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Lattice gluodynamics at negative g^2

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We consider Wilson's SU(N) lattice gauge theory (without fermions) at negative values of $\beta = 2N/g^2$ and for $N = 2$ or 3. We show that in the limit $\beta \rightarrow -\infty$, the path integral is dominated by configurations where links variables are set to a nontrivial element of the center on selected nonintersecting lines. For $N = 2$, these configurations can be characterized by a unique gauge invariant set of variables, while for $N = 3$ a multiplicity growing with the volume as the number of configurations of an Ising model is observed. In general, there is a discontinuity in the average plaquette when g^2 changes its sign which prevents us from having a convergent series in g^2 for this quantity. For $N = 2$, a change of variables relates the gauge invariant observables at positive and negative values of β. For $N = 3$, we derive an identity relating the observables at β with those at β rotated by $\pm 2\pi/3$ in the complex plane and show numerical evidence for a Ising like first order phase transition near $\beta = -22$. We discuss the possibility of having lines of first order phase transitions ending at a second order phase transition in an extended bare parameter space.

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I. INTRODUCTION

It has been known from the early days of QED that perturbative series have a zero radius of convergence [1]. This has not prevented Feynman diagrams to become an essential tool in particle physics. However, perturbative series need to be used with caution. The divergent nature of QED series was foreseen by Dyson as a consequence of the apparently pathological nature [1] of the ground state in a fictitious world with negative e^2. Like charges then attract and pair creation can be invoked to produce states where electrons are brought together in a given region and positrons in another. Dyson concludes that as this process sees no end, no stable vacuum can exist.

For Euclidean lattice models, related situations are encountered. For scalar field theory with $\lambda\phi^4$ interactions, configurations with large field values make the path integral ill-defined when $\lambda < 0$ (provided that no higher even powers of ϕ appear in the action with a positive sign and that the path of integration is not modified). Modified series with a finite radius of convergence can be obtained by introducing a large field cutoff [2,3]. We are then considering a slightly different problem. In simple situations [4], it is possible to determine an optimal value of the field cutoff that, at a given order in perturbation, minimizes or eliminates the discrepancy. For nonabelian gauge theories in the continuum Hamiltonian formulation, the substitution $g \rightarrow ig$ makes the quartic part unbounded from below and the cubic part nonhermitian.

It should be noted that in quantum mechanics [5], it is possible to change the boundary conditions of the Schrödinger equation in such a way that the spectrum of an harmonic oscillator with a perturbation of the form ix^3 or $-x^4$ stays real and positive. The procedure can be extended to scalar field theory in order to define a sensible $i\phi^3$ theory [6]. Even though conventional Monte Carlo calculations would fail for these models, complex Langevin methods can be used to calculate Green's functions [7].

In the case of lattice gauge theory with *compact* gauge groups, the action per unit of volume is bounded from below and there is no large field problem. Consequently, these models have well-defined expectation values when $g^2 < 0$ and we can consider the limit $g^2 \rightarrow 0^-$. In this article, we discuss the behavior of Wilson loops for SU(N) lattice gauge theory with $g^2 < 0$. This work is motivated by the need to understand the unexpected behavior of the lattice perturbative series for the 1×1 plaquette calculated up to order 10 [8–10]. An analysis of the successive ratios [11,12] may suggest that the series has a finite radius of convergence and a nonanalytic behavior near $\beta \simeq 5.7$ in contradiction with the general expectations that the series should be asymptotic and the transition from weak to strong coupling smooth.

We consider here pure (no fermions) gauge models with a minimal lattice action [13]. For definiteness this model and our notations are defined in Sec. II. The extrema of the action are discussed in III and enumerated for SU(2) and SU(3). We then discuss (Sec. IV) the case of SU(2) and show that planar Wilson loops with an area A (in plaquette units) pick up a factor $(-1)^A$ when g^2 becomes negative and the behavior for $g^2 < 0$ is completely determined by the behavior with $g^2 > 0$. As the Wilson loops are nonzero when $g^2 \rightarrow 0^+$, the ones with an odd area have a discontinuity which invalidates the idea of a regular perturbative series.

The case of SU(3) is discussed in Sec. V where we derive identities involving Wilson loops calculated with a coupling rotated by $\pm 2\pi/3$ in the complex plane. Consequently, for $N = 3$, we have no a-priori knowledge

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regarding the behavior of Wilson loops when $g^2 < 0$. We report numerical evidence for a first order phase transition near $\beta = 6/g^2 \simeq -22$ using methods similar to Ref. [14]. The implications of our findings are summarized in the conclusions.

