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Noncommutative isotropic harmonic oscillator

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The energy spectrum of an isotropic harmonic oscillator as a function of the noncommutativity parameter Θ is studied. It is shown that for a dense set of values of Θ the spectrum is degenerated and the algebra responsible for degeneracy can always be chosen to be SU(2). The generators of the algebra are constructed explicitly.

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It is well known that the noncommutative geometry plays an important role in string theory and M theory [1]. It has been found that, in a certain limit, string theory reduces to a gauge theory on noncommutative space. Since then quantum theory on noncommutative spaces has become a subject of intensive research.

The simplest example of such a theory is nonrelativistic quantum mechanics of a fixed number of particles. There are indications that some testable predictions can be found even in this case [2]. There appeared many papers dealing with quantum mechanics on noncommutative spaces [3–24].

In particular, several authors studied the energy spectrum of natural Hamiltonians on two-dimensional noncommutative space defined by the commutation rule

$$[\hat{x}_i, \hat{x}_j] = i\Theta\varepsilon_{ij}, i, j = 1, 2. \tag{1}$$

Unfortunately, most of these models cannot be solved exactly so only perturbative results are available. They show that the Θ dependence of physical quantities is quite involved.

In particular, it has been suggested [13,25] that, in general, the limit $\Theta \to 0$ cannot be taken directly.

In the present paper we show that, even if the limit $\Theta \to 0$ *can* be taken directly, it is sometimes still highly nontrivial (although numerically simple). This happens if the $\lim_{\Theta\to 0} E_n(\Theta)$ is not uniform in n.

More specifically, we consider an isotropic harmonic oscillator on a noncommutative plane (1). Both the Hamiltonian and the commutation rule (1) are invariant under rotations. However, in two dimensions the rotation group is Abelian so it does not imply energy degeneracy. In the commutative case the energy spectrum is degenerate due to the existence of dynamical SU(2) symmetry. For $\Theta \neq 0$ the situation appears to be more complicated. For a dense set of values of Θ the energy spectrum *is* degenerate and the dynamical SU(2) symmetry is still responsible for degeneracy; however, the structure of SU(2) multiplets changes abruptly with Θ. The complement set, on which the Hamiltonian has a simple spectrum, is also dense. Therefore, with Θ going to zero, one observes a complicated pattern of nondegenerate and degenerate states, the latter being organized in SU(2) multiplets varying with Θ.

We start with the Hamiltonian for an isotropic oscillator on a noncommutative plane

$$\hat{H} = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{m\omega^2}{2}(\hat{x}_1^2 + \hat{x}_2^2). \tag{2}$$

Let $x_i, p_i, i = 1, 2$ be the standard canonical variables, $[x_i, x_j] = 0$, $[p_i, p_j] = 0$, $[x_i, p_j] = i\hbar\delta_{ij}$. Then the commutation rules (1) can be solved in terms of canonical variables as follows:

$$\hat{x}_i = x_i - \frac{\Theta}{2\hbar}\varepsilon_{ij}p_j, \qquad \hat{p}_i = p_i. \tag{3}$$

With the above Ansatz Eq. (2) takes the form

$$\hat{H} = \frac{1}{2M}(p_1^2 + p_2^2) + \frac{M\Omega^2}{2}(x_1^2 + x_2^2) - \frac{\Theta M\Omega^2}{2\hbar}L \tag{4}$$

where

$$L \equiv x_1 p_2 - x_2 p_1 \tag{5}$$

is the angular momentum and

$$M = \frac{m}{1 + \frac{m^2\omega^2\Theta^2}{4\hbar^2}}, \qquad \Omega = \omega\sqrt{1 + \frac{m^2\omega^2\Theta^2}{4\hbar^2}}. \tag{6}$$

