Back-reaction and the trans-Planckian problem of inflation reexamined

Robert H. Brandenberger*

Department of Physics, McGill University, Montréal, Québec, H3A 2T8, Canada
Perimeter Institute for Theoretical Physics, Waterloo, N2J 2W9, Canada
Department of Physics, Brown University, Providence, Rhode Island 02912, USA

Jérôme Martin†

Institut d'Astrophysique de Paris, GReCO, 98bis Boulevard Arago, 75014 Paris, France
(Received 22 October 2004; published 6 January 2005)

It has recently been suggested that Planck scale physics may effect the evolution of cosmological fluctuations in the early stages of cosmological inflation in a nontrivial way, leading to an excited state for modes whose wavelength is super-Planck but sub-Hubble. In this case, the issue of how this excited state back-reacts on the background space-time arises. In fact, it has been suggested that such back-reaction effects may lead to tight constraints on the magnitude of possible deviations from the usual predictions of inflation. In this note we discuss some subtle aspects of this back-reaction issue and point out that rather than preventing inflation, the back-reaction of ultraviolet fluctuations may simply lead to a renormalization of the cosmological constant driving inflation.

DOI: 10.1103/PhysRevD.71.023504 PACS numbers: 98.80.Cq

I. INTRODUCTION

The most important success of the inflationary Universe scenario [1] is that it provides a causal mechanism for the origin of the observed density fluctuations and microwave background anisotropies [2] (see also Refs. [3,4]). Key to this success is the fact that the physical wavelength corresponding to a fixed comoving scale is exponentially stretched during the period of inflation. Thus, provided that the period of inflation lasts sufficiently long, fluctuations on scales of cosmological interest today originate on sub-Hubble scales during inflation. Since inflation redshifts all initial classical fluctuations, it is reasonable to assume that matter starts out in a quantum vacuum state (in the frame set by the background cosmology, see, e.g., Ref. [5] for a discussion). Each fluctuation mode thus starts out in its vacuum state at the time that the initial conditions are set up (e.g., the beginning of the period of inflation), it undergoes quantum vacuum oscillations while the wavelength is smaller than the Hubble radius, but freezes out when the wavelength equals the Hubble radius (see, e.g., Ref. [6] for comprehensive reviews of the theory of cosmological fluctuations). Subsequently, the quantum state of the fluctuations undergoes squeezing on super-Hubble scales, and reenters the Hubble radius during the post-inflationary Friedman-Lemaître-Robertson-Walker (FLRW) phase as a highly squeezed and effectively classical state (see, e.g., Ref. [7] for a discussion of the classicalization of the state).

However, as first pointed out in Ref. [8], this success of inflationary cosmology leads to an important conceptual problem, the *trans-Planckian problem*. Since the period of inflation in typical scalar-field-driven inflationary models is very long (see, e.g., Ref. [9] for a review), the scales of cosmological interest today are not only sub-Hubble, but in fact sub-Planck at the beginning of inflation. Thus, the formalism used to calculate the evolution of fluctuations is in fact not justified. It is possible that the unknown trans-Planckian physics will lead to an evolution of the fluctuations on sub-Planckian scales which from the point of view of free scalar-field theory coupled to general relativity looks nonadiabatic. In Ref. [10] (see also Ref. [11]), toy models for such an evolution were constructed making use of modified dispersion relations which were assumed to describe the physics on sub-Planckian scales. Since the time interval spent in the trans-Planckian domain may depend on the wavelength, such models may lead to changes in the spectral index of the fluctuations.

Subsequently, other approaches to the trans-Planckian problem were suggested, e.g., analyses based on space-space noncommutativity [12], space-time noncommutativity [13], minimal length uncertainty relation [14], effective field theory [15], minimal trans-Planckian assumptions (starting each mode in some vacuum state at the time when its wavelength equals the Planck length) [16,17], and boundary renormalization group (RG) flow [18]. These analyses typically give that trans-Planckian corrections to the predictions for cosmological fluctuations are proportional to $(H_{\rm inf}/m_{\rm C})^n$ where $H_{\rm inf}$ is the Hubble parameter during inflation, $m_{\rm C}$ a new scale at which nonstandard physical effects show up and n a number which depends on the initial state assumed at the time of "creation." On the other hand, analyses based on modified dispersion relations give a correction proportional to the time spent by the physical modes in the region where adiabaticity is violated, see, e.g., [19] for a recent review.

*Electronic address: rhb@het.brown.edu
†Electronic address: jmartin@iap.fr

Tanaka [20] and Starobinsky [21] and Porrati [22] (see also Ref. [23]) have, however, raised an important concern regarding the possible amplitude of trans-Planckian corrections (see also Ref. [24] for unrelated concerns): if trans-Planckian physics leads to an excited state for fluctuation modes on sub-Hubble but super-Planck scale during the period of inflation, the back-reaction of these excitations on the background must be considered. In the case where the trans-Planckian effects are modeled by a modified dispersion relation, a simple estimate of the energy density carried in these ultraviolet modes

$$\langle \rho \rangle_{\rm UV} = \int_{k_{\rm phys}=H_{\rm inf}}^{k_{\rm phys}=m_{\rm C}} {\rm d}^3 k_{\rm phys} \omega_{\rm phys}(k_{\rm phys}) n_{k_{\rm phys}}, \qquad (1)$$

where $k_{\rm phys}$ is the physical wave number, $n_{k_{\rm phys}}$ is the occupation number, and $\omega_{\rm phys}$ is the frequency of the mode, leads to the conclusion that $\langle \rho \rangle_{\rm UV}$ will exceed the background density unless $n_{k_{\rm phys}}$ is smaller than $(H_{\rm inf}/m_{\rm Pl})^2$ [see also Eq. (7) below], thus constraining the possible effects of trans-Planckian physics on the spectrum of fluctuations (in the above, we have assumed that the ultraviolet cutoff is the usual Planck scale $m_{\rm Pl}$ which is not mandatory at all).

In this article, we point out some subtleties with the above back-reaction argument which may change the conclusions dramatically. If we assume, as is conventionally done in analyzing quantum fields in curved space-time, that the ultraviolet cutoff scale is time independent in terms of physical length, then in an exponentially expanding background geometry, the contribution of ultraviolet (i.e., sub-Hubble) modes to the energy density is constant in time, as, in fact, follows directly from the time-translation invariance of the physics. Moreover, the corresponding equation of state, due to the fact that the dispersion relation is modified, can strongly differ from that of ultrarelativistic particles and, as we demonstrate below, tends to that of the vacuum. Hence, our main conclusion is that, instead of preventing inflation, the ultraviolet modes may in fact simply renormalize the value of the cosmological constant driving inflation.

This article is organized as follows. In the next section, Sec. II, we describe the arguments that have been put forward to claim that there is a back-reaction problem and we criticize them. Then, in Sec. III, we present an explicit calculation of the equation of state of a scalar field with a modified dispersion relation. We show that the ultraviolet modes possess an equation of state which is almost that of a cosmological constant. Finally, in Sec. IV, we point out problems with our approach, indicate directions for further investigations and present our general conclusions.

II. THE BACK-REACTION PROBLEM

As explained in the introduction, it is possible that the trans-Planckian effects affect the standard inflationary predictions. In this paper, for the sake of illustration, we model physics at very short scales by a nonlinear dispersion relation $\omega_{\rm phys}(k_{\rm phys})$. For wave numbers such that $k_{\rm phys} \ll m_{\rm C}$ where $m_{\rm C}$ is a new scale at which nonstandard physical effects show up, the dispersion relation is linear for obvious phenomenological reasons. On the contrary, for modes such that $k_{\rm phys} \gg m_{\rm C}$, the shape of $\omega_{\rm phys}(k_{\rm phys})$ is *a priori* unknown. It is has been shown that a nonadiabatic evolution of the mode function in the trans-Planckian region necessarily implies a modification of the inflationary predictions, in particular, a modification of the power spectrum. To be more precise, in cosmology, the dispersion relation becomes time-dependent and equal to $\omega = a\omega_{\rm phys}(k/a)$ (ω and k denote comoving frequency and wave number, respectively). Then, the Wentzel-Kramers-Brillouin (WKB) approximation is satisfied provided that $|Q/\omega^2| \ll 1$, where the quantity Q is defined by $Q = 3(\omega')^2/(4\omega^2) - \omega''/(2\omega)$ (a prime stands for the derivative with respect to conformal time), see also Ref. [25]. If the previous condition is worked out, then one sees that the WKB approximation is violated if $\omega_{\rm phys} < H_{\rm inf}$ and that corrections to the standard result can occur in this case.

This conclusion has been criticized in Refs. [20,21] and many reasons why a modification of the inflationary power spectrum would be unlikely have been provided in these articles. In the following, we will examine each of them.