II. THE MODEL, NOTATIONS

In the following, we consider the minimal, unimproved, lattice gauge model originally proposed by K. Wilson [13]. Our conventions and notations are introduced in this section for definiteness. We consider a cubic lattice in D dimensions. A SU(N) group element is attached to each link l and U_l denotes its fundamental representation. U_p denotes the conventional product of U_l (or hermitian conjugate) along the sides of a 1×1 plaquette p. The minimal action reads

$$S = \beta \sum_p [1 - (1/N) ReTr(U_p)], \tag{1}$$

with $\beta = 2N/g^2$. The lattice functional integral or partition function is

$$Z = \prod_l \int dU_l \mathrm{e}^{-S} \tag{2}$$

with dU_l the SU(N) invariant Haar measure for the group element associated with the link l. The average value of any quantity $\mathfrak{Q}$ is defined as usual by inserting $\mathfrak{Q}$ in the integral and dividing by Z.

In the following, we consider symmetric finite lattice with L^D sites and periodic boundary conditions. For reasons that will become clear in the next sections, L will always be even. The total number of 1×1 plaquettes is denoted

$$\mathcal{N}_p \equiv L^D D(D-1)/2. \tag{3}$$

Using

$$f \equiv -(1/\mathcal{N}_p) \ln Z, \tag{4}$$

we define the average density

$$P(\beta) \equiv \partial f/\partial \beta = (1/\mathcal{N}_p) \langle \sum_p [1 - (1/N) ReTr(U_p)] \rangle. \tag{5}$$

In statistical mechanics, f would be the free energy density multiplied by β and P the energy density. In analogy we can also define the constant volume specific heat per plaquette

$$C_V = -\beta^2 \partial P/\partial \beta. \tag{6}$$

III. THE LIMIT $\beta \to -\infty$

In the limit $\beta \to -\infty$, we expect the functional integral to be dominated by configurations which *maximize* $\sum_P [1 - (1/N) ReTr(U_P)]$. In the opposite limit ($\beta \to +\infty$), the same quantity needs to be minimized which can be accomplished by taking U_l as the identity everywhere.

We first consider the question of finding the extrema of TrU. For our study of the behavior when $\beta \to -\infty$, we are particularly interested in finding absolute minima of TrU. Using $TrU = Tr(VUV^\dagger)$ for V unitary, $U = \mathrm{e}^{iH}$ with H traceless and hermitian, and $V\mathrm{e}^{iH}V^\dagger = \mathrm{e}^{iVHV^\dagger}$, we can diagonalize H and write

$$ReTrU = \sum_{i=1}^{N-1} \cos(\phi_i) + \cos\left(\sum_{i=1}^{N-1} \phi_i\right). \tag{7}$$

The extremum condition then reads

$$\sin(\phi_i) + \sin\left(\sum_{i=1}^{N-1} \phi_i\right) = 0, \tag{8}$$

for $i = 1, \ldots N-1$. The trivial solution is all $\phi_i = 0$. We then have $ReTrU = N$ which is an absolute maximum.

For $N = 2$, we have only one nontrivial solution $\phi_1 = \pi$, which corresponds to the nontrivial element of the center $U = -\mathbb{1}$. We then have $ReTrU = -2$ which is an absolute minimum.

For $N = 3$, we have five nontrivial solutions. Two correspond to the nontrivial elements of the center ($\phi_1 = \phi_2 = \pm 2\pi/3$). The matrix of second derivatives has two positive eigenvalues and these two solutions correspond to a minimum. We use the notation $\Omega \equiv \mathrm{e}^{i2\pi/3}\mathbb{1}$ on the diagonal. We have $ReTr\Omega = -3/2$, which we will see is an absolute minimum. The other three solutions are ($\phi_1 = \pi, \phi_2 = 0$), ($\phi_1 = 0, \phi_2 = \pi$) and ($\phi_1 = \pi, \phi_2 = \pi$). They correspond to elements conjugated to diagonal matrices belonging to the three canonical SU(2) subgroups with the $SU(2)$ element being the non trivial center element. These three solutions have matrices of second derivatives with eigenvalues of opposite signs and correspond to saddle points rather than minimum or maximum. In the three cases $ReTrU = -1$.

For general N, it is clear that we can always find at least one group element U such that $ReTrU$ is an absolute minimum. In particular, for N even, $U = -1$ is such a group element, with $ReTrU = -N$ (the individual matrix elements must have a complex norm less then one). For $N \geq 3$ and odd, it is easy to check that all $\phi_i = (N-1)\pi/N$ is a solution of the extremum condition Eq. (8). For this choice, $ReTrU = -N|\cos[(N-1)\pi/N]|$ which is clearly negative and tends to $-N$ as N becomes large. This solution (the element of the center the closest to $-\mathbb{1}$) gives an absolute minimum of TrU for $N = 3$ and we conjecture that it is also the case for larger N.

