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Adventures in thermal duality. II. Towards a duality-covariant string thermodynamics

Keith R. Dienes* and Michael Lennek†

Department of Physics, University of Arizona, Arizona 85721, USA

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In a recent companion paper, we observed that the rules of ordinary thermodynamics generally fail to respect thermal duality, a symmetry of string theory under which the physics at temperature T is related to the physics at the inverse temperature $1/T$. Even when the free energy and internal energy exhibit the thermal duality symmetry, the entropy and specific heat are defined in such a way that this symmetry is destroyed. In this paper, we propose a modification of the traditional definitions of these quantities, yielding a manifestly duality-covariant thermodynamics. At low temperatures, these modifications produce corrections to the standard definitions of entropy and specific heat which are suppressed by powers of the string scale. These corrections may nevertheless be important for the full development of a consistent string thermodynamics. We find, for example, that the string-corrected entropy can be smaller than the usual entropy at high temperatures, suggesting a possible connection with the holographic principle. We also discuss some outstanding theoretical issues prompted by our approach.

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I. INTRODUCTION

In a recent companion paper [1], we observed that the rules of ordinary thermodynamics generally fail to respect the thermal duality symmetry of string theory under which the physics at temperature T is related to the physics at temperature T_c^2/T, where T_c is a critical (or self-dual) temperature related to the string scale. The reason for this failure is simple: Even though the string vacuum amplitude $\mathcal{V}(T)$ might exhibit an invariance under this symmetry, with $\mathcal{V}(T) = \mathcal{V}(T_c^2/T)$, the subsequent temperature derivatives d/dT that are needed in order to calculate other thermodynamic quantities generally destroy this symmetry. This then results in quantities such as entropy and specific heat which fail to exhibit thermal duality symmetries.

It is, of course, entirely possible that thermal duality should be viewed only as an "accidental" symmetry of the string vacuum amplitude; we thus would have no problem with the loss of this symmetry when calculating other thermodynamic quantities. However, given that the thermal duality symmetry of string theory follows immediately from T duality and Lorentz invariance, this symmetry appears to be every bit as deep as the dualities that occur at zero temperature. Thus, it seems more natural to consider thermal duality as a fundamental property of a consistent string theory, and demand that this symmetry hold for *all* physically relevant thermodynamic quantities.

This problem was considered in Ref. [1], where it was shown that there exist certain special vacuum amplitudes $\mathcal{V}(T)$ for which all thermodynamic quantities exhibit the thermal duality symmetry. Moreover, it was shown that these solutions for $\mathcal{V}(T)$ correspond to highly symmetric string modular integrals in which the time/temperature direction is compactified on S^1 (a circle) or $S^1/\mathbb{Z}_2$ (a line segment). Thus, for such constructions, there is no loss of thermal duality. In fact, as discussed in Ref. [1], the constraint of thermal duality in such cases may be of sufficient strength to enable an exact, closed-form evaluation of the relevant thermodynamic quantities.

However, this method of restoring thermal duality is less than satisfactory. Because this approach applies only for certain selected ground states, it lacks the generality that should apply to a fundamental symmetry. If thermal duality is to be considered an intrinsic property of finite-temperature string theory (akin to T duality), then the formulation of the theory itself—including its rules of calculation—should respect this symmetry regardless of the specific ground state.

This argument should apply even if the specific string ground state in question does not exhibit thermal duality (such as may occur in finite-temperature string constructions utilizing temperature-dependent Wilson lines). Indeed, even when thermal duality is "spontaneously broken" in this way, the theoretical definitions of all relevant physical thermodynamic quantities should still reflect this duality symmetry. After all, it is certainly acceptable if the entropy or specific heat fail to exhibit thermal duality because the ground state fails to yield a duality-symmetric vacuum amplitude $\mathcal{V}(T)$. However, it is not acceptable if this failure arises because the *definitions* of the entropy or specific heat in terms of $\mathcal{V}(T)$ are themselves not duality covariant.