In Ref. [21], it has been claimed that if $\omega_{\rm phys}(k_{\rm phys})$ is such that the WKB approximation is violated for $k_{\rm phys} > m_{\rm C}$ then there are no preferred initial conditions. This is certainly correct for the class of dispersion relations considered in Ref. [26], as discussed in Ref. [27], but not true in general. An explicit counterexample has been provided in Ref. [27] and is studied in the present paper. The corresponding dispersion relation is sketched in Fig. 1. On this plot, one notices that in the region where $k_{\rm phys} > m_{\rm C}$ there is an interval of finite range, namely $k_{\rm phys} \in [\Lambda_1, \Lambda_2]$, in which the WKB approximation is violated (i.e., $\omega_{\rm phys} < H_{\rm inf}$). Important for our analysis is also the fact that for $k_{\rm phys} \to +\infty$ the adiabatic approximation is restored, which follows since we have $\omega_{\rm phys} > H_{\rm inf}$. In this latter regime, the adiabatic vacuum is obviously the preferred initial state.

The other arguments involve the calculation of the stress-energy tensor and are as follows. In the standard scenario, the initial conditions are fixed for all wave numbers at some initial time. If the number of e-foldings of inflation is greater than about 70, the physical wavelength of modes which are currently probed in cosmic microwave experiments is smaller than the Planck length at the beginning of inflation. One usually assumes that the evolution starts out from the adiabatic (Bunch-Davis) vacuum. If the

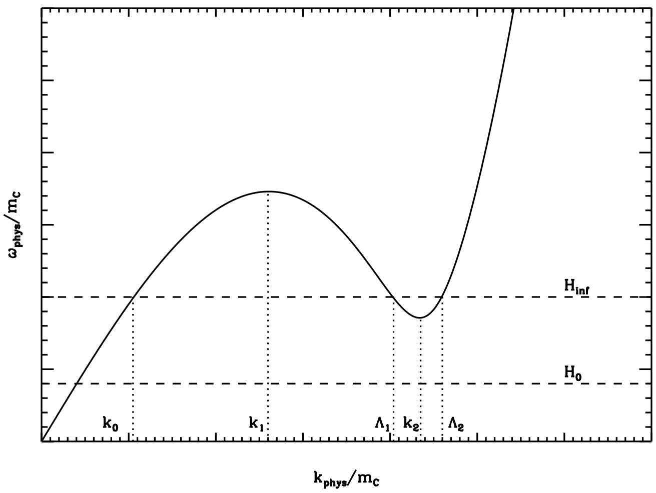

FIG. 1. Sketch of the dispersion used for the study of the toy model. The five scales k_0, k_1, k_2, Λ_1, and Λ_2 are defined in the text. $H_{\rm inf}$ is the Hubble parameter during inflation while H_0 is the Hubble parameter today. Clearly, this is not a scaled figure since H_0 should be much smaller than is represented on the plot.

evolution is nonadiabatic in the region $k_{\rm phys} > m_{\rm C}$, then the state in the region $H_{\rm inf} < k_{\rm phys} < m_{\rm C}$ will differ from the usual adiabatic vacuum. Therefore, if one concentrates only on what happens in the region $H_{\rm inf} < k_{\rm phys} < m_{\rm C}$, the trans-Planckian effects boil down to a modification of the initial conditions. This last argument has been used in Refs. [20,21] as follows. Roughly speaking, a nonvacuum state means a nonvanishing energy density and there is now the danger that this dominates over the energy density of the inflationary background which is $m_{\rm Pl}^2 H_{\rm inf}^2$. According to Refs. [20,21], this is actually what happens unless the level of excitation of the initial state compared to the adiabatic vacuum is very small, leading to unmeasurably small trans-Planckian effects on the spectrum of fluctuations. Since observational evidence seems to indicate that inflation is the correct theory of the very early universe, Refs. [20,21] conclude that trans-Planckian effects of significant importance are in fact not possible.

In order to understand the above argument in more detail, let us be more accurate about what has actually been done in Refs. [20,21]. It is well known that cosmological perturbations (density fluctuations and gravitational waves) can, in some contexts, be viewed as a free scalar field $\varphi(\eta, \mathbf{x})$ on a time-dependent background space-time. In the case of gravitational waves, the correspondence is exact, for scalar metric fluctuations (density perturbations), the correspondence is only exact if the equation of state of the background is time independent. In the general case, the squeezing factor for the density fluctuations is given not by the FLRW scale factor $a(t)$, but by a function $z(t)$ which depends both on the background geometry and the background matter - for details see, e.g., [6]. Then, the corresponding energy density and pressure are given by the mean values of the stress-energy tensor $\langle T_{\mu\nu}\rangle$. In an excited state characterized by the mode distribution function $n = n(k)$, one has

$$\langle\rho\rangle = \frac{1}{4\pi^2 a^4}\int_0^{+\infty} {\rm d}k k^2\left[\frac{1}{2} + n(k)\right] \times\left[a^2\left|\left(\frac{\mu_k}{a}\right)'\right|^2 + k^2|\mu_k|^2\right], \tag{2}$$

$$\langle p\rangle = \frac{1}{4\pi^2 a^4}\int_0^{+\infty} {\rm d}k k^2\left[\frac{1}{2} + n(k)\right] \times\left[a^2\left|\left(\frac{\mu_k}{a}\right)'\right|^2 - \frac{k^2}{3}|\mu_k|^2\right], \tag{3}$$

where μ_k is the rescaled Fourier amplitude, i.e., $\mu_k \equiv a(\eta)\varphi_k(\eta)$ and normalized such that $\mu_k \simeq 1/\sqrt{2k}$. In the above, η denotes conformal time, and a prime the derivative with respect to η.

In the above expressions, the terms proportional to the factor $1/2$ are divergent in the ultraviolet regime, i.e., $k \to +\infty$, and represent the quantum vacuum contribution. This means that, in order to give sense to the above expressions, the stress-energy tensor should be first properly renormalized, for instance by adiabatic regularization [28]. In this paper, we will simply subtract the contribution of the quantum vacuum energy since this is what has been done in Refs. [20,21]. Another justification is that we are in fact mainly interested in the terms proportional to $n(k)$ which describe the contributions originating from the excited quanta.

In Refs. [20,21], considerations have been restricted to physical modes such that $H_{\rm inf} < k_{\rm phys} < m_{\rm C}$. In this region, the dispersion relation is linear and, since the WKB approximation is satisfied, the mode function can be written as

$$\mu_k = \frac{\alpha_k}{\sqrt{2k}}{\rm e}^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}}{\rm e}^{+ik\eta}, \tag{4}$$

where $|\alpha_k|^2 - |\beta_k|^2 = 1$. Inserting this mode function into the vacuum expressions of the energy density and pressure, one finds

$$\langle\rho\rangle_{\rm UV} = \frac{1}{2\pi^2 a^4}\int_{aH_{\rm inf}}^{am_{\rm C}} \frac{{\rm d}k}{k} k^4|\beta_k|^2, \tag{5}$$

$$\langle p\rangle_{\rm UV} = \frac{1}{2\pi^2 a^4}\frac{1}{3}\int_{aH_{\rm inf}}^{am_{\rm C}} \frac{{\rm d}k}{k} k^4|\beta_k|^2, \tag{6}$$

We see that the coefficient $|\beta_k|^2$ represents the number of particles, $n(k) = |\beta_k|^2$. This describes the modification of the standard initial conditions due to the trans-Planckian effects (let us remind that the usual adiabatic initial conditions correspond to $\alpha_k = 1$ and $\beta_k = 0$). Then, a back-of-the-envelope calculation shows that $\langle\rho\rangle_{\rm UV} \simeq m_{\rm C}^4|\beta_k|^2$ where we have used the fact that, in de Sitter space-time, the coefficient β_k is scale-independent (time-translation invariance). If we require that the energy density of the test scalar field be smaller than the background density, then it follows that

$$|\beta_k|^2 < \frac{m_{\rm Pl}^2 H_{\rm inf}^2}{m_{\rm C}^4}. \qquad (7)$$

A similar expression has been obtained in Ref. [21], with $m_{\rm C} = m_{\rm Pl}$ and, crucially, $H_{\rm inf}$ replaced with H_0, the present value of the Hubble parameter. It is clear that the constraint on β_k is completely different (and much more difficult to satisfy) if one uses $H_0 \simeq 10^{-61} m_{\rm Pl}$ in the above equation rather than $H_{\rm inf} \simeq 10^{-5} m_{\rm Pl}$ (we also notice that $m_{\rm C}$ needs not be the Planck mass). The reason for this difference is again (see above the discussion of the preferred initial conditions) that, in Ref. [21], it was assumed that violation of the WKB approximation in the trans-Planckian region necessarily implies that $\omega(k) \to 0$ as $k \to +\infty$, as for the dispersion relation envisaged in Ref. [26]. In this case, the adiabatic condition is violated today for a range of trans-Planckian modes, and one should indeed replace $H_{\rm inf}$ by H_0 as done in Ref. [21]. This results in a very strong constraint on β_k. This was in fact the essence of the criticism made in Ref. [27] against the dispersion relation considered in Ref. [26]. However, again, the argument does not apply for dispersion relations of the type shown in Fig. 1 and, therefore, is not true in general. The reason can be very easily understood from Fig. 1. Since, after inflation, the Hubble parameter decreases and since, at some point, its value becomes smaller than the minimum of the dispersion relation, the adiabatic condition is restored at late times for all modes, and particle production stops. Therefore, in this case, the calculation should be done with $H_{\rm inf}$ and not with H_0 as done in Ref. [21]. In Refs. [19,27], it has been shown that there is a window for which the modification of the power spectrum is not completely negligible and for which there is no back-reaction problem, see for instance the discussion after Eq. (68) in Ref. [27].