We can now obtain an absolute minimum of the action if we can build a configuration such that $ReTrU_P$ takes its absolute minimum value for every plaquette. This can be accomplished by the following construction. In the

Appendix, we show that (at least for $D \le 4$) and for L even, it is possible to construct a set of lines on the lattice such that every plaquette shares *one and only one* link with this set of lines. We call such a set of links $\mathfrak{L}$. One can then put an element which gives an absolute minimum of $ReTrU$ on the links of $\mathfrak{L}$ and the identity on all the other links. For $SU(2)$, there is only one possible choice that minimizes $ReTrU$, namely $U = -\mathbb{1}$. For $SU(3)$, there are two possible choices $U = \Omega$ or $U = \Omega^\dagger$. We emphasize that the construction only works for L even. If L is odd, there will be lines of frustration in every plane.

The set of links $\mathfrak{L}$ is not unique. Starting with a given set, we can generate another one by translating the lines by one lattice spacing or rotating them by $\pi/2$ about the lattice axes. By direct inspection, it is easy to show that for $D = 2$ there are 4 such a sets of lines while for $D = 3$ there are eight of them.

Enumerating all the gauge inequivalent minima of the action at negative β for arbitrary D and N appears as a nontrivial problem. In the rest of this section, we specialize the discussion to $N = 2$ or 3. In order to discriminate among gauge inequivalent configurations, it is useful to make the following (gauge invariant) argument: in order to have an absolute minimum of the action, for every 1×1 plaquette p, the product $U_p(\mathbf{n})$ of the U_l along p starting at any site $\mathbf{n}$ of p, is a nontrivial element of the center. Under a local gauge transformation, $U_p(\mathbf{n}) \to V(\mathbf{n})U_p(\mathbf{n})V(\mathbf{n})^\dagger = U_p(\mathbf{n})$ since $U_p(\mathbf{n})$ commutes with any $SU(N)$ matrix. For the same reason, changing $\mathbf{n}$ along the plaquette amounts to a $VUV^\dagger$ conjugation and has no effect on the center. Consequently, configurations with a different set of $\mathfrak{U} = \{U_p\}$ are not gauge equivalent. One can think of $\mathfrak{U}$ as a set of electric and magnetic field configurations.

For SU(2), there is only one, uniform, set $\mathfrak{U}$ where all the elements $U_p = -\mathbb{1}$. For $D = 2$, this can be realized in four different ways by putting $-\mathbb{1}$ on the four distinct sets $\mathfrak{L}$. These four configurations are all gauge equivalent. The gauge transformations that map these four configurations into each others can be obtained by taking $V = -\mathbb{1}$ on every other sites of the lines of $\mathfrak{L}$. For $D = 3$, it is also possible to show that the eight configurations that can be constructed with a similar procedure can also be shown to be gauge equivalent. The gauge transformation can be obtained by taking $V = -\mathbb{1}$ on every other sites of the lines of $\mathfrak{L}$ pointing in two particular directions and in such way that one half of the lines created by the gauge transformation associated with one direction "annihilate" with one half of the lines created by the gauge transformation associated with the other direction. We conjecture that in higher dimensions, the configurations that minimize the action for SU(2) are also related by gauge transformations.

For SU(3) the situation is quite different because for every link of a particular $\mathfrak{L}$, we have two possible nontrivial element of the center. Since there are $\mathcal{N}_p = L^D D(D-1)/2$ plaquettes on a L^D lattice and one link of $\mathfrak{L}$ per plaquette, shared by $2(D-1)$ plaquette, we have $DL^D/4$ links in any $\mathfrak{L}$. Picking a particular $\mathfrak{L}$, it possible to construct $2^{DL^D/4}$ distinct $\mathfrak{U}$. Consequently, there are at least $2^{DL^D/4}$ gauge inequivalent minima of the action for SU(3). Note that $2^{DL^D/4}$ always is an integer for L even, which has been assumed. In the case $D = 4$, the degeneracy is simply 2^{L^4} which is the same as the number of configurations of an Ising model on a L^4 lattice.

In summary, we predict a discontinuity in P as g^2 changes sign. In the limit $\beta \to +\infty$, we have $P \to 0$, while in the limit $\beta \to -\infty$, we expect $P \to 2$ for N even, and $1 + |\cos[(N-1)\pi/N]|$ for N odd.