For this reason, we are motivated to develop an alternative, fully covariant string thermodynamics in which thermal duality is manifest. This is the goal of the present paper (a short outline of our basic ideas also appears in Ref. [2]). As we shall see, this new duality-covariant

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*Electronic address: dienes@physics.arizona.edu

†Electronic address: mlennek@physics.arizona.edu

framework will preserve the definitions of free energy and internal energy, but will lead naturally to modifications in the usual thermodynamics definitions for other quantities such as entropy and specific heat. At low temperatures, these modifications produce "corrections" to the standard definitions of entropy and specific heat which are suppressed by powers of the string scale. These corrections are therefore unanticipated from the low-energy (low-temperature) point of view. At higher temperatures, however, these modifications are significant, and may be important for a full understanding of string thermodynamics at or near the self-dual temperature T_c. In fact, we shall find that our new, string-corrected entropy is often smaller than the usual entropy, with the suppression becoming increasingly severe as the temperature approaches the string scale. This suggests an intriguing connection with the holographic principle, and leads to some novel speculations concerning the physics near the critical temperature.

II. THERMAL DUALITY AND TRADITIONAL THERMODYNAMICS

Thermal duality [3–8] is a symmetry which relates string thermodynamics at temperature T to string thermodynamics at the inverse temperature T_c^2/T. Here T_c is a critical, self-dual temperature which is ultimately set by the string scale. It is easy to see how thermal duality emerges. In string theory (just as in ordinary quantum field theory), finite-temperature effects can be incorporated [8,9] by compactifying an additional time dimension on a circle of radius $R_T = (2\pi T)^{-1}$. However, Lorentz invariance guarantees that the properties of this extra time dimension should be the same as those of the original space dimensions, and T duality [10–12] tells us that the closed string theory on a compactified space dimension of radius R is indistinguishable from that on a space of radius R_c^2/R, where $R_c \equiv \sqrt{\alpha'} = M_{\text{string}}^{-1}$ is a critical, self-dual radius. Together, these two symmetries thus imply a thermal duality symmetry under which the physics (specifically the one-loop string partition function Z_{string}) should be invariant with respect to the thermal duality transformation $T \to T_c^2/T$:

$$Z_{\text{string}}(T_c^2/T) = Z_{\text{string}}(T), \tag{2.1}$$

where $T_c \equiv M_{\text{string}}/2\pi$. Note that this symmetry holds to all orders in perturbation theory [6].

All thermodynamic quantities of interest are generated from this partition function. The one-loop vacuum amplitude $\mathcal{V}(T)$ is given by [3,9,13]

$$\mathcal{V}(T) \equiv -\frac{1}{2}\mathcal{M}^{D-1}\int_{\mathcal{F}}\frac{d^2\tau}{(\text{Im}\,\tau)^2}Z_{\text{string}}(T), \tag{2.2}$$

where $\mathcal{M} \equiv M_{\text{string}}/2\pi$ is the reduced string scale; D is the spacetime dimension; τ is the complex modular parameter describing the shape of the one-loop toroidal world sheet; and $\mathcal{F} \equiv \{\tau : |\text{Re}\,\tau| \leq \frac{1}{2}, \text{Im}\,\tau > 0, |\tau| \geq 1\}$ is the fundamental domain of the modular group. In general, $\mathcal{V}(T)$ plays the role normally assumed by $-\ln Z$, where Z is the usual thermodynamic partition function in the canonical ensemble. Given this definition for $\mathcal{V}$, the free energy F, internal energy U, entropy S, and specific heat c_V follow from the usual thermodynamic definitions:

$$\begin{aligned} F &= T\mathcal{V}, & U &= -T^2\frac{d}{dT}\mathcal{V}, \\ S &= -\frac{d}{dT}F, & c_V &= \frac{d}{dT}U. \end{aligned} \tag{2.3}$$

It follows directly from these definitions that $U = F + TS$ and that $c_V = TdS/dT$. Note that $\Lambda \equiv \lim_{T\to 0}F(T)$ is the usual one-loop zero-temperature cosmological constant.