The other arguments presented against possible trans-Planckian modifications of the inflationary power spectrum involve the calculation of the equation of state (which is characterized by the parameter $\omega_{\rm st} = p/\rho$). The main argument is that the "dangerous" created particles behave as a radiation field (or as ultrarelativistic particles) and, therefore, that their energy density scales as a^{-4} and not as the vacuum, see Ref. [20]. We now explain why this line of reasoning is problematic.

First of all, using the stress-energy tensor of a test scalar field is questionable since one wants in fact to calculate the equation of state of the cosmological perturbations, not that of a test scalar field. It has been shown in Refs. [29,30] that the equation of state of the effective stress-energy tensor of the cosmological perturbations differs on super-Hubble scales (which is a region in which the adiabatic condition is not valid and thus has similarities to the trans-Planckian interval where the adiabatic condition is violated) from the equation of state of a test field. It is $\omega_{\rm st} = -1$ instead of $\omega_{\rm st} = -1/3$. The difference is clearly of utmost importance in the present context. In fact, what should be done is to calculate the stress-energy tensor of cosmological perturbations (which is second order in the perturbed metric) in the case where the dispersion relation is modified. To our knowledge, this calculation has never been performed, and is very complicated. Therefore, this is beyond the scope of the present article.

Keeping the previous point in mind, let us come back to the calculation of the stress-energy tensor of a test scalar field. As already mentioned in the introduction [and demonstrated explicitly in Eq. (28)], $\langle\rho\rangle_{\rm UV}$ is in fact constant and does not scale as a^{-4} despite the fact that $p/\rho = 1/3$. The nonconservation of the energy-momentum tensor can be understood as follows. Using the expression of the energy density, it is easy to establish that

$$\frac{1}{a^4}\frac{{\rm d}}{{\rm d}t}(a^4\langle\rho\rangle_{\rm UV}) = \frac{H_{\rm inf}}{2\pi^2}[m_{\rm C}^4|\beta_{k=am_{\rm C}}|^2 - H_{\rm inf}^4|\beta_{k=aH_{\rm inf}}|^2]. \qquad (8)$$

The two terms in the right-hand side of the above expression, responsible for the nonconservation, originate from the time-dependent limits of integration in Eq. (5). The first one comes from the upper limit while the second one originates from the lower limit. The corresponding physical interpretation is clear: due to the expansion of the background, there is a flow of modes coming from the trans-Planckian region and entering the region $H_{\rm inf} < k_{\rm phys} < m_{\rm C}$ and there is also a flow of modes leaving the region $H_{\rm inf} < k_{\rm phys} < m_{\rm C}$ while they are becoming super-Hubble modes. Equation (8) is similar to Eq. (4) of Ref. [21]. The only difference is the absence in Ref. [21] of the second term on the right-hand side (describing the outgoing flow of modes). This term is necessarily present because the integral in Eq. (5) cannot be computed with a vanishing lower integral since, for $k_{\rm phys} < H_{\rm inf}$, the mode function is no longer given by Eq. (4) but rather by $\mu_k \simeq a(\eta)$. However, the second term in (8) is clearly very small in comparison with the first one and, therefore, can be safely neglected.

Finally, maybe the most important reason why calculating the back-reaction can be more subtle than previously thought is the following. The conclusion that $p/\rho = 1/3$ is in fact obtained from an inconsistent procedure since it does not take into account the fact that the dispersion relation is modified [let us recall that the previous considerations are based on Eqs. (5) and (6) that have been obtained under the assumption that $\omega_{\rm phys} = k_{\rm phys}$]. In the following, we shall study the equation of state of the ultraviolet terms (5) and (6) in a toy model for trans-Planckian physics in which we can describe the excitation of the mode functions in the far ultraviolet range from well-defined vacuum initial conditions in a mathematically consistent way. We shall show that the fact that the dispersion relation is modified can change the equation of state, a conclusion also reached in Ref. [31] in a slightly different context.

To summarize, the calculation of the back-reaction must be performed with the trans-Planckian corrections taken into account (i.e., in the present context with a modified dispersion relation). It is clearly inconsistent to calculate the modified power spectrum with the trans-Planckian corrections on one hand and, on the other hand, to evaluate the corresponding back-reaction without these corrections. This can change, in a crucial way, the calculation of the energy density and/or the equation of state.

III. THE TOY MODEL

A. Description of the dispersion relation

We model the trans-Planckian effect by means of the following nonstandard dispersion relation [27]

$$\omega_{\rm phys}^2(k_{\rm phys}) = k_{\rm phys}^2 - 2b_{11}k_{\rm phys}^4 + 2b_{12}k_{\rm phys}^6. \tag{9}$$

This dispersion relation is chosen such that the modes evolve adiabatically for extremely high wave numbers, but, given an appropriate choice of the constants b_{11} and b_{12}, there is an intermediate region of wave numbers in which the mode evolution is not adiabatic. We will start the modes in their adiabatic vacuum in the extreme ultraviolet and calculate how they are excited during the phase in which the evolution violates the adiabaticity condition.

If we introduce the new dimensionless coefficients α and β such that $\alpha \equiv 2b_{11}m_{\rm C}^2$ and $\beta \equiv 2b_{12}m_{\rm C}^4$, where $m_{\rm C}$ is a new free energy scale to be specified later on, then the dispersion relation can be rewritten as

$$\left(\frac{\omega_{\rm phys}}{m_{\rm C}}\right)^2 = \left(\frac{k_{\rm phys}}{m_{\rm C}}\right)^2 - \alpha\left(\frac{k_{\rm phys}}{m_{\rm C}}\right)^4 + \beta\left(\frac{k_{\rm phys}}{m_{\rm C}}\right)^6. \tag{10}$$

It is represented in Fig. 1. This relation is in fact characterized by one parameter, the "shape parameter" Υ defined by $\Upsilon \equiv 3\beta/\alpha^2$. The derivative of the dispersion relation vanishes at $k_{1,2}/m_{\rm C} = \sqrt{\alpha/(3\beta)}\sqrt{1 \mp \sqrt{1-\Upsilon}}$ which shows that $\Upsilon < 1$ and the requirement that the dispersion relation stays positive implies that $\Upsilon > 3/4$. To summarize, one has

$$\tfrac{3}{4} < \Upsilon < 1. \tag{11}$$

Obviously, the scales k_1 and k_2 only depend on the shape of the dispersion relation, i.e., only on the parameters α and β. It is more convenient to express everything in terms of α and Υ. This gives

$$\frac{k_1^2}{m_{\rm C}^2} = \frac{1}{\alpha\Upsilon}\left(1 - \sqrt{1-\Upsilon}\right), \tag{12}$$

$$\frac{k_2^2}{m_{\rm C}^2} = \frac{1}{\alpha\Upsilon}\left(1 + \sqrt{1-\Upsilon}\right). \tag{13}$$

In this paper, for simplicity we restrict ourselves to the case where the background space-time is de Sitter, characterized by the constant Hubble parameter $H_{\rm inf}$. Then, k_0, Λ_1 and Λ_2 are the two scales for which $\omega_{\rm phys} = H_{\rm inf}$, see Fig. 1. They depend on the parameters α and β but, clearly, also on $H_{\rm inf}/m_{\rm C}$. In fact, their explicit expressions can easily be derived. For this purpose, let us define the coefficients Q and R by

$$Q \equiv \frac{1}{\alpha^2\Upsilon^2}(\Upsilon - 1), \tag{14}$$

$$R \equiv \frac{1}{\alpha^3\Upsilon^3}\left[1 - \frac{3}{2}\Upsilon + \frac{3}{2}\alpha\Upsilon^2\left(\frac{H_{\rm inf}}{m_{\rm C}}\right)^2\right]. \tag{15}$$

Let us notice that $H_{\rm inf}$ crosses the dispersion relation 3 times only if $Q^3 + R^2 < 0$. This implies that $H_{\rm inf}$ should be chosen such that

$$H_{\rm min} < H_{\rm inf} < H_{\rm max}, \tag{16}$$

with $H_{\rm min}$ and $H_{\rm max}$ given by the following expressions (which, obviously, only depend on the shape of the dispersion relation)

$$\frac{H_{\rm min}}{m_{\rm C}} \equiv \frac{1}{\sqrt{\alpha}\Upsilon}\sqrt{\Upsilon - \frac{2}{3} - \frac{2}{3}(1-\Upsilon)^{3/2}}, \tag{17}$$

$$\frac{H_{\rm max}}{m_{\rm C}} \equiv \frac{1}{\sqrt{\alpha}\Upsilon}\sqrt{\Upsilon - \frac{2}{3} + \frac{2}{3}(1-\Upsilon)^{3/2}}. \tag{18}$$