IV. N=2

In this section we discuss SU(2) gauge theories at negative β. The basic idea is that it is possible to change $\beta ReTrU_p$ into $-\beta ReTrU_p$ by making the *change of variables* $U_l \to -U_l$ for every link l of a particular $\mathfrak{L}$. Since $-\mathbb{1}$ is an element of SU(2) and since the Haar measure is invariant under left or right multiplication by a group element, this does not affect the measure of integration. Consequently, we have

$$Z(-\beta) = \mathrm{e}^{2\beta\mathcal{N}_p} Z(\beta). \tag{9}$$

Taking the logarithmic derivative as in Eq. (5), we obtain

$$P(\beta) + P(-\beta) = 2. \tag{10}$$

This identity can be seen in the symmetry of the curve $P(\beta)$ shown in Fig. 1. The validity of Eq. (10) can be further checked by calculating the difference

$$\Delta(\beta) \equiv |P(\beta) + P(-\beta) - 2|, \tag{11}$$

which should be zero except for statistical fluctuations. Figure 2 illustrates this statement and shows that the statistical errors of our calculations are of order 10^{-4} or less.

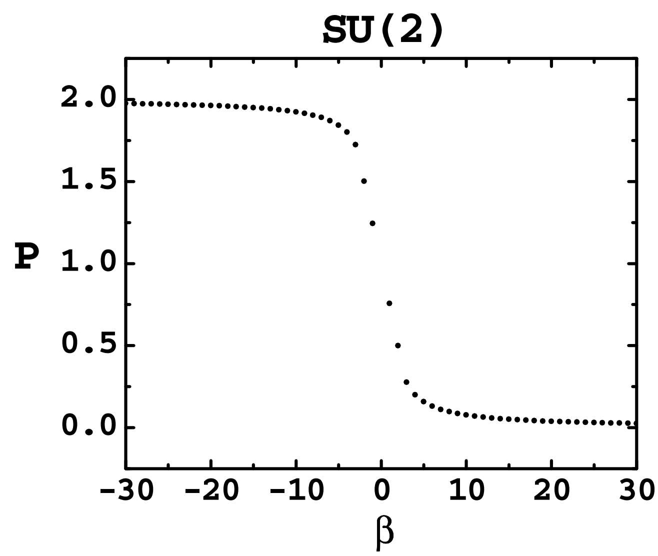

FIG. 1. The average action density $P(\beta)$ for SU(2).

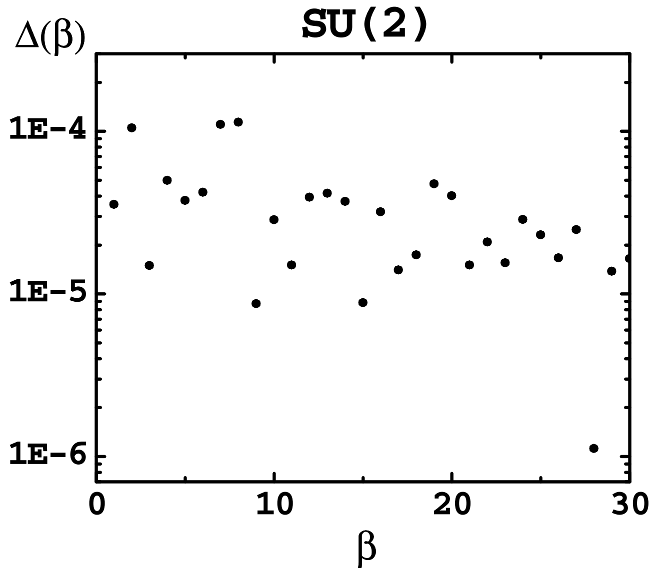

FIG. 2. $\Delta(\beta)$ defined in Eq. (11) versus β.

The relation between $P(\pm\beta)$ of Eq. (10) together with the assumption that $P(+\infty) = 0$, is in agreement with the statement made in Sec. III that P seen as a function of $g^2 = 2N/\beta$, jumps discontinuously by two as g^2 becomes negative. This invalidates the idea that P could have a regular expansion about $g^2 = 0$ with a nonzero radius of convergence.

This relation can also be used in the opposite limit and expanded about $\beta = 0$. The odd terms cancel automatically. The even terms of order two and higher add and cannot cancel. Consequently, the even coefficients of the strong coupling expansion of $P(\beta)$ and the odd coefficients of the free energy should vanish, in agreement with explicit calculations [15].

