Volume integral theorem for exotic matter

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We answer an important question in general relativity about the volume integral theorem for exotic matter by suggesting an exact integral quantifier for matter violating Averaged Null Energy Condition (ANEC). It is checked against some well-known static, spherically symmetric traversable wormhole solutions of general relativity with a sign reversed kinetic term minimally coupled scalar field. The improved quantifier is consistent with the principle that traversable wormholes can be supported by arbitrarily small quantities of exotic matter.

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Traversable wormholes are just as good a prediction of Einstein's general relativity as are black holes. The notion of exotic matter required to construct such wormholes has found a novel justification in the role of dark matter on a cosmological scale [1]. There arises a natural enquiry as to how much of such exotic matter, violating specifically the Averaged Null Energy Condition (ANEC), is required to support a traversable Lorentzian wormhole on a local scale. This question has been recently addressed by Visser, Kar, and Dadhich [2]. Their key result is the proposal of quantifying the total ANEC-violating matter in terms of a volume-integral theorem. Such a theorem is of paramount importance as it has potential implications for topological censorship or positive mass theorems of general relativity. Moreover, given the widespread interest in traversable wormholes in recent years, especially among the physics community, it is imperative that the volume-integral in question be properly identified. The need for a correct quantifier has also been recognized in Ref. [3].

The work in Ref. [2] concluded with a *qualitative* statement about the total amount of exotic matter which states that appropriately chosen traversable wormholes can be supported by arbitrarily small quantities of exotic matter. Let us make it very clear at the outset that we do not contend this important conclusion. Our interest here is different, that is, to know the *exact quantity* of exotic matter present in a given spacetime and we feel that a more reasonable approach could be to first identify the corresponding volume integral and then draw qualitative conclusions (i.e., large or small quantity) as corollaries. The main purpose of this Brief Report is an attempt to do just this. We suggest an improved volume quantifier that checks well against some known examples of static, spherically symmetric traversable wormholes in the Einstein minimally coupled scalar field theory. That is to say, the integral satisfies an important physical criterion in reproducing, to first order, the scalar "charge" present in the solution, and hence the exact quantity of exotic matter.

Let us begin with the volume integration measure dV, viz., "$4\pi R^2 dR$". It is a four dimensional natural measure in the Reissner-Nordström type of solution in curvature coordinates where, strictly, $g_{tt}g_{RR} = -1$. But, if the same measure "$4\pi R^2 dR$" is applied to different spherically symmetric solutions (where $g_{tt}g_{RR} \neq -1$), like the ones we are going to consider, the integral $4\pi \int \rho R^2 dR$ does not reproduce the exotic mass (scalar charge in our case). Of course, it may still approximate the exact value in some way, but, as mentioned, our interest here is in exact values. Similar comments apply also to the measure "$\sqrt{g_3}d^3x$" (where g_3 is the determinant of the spatial part of the metric). All these will be evident in what follows.

Consider the Morris-Thorne-Yurtsever (MTY) [4] form of a static, spherically symmetric wormhole in the curvature coordinates (t, R, θ, φ) (We take $G = c = 1$):

$$ds^2 = -\exp[2\phi]dt^2 + \frac{dR^2}{1 - b(R)/R} + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

The throat of the wormhole occurs at $R = R_0$ such that $b(R_0) = R_0$, and we assume $\exp[2\phi(R_0)] \neq 0$. The density and pressures can be calculated in the static orthonormal frame. Then, using these, one has

$$\Omega = 4\pi \times \int_{R_0}^{\infty}[\rho + p_R] \times R^2 dR \equiv \left[(R - b)\ln\left(\frac{\exp[2\phi]}{1 - b/R}\right)\right]_{R_0}^{\infty} - \int_{R_0}^{\infty}(1 - b')\left[\ln\left(\frac{\exp[2\phi]}{1 - b/R}\right)\right]dR, \tag{2}$$

where the prime denotes differentiation with respect to R. Additionally, the boundary term has been assumed to be zero and the second (integral) part has been proposed in

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Ref. [2] as the volume-integral theorem that provides information about the total amount of ANEC-violating matter in the spacetime. The boundary term can be made to vanish in several ways. One possibility is that, $\phi(R)$ and $b(R)$ be asymptotically Schwarzschild [2], i.e., $\phi(R) \sim -m/R + O(R^{-2})$ and $b(R) \sim 2m + O(R^{-1})$. An useful alternative could be $\phi(R) \sim O(R^{-2})$ and $b(R) \sim O(R^{-1})$, given the possibility of a host of traversable wormholes one is free to construct. However, for our purposes, we require that the spacetime be only asymptotically flat although, just incidentally, the examples we are going to consider follow the first set of asymptotic Schwarzschild behavior.