Then the three solutions can be found explicitly since they are in fact solutions of a third order polynomial equation (more precisely, $\omega_{\rm phys}^2 = H_{\rm inf}^2$ is a sixth order polynomial equation that can be reduced to a third order equation in the variable $k_{\rm phys}^2$). The three solutions can be written as

$$\frac{k_0^2}{m_{\rm C}^2} = \frac{1}{\alpha\Upsilon}\left[1 + 2\sqrt{1-\Upsilon}\cos\left(\frac{\theta + 2\pi}{3}\right)\right], \tag{19}$$

$$\frac{\Lambda_1^2}{m_{\rm C}^2} = \frac{1}{\alpha\Upsilon}\left[1 + 2\sqrt{1-\Upsilon}\cos\left(\frac{\theta + 4\pi}{3}\right)\right], \tag{20}$$

$$\frac{\Lambda_2^2}{m_{\rm C}^2} = \frac{1}{\alpha\Upsilon}\left[1 + 2\sqrt{1-\Upsilon}\cos\left(\frac{\theta}{3}\right)\right], \tag{21}$$

where $\theta \equiv \cos^{-1}(R/\sqrt{-Q^3})$. On can check that, if $3/4 < \Upsilon < 1$, then $k_0 < \Lambda_1 < \Lambda_2$ as required.

Let us now consider a scalar field the dispersion relation of which is given by Eq. (10). Then, as demonstrated in

Ref. [27], the vacuum expectation value of the energy density and pressure are given by

$$\langle\rho\rangle = \frac{1}{4\pi^2 a^4}\int_0^{+\infty} \mathrm{d}k k^2\left[a^2\left|\left(\frac{\mu_k}{a}\right)'\right|^2 + \omega^2(k)|\mu_k|^2\right], \tag{22}$$

$$\langle p\rangle = \frac{1}{4\pi^2 a^4}\int_0^{+\infty} \mathrm{d}k k^2\left[a^2\left|\left(\frac{\mu_k}{a}\right)'\right|^2 + \left(\frac{2}{3}k^2\frac{\mathrm{d}\omega^2}{\mathrm{d}k^2} - \omega^2\right)|\mu_k|^2\right]. \tag{23}$$

This stress-energy tensor is conserved, i.e.,$\langle\rho\rangle' + 3\mathcal{H}\langle\rho + p\rangle = 0$, with $\mathcal{H} \equiv a'/a$ as shown explicitly in Ref. [32]. The modification of the energy density has exactly the expected form while the modification of the pressure is more complicated, involving the derivative of the dispersion relation. One can easily checked that, for the linear dispersion relation, the above formulas reduce to the standard ones.

B. Near ultraviolet region

Let us now try to evaluate these expressions explicitly. If we are in a region where the WKB approximation holds, then the mode function can be written as

$$\mu_k(\eta) \simeq \frac{\alpha_k}{\sqrt{2\omega(k,\eta)}}\exp\left[-i\int^\eta \omega(k,\tau)\mathrm{d}\tau\right] + \frac{\beta_k}{\sqrt{2\omega(k,\eta)}}\exp\left[i\int^\eta \omega(k,\tau)\mathrm{d}\tau\right], \tag{24}$$

with $|\alpha(k)|^2 - |\beta(k)|^2 = 1$ from the Wronskian normalization condition. Inserting this expression into the formula giving the energy density, one obtains

$$\langle\rho\rangle = \frac{1}{4\pi^2 a^4}\int \mathrm{d}k k^2\left\{\frac{1}{2\omega}[\omega^2 + |\gamma|^2] + \frac{|\beta_k|^2}{\omega}[\omega^2 + |\gamma|^2] + \frac{\alpha_k\beta_k^*}{2\omega}[\omega^2 + \gamma^2]\mathrm{e}^{-2i\int^\eta \omega(k,\tau)\mathrm{d}\tau} + \frac{\alpha_k^*\beta_k}{2\omega}[\omega^2 + (\gamma^*)^2]\mathrm{e}^{2i\int^\eta \omega(k,\tau)\mathrm{d}\tau}\right\}, \tag{25}$$

where we have used $|\alpha_k|^2 = 1 + |\beta_k|^2$. In the above expression, the quantity γ is defined as follows

$$\gamma(k,\eta) \equiv \left[\frac{\omega'(k,\eta)}{2\omega(k,\eta)} + i\omega(k,\eta) + \frac{a'}{a}\right]. \tag{26}$$

In a situation where WKB is a good approximation, we have $\gamma/\omega \simeq i$ and the previous expression reduces to

$$\langle\rho\rangle = \frac{1}{2\pi^2 a^4}\int_{\mathcal{K}} \mathrm{d}k k^2\left(\frac{1}{2} + |\beta_k|^2\right)\omega(k), \tag{27}$$

where the domain of integration $\mathcal{K}$ corresponds to the region where the WKB approximation is valid. Note that in order to remove the two oscillatory terms, no procedure of time averaging is needed in contrast with what was done in Ref. [20].

We now demonstrate that the energy density of Eq. (27) is constant in time. Replacing the time-dependent comoving frequency $\omega(k)$ by its expression in terms of the physical frequency, namely $\omega = a\omega_{\rm phys}(k/a)$, one gets

$$\langle\rho\rangle = \frac{|\beta_k|^2}{2\pi^2}\int_{\rm k_0}^{\Lambda_1} \mathrm{d}k_{\rm phys}k_{\rm phys}^2\omega_{\rm phys}(k_{\rm phys}), \tag{28}$$

where we have used that β_k is scale-independent in the case of a de Sitter background. We have also specified the domain of integration $\mathcal{K}$. It is of course crucial that this domain be defined in terms of physical wave numbers. As announced before, $\langle\rho\rangle$ is time independent. Of course, this does not imply that the corresponding equation of state is necessarily -1 because, since we consider only a limited range of wave numbers, the energy density is not conserved. Only the total energy density, integrated over wave numbers from 0 to $+\infty$, is conserved.

Let us now evaluate the pressure. Repeating the same steps as before, a long but straightforward calculation gives

$$\langle p\rangle = \frac{1}{4\pi^2 a^4}\int \mathrm{d}k k^2\left\{\frac{1}{2\omega}\left[\frac{2}{3}k^2\frac{\mathrm{d}\omega^2}{\mathrm{d}k^2} - \omega^2 + |\gamma|^2\right] + \frac{|\beta_k|^2}{\omega}\left[\frac{2}{3}k^2\frac{\mathrm{d}\omega^2}{\mathrm{d}k^2} - \omega^2 + |\gamma|^2\right] + \frac{\alpha_k\beta_k^*}{2\omega}\left[\frac{2}{3}k^2\frac{\mathrm{d}\omega^2}{\mathrm{d}k^2} - \omega^2 + \gamma^2\right]\mathrm{e}^{-2i\int^\eta \omega(k,\tau)\mathrm{d}\tau} + \frac{\alpha_k^*\beta_k}{2\omega}\left[\frac{2}{3}k^2\frac{\mathrm{d}\omega^2}{\mathrm{d}k^2} - \omega^2 + (\gamma^*)^2\right]\mathrm{e}^{2i\int^\eta \omega(k,\tau)\mathrm{d}\tau}\right\}, \tag{29}$$

This time, in order to remove the oscillatory terms, we can take the time average of the previous expression, as done for the energy density in Ref. [20]. In the following we denote the corresponding double average by the symbol $\langle\langle\cdots\rangle\rangle$. This yields

$$\langle\langle p\rangle\rangle = \frac{1}{3}\frac{1}{2\pi^2 a^4}\int_{\mathcal{K}} \mathrm{d}k k^2\left(\frac{1}{2} + |\beta_k|^2\right)\omega(k)\frac{\mathrm{d}\ln\omega^2}{\mathrm{d}\ln k^2}. \tag{30}$$

The expression for the pressure can be evaluated explicitly. We use the fact that the coefficient β_k is scale-independent, thanks to the time-translation invariance. Then

$$\langle\langle p\rangle\rangle = \frac{1}{3}\frac{m_{\rm C}^4|\beta_k|^2}{2\pi^2}\left[\mathcal{P}\left(\frac{\Lambda_1}{m_{\rm C}}\right) - \mathcal{P}\left(\frac{k_0}{m_{\rm C}}\right)\right], \tag{31}$$

where the function $\mathcal{P}(z)$ is defined by the following expression

$$\mathcal{P}(z) \equiv \frac{3}{16\alpha^2\Upsilon^2}\frac{\omega(z)}{z}\left[(9-8\Upsilon)+2\alpha\Upsilon z^2+\frac{8}{3}\alpha^2\Upsilon^2 z^4\right] + \frac{27}{32\alpha^2\Upsilon^2}\sqrt{\frac{3}{\Upsilon}}\left(1-\frac{4}{3}\Upsilon\right)\ln\left[\sqrt{\frac{3}{\Upsilon}}\left(\frac{2}{3}\alpha\Upsilon z^2-1\right) + 2\frac{\omega(z)}{z}\right], \quad (32)$$

with $\omega(z) \equiv \sqrt{z^2-\alpha z^4+\alpha^2\Upsilon z^6/3}$.