The discontinuity at $g^2 = 0$ can be extended to Wilson loops of odd area (in plaquette units). To see this, let us consider a Wilson loop $W(C)$ with C a contour that is the boundary of an area made out of A plaquettes. For simplicity, let also assume that this area is connected and has no self-intersections. Under the change of variables $U_l \to -U_l$ for every link l of an arbitrary set $\mathfrak{L}$, we have $W(C) \to (-1)^A W(C)$. This follows from the fact that for any line, the parity of the number of links of C shared with this line, is the same as the number of plaquettes of the area in contact with this lines. Since $\mathfrak{L}$ shares a link with every plaquette, we obtain the desired result. This can be summarized as

$$\langle W(C)\rangle_{-\beta} = (-1)^A \langle W(C)\rangle_\beta. \tag{12}$$

We can now try to interpret the change of the Wilson loop with the area in a term of a potential. We consider a rectangular $R \times T$ contour C and write

$$W(R, T, \beta) \equiv \langle W(C)\rangle_\beta \propto \mathrm{e}^{-E(R,\beta)T}. \tag{13}$$

From Eq. (12) this implies

$$E(R, -|\beta|) = E(R, |\beta|) + i\pi R. \tag{14}$$

This property can be related to the fact that the configurations of minimum action are invariant under translations by two lattice spacings but not under translations by one lattice spacing. This also confirms our expectation that the hamiltonian develops a nonhermitian part.

V. N=3

For $N = 3$, $-\mathbb{1}$ is *not* a group element and the closest thing to the change of variables used for $N = 2$ that we can invent is a multiplication by a nontrivial element of the center $\zeta\mathbb{1}$ for the links of a particular set $\mathfrak{L}$. We then obtain

$$Z(\zeta\beta) = \mathrm{e}^{(1-\zeta)\beta\mathcal{N}_p} Z(\beta) \times \left\langle \mathrm{e}^{(\beta/3)\sum_p (\zeta Re \zeta^\star Tr U_p - Re Tr U_p)} \right\rangle_\beta. \tag{15}$$

In the case $N = 2$, ζ is replaced by -1, $\langle ...\rangle_\beta$ becomes 1 and we recover Eq. (9). In the case of SU(3), the factor $\langle ...\rangle_\beta$ prevents us from deriving an exact identity analog to Eq. (10) for SU(2). It is however possible to obtain an approximate generalization which is a good approximation for small β. Setting $\beta = \zeta^\star x$, taking the logarithmic derivative with respect to x and setting $x = \zeta\beta$, we obtain

$$P(\zeta\beta) = 1 - \zeta^\star + \zeta^\star P(\beta) + \mathcal{O}(\beta). \tag{16}$$

Taking the real part and using $1 + \zeta + \zeta^2 = 0$, we obtain

$$P(\beta) + P(\zeta\beta) + P(\zeta^2\beta) = 3 + \mathcal{O}(\beta^6), \tag{17}$$

which can be seen as an approximate $SU(3)$ version of Eq. (10). The cancellation of the terms of order 1, 2, 4 and 5 occurs independently of the values of the coefficients at these orders. The absence of contribution of order 3 and the presence of a nonzero contribution at order six comes from the fact [16] that $ln(Z)/\mathfrak{N}_p$ has a zero (nonzero) contribution at order 4 (7).

As it does not seem possible to obtain $P(\beta)$ for β real and negative from our knowledge at β real and positive, we have to resort to a direct numerical approach. The results are shown in Fig. 3. A discontinuity near $\beta = -22$ is clearly visible. This indicates a first order phase transition.

The metastable branches have been studied following the approach of M. Creutz [14] used to study of a fist order transition for SU(5). As β becomes more and more negative, the system becomes more ordered but has also higher average energy P, the supercooled/heated terminology may be confusing and will be avoided.

We have run Monte Carlo simulations on a 8^4 lattice at $\beta = -22$ with four different initial configurations. Our best estimate of the critical β for this volume is -22.09. The first initial configuration was completely ordered (in the $\beta \to -\infty$ sense, with $P = 1.5$) by putting a nontrivial element of the center on a set of lines $\mathfrak{L}$. As we set $\beta =$

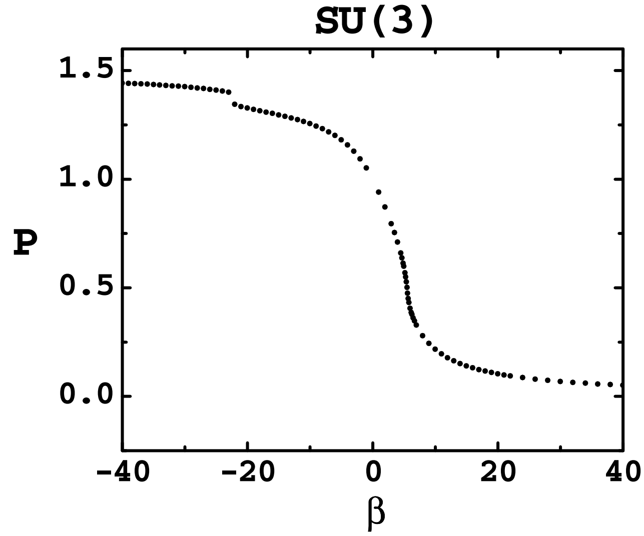

FIG. 3. The average action density $P(\beta)$ for SU(3)