It is straightforward to determine the extent to which the thermal duality exhibited by Z_{string} in Eq. (2.1) is inherited by its descendants in Eqs. (2.2) and (2.3). Since $\mathcal{V}$ is nothing but the modular integral of Z_{string}, the invariance of Z_{string} under the thermal duality transformation immediately implies the invariance of $\mathcal{V}$:

$$\mathcal{V}(T_c^2/T) = \mathcal{V}(T). \tag{2.4}$$

Similarly, from its definition in Eq. (2.3), we find that F transforms *covariantly* under thermal duality:

$$F(T_c^2/T) = (T_c/T)^2F(T). \tag{2.5}$$

Thus, F also respects the thermal duality symmetry. Finally, it is easy to verify that the internal energy U also transforms covariantly under thermal duality:

$$U(T_c^2/T) = -(T_c/T)^2U(T). \tag{2.6}$$

The overall minus sign in this duality transformation has the net effect of fixing a zero for the internal energy such that it vanishes at the self-dual temperature, $U(T_c) = 0$. Since $dU/dT > 0$ for all $T < T_c$, this zero energy requires $U(T) < 0$ for $T < T_c$.

Unfortunately, the entropy and specific heat fail to transform either invariantly or covariantly under the duality transformation. In other words, these quantities fail to transform as *bona fide* representations of the thermal duality symmetry. If thermal duality is indeed a fundamental property of string theory, the failure of the entropy and specific heat to transform covariantly under thermal duality suggests that these quantities are improperly defined from a string-theoretic standpoint. At best, they are not the proper "eigenquantities" which should correspond to physical observables.

It is straightforward to determine the source of the difficulty. Even though Z_{string} and $\mathcal{V}$ are thermal duality invariant, the passage to the remaining thermodynamic quantities involves the mathematical operations of multiplication by, and differentiation with respect to, the temperature T. While multiplication by T preserves covariance under the thermal duality symmetry, differentiation with respect to T generally does not. Indeed, although the derivative in the definition for $U(T)$ happens to preserve thermal duality covariance, this covariance is broken by the subsequent differentiations which are needed to construct the entropy and specific heat.

The problem of a derivative failing to preserve a symmetry is an old one in physics; the solution is to construct the analogue of a covariant derivative. This procedure is well known in gauge theories, where the need to construct a covariant derivative respecting the local gauge symmetry requires the introduction of an entirely new degree of freedom, namely, the gauge field. Fortunately, in the present case of the thermal duality, the situation is far simpler.

III. MODULAR INVARIANCE AND THRESHOLD CORRECTIONS: AN ANALOGY AND SOME HISTORY

As a digression, let us first consider an analogous case involving modular invariance. This case will be mathematically similar to the case of thermal duality transformations. In general, a modular-covariant function $f(\tau)$ is one for which

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau), \tag{3.1}$$

for all $a, b, c, d \in \mathbb{Z}$ with $ad-bc=1$. The quantity k is called the modular weight of f. Note that the special case with $(a, b, c, d)=(0,-1,1,0)$ yields the modular transformation $\tau \longrightarrow -1/\tau$, which is very similar to the thermal duality transformation. However, if f is modular covariant with weight k, it is easy to verify that $df/d\tau$ is not modular covariant; in other words, $d/d\tau$ is not a modular-covariant derivative. Instead, the appropriate modular-covariant derivative acting on a modular function of weight k is

$$D_\tau \equiv \frac{d}{d\tau}-\frac{ik}{2\mathrm{Im}\,\tau}. \tag{3.2}$$

This ensures that if f is a modular-covariant function of weight k, then $D_\tau f$ is also modular covariant, with weight $k+2$.

The existence of this modular-covariant derivative is not merely a mathematical nicety: It turns out to play an important role in calculating string threshold corrections to low-energy gauge couplings [14]. Recall that, in string theory, the partition function $Z_{\text{string}}(\tau)$ is a modular-invariant trace over all states in the string Fock space, where τ is the complex parameter describing the shape of the torus (one-loop diagram); the final result can generally be written as a sum of products of modular-covariant functions $f(\tau)$ and their complex conjugates. However, in order to calculate threshold corrections to the running of the low-energy gauge couplings due to the infinite towers of massive string states, the rules of ordinary quantum field theory instruct us to calculate a slightly different trace over the Fock space in which the contribution from each state is now multiplied by its squared gauge charge [15]. However, it turns out that multiplication by the squared gauge charge in the trace is mathematically equivalent to replacing certain occurrences of f in the final partition function with the derivative $df/d\tau$, thereby breaking the underlying modular invariance of the theory. Thus, it appears that the usual calculations inherited from quantum field theory lead to results which fail to respect the underlying string symmetries.