L, as defined by Eq. (5), gives the proper transformation rules for x_i and p_i, $[L, x_i] = i\hbar\varepsilon_{ij}x_j$, $[L, p_i] = i\hbar\varepsilon_{ij}p_j$. In turn, the relations (3) are $O(2)$-covariant so L generates also the proper $O(2)$-transformation rules for initial dynamical variables $\hat{x}_i$, $\hat{p}_i$. Note, however, that L, when expressed in terms of the latter, acquires an additional term,

$$L = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 + \frac{\Theta}{2\hbar}(\hat{p}_1^2 + \hat{p}_2^2).$$

It is straightforward to find the spectrum of $\hat{H}$. To this end we define the relevant creation and annihilation operators,

$$a_\pm \equiv \frac{1}{2\sqrt{M\Omega\hbar}}(p_1 \pm ip_2) - \frac{i}{2}\sqrt{\frac{M\Omega}{\hbar}}(x_1 \pm ix_2); \tag{7}$$

then

$$[a_\alpha, a_\beta] = 0, \qquad [a_\alpha^+, a_\beta^+] = 0, \qquad [a_\alpha, a_\beta^+] = \delta_{\alpha\beta} \tag{8}$$

and $\hat{H}$ takes the form [8]

$$\hat{H} = \hbar\Omega_+\left(N_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(N_- + \frac{1}{2}\right). \tag{9}$$

$N_\pm$ are the standard particle-number operators, $N_\alpha \equiv a_\alpha^+ a_\alpha$, while

$$\Omega_\pm \equiv \Omega \mp \frac{M\Omega^2\Theta}{2\hbar}. \tag{10}$$

Note that, due to (6), $\Omega_\pm > 0$. The eigenvalues of $\hat{H}$ read

$$E_{n_+n_-} = \hbar\Omega_+\left(n_+ + \frac{1}{2}\right) + \hbar\Omega_-\left(n_- + \frac{1}{2}\right) \tag{11}$$

and the relevant eigenvectors are

$$| n_+n_- > = \frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}(a_+^\pm)^{n_+}(a_-^\pm)^{n_-} \mid 0 > . \tag{12}$$

The properties of the spectrum depend on the ratio Ω_+/Ω_-. For irrational Ω_+/Ω_- the spectrum is nondegenerate while rational Ω_+/Ω_- leads to degeneracy. Consider the latter case. Assume that

$$\frac{\Omega_+}{\Omega_-} = \frac{k}{l} \tag{13}$$

where k, l are relatively prime. Equation (13) implies

$$\Theta = \frac{2\hbar}{m\omega}\frac{|\frac{l-k}{l+k}|}{\sqrt{1 - (\frac{l-k}{l+k})^2}}. \tag{14}$$

Let us put

$$\Omega_+ = k\sigma, \qquad \Omega_- = l\sigma, \qquad \sigma \equiv \frac{\omega}{\sqrt{lk}}. \tag{15}$$

Then

$$E_{n_+n_-} = \hbar\sigma(kn_+ + ln_-) + \hbar\sigma\left(\frac{l+k}{2}\right). \tag{16}$$

The last term on the right-hand side is an overall constant. It follows immediately from Eq. (16) that the spectrum is degenerate, the level of degeneracy being equal to the number of natural solutions $n_\pm$ to the equation $kn_+ + ln_- = \text{const}$. Surprisingly enough, the symmetry algebra responsible for degeneracy is always SU(2), like for the isotropic case. To see this we classify all pairs (n_+, n_-) according to their congruence properties [26],

$$\begin{aligned} n_+ &= pl + r_+, \qquad 0 \le r_+ \le l - 1, \\ n_- &= qk + r_-, \qquad 0 \le r_- \le k - 1. \end{aligned} \tag{17}$$

Let us fix $r \equiv (r_+, r_-)$ and let X_r be the subspace spanned by the vectors $| n_+n_- >$ such that n_+, n_- are congruent to r_+, r_- modulo l and k, respectively. Using (17) one can write the energy spectrum of $\hat{H}$, when restricted to X_r, in the form

$$E_{pq} = \hbar kl\sigma(p + q) + \hbar\sigma\left[k\left(r_+ + \frac{1}{2}\right) + l\left(r_- + \frac{1}{2}\right)\right]. \tag{18}$$

Therefore, in each X_r the energy spectrum coincides, up to an additive constant, with that of the isotropic oscillator. Moreover, the degeneracy is possible only among the states belonging to the same X_r [26].