-22, we expect to stay on the upper branch and end up with $P \simeq 1.39$ (black dots in Fig. 4) for many iterations. The second configuration was completely random (empty circles) and stayed on the lower branch when β was set to -22, to end up at $P \simeq 1.34$. The third configuration (empty squares) was initially random, we then temporarily set $\beta = -27$ letting P go up to 1.38, expecting to reach lower metastable branch. When β is set to -22, P stabilized to the lower value 1.34. Finally, we prepared a fourth initial configuration (empty triangles) by first setting a nontrivial element of the center on a given $\mathfrak{L}$ and then temporarily setting $\beta = -17$ until P is near 1.35 expecting to reach the upper metastable part. When β is finally set to -22, we reach the upper branch value $P = 1.39$. Figure 4 is quite similar to Fig. 1 of Ref. [14] and has the same type of crossings.

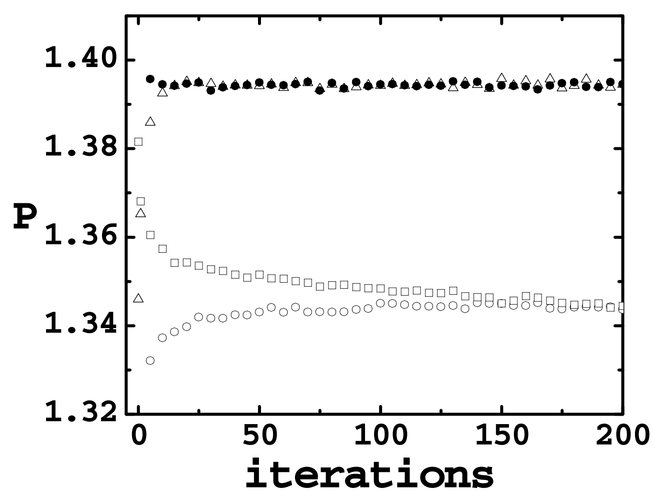

FIG. 4. P as a function of iterations for the four initial configurations described in the text.

We believe that the first order transition observed above is similar to the one observed [17] for the $D = 4$ Z_2 gauge theory. This model is dual to a nearest neighbor Ising model. In Fig. 5, we show histograms of the distribution of ImU below, near and above the transition. ImU allows to separate the two nontrivial elements of the center Ω and $\Omega^\dagger$. As β becomes more negative and goes through the