This state of affairs persisted for almost a decade until it was found [16] that a full *string* calculation performed in the presence of a suitable infrared regulator introduces additional unexpected contributions to the threshold corrections. Remarkably, these extra contributions correspond to adding the second term in Eq. (3.2) to each occurrence of $df/d\tau$, thereby elevating the noncovariant derivative $d/d\tau$ into the full covariant derivative D_τ. These extra contributions are intrinsically gravitational in origin, arising from spacetime curvature backreactions and dilaton tadpoles, and thus would not have been anticipated from a straightforward field-theoretic derivation. However, these extra contributions are precisely what are needed to restore modular invariance to the full string threshold calculation, as expected from the string perspective. A review of this situation can be found in Ref. [14].

The lesson from this example is clear: Although field-theoretic considerations may suggest the existence of certain derivatives in the definitions of physically relevant quantities, a full string calculation of these quantities should involve only those covariant forms of these derivatives which respect the underlying string symmetries. What we are proposing, then, is to follow this example in the case of the temperature derivatives appearing in traditional string thermodynamics, using thermal duality covariance as our guide.

IV. THERMAL DUALITY-COVARIANT DERIVATIVES

We shall now proceed to construct our thermal duality-covariant derivatives. We begin with a mathematical definition: If function $f(T)$ has the duality transformation

$$f(T_c^2/T)=\gamma(T_c/T)^k f(T), \tag{4.1}$$

with $\gamma=\pm1$, we shall say that $f(T)$ is a thermal duality-covariant function with "weight" k and sign ±1 ("even" or "odd"). Note that $\gamma=\pm1$ are the most general coefficients which preserve the $\mathbb{Z}_2$ nature of the thermal duality transformation.

It is easy to verify that multiplication by T is a covariant operation, resulting in a function with weight $k+2$ and the same sign for γ. Our goal, however, is to construct a thermal duality-covariant derivative. Towards this end, let us imagine that this derivative takes the general form

$$D_T = \frac{d}{dT} + \frac{g(T)}{T}, \tag{4.2}$$

where $g(T)$ is a function of T and T_c. Explicitly evaluating $[D_T f](T_c^2/T)$ using Eqs. (4.1) and (4.2), we then find that $D_T f$ will be duality covariant with weight $k-2$ and sign $-\gamma$, i.e.,

$$[D_T f](T_c^2/T) = -\gamma(T_c/T)^{k-2}[D_T f](T), \tag{4.3}$$

only if $g(T)$ satisfies the constraint

$$g(T) + g(T_c^2/T) = -k. \tag{4.4}$$

Note that $g(T) = 0$ is *not* a solution if $k \neq 0$; in other words, for nonzero k, we *must* make an additional contribution to the ordinary temperature derivative in order to preserve duality covariance. Indeed, since Eq. (4.4) must hold for all T, the function $g(T)$ must be proportional to the weight k. Our task is then to find a suitable function $g(T)$.

In principle, there may be many functions $g(T)$ which satisfy Eq. (4.4). However, again taking duality covariance as our guide, let us suppose that $g(T)$ is itself a duality-covariant function with weight α and sign γ_g:

$$g(T_c^2/T) = \gamma_g (T_c/T)^\alpha g(T). \tag{4.5}$$

Substituting this into Eq. (4.4), we then obtain a solution for $g(T)$:

$$g(T) = -\frac{k}{1+\gamma_g (T_c/T)^\alpha} = -\frac{kT^\alpha}{T^\alpha + \gamma_g T_c^\alpha}. \tag{4.6}$$

Thus, this solution for $g(T)$ ensures a duality-covariant derivative for all α and γ_g.

Thus far, the values of α and γ_g are unfixed. In certain circumstances, however, we can impose various physical constraints in order to narrow the range of possibilities. For example, we might wish to demand that our covariant derivative reduce to the usual derivative as $T/T_c \to 0$, with only small corrections suppressed by inverse powers of T_c. In other words, we wish to demand

$$\frac{g(T)f}{T} \ll \frac{df}{dT} \quad \text{as } T/T_c \to 0. \tag{4.7}$$

With $g(T)$ given by Eq. (4.6), this generally restricts us to the cases with $\alpha > 1$, although this constraint can be evaded or strengthened depending on the specific function f. Likewise, if we wish to retain the usual symmetry under which the temperature derivative is odd under $T \to -T$, we should require $\alpha \in 2\mathbb{Z}$, although once again this constraint is not mandatory. Finally, we would like our covariant derivatives to remain finite as $T \to T_c$. Thus, we shall restrict our attention to the cases with $\gamma_g = +1$, deferring our discussion of the $\gamma_g = -1$ case to Sec. VII. We shall, however, leave α as a free (positive) parameter.