From the above expression, the energy density can be evaluated very simply if one notices that

$$\int_{\mathcal{K}} \mathrm{d}k k^2\omega(k)\frac{\mathrm{d}\ln\omega^2}{\mathrm{d}\ln k^2} = k^3\omega(k)|_{\mathcal{K}} - 3\int_{\mathcal{K}}\mathrm{d}k k^2\omega(k). \quad (33)$$

Then the equation of state $\omega_{\mathrm{st}} \equiv p/\rho$ can be expressed as

$$\omega_{\mathrm{st}} = \left[\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)\frac{(\Lambda_1/m_{\mathrm{C}})^3-(k_0/m_{\mathrm{C}})^3}{\mathcal{P}(\Lambda_1/m_{\mathrm{C}})-\mathcal{P}(k_0/m_{\mathrm{C}})}-1\right]^{-1}. \quad (34)$$

A typical example is represented in Fig. 2. The striking feature of this plot is that, for $H_{\mathrm{inf}}/m_{\mathrm{C}} \ll 1$, the equation of state goes to -1. Let us emphasize that this regime corresponds to a physical requirement that should be met if one wants to be in a realistic situation. In order to understand better how this happens, we perform the following perturbative treatment.

If the Hubble parameter is small in comparison with the new scale m_{C}, then the minimum of the dispersion relation is close to zero which in turn means that $\Upsilon \simeq 3/4$. Admittedly, this is a fine-tuning of the shape of the dispersion relation. The above considerations suggest that that Taylor expansion in $\Upsilon - 3/4$ can be performed. Then, one finds [see Eqs. (14) and (15)]

$$\frac{R}{\sqrt{-Q^3}} = -1 + \frac{27}{4}\alpha\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)^2 + \mathcal{O}\left(\Upsilon-\frac{3}{4}\right). \quad (35)$$

The next step is to expand the $\cos^{-1}$ function which appears in the paragraph following (21) with the above argument. At this point, since we expect $H_{\mathrm{inf}} \ll m_c$, one can treat $\alpha(H_{\mathrm{inf}}/m_{\mathrm{C}})^2$ as a small parameter and Taylor expand our expressions, e.g., Eq. (35), in this parameter. However, this approximation will break down when H_{inf} becomes large, especially far from H_{min}. This is true even if $\Upsilon = 3/4$ since then $\alpha(H_{\mathrm{max}}/m_{\mathrm{C}})^2 = 8/27 \simeq 0.3$ which represents an error of $\simeq 30\%$. In the vicinity of H_{min} the approximation is of course much better. Then, working at zeroth order in $\Upsilon - 3/4$, it is easy to show that the angle θ introduced in the paragraph after (21) is

$$\theta = \pi - \sqrt{\frac{27}{2}\alpha\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)^2} + \mathcal{O}\left\{\left[\alpha\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)^2\right]^{3/2}\right\} + \mathcal{O}\left(\Upsilon-\frac{3}{4}\right). \quad (36)$$

If one inserts the above expression into the formulas giving k_0 and Λ_1, one obtains

$$\left(\frac{k_0}{m_{\mathrm{C}}}\right)^2 = \frac{4}{3\alpha}\left\{\frac{3}{4}\alpha\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)^2 + \mathcal{O}\left[\alpha^2\left(\frac{H_{\mathrm{inf}}}{m_{\mathrm{C}}}\right)^4\right]\right\} + \mathcal{O}\left(\Upsilon-\frac{3}{4}\right), \quad (37)$$

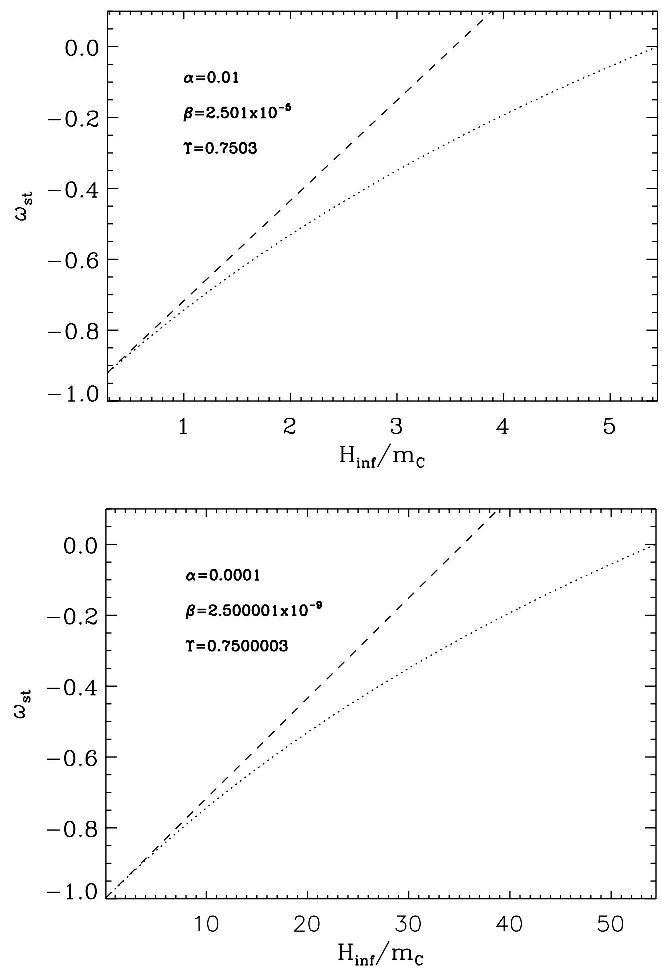

FIG. 2. Top panel: Equation of state versus $H_{\mathrm{inf}}/m_{\mathrm{C}}$ for $\alpha = 0.01$ and $\Upsilon = 0.7003$ (dotted line) computed according to Eq. (34). The dashed line is an approximation of the equation of state, valid for small values of $H_{\mathrm{inf}}/m_{\mathrm{C}}$, and derived in Eq. (41). Bottom panel: same as the top panel but with $\alpha = 0.0001$ and $\Upsilon = 0.700\,003$.

$$\left(\frac{\Lambda_1}{m_C}\right)^2 = \frac{2}{\alpha}\left\{1 - \sqrt{\frac{\alpha}{2}\left(\frac{H_{\rm inf}}{m_C}\right)^2} - \frac{\alpha}{4}\left(\frac{H_{\rm inf}}{m_C}\right)^2 + \mathcal{O}\left[\alpha^{3/2}\left(\frac{H_{\rm inf}}{m_C}\right)^3\right]\right\} + \mathcal{O}\left(\Upsilon - \frac{3}{4}\right). \qquad (38)$$

The first expression is expected since it says that at leading order $k_0 \simeq H_{\rm inf}$. This is because the usual transition between sub and super-Hubble modes occurs in the region where the dispersion relation is almost linear. In the same manner, one has

$$\mathcal{P}\left(\frac{k_0}{m_C}\right) = \frac{1}{\alpha^2}\left\{1 + \frac{1}{2}\alpha\left(\frac{H_{\rm inf}}{m_C}\right)^2 + \mathcal{O}\left[\alpha^2\left(\frac{H_{\rm inf}}{m_C}\right)^4\right]\right\} + \mathcal{O}\left(\Upsilon - \frac{3}{4}\right), \qquad (39)$$

$$\mathcal{P}\left(\frac{\Lambda_1}{m_C}\right) = \frac{1}{\alpha^2}\left(\frac{H_{\rm inf}}{\Lambda_1}\right)\left\{4 - 5\sqrt{\frac{\alpha}{2}\left(\frac{H_{\rm inf}}{m_C}\right)^2} - \frac{1}{4}\alpha\left(\frac{H_{\rm inf}}{m_C}\right)^2 + \mathcal{O}\left[\alpha^{3/2}\left(\frac{H_{\rm inf}}{m_C}\right)^3\right]\right\} + \mathcal{O}\left(\Upsilon - \frac{3}{4}\right). \qquad (40)$$

Finally, putting everything together, at leading order in $\Upsilon - 3/4$ and in $\alpha(H_{\rm inf}/m_C)^2$, we obtain a simple equation in the regime of interest, namely

$$\omega_{\rm st} \simeq -1 + 2\sqrt{2\alpha\left(\frac{H_{\rm inf}}{m_C}\right)^2}. \qquad (41)$$

It is represented by a dashed line in Fig. 2. We are now in a position where the behavior of the equation of state can be understood better. In the limit $\Upsilon \to 3/4$ and $H_{\rm inf} \to H_{\rm min}$, the equation of state goes to -1. Let us notice that, since we have $H^2_{\rm min} = 8(\Upsilon - 3/4)/(3\alpha) + \cdots$, the limit $H \to H_{\rm min}$ corresponds, in this case, to $H_{\rm inf} \to 0$. We conclude that, in the physical regime of interest, namely $H/m_C \ll 1$ the equation of state is extremely close to that of the vacuum. Far from $H_{\rm min}$ the approximation used above breaks down as is apparent from Fig. 2. When $H_{\rm inf} \to H_{\rm max}$, it is clear that $k_0 \to \Lambda_1$ and the expression giving the equation of state becomes ambiguous. From the plots, we see that $\omega_{\rm st}$ goes in fact to zero. Therefore, even if the Hubble constant is not small in comparison with m_C, the equation of state $\omega_{\rm st}$ remains negative.