To get an idea of the value of the charge we want to retrieve, consider the Einstein minimally coupled scalar field theory given by the field equations

$$R_{\mu\nu} = -2\Phi_{,\mu}\Phi_{,\nu} \tag{3}$$

$$\Phi^{;\mu}_{;\mu} = 0, \tag{4}$$

where $\mu, \nu = 0, 1, 2, 3$; Φ is the scalar field, $R_{\mu\nu}$ is the Ricci tensor and the semicolon denotes covariant derivatives with respect to the metric $g_{\mu\nu}$. The minus sign on the right implies that the scalar field has a negative kinetic energy so that the stresses violate energy conditions (ghost scalar field) [5]. These field equations are just the vacuum Jordan frame Brans-Dicke equations rewritten in the conformally rescaled Einstein frame. (They also follow from the vacuum low energy string theory in four dimensions). Scheel, Shapiro, and Teukolsky [6] have shown that the general asymptotically flat, static solution has the asymptotic form

$$\begin{aligned} g_{00} &= -1 + \frac{2M_T}{r}, \qquad g_{0i} = 0, \\ g_{ij} &= 1 + \delta_{ij}\left(\frac{2M_T}{r}\right), \qquad \Phi = 1 + \frac{2M_S}{r}, \end{aligned} \tag{5}$$

where $i, j = 1, 2, 3$; r is the isotropic radial variable, M_T and M_S are the tensor and (exotic) scalar masses, respectively. M_S may be termed as scalar charge on one side of the wormhole. Our viewpoint is that it is this M_S that the desired quantifier should first reproduce, and thereby justify itself, before it can be employed to assess the total ANEC-violating matter. To this end, we now state our ANEC volume-integral

$$\Omega_{\rm ANEC} = \int_{x_{th}}^{\infty} \iint [T_{\mu\nu}k^\mu k^\nu]\sqrt{-g_4}d^3x, \tag{6}$$

for null k^μ, stress tensor $T_{\mu\nu}$, $g_4 \equiv \det|g_{\mu\nu}|$ and the throat at x_{th}. Some additional comments are in order here. We have picked up the integration measure "$\sqrt{-g_4}d^3x$" from the general relativity conservation law with the difference that the integration is taken from x_{th} to ∞ because of the allowed coordinate range in wormhole geometry. It should be applicable to any spacetime that is asymptotically Minkowskian. In the simple case of spherical symmetry, assuming that the ANEC-violating matter is related only to p_r, and not to the transverse components [2], we have

$$\Omega_{\rm ANEC} = \int_{r_0}^{\infty}\int_0^{\pi}\int_0^{2\pi}[\rho + p_r]\sqrt{-g_4}drd\theta d\varphi, \tag{7}$$

where r_0 is the throat radius. We want to try Eq. (7) with some examples below.

The form of a certain exact general class of solutions of the Eqs. (3) and (4) is given in isotropic coordinates (t, r, θ, φ) by:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -e^{2\phi(r)}dt^2 + e^{-2\psi(r)}[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2], \end{aligned}$$