Thus, combining our results and taking $\gamma_g = +1$, we obtain a thermal duality-covariant derivative given by

$$D_T = \frac{d}{dT} - \frac{k}{T}\frac{T^\alpha}{T^\alpha + T_c^\alpha}. \tag{4.8}$$

In this derivative, the second term functions as a "correction" term which is suppressed when $T \ll T_c$, but which grows large as the temperature approaches the string scale. Indeed, α essentially governs the *rate* at which our correction term becomes significant as $T \to T_c$. Of course, the presence of this correction term is critical, ensuring that if f is covariant with weight k and sign γ, then $D_T f$ is also covariant, with weight $k-2$ and sign $-\gamma$. Note that, unlike the case with modular transformations, there is no thermal duality-covariant derivative which preserves the sign of γ.

It may seem strange that our covariant derivative depends on k, which is a property of the function upon which the derivative operates. However, this is completely analogous to the situation we have just discussed for modular invariance in Sec. III. Indeed, even in gauge theory, the gauge-covariant derivative depends on the gauge charge of the state on which it operates. In this analogy, k functions as the duality "charge" of the function $f(T)$, and the remaining factor $T^\alpha/(T^\alpha + T_c^\alpha)$ functions as the duality "gauge field" (i.e., as the connection).

In principle, the value of α is unconstrained as long as $\alpha > 1$. We note, however, that in the limit as $T \to T_c$, the covariant derivative in Eq. (4.8) takes the limiting form

$$D_T \to \frac{d}{dT} - \frac{k}{2T}. \tag{4.9}$$

This is the direct analogue of Eq. (3.2), and is equivalent to the general derivative in Eq. (4.8) with $\alpha = 0$. Thus, the $\alpha = 0$ case will continue to have relevance at the critical temperature T_c. Moreover, as we shall see in Sec. VII, this derivative has another important property as well.

We stress that this form for the covariant derivative is not unique. In principle, any function $g(T)$ satisfying Eq. (4.4) could serve in the construction of a covariant derivative. Of course, physically sensible solutions for $g(T)$ must have the property that $g(T)/T \to 0$ as $T/T_c \to 0$, so that our corrections vanish at small temperatures and traditional thermodynamics is restored. Likewise, at the other extreme, we see directly from Eq. (4.4) that there are only two possibilities as $T \to T_c$: Either $g(T)$ remains finite, in which case we must have $g(T) \to -k/2$, or $g(T)$ diverges, in which case we must have $g(T) \to \pm\infty$ as $T \to T_c^{\mp}$. In the former case, we necessarily obtain the covariant derivative (4.9) as $T \to T_c$, regardless of the specific solution for $g(T)$. The specific solution for $g(T)$ therefore serves only to interpolate between the fixed $T \to 0$ and $T \to T_c$ limits.

Presumably, the specific form of $g(T)$ [and if $g(T)$ is covariant, the specific value of α] can be determined through a full string calculation including gravitational backreactions (analogous to the calculation performed in Ref. [16]) in which this covariant derivative is obtained from first principles. However, the important point from our analysis is that there is *necessarily* a string-suppressed correction term which must be added to the usual temperature derivative, and that its form is already significantly constrained, especially in the $T \to T_c$ limit. Thus, we shall continue to use the covariant derivative (4.8) in the following, even though we must bear in mind that other solutions for $g(T)$ may exist.

V. A DUALITY-COVARIANT STRING THERMODYNAMICS

Given this covariant derivative, we can now construct a manifestly covariant thermodynamics: Our procedure is simply to replace all derivatives in Eq. (2.3) with the duality-covariant derivative in Eq. (4.8). We thus obtain

$$\begin{aligned} \tilde{F} &= T\mathcal{V}, & \tilde{U} &= -T^2 D_T \mathcal{V}, \\ \tilde{S} &= -D_T \tilde{F}, & \tilde{c}_V &= D_T \tilde{U}. \end{aligned} \tag{5.1}$$

The tildes are inserted to emphasize that the new quantities we are defining need not, *a priori*, be the same as their traditional counterparts.