C. Far ultraviolet region

Let us now consider the far ultraviolet region. We fix the initial conditions in the region where the WKB approximation holds by selecting positive frequency modes, which corresponds to the choice of the Bunch-Davies adiabatic vacuum state for the field,

$$\mu_k(\eta) = \frac{1}{\sqrt{2\omega(k,\eta)}}\exp\left[-i\int^{\eta}\omega(k,\tau){\rm d}\tau\right]. \qquad (42)$$

At some time $\eta_2(k)$, the comoving mode enters the far ultraviolet region, i.e., when the physical wavelength equal Λ_2, and μ_k and its first derivative must be matched to the above solution, which gives

$$\mu_k(\eta) = \frac{1}{\sqrt{2\omega[k,\eta_2(k)]}}{\rm e}^{-i\int_{\eta_{\rm i}}^{\eta_2(k)}\omega(k,\tau){\rm d}\tau}\frac{a(\eta)}{a(\eta_2)} \times\left\{1 - \gamma(k,\eta_1)\int_{\eta_2(k)}^{\eta}\left[\frac{a(\eta_2)}{a(\tau)}\right]^2{\rm d}\tau\right\}, \qquad (43)$$

where the quantity γ has been defined previously, see Eq. (26). Then, one can safely neglect the second term which is the decaying mode and express everything in terms of physical quantities, using that the scale factor can be written as $a(\eta) = -1/(H_{\rm inf}\eta)$. This gives

$$|\mu_k(\eta)|^2 \simeq \frac{1}{\omega[k,\eta_2(k)]}\frac{\Lambda_2^2}{k^2_{\rm phys}(\eta)}. \qquad (44)$$

But, one has $\omega[k,\eta_2(k)] = a(\eta_2)\omega_{\rm phys}[k/a(\eta_2)] = a(\eta_2)\omega_{\rm phys}(\Lambda_2) = a(\eta_2)H_{\rm inf}$. Since $a[\eta_2(k)] = k/\Lambda_2$, one finally arrives at

$$|\mu_k(\eta)|^2 = \frac{1}{2k}\frac{\Lambda_2^3}{H_{\rm inf}k^2_{\rm phys}(\eta)}. \qquad (45)$$

Then, it is straightforward to calculate the energy density and the pressure. This gives

$$\langle\rho\rangle = \frac{1}{16\pi^2}\frac{\Lambda_2^3 m_C^2}{H_{\rm inf}}\left\{\left[\left(\frac{\Lambda_2}{m_C}\right)^2 - \left(\frac{\Lambda_1}{m_C}\right)^2\right] - \frac{\alpha}{2}\left[\left(\frac{\Lambda_2}{m_C}\right)^4 - \left(\frac{\Lambda_1}{m_C}\right)^4\right] + \frac{\beta}{3}\left[\left(\frac{\Lambda_2}{m_C}\right)^6 - \left(\frac{\Lambda_1}{m_C}\right)^6\right]\right\}, \qquad (46)$$

$$\langle p\rangle = -\frac{1}{48\pi^2}\frac{\Lambda_2^3 m_C^2}{H_{\rm inf}}\left\{\left[\left(\frac{\Lambda_2}{m_C}\right)^2 - \left(\frac{\Lambda_1}{m_C}\right)^2\right] + \frac{\alpha}{2}\left[\left(\frac{\Lambda_2}{m_C}\right)^4 - \left(\frac{\Lambda_1}{m_C}\right)^4\right] - \beta\left[\left(\frac{\Lambda_2}{m_C}\right)^6 - \left(\frac{\Lambda_1}{m_C}\right)^6\right]\right\}. \qquad (47)$$

In the standard case where $\alpha = \beta = 0$, one recovers that $p/\rho = -1/3$ as required for superhorizon modes (see, e.g., [30]). Then, straightforward algebraic manipulations show that

$$\langle\rho\rangle = -\langle p\rangle + \frac{2}{3}\frac{\Lambda_2^3 m_C^2}{H_{\rm inf}}\left\{\left[\left(\frac{\Lambda_2}{m_C}\right)^2 - \left(\frac{\Lambda_1}{m_C}\right)^2\right] - \alpha\left[\left(\frac{\Lambda_2}{m_C}\right)^4 - \left(\frac{\Lambda_1}{m_C}\right)^4\right] + \beta\left[\left(\frac{\Lambda_2}{m_C}\right)^6 - \left(\frac{\Lambda_1}{m_C}\right)^6\right]\right\} \qquad (48)$$

$$= -\langle p\rangle + \frac{2}{3}\frac{\Lambda_2^3 m_{\rm C}^2}{H_{\rm inf}}[\omega_{\rm phys}^2(\Lambda_2) - \omega_{\rm phys}^2(\Lambda_1)] = -\langle p\rangle. \tag{49}$$

Therefore, in the far ultraviolet region, the equation of state is nothing but the vacuum equation of state, i.e., -1.

Can we understand this result better? For this purpose let us consider Eqs. (22) again. The terms $(\mu_k/a)'$ vanish because $\mu_k \propto a$ in the region under consideration. Then, the link between the pressure and the energy density can be rewritten as

$$\langle p\rangle = -\langle \rho\rangle + \frac{2}{3}\frac{1}{4\pi^2 a^4}\int_{\mathcal{K}} \mathrm{d}k k^4 \frac{\mathrm{d}\omega^2}{\mathrm{d}k^2}|\mu_k|^2. \tag{50}$$

Now, if $|\mu_k|^2$ scales as $|\mu_k|^2 \propto 1/k^3$ as indicated by Eq. (45) which is the consequence of having matched the mode function in the far ultraviolet region to the initial Bunch-Davis vacuum, then the above expression can be rewritten as

$$\langle p\rangle + \langle \rho\rangle \propto \frac{2}{3}\int_{\mathcal{K}} \mathrm{d}k \frac{\mathrm{d}\omega^2}{\mathrm{d}k}. \tag{51}$$

Clearly, if the interval $\mathcal{K}$ is such that the frequency is the same at its boundaries then we obtain the equation of state of the vacuum. This conclusion does not depend on the detailed shape of the dispersion relation in this region. It is also obvious that the calculations done previously for a specific dispersion relation are fully compatible with the above considerations, in particular, the difference between $\langle p\rangle$ and $-\langle \rho\rangle$ in Eq. (48) is given by a term equal to $2/3$ times the difference between the square of the effective frequency at the boundaries of the far ultraviolet region.

To conclude this section, one can estimate the equation of state of the created particles coming from the whole ultraviolet region. In the near ultraviolet region, one has

$$\langle p\rangle_{\rm near-UV} = \left[-1 + \mathcal{O}\left(\frac{H_{\rm inf}}{m_{\rm C}}\right)\right]\langle \rho\rangle_{\rm near-UV}, \tag{52}$$

while in the far ultraviolet region

$$\langle p\rangle_{\rm far-UV} = -\langle \rho\rangle_{\rm far-UV}. \tag{53}$$

In the "ultrafar" region ($k_{\rm phys} > \Lambda_2$), i.e., in the region where the initial condition are fixed, we have by definition the adiabatic vacuum and, therefore, no created particles. Hence, we do not need to take into account this region. As a consequence, the equation of state of the whole ultraviolet region can be written as

$$\omega_{\rm st-UV} \equiv \frac{\langle p\rangle_{\rm near-UV} + \langle p\rangle_{\rm far-UV}}{\langle \rho\rangle_{\rm near-UV} + \langle \rho\rangle_{\rm far-UV}} \simeq -1 + \mathcal{O}\left(\frac{H_{\rm inf}}{m_{\rm C}}\right), \tag{54}$$

where we have used Eqs. (52) and (53). Since the total energy density

$$\langle \rho\rangle_{\rm UV} \equiv \langle \rho\rangle_{\rm near-UV} + \langle \rho\rangle_{\rm far-UV} \tag{55}$$

is constant in time, we see that the energy density of the created particles almost behaves as a positive cosmological constant. The slight nonconservation is due to the fact that we have not taken into account the infrared region which also contributes (let us recall that the total energy-momentum tensor is conserved exactly). The same calculation performed with a linear dispersion relation would have led to the result $\omega_{\rm st-UV} = 1/3$. We have thus demonstrated explicitly that taking into account the trans-Planckian corrections is important when one evaluates the back-reaction.