$$\phi(r) = \psi(r) = -\frac{M}{r}, \qquad \Phi(r) = 1 - \frac{M}{r}. \tag{8}$$

This solution was actually proposed by Yilmaz [7] decades ago, in fact a few years earlier than the advent of Brans-Dicke theory. However, it follows also from the Brans-Dicke theory under conformal rescaling. For this solution $M_T = M$ and $M_S = -M/2$. The metric in Eqs. (8) exactly coincides up to second order with the Schwarzschild metric in isotropic coordinates. That is, the solution describes all the weak field tests of general relativity just as exactly as the Schwarzschild metric does for $r > M/2$. It is actually a singularity free solution as the curvature scalars are all zero at $r = 0$ and at $r = \infty$, and thus the solution has two asymptotically flat regions. The tidal forces are finite everywhere. In fact, it satisfies all the five conditions laid down by Visser [8] for any isotropic form to qualify as a traversable wormhole (see Ref. [9] for more details). The throat appears at $r_0 = M$. Calculations of the energy density (ρ) and pressures (p_r, p_θ, p_φ) give $\rho = -f$, $p_r = -f$, $p_\theta = p_\varphi = f$ where $f \equiv (\frac{1}{8\pi})M^2r^{-4}e^{-2M/r} > 0$. That is, both the Weak Energy Condition ($\rho \geq 0$) and NEC ($\rho + p_r \geq 0$) are violated, as expected in a spacetime containing wormholes. With these expressions, the integral (7) converges and immediately gives the values for the scalar charge $\Omega^{p_r=0}_{\rm ANEC} = -M/2 \equiv M_S$, and the total ANEC-violating mass $\Omega_{\rm ANEC} = -M$, no matter whatever coordinate network we use. These results fundamentally confirm the validity of our integral. Returning to the MTY form via the transformation $R = r\exp[M/r]$ (Note that for both $r \to 0$ and $r \to \infty$, we have $R \to \infty$ and the throat now occurs at $R_0 = Me$), and calculating with (2), we find $\Omega^{p_r=0} = M(1 - e/2) \neq -M/2$ and $\Omega = M(2 - e) \neq -M$. The use of $\sqrt{g_3}d^3x$ measure in (2) instead of "$4\pi R^2dR$" measure gives $\Omega = M(1 - e) = -1.71M$, none of which obviously coincides with the desired value. Now we do have here a traversable wormhole with ANEC-violating mass $\Omega_{\rm ANEC} = -M$, but how can we make it arbitrarily small? We can let $M \to 0$ to achieve it, but that would mean that we approach the trivial Minkowski spacetime!

Let us consider a second, but qualitatively different example provided by another class of exact solutions of the set (3) and (4):

$$\phi(r) = \beta \ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right],$$
$$\psi(r) = (\beta - 1)\ln\left(1 - \frac{m}{2r}\right) - (\beta+1)\ln\left(1+\frac{m}{2r}\right), \tag{9}$$
$$\Phi(r) = 1 + [\beta^2 - 1]^{1/2}\ln\left[\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right].$$

It was proposed in that form by Buchdahl [10] long ago, but it can also be obtained from the Brans-Dicke solution by conformal rescaling. The two undetermined constants m and β are related to the source strengths of the gravitational and scalar parts of the configuration. The tensor mass responsible for known gravitational effects appears always as a product $M_T = m\beta$ so that weak field effects can not separately measure the components. Once the scalar component is set to a constant value ($\Phi = 1 \Rightarrow \beta = 1$), the solutions (9) reduce to the Schwarzschild black hole in accordance with Wheeler's "no scalar hair" conjecture. Physically, this indicates the possibility that the scalar field could be radiated away during collapse and the end result is a Schwarzschild black hole. But for $\beta \neq 1$, the solution has a naked singularity at $r_{\mathrm{NS}} = m/2$. However, the throat occurs at $r_0^+ = \frac{m}{2}[\beta + (\beta^2-1)^{1/2}] > r_{\mathrm{NS}}$ and it is known that the solution represents a traversable wormhole as it also shows $\rho = -h$, $p_r = -h$, $p_\theta = p_\varphi = h$ with corresponding expression for $h > 0$ [11]. The scalar field expands like: $\Phi \approx 1 - (m/r)\sqrt{\beta^2-1} + O(1/r^2)$ and provides a charge $M_S = -(m/2)\sqrt{\beta^2-1}$. Using (7), we find,

$$\Omega_{\mathrm{ANEC}}^{p_r=0} = -\left(\frac{m}{4}\right) \times (\beta^2-1) \times \ln\left[\frac{1+1/\beta}{1-1/\beta}\right]$$
$$\approx -\left(\frac{m}{2}\right)\sqrt{\beta^2-1} \times \left(1 - \frac{1}{2\beta^2}\right), \tag{10}$$

