\dot{x}^a - 2g_{ac}\ddot{x}^a \\
&= g_{ac}\ddot{x}^a + \left[\frac{1}{2}(g_{ac,b} + g_{bc,a} - g_{ab,c}) - \frac{\alpha_{,c}}{2\alpha} g_{ab}\right]\dot{x}^a\dot{x}^b + 2\frac{\dot{\alpha}}{\alpha} g_{ac}\dot{x}^a \\
&= g^{cd} g_{ab}\ddot{x}^a + \left[\frac{1}{2} g^{cd}(g_{ac,b} + g_{bc,a} - g_{ab,c}) - \frac{\alpha_{,c}}{2\alpha} g^{cd} g_{ab}\right]\dot{x}^a\dot{x}^b + 2\frac{\dot{\alpha}}{\alpha} g^{cd} g_{ac}\dot{x}^a \\
&\rightarrow \ddot{x}^a + \left[\Gamma^a_{bc} - \frac{\alpha_a}{2\alpha} g_{bc}\right]\dot{x}^b\dot{x}^c + \frac{2\dot{\alpha}}{\alpha}\dot{x}^a = 0.
\end{aligned} \tag{A2}
$$

In the isotropic, Newtonian case, one has $\alpha = \alpha(r, \tau \sim t)$, and thus

$$
\vec{a} = \vec{a}_N + \frac{\alpha'}{2\alpha} g_{bc}\dot{x}^b\dot{x}^c\vec{u}_r - \frac{2\dot{\alpha}}{\alpha}\vec{v}, \tag{A3}
$$

where the prime denotes derivative with respect to the radial coordinate.

Using $g_{ab} = \mathrm{diag}(1, -1, -1, -, 1)$ so that $g_{bc}\dot{x}^b\dot{x}^c = 1 - v^2 \simeq 1$, one obtains a radial anomalous acceleration plus a time-dependent drag force:

$$
a_A = \frac{\alpha_{,r}}{2\alpha} < 0, \qquad a_D = -\frac{2\dot{\alpha}}{\alpha} < 0. \tag{A4}
$$

[1] See, e.g., A. Linde, hep-th/0402051; K. A. Olive, Phys. Rev. **190**, 307 (1990).

[2] O. Bertolami, Nuovo Cimento Soc. Ital. Fis. **93B**, 36 (1986); Fortschr. Phys. **34**, 829 (1986); M. Ozer and M. O. Taha, Nucl. Phys. **B287**, 776 (1987); B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988); Ap. J. Lett. **325**, 117 (1988); C. Wetterich, Nucl. Phys. **B302**, 668 (1988); R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); P. G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998); I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 986 (1999); P. Binétruy, Phys. Rev. D **60**, 063502 (1999); J. E. Kim, J. High Energy Phys. 05 (1999) 022; J. P. Uzan, Phys. Rev. D **59**, 123510 (1999); T. Chiba, Phys. Rev. D **60**, 083508 (1999); L. Amendola, Phys. Rev. D **60**, 043501 (1999); O. Bertolami and P. J. Martins, Phys. Rev. D **61**, 064007 (2000); A. Albrecht and C. Skordis, Phys. Rev. Lett. **84**, 2076 (2000); N. Banerjee and D. Pavón, Classical Quantum Gravity **18**, 593 (2001); A. A. Sen, S. Sen, and S. Sethi, Phys. Rev. D **63**, 107501 (2001); M. C. Bento, O. Bertolami, and N. C. Santos, Phys. Rev. D **65**, 067301 (2002).

[3] A. Y. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B **511**, 265 (2001); M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **66**, 043507 (2002); **67**, 063003 (2003); Phys. Lett. B **575**, 172 (2003); O. Bertolami, A. A. Sen, S. Sen, and P. T. Silva, Mon. Not. R. Astron. Soc. **353**, 329 (2004); N. Bilic, G. B. Tupper, and R. D. Viollier, Phys. Lett. B **535**, 17 (2002).

[4] J. A. Friedman and B. A. Gradwohl, Phys. Rev. Lett. **67**, 2926 (1991); J. McDonald, Phys. Rev. D **50**, 3637 (1994); O. Bertolami and F. M. Nunes, Phys. Lett. B **452**, 108 (1999); P. J. E. Peebles, astro-ph/0002495; J. Goodman, astro-ph/0003018; M. C. Bento, O. Bertolami, R. Rosenfeld, and L. Teodoro, Phys. Rev. D **62**, 041302 (2000); M. C. Bento, O. Bertolami, and R. Rosenfeld, Phys. Lett. B **518**, 276 (2001); T. Matos and L. A. Ureña-Lopez, Phys. Lett. B **538**, 246 (2002).

[5] O. Bertolami and J. Páramos, Classical Quantum Gravity **21**, 3309 (2004).

[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999); G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000); R. Gregory, V. A. Rubakov, and S. M. Sibiryakov, Phys. Rev. Lett. **84**, 5928 (2000); I. I. Kogan, astro-ph/0108220.

[7] E. Fischbach and C. L. Talmadge, *The Search for Non-Newtonian Gravity* (Springer, New York, 1999).

[8] O. Bertolami and F. M. Nunes, Classical Quantum Gravity **20**, L61 (2003).

[9] J. D. Anderson, P. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto, and S. G. Turyshev, Phys. Rev. D **65**, 082004 (2002).
[10] J. I. Katz, Phys. Rev. Lett. **83**, 1892 (1999).
[11] E. M. Murphy, Phys. Rev. Lett. **83**, 1890 (1999).
[12] S. R. Beane, Gen. Relativ. Gravit. **29**, 945 (1997); O. Bertolami, Classical Quantum Gravity **14**, 2785 (1997).
[13] G. W. Anderson and S. M. Carroll, astro-ph/9711288.
[14] U. França and R. Rosenfeld, Phys. Rev. D **69**, 063517 (2004).
[15] V. B. Bhatia, *Textbook of Astronomy and Astrophysics with Elements of Cosmology* (Narosa Publishing House, Delhi, 2001).
[16] T. Padmanabhan, *Theoretical Astrophysics: Stars and Stellar Systems* (Cambridge University Press, Cambridge, 2001).
[17] J. N. Bahcall, Phys. Rev. D **33**, 47 (2000).
[18] J. D. Anderson, P. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto, and S. G. Turyshev, Phys. Rev. Lett. **81**, 2858 (1998).