We end this section with the following remark. It is important to keep in mind that the previous calculation is not the calculation of the equation of state of cosmological perturbations. Therefore, one cannot claim that the equation of state of the cosmological perturbations with a modified dispersion relation is the one of a cosmological constant. What has been calculated is just the equation of state of a test field. Nevertheless, we have shown with the help of the previous toy model that any attempt to evaluate the equation of state of the cosmological perturbations in a regime where the dispersion relation is modified must take into account these trans-Planckian corrections.

IV. DISCUSSION AND CONCLUSIONS

Now that we have determined the energy density and pressure of the ultraviolet modes, we will study their back-reaction on the background space-time and matter. We work in the context of the toy model studied in the previous section. We assume that at early times all modes with wave number larger than Λ_2 start out in their adiabatic vacuum. Thus, after subtraction of the quantum vacuum terms, these modes do not contribute to the energy-momentum tensor, and thus there will be no remaining ultraviolet divergences.

We now follow a mode with fixed comoving wavelength which starts out deep in the ultraviolet region ($k_{\rm phys} > \Lambda_2$) in its adiabatic vacuum. The mode will then spend a finite time interval in the intermediate frequency range $\Lambda_1 < k_{\rm phys} < \Lambda_2$ during which the adiabaticity condition for mode evolution is violated, the state gets squeezed, and, from the point of view of the vacuum state for $k_{\rm phys} < \Lambda_1$, a nonvanishing occupation number n_k is generated. This occupation number remains constant when $k_0 < k_{\rm phys} < \Lambda_1$, since in this frequency range the adiabaticity condition is restored.

Given this setup, we consider the energy density $\rho_{\rm UV}$ of ultraviolet modes (modes with wavelength smaller than the Hubble radius) which is the sum of (27) and (46), see Eq. (55). By time-translation invariance of the background, both terms are independent of time in a de Sitter background in which $H_{\rm inf}$ is constant. Thus,

$$\frac{\mathrm{d}\rho_{\rm UV}}{\mathrm{d}t} = 0. \tag{56}$$

This seems to indicate that the ultraviolet energy density evolves like a cosmological constant, and its back-reaction, rather than preventing inflation, will simply lead to a renormalization of the cosmological constant.

Naive intuition, namely, treating the equation of state as radiative, i.e., $p_{\rm UV} = \rho_{\rm UV}/3$, would have led to a problem with this conclusion, namely, an extreme nonconservation of the energy-momentum tensor of the ultraviolet modes, a nonconservation on the energy density scale of Λ_1^4 (in this context the difference between Λ_1 and Λ_2 is not relevant). However, our analysis of the previous section has shown that in fact the energy-momentum tensor of the ultraviolet modes is that of a cosmological constant, up to correction terms which are suppressed by a factor of $H_{\rm inf}/m_{\rm C}$ compared to the dominant terms.

Let us take a look at the equations of back-reaction. Following the method discussed in Refs. [33,34] in the context of the problem of the back-reaction of infrared cosmological fluctuations. The back-reaction effect of interest here is the fact that linear cosmological fluctuations effect the background metric and matter if one works out the Einstein equations to quadratic order in the amplitude of the primordial perturbations. To be specific, in the following we shall consider a homogeneous, isotropic and spatially flat background, and will take matter to be a scalar field φ with a quadratic potential given by the scalar-field mass m. For the fluctuations we parameterize the equation of state as

$$p_{\rm UV} \equiv (-1+\alpha)\rho_{\rm UV}, \tag{57}$$

where (by the above discussion) α is a positive constant expected to be of the order $H_{\rm inf}/m_{\rm C}$.

Our starting point consists of taking the FLRW equations

$$H^2 = \frac{\kappa}{3}\rho, \quad \frac{{\rm d}\rho}{{\rm d}t} = -3H(\rho + p), \tag{58}$$

where $\kappa \equiv 8\pi/m_{\rm Pl}^2$. The presence of the fluctuations produces a back-reaction effect on the background which is quadratic in the amplitude of the fluctuations [29,30] and leads to a correction of both metric and matter, i.e., to corrections δH of the Hubble expansion rate and $\delta\varphi$ of the scalar field

$$H = H_{\rm inf} + \delta H, \quad \varphi = \varphi_{\rm inf} + \delta\varphi, \tag{59}$$

where the subscripts "inf" stand for the quantities evaluated in the unperturbed background. The back-reaction of the linear fluctuations on the background is described by contributions $\rho_{\rm br} = \rho_{\rm UV}$ and $p_{\rm br} = p_{\rm UV}$ (and we use the ultraviolet energy densities and pressures as the back-reaction quantities) to the energy density and pressure. If we insert the back-reaction ansatz (59) into the Eqs. (58) (written in the slow-roll approximation) and linearize in δH and $\delta\varphi$, we obtain the following equations

$$2H_{\rm inf}\delta H = \frac{\kappa}{3}(\rho_{\rm UV} + m^2\varphi_{\rm inf}\delta\varphi),$$

$$m^2\left(\frac{{\rm d}\varphi_{\rm inf}}{{\rm d}t}\right)\delta\varphi + m^2\varphi_{\rm inf}\left(\frac{{\rm d}\delta\varphi}{{\rm d}t}\right) = -3H_{\rm inf}(\rho_{\rm UV} + p_{\rm UV}) - \frac{{\rm d}\rho_{\rm UV}}{{\rm d}t}. \tag{60}$$

We have assumed that the background model has a quadratic potential, i.e., $V(\varphi) = m^2\varphi^2/2$. The back-reaction of infrared modes (modes with wavelength larger than the Hubble radius) was analyzed in [29,30] (see also Ref. [35]) and was shown to correspond to a negative cosmological constant whose absolute value increases in time (see Ref. [36] for resulting speculations on how this effect might be used to address the cosmological constant problem). Here, we will study the back-reaction effects of the ultraviolet modes for which ${\rm d}\rho_{\rm br}/{\rm d}t = {\rm d}\rho_{\rm UV}/{\rm d}t = 0$ and for which the equation of state is given by (57). In this case, the linearized Klein-Gordon equation can be simplified to yield

$$\frac{{\rm d}\delta\varphi}{{\rm d}t} + \frac{1}{\varphi_{\rm inf}}\left(\frac{{\rm d}\varphi_{\rm inf}}{{\rm d}t}\right)\delta\varphi = -\frac{3\alpha H_{\rm inf}}{m^2\varphi_{\rm inf}}\rho_{\rm UV}. \tag{61}$$

The solution of Eqs. (61) and (60) is

$$\delta\varphi = -3\sqrt{\frac{\kappa}{6}}\left(\frac{\rho_{\rm UV}}{m}\right)\alpha t, \tag{62}$$

$$\delta H = \frac{\kappa}{6}\frac{\rho_{\rm UV}}{H_{\rm inf}}(1 - 3\alpha H_{\rm inf}t). \tag{63}$$

As expected $\delta\varphi$ and δH vanish if $\rho_{\rm UV} = 0$. As stressed in [37] and later analyzed in detail in [38], it is important to express the result in terms of physical observables instead of in terms of the nonmeasurable background time coordinate t. The obvious clock in our simple system is the scalar field $\varphi \equiv \varphi_{\rm inf} + \delta\varphi$ itself. After a simple calculation it follows that the back-reaction effect δH measured in terms of φ is given by

$$H_{\rm inf} + \delta H = \sqrt{\frac{\kappa}{6}}\left(m\varphi + \sqrt{\frac{\kappa}{6}}\frac{\rho_{\rm UV}}{H_{\rm inf}}\right). \tag{64}$$

Thus, the effect of the ultraviolet back-reaction terms in a de Sitter background corresponds to a positive renormalization of the cosmological constant. This result does not depend on the value of α.

So far, we have shown that a large ultraviolet back-reaction does not prevent inflation, in contrast to naive expectations. Instead, it leads to a renormalization of the cosmological constant. On the other hand, as is evident from Eq. (62), the back-reaction terms can lead to a faster rolling of the scalar field $\varphi_{\rm inf}$. However, for the small values of α of the order of $H_{\rm inf}/m_{\rm C}$ indicated by our analysis, the increase in the rolling speed does not prevent a phase of inflation of sufficient length (i.e., having $|\delta\varphi| < |\varphi_{\rm inf}|$ for a time interval $t \simeq H_{\rm inf}^{-1}$) as long as

$$n_k \lesssim \frac{H_{\rm inf}}{m_{\rm Pl}}, \tag{65}$$

which is a less stringent constraint than the one derived by Tanaka, and still allows observable effects on cosmic microwave fluctuations from trans-Planckian physics.