Let us now determine the implications of these definitions. Of course, since $\mathcal{V}$ is duality *invariant* (i.e., $\mathcal{V}$ has $k = 0$ with $\gamma = +1$), we see that $\tilde{F}$ continues to be covariant with $k = 2$ and $\gamma = +1$. Thus, the free energy $F(T)$ is unaltered: $\tilde{F} = F$. This is expected, since we saw in Eq. (2.5) that F is already thermal duality covariant.

A similar situation exists for the internal energy $\tilde{U}(T)$. Since $\mathcal{V}$ is covariant with $k = 0$, we see that the covariant derivative D_T in this special case is exactly the same as the usual derivative d/dT. Thus, the internal energy is also unaffected: $\tilde{U} = U$. Of course, this also makes sense, since $U(T)$ was already seen to be covariant in Eq. (2.6), with $k = 2$ and $\gamma = -1$. However, this example illustrates that any duality-covariant quantity can (and should) be expressed in terms of covariant derivatives. Thus, the internal energy U continues to fit into our overall framework involving only those derivatives.

We now turn our attention to $\tilde{S}$ and $\tilde{c}_V$. It is in these cases that new features arise. Since $\tilde{F} = F$ is already covariant with $k = 2$ and $\gamma = +1$, we find that

$$\tilde{S} = -D_T F = S + \frac{2T^{\alpha-1}F}{T^\alpha + T_c^\alpha} = S + \frac{2T^\alpha}{T^\alpha + T_c^\alpha}\mathcal{V}. \tag{5.2}$$

Thus, we see that the "corrected" entropy $\tilde{S}$ differs from the usual entropy S by the addition of an extra string-suppressed term proportional to the free energy F. Indeed, it is this corrected entropy $\tilde{S}$ which is thermal duality covariant, transforming with $k = 0$ and $\gamma = -1$. Interestingly, since $\tilde{S}$ is finite and odd, we see that the corrected entropy has a zero at the critical temperature: $\tilde{S}(T_c) = 0$. This resembles the situation with the internal energy U, which also vanishes at $T = T_c$; in fact, we find

$$\tilde{S} \to S + \frac{F}{T} = \frac{U}{T} \quad \text{as } T \to T_c. \tag{5.3}$$

Of course, both of these properties differ significantly from our usual expectations.

Since our corrections to the entropy are suppressed by powers of the string scale, we see that $\tilde{S}$ continues to obey the third law of thermodynamics, with $\tilde{S} \to 0$ as $T \to 0$ in situations with a massless unique ground state. As discussed in Sec. IV, this is the result of requiring $\alpha > 1$. However, imposing our general condition in Eq. (4.7), we find that we must actually restrict ourselves to values of α for which

$$\frac{2T^{\alpha-1}F}{T^\alpha + T_c^\alpha} \ll S \quad \text{as } T/T_c \to 0. \tag{5.4}$$

In general, depending on the particular thermodynamic system under study, this can yield constraints which are stronger than $\alpha > 1$.

Finally, the corrected specific heat is given by

$$\tilde{c}_V = D_T U = c_V - \frac{2T^{\alpha-1}U}{T^\alpha + T_c^\alpha} = c_V + \frac{2T^{\alpha+1}}{T^\alpha + T_c^\alpha}\frac{d\mathcal{V}}{dT}. \tag{5.5}$$

Thus, the difference between the uncorrected and corrected specific heats is a string-suppressed term proportional to the internal energy U. Since U vanishes at $T = T_c$ as a result of thermal duality, the corrected specific heat $\tilde{c}_V$ approaches the uncorrected specific heat c_V both as $T \to 0$ and as $T \to T_c$; indeed, in the latter limit, we find

$$\tilde{c}_V \to c_V - \frac{U}{T} = c_V - \tilde{S} \quad \text{as } T \to T_c. \tag{5.6}$$

Note that, in general, $\tilde{c}_V$ is duality invariant and even, just like $\mathcal{V}$. Once again, for a consistent low-temperature limit which reproduces ordinary thermodynamics, we must choose α such that

$$\frac{2T^{\alpha-1}U}{T^\alpha + T_c^\alpha} \ll c_V \quad \text{as } T/T_c \to 0. \tag{5.7}$$

This constraint typically yields bounds on α which are the same as those stemming from Eq. (5.4).