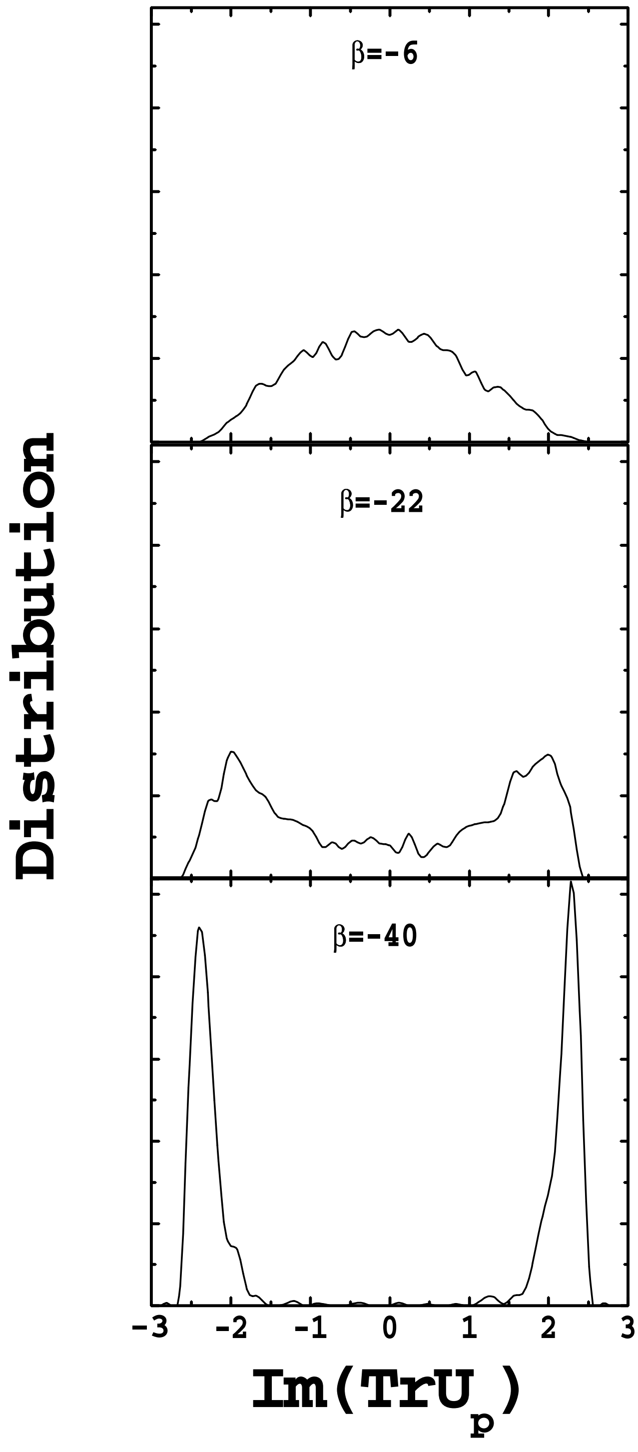

FIG. 5. Distribution of $ImTrU$ for three values of β.

transition, a broad distribution around 0 develops two bumps which keep separating and sharpening as one would observe in an Ising model.

The transition can also be seen as a singularity in the specific heat defined in Eq. (6) as shown in Fig. 6. As expected the height of the peak increases with the volume. The location of the transition sightly moves left as the volume increases.

Finally, we would like to compare the decay of the Wilson loop at negative β for SU(2) and SU(3). In Fig. 7, we have plotted the Wilson loop $\langle W(1, R)\rangle$ for these two groups. For SU(2) we observe the same decay as at positive β but with alternated signs as predicted in Sec. IV. For $SU(3)$, the decay is much faster than at the positive value of β and show that the sign alternates as long as the signal is larger than the statistical fluctuations (namely for $R \leq 6$).

The sign alternates at relatively low values of $|\beta|$. This agrees with the strong coupling expansion which predicts a $(-|\beta|/18)^{(R\times 1)}$ behavior for β negative and small in absolute value.

VI. CONCLUSIONS

We have studied lattice gauge theories at negative β. Wilson loops are well-defined and calculable with the Monte Carlo method. However, the limits $\beta \to \pm\infty$ of P differ and an expansion in $g^2 \propto 1/\beta$ cannot have a finite radius of convergence. This statement has been substantiated for $N = 2$ and 3, but from the discussion of Sec. III, it seems clear that it should extend to general N.

We found a first order phase transition near $\beta = -22$. At this point, it seems unrelated to the known transition near $\beta \simeq +6$ and branch cuts in the complex plane discussed by J. Kogut [18]. However, a more complete picture may appear if we study P for a larger class of action. It is conceivable that by introducing a linear combination of terms involving larger contours than the 1×1 plaquette, multiplied by one free parameter, we could create a line of first order phase transition ending at a second order phase transition. Another possibility would be to add an adjoint term as in Ref. [19]. Finding new second order phase transitions would allow us to define a nontrivial continuum limit.

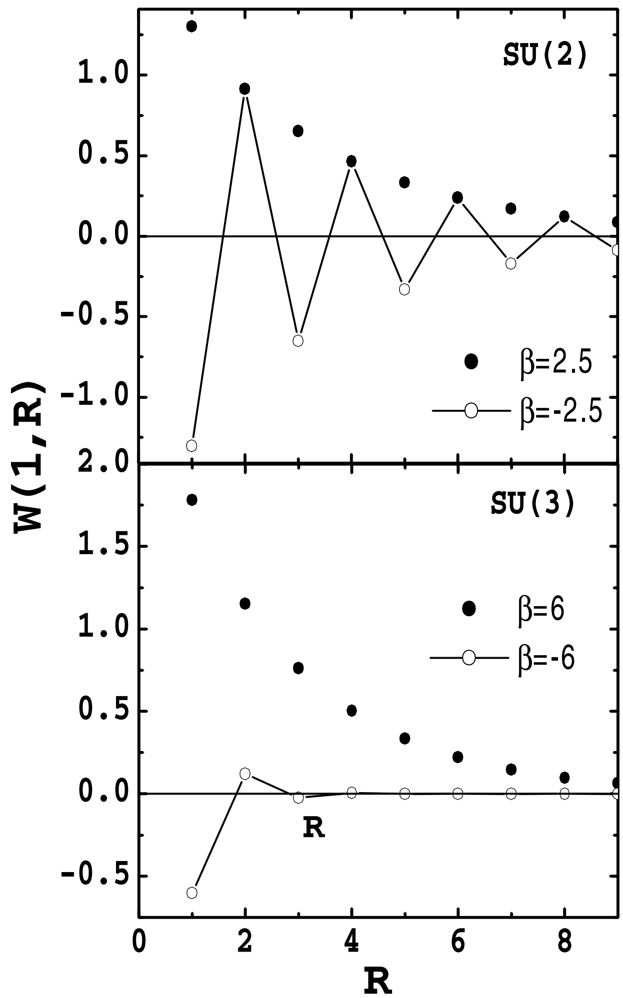

FIG. 7. The Wilson loop $\langle W(1, R)\rangle$ at $\beta = \pm 2.5$ for SU(2) and $\beta = \pm 6$ for SU(3).