It is also not difficult to find the relevant SU(2) symmetry algebra responsible for degeneracy in each X_r. To this end let us note that each X_r is the Fock space if one makes an identification $| n_+n_- > \cong | p, q>$ and defines

$$\begin{aligned} b_{r_+} | p, q> &= \sqrt{p} \, | p - 1, q>, \\ b_{r_-} | p, q> &= \sqrt{q} \, | p, q - 1>, \\ b_{r_+}^+ | p, q> &= \sqrt{p + 1} \, | p + 1, q>, \\ b_{r_-}^+ | p, q> &= \sqrt{q + 1} \, | p, q + 1 > . \end{aligned} \tag{19}$$

New operators are obviously expressible in terms of a_α, a_α^+. The relevant formulas are slightly complicated and read [26]

$$\begin{aligned} b_{r_+} &= \sqcap_{s=1}^{l}(N_+ + s)^{-(1/2)}\left(\frac{N_+ - r_+}{l}\right)^{(1/2)} a_+^l, \\ b_{r_-} &= \sqcap_{s=1}^{k}(N_- + s)^{-(1/2)}\left(\frac{N_- - r_-}{k}\right)^{1/2} a_-^k. \end{aligned} \tag{20}$$

In spite of their appearance these operators are well defined.

Having constructed new creation-annihilation operators one easily finds symmetry operators. In fact, defining (here σ_i are Pauli matrices)

$$T_{ri} \equiv \frac{1}{2} b_{r\alpha}^+(\sigma_i)_{\alpha\beta} b_{r\beta}, \qquad i = 1, 2, 3 \tag{21}$$

one checks that the following relations hold in χ_r:

$$[T_{ri}, \hat{H}] = 0, \qquad [T_{ri}, T_{rj}] = i\varepsilon_{ijk}T_{rk}. \tag{22}$$

Now, the total Hilbert space is the orthogonal sum of the X_r, $X = \oplus_r X_r$. Let P_r be the projection operator on X_r; define

$$T_i = \sum_r P_r T_{ri} P_r \equiv \sum_r T_{ri} P_r. \tag{23}$$

Then Eqs. (22) extend to the whole Hilbert space χ,

$$[T_i, \hat{H}] = 0, \qquad [T_i, T_j] = i\varepsilon_{ijk}T_k \tag{24}$$

which proves that the symmetry algebra is always SU(2) provided Eq. (14) holds.

What remains is to construct P_r explicitly. Again, it is not difficult to verify that [27]

$$P_r = \left(\frac{1}{l}\sum_{s=0}^{l-1} e^{(2i\pi s/l)(N_+ - r_+)}\right)\left(\frac{1}{k}\sum_{t=0}^{k-1} e^{(2i\pi t/k)(N_- - r_-)}\right). \quad (25)$$

Let us summarize our results. For $\Theta \to 0$ all energy levels tend to their undeformed values. However, this limit is not uniform in quantum numbers n_+, n_-. This results in quite involved Θ behavior of the system. For a dense set of values the energy spectrum is nondegenerate and the Hamiltonian is essentially the only independent operator in the sense that any operator commuting with $\hat{H}$ is a function of $\hat{H}$. On the other hand, for the complement dense set of Θ's obeying Eq. (14) the energy spectrum is degenerate and the symmetry algebra responsible for this degeneracy is always SU(2). However, the structure of symmetry operators and irreducible SU(2) multiplets change very rapidly with the change of Θ. In fact, a slight change of Θ can produce an enormous change of k, l which determine the decomposition of X into the sum of X_r's and the form of symmetry algebra.

This phenomenon has its classical counterpart. If the symplectic structure is modified by imposing $\{x_i, x_j\} = \Theta\varepsilon_{ij}$, the Hamiltonian continues to be integrable for all values of Θ. However, it becomes superintegrable for a dense set of values of Θ while it is not superintegrable for the complementary dense set.

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