We have presented an attempt to study the effects of back-reaction on the trans-Planckian problem of inflationary cosmology. The question initially posed in Refs. [20,21] is whether the back-reaction of a state which consists of excited modes during the inflationary phase on scales smaller than the Hubble radius will prevent inflation. We have seen that the analysis is more subtle than it initially appears from the above works. We have shown that the back-reaction of an excited state preserving time-translation invariance does not prevent inflation, but simply leads to a renormalization of the Hubble constant. However, back-reaction will lead to a slightly faster rolling of the scalar field. As long as the occupation numbers n_k are smaller than $H_{\rm inf}/m_{\rm Pl}$, the rolling will be consistent with the standard inflationary paradigm. Such occupation numbers can lead to observable effects on the cosmic microwave background.

To render the analysis well defined, we have considered a dispersion relation for which the violation of adiabaticity is concentrated in a finite range of physical wave numbers $\Lambda_1 < k_{\rm phys} < \Lambda_2$. We have assumed that all modes start off in their adiabatic vacuum state when $k_{\rm phys} > \Lambda_2$. They are squeezed while $\Lambda_1 < k_{\rm phys} < \Lambda_2$, and then emerge as excited states when $k_{\rm phys} < \Lambda_1$. Let us also notice that by making the wavelength interval $\Lambda_1 < k_{\rm phys} < \Lambda_2$ small, the energy density in the far ultraviolet modes can always be made small compared to the cutoff energy density Λ_1^4 (or Λ_2^4). Thus, in our approach the Planck energy density problem recently discussed in Ref. [39] in an approach to quantum field theory on a growing lattice, in which the number of fundamental field modes is increasing in time, does not arise (the analysis of Ref. [39] finds that the continual creation of modes at the cutoff scale yields an energy density which is of Planck scale). In our analysis, the Hilbert space of modes is time independent, but as time proceeds an increasing subset of this space gets populated. This mechanism for continual excitation of new modes (see [40] for original work along these lines and [41] for resulting speculations concerning the cosmological constant) appears to be more smooth than the lattice approach of Ref. [39].

There are several important deficiencies in our analysis. First, we have used an *ad hoc* regularization and renormalization prescription which consists of imposing effectively an abrupt cutoff in momentum space, and of subtracting the ground state energy of each field Fourier mode. This procedure is not covariant. It would be of interest to study our problem using a mathematically more rigorous regularization prescription, such as adiabatic regularization (see, e.g., Ref. [42] and references therein).

Another serious concern is that we have considered matter fluctuations without taking into account the induced metric fluctuations. It is well known that the inclusion of metric fluctuations leads to dramatically different results for super-Hubble-scale perturbations [29,30]. Thus, one might expect that the gravitational fluctuations could play an important role in the far ultraviolet region where $\omega_{\rm phys} < H_{\rm inf}$. However, at the present time we are not able to study this issue because the nonstandard dispersion relation has been set up for the matter sector only. It is a challenge for future research to include the presence of nonstandard dispersion relations consistently in both the gravitational and matter sector.

ACKNOWLEDGMENTS

We wish to thank Ted Jacobson and Martin Lemoine for interesting discussions. During the course of this work, R. B. was supported by the Perimeter Institute for Theoretical Physics, and (at Brown) by the US Department of Energy under Contract No. DE-FG02-91ER40688, TASK A. J. M. thanks the High Energy Group of Brown University and the Perimeter Institute for Theoretical Physics for warm hospitality.

[1] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[2] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981); Pis'ma Zh. Eksp. Teor. Fiz. **33**, 549 (1981).
[3] W. H. Press, Phys. Scr. **21**, 702 (1980).
[4] V. N. Lukash, Sov. Phys. JETP **52**, 807 (1980); Zh. Eksp. Teor. Fiz. **79**, 1601 (1980).
[5] R. H. Brandenberger, Nucl. Phys. **B245**, 328 (1984).
[6] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992); J. Martin, hep-th/0406011.
[7] D. Polarski and A. A. Starobinsky, Classical Quantum Gravity **13**, 377 (1996).
[8] R. H. Brandenberger, hep-ph/9910410.
[9] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Academic, Chur, 1990).
[10] R. H. Brandenberger and J. Martin, Mod. Phys. Lett. A **16**, 999 (2001); J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001).
[11] J. C. Niemeyer, Phys. Rev. D **63**, 123502 (2001); S. Shankaranarayanan, Classical Quantum Gravity **20**, 75 (2003); J. C. Niemeyer and R. Parentani, Phys. Rev. D **64**, 101301 (2001).
[12] C. S. Chu, B. R. Greene, and G. Shiu, Mod. Phys. Lett. A

16, 2231 (2001); R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D **64**, 103502 (2001); R. Easther, B. R. Greene, W. H. Kinney, and G. Shiu, Phys. Rev. D **67**, 063508 (2003); F. Lizzi, G. Mangano, G. Miele, and M. Peloso, J. High Energy Phys. 06 (2002) 049; S. F. Hassan and M. S. Sloth, Nucl. Phys. **B674**, 434 (2003).

[13] R. Brandenberger and P. M. Ho, Phys. Rev. D **66**, 023517 (2002); AAPPS Bull. **12N1**, 10 (2002).

[14] A. Kempf and J. C. Niemeyer, Phys. Rev. D **64**, 103501 (2001);

[15] C. Burgess *et al.*, J. High Energy Phys. 2 (2003) 048; C. Burgess, J. Cline, and R. Holman, J. Cosmol. Astropart. Phys. 0310 (2003) 004.

[16] U. H. Danielsson, Phys. Rev. D **66**, 023511 (2002).

[17] V. Bozza, M. Giovannini, and G. Veneziano, J. Cosmol. Astropart. Phys. 05 (2003) 001; J. C. Niemeyer, R. Parentani, and D. Campo, Phys. Rev. D **66**, 083510 (2002).

[18] K. Schalm, G. Shiu, and J. P. van der Schaar, J. High Energy Phys. 4 (2004) 076.

[19] J. Martin and R. Brandenberger, Phys. Rev. D **68**, 063513 (2003).

[20] T. Tanaka, astro-ph/0012431.

[21] A. A. Starobinsky, Pis'ma Zh. Eksp. Teor. Fiz. **73**, 415 (2001); JETP Lett. **73**, 371 (2001).

[22] M. Porrati, Phys. Lett. B 596 (2004) 306.

[23] N. Kaloper, M. Kleban, A. Lawrence, S. Shenker, and L. Susskind, J. High Energy Phys. 11 (2002) 037; M. Porrati, Phys. Lett. B **596**, 306 (2004).

[24] A. A. Starobinsky and I. I. Tkachev, JETP Lett. **76**, 235 (2002); Pis'ma Zh. Eksp. Teor. Fiz. **76**, 291 (2002).

[25] J. Martin and D. J. Schwarz, Phys. Rev. D **67**, 083512 (2003).

[26] L. Mersini, M. Bastero-Gil, and P. Kanti, Phys. Rev. D **64**, 043508 (2001).

[27] M. Lemoine, M. Lubo, J. Martin, and J. P. Uzan, Phys. Rev. D **65**, 023510 (2002).

[28] L. Parker and S. A. Fulling, Phys. Rev. D **9**, 341 (1974).

[29] V. F. Mukhanov, L. R. W. Abramo, and R. H. Brandenberger, Phys. Rev. Lett. **78**, 1624 (1997).

[30] L. R. W. Abramo, R. H. Brandenberger, and V. F. Mukhanov, Phys. Rev. D **56**, 3248 (1997).

[31] T. Jacobson and D. Mattingly, Phys. Rev. D **63**, 041502 (2001).

[32] M. Lemoine, J. Martin, and J. P. Uzan, Phys. Rev. D **67**, 103520 (2003).

[33] L. R. W. Abramo, Phys. Rev. D **60**, 064004 (1999).

[34] L. R. W. Abramo, gr-qc/9709049.

[35] N. C. Tsamis and R. P. Woodard, Phys. Lett. B **301**, 351 (1993); N. C. Tsamis and R. P. Woodard, Nucl. Phys. **B474**, 235 (1996).

[36] R. H. Brandenberger, "On the Nature of Dark Energy: Observational and Theoretical Results on the Accelerating Universe," Proceedings of the 18th IAP Colloquium, Paris, France, 2002; hep-th/0210165.

[37] W. Unruh, astro-ph/9802323.

[38] G. Geshnizjani and R. H. Brandenberger, Phys. Rev. D **66**, 123507 (2002); G. Geshnizjani and R. H. Brandenberger, hep-th/0310265.

[39] B. Z. Foster and T. Jacobson, J. High Energy Phys. 08 (2004) 024.

[40] A. Kempf, Phys. Rev. D **63**, 083514 (2001).

[41] A. Kempf, "On the Nature of Dark Energy: Observational and Theoretical Results on the Accelerating Universe," Proceedings of the 18th IAP Colloquium, Paris, France, 2002; gr-qc/0210077.

[42] F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, Phys. Rev. D **65**, 103521 (2002).