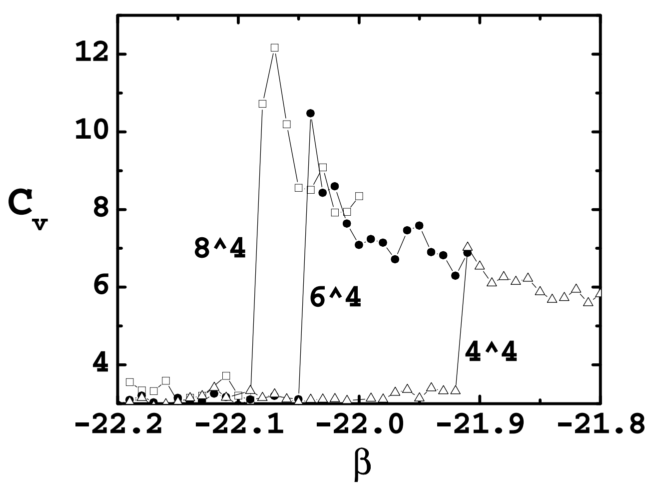

FIG. 6. The specific heat C_V versus β near the first order phase transition.

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APPENDIX: MAXIMAL SETS OF NON-INTERSECTING OF LINES ON A CUBIC LATTICE

In this Appendix, we consider a D dimensional cubic lattice. We restrict the use of ‘‘line’’ to collections of links along the D principal directions of the lattice and the use of "plane" to collections of plaquettes along the $D(D-1)/2$ principal orientations. In other words, these objects are lines and planes in the usual sense, but we exclude some "oblique" sets that can be constructed out of the sites.

We now try to construct a set of lines such that every plaquette shares one and only one link with this set. It is obvious that these lines cannot intersect, otherwise, at the point of intersection and in the plane defined by the two lines, we could fit four plaquettes, each sharing two links with the lines. These lines cannot be obtained from each other by a translation of one lattice spacing in one single direction, otherwise the set of lines would share two opposite links on the plaquettes in between the two lines.

For $D = 2$, the problem has obvious solutions, we can pick for instance a set of vertical lines separated by two lattice spacings. Using translation by one lattice spacing and rotation by $\pi/2$, it is possible to obtain three other solutions. For $D > 2$, it is sufficient to show that for every plane (in the restricted sense defined above), we have a $D = 2$ solution. As this restricted set of planes contains all the plaquette ounce, we would have then succeeded in proving the assertion. If such a solution exists, it is invariant by a translation by two lattice spacing in any direction. Consequently, we only need to prove the existence of the set of lines on a 2^D lattice with periodic boundary conditions. The full solution is then obtained by translation of the 2^D solution. If the lattice is finite, this only works if L is even, an assumption we have maintained in this article.

On a 2^D lattice, the lines (as defined above) are constructed by fixing $D-1$ coordinates values to be 0 or 1 and leaving the remaining coordinate arbitrary. For instance, for $D = 3$, a line in the third direction coming out of the origin will be denoted $(0, 0, A)$ where A stands for arbitrary and means 0 or 1. In general D, there are $D2^{D-1}$ such lines. Consequently, there are $D2^D$ links, each shared by $2(D-1)$ plaquettes. There are thus $D2^D 2(D-1)/4 = D(D-1)2^{D-1}$ plaquettes. A set of lines which has exactly one link in common with every plaquette, has $D(D-1)2^{D-1}/[2(D-1)] = D2^{D-2}$ links in other words it must contain $D2^{D-3}$ lines. For $D = 2$, such a set has only one line and there are four possible choices. For $D = 3$, an example of solution is $\{(A, 0, 0), (0, A, 1), (1, 1, A)\}$. It is not difficult to show that there are eight distinct solutions of this type. For $D = 4$, a solution consists in eight lines. An example of solution is

$$\{(A, 0, 0, 0), (0, A, 0, 1), (0, 1, A, 0), (0, 0, 1, A), (1, 1, 0, A), (1, 0, A, 1), (1, A, 1, 0), (A, 1, 1, 1)\}.$$

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