from which one can read off the scalar charge. Also, like the first example, $\Omega_{\mathrm{ANEC}} = 2\Omega_{\mathrm{ANEC}}^{p_r=0}$. An interesting corollary from Eq. (10) is the following: Consider the total energy denoted by, say, $\overline{M} = M_T + \Omega_{\mathrm{ANEC}}^{p_r=0}$. At $\beta = 1$, of course, $\overline{M} = m$, but it turns out that, as β increases from the value 1, the quantity $\overline{M}$ decreases to a minimum value $\overline{M} \approx 0.93m$ at $\beta \approx 1.16$, and again *increases* to $\overline{M} = m$ at around $\beta \approx 1.51$. Thereafter, $\overline{M}$ continues to grow beyond the value m almost linearly with increasing β. These informations allow us to visualize how the total mass changes as one increases the component of ghost energy in the configuration. However, returning to our topic, the metric in (9) can be transformed to MTY form under $R = re^{-\psi}$ and we can compute (2) with the "$4\pi R^2 dR$" or any other measure but that would not give us (10). Now, one can make $\Omega_{\mathrm{ANEC}} \to 0$ by tuning $\beta \to 1+$, in which case, the solution gradually approximates to the vacuum Schwarzschild solution.

Finally, keeping in mind that the volume integral in (7) is neatly supported by our known wormhole examples, it is curious to see what result it gives for the "$R = 0$" self-dual wormhole [2,12] for which $\rho = 0$. It is helpful to have the solution in view:

$$ds^2 = -\left[\varepsilon + \lambda\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2\right]dt^2 + \left(1+\frac{m}{2r}\right)^4[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2], \tag{11}$$

where ε and λ are arbitrary constants. The Schwarzschild solution is recovered at the value $\varepsilon = 0$. The Eq. (7) works out simply to

$$\Omega_{\mathrm{ANEC}}^{\rho=0} = -2m\varepsilon \ln r]_{m/2}^{\infty} \tag{12}$$

If aesthetics is any guiding principle, Eq. (12) amply satisfies it when contrasted with the expressions one obtains otherwise. It shows that it is ε that controls the amount of ANEC-violating matter. Unfortunately, Eq. (12) together with similar expressions computed from the metric (11) show an asymptotic logarithmic divergence. What does it tell us? One possibility immediately suggests itself: Set ε identically to zero, that is, conclude that an asymptotically flat spacetime with $\rho = 0$ can only be a Schwarzschild vacuum ($\Omega_{\mathrm{ANEC}}^{\rho=0} = 0$). A more interesting possibility is to truncate the spacetime such that the exotic matter lies only within the fixed radii $(\frac{m}{2}, a]$ beyond which the spacetime is exactly Schwarzschild [2]. With Eq. (12), the limiting arguments appear simpler and transparent. We have

$$\Omega_{\mathrm{ANEC}}^{\rho=0} = -2m\varepsilon \ln\left[\frac{2a}{m}\right], \tag{13}$$

so that $\Omega_{\mathrm{ANEC}}^{\rho=0} \to 0$ as $a \to m/2$ and/or $\varepsilon \to 0$.

To summarize, our key suggestion is the volume quantifier given in Eq. (6): In the simplest case of spherical symmetry, it has justified itself by retrieving the exact quantity of scalar charge in the first example. Its use in the second example has thrown up an expression for the exotic mass, viz., Eq. (10), which is not obvious *a priori* and it also provides some new insights into the behavior of total mass. When applied to the self-dual case, Eq. (6) yields a very sensible result. Finally, as a corollary, it is found to be consistent with the principle that the ANEC-violating matter can be made arbitrarily small [13]. It would be worthwhile to examine the integral (6) in non-spherically symmetric cases. This is a task for the future.

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[13] It would be very desirable to have a formulation of a *minimum* principle with Eq. (6), such that $\delta\Omega_{\mathrm{ANEC}} = 0$, that does not rely on specific geometries, but relies only on the general wormhole behavior of ϕ and b.