These new, corrected definitions for entropy and specific heat restore a certain similarity between the pairs of thermodynamic quantities $(\tilde{F}, \tilde{U})$ and $(\tilde{S}, \tilde{c}_V)$. Members of each pair share the same duality weight k and have opposite sign for γ. Of course, the first pair has weight $k = 2$ while the second pair has $k = 0$.

Given these results, we can also see explicitly why the usual uncorrected entropy S and specific heat c_V fail to transform correctly under the thermal duality transformations. From Eqs. (5.2) and (5.5), we see that S and c_V can each be reexpressed as admixtures of $k = 0$ quantities which have opposite parities (even or odd) under the thermal duality transformation. For example, S is a linear combination of $\tilde{S}$ (which is odd) and $T^{\alpha-1}F/(T^\alpha + T_c^\alpha)$ (a small correction term which is even). Only the proper corrected linear combinations inherent in $(\tilde{S}, \tilde{c}_V)$ disentangle this behavior.

Thus, we conclude that the "natural" duality weights and signs for the entropy and specific heat are $k = 0$ and $\gamma = \mp 1$ respectively, with the corrections in Eqs. (5.2) and (5.5) having the net effect of restoring these properties to an otherwise noncovariant S and c_V. Moreover, as we have seen, these transformation properties also make sense from the standpoint of the usual thermodynamic identities. Of course, these conclusions hold only to the extent that our functions are considered to be completely general. For example, as discussed in Ref. [1], it is possible to construct special vacuum amplitudes $\mathcal{V}(T)$ such that the *uncorrected* entropy $S(T)$ turns out to be "accidentally" covariant with a nonzero weight. However, even these functions are unsatisfactory because they are the results of definitions which fail to respect the thermal duality symmetry. Thus, such accidentally covariant entropies should still be corrected in the manner described here, thereby restoring the proper weights and signs to these thermodynamic quantities. We shall see explicit examples of this below.

In most realistic examples, the free energy F and the uncorrected entropy S have opposite overall signs. Thus, our string-theoretic corrections to the entropy in Eq. (5.2) generally tend to *decrease* the entropy,

$$\tilde{S} \leq S, \tag{5.8}$$

with the suppression becoming increasingly severe as $T \rightarrow T_c$. Likewise, in the range $T < T_c$, the internal energy U and the specific heat c_V also typically have opposite signs. [Recall that $U < 0$ for $T < T_c$, as discussed below Eq. (2.6).] We therefore find that

$$\tilde{c}_V \geq c_V, \tag{5.9}$$

with the bound saturating both at $T = 0$ [where $g(T) = 0$] and at $T = T_c$ [where $U(T) = 0$]. As we shall see, these inequalities will be extremely important in the following.

VI. AN EXPLICIT DUALITY-COVARIANT EXAMPLE

In this section, we shall calculate the string-corrected entropy and specific heat within the context of a specific example displaying thermal duality. Towards this end, let us consider the vacuum amplitude [1]

$$\mathcal{V}^{(D)}(T) = -\frac{(T^D + T_c^D)^{2/D}}{TT_c}, \tag{6.1}$$

where $D \geq 1$ is an arbitrary exponent. (Here and henceforth, we shall express all thermodynamic quantities in units of the reduced string scale $\mathcal{M} \equiv M_{\text{string}}/2\pi$.) As discussed in Ref. [1], this function $\mathcal{V}(T)$ emerges as the vacuum amplitude corresponding (either exactly, or approximately and highly accurately) to finite-temperature string constructions in which the time/temperature dimension is compactified on a circle. The parameter D is the spacetime dimension of the zero-temperature string model. Note that this functional form for $\mathcal{V}$ has the property that the resulting entropy S is accidentally covariant with weight D and sign $+1$; however, this will play no role in the following. Indeed, the corresponding specific heat is noncovariant for all $D > 2$.

A. String-corrected entropy

Given this functional form for $\mathcal{V}(T)$, it is straightforward to calculate both the traditional entropy $S(T)$ and the corrected entropy $\tilde{S}(T)$ as functions of temperature, obtaining

$$S^{(D)}(T) = 2\frac{T^{D-1}}{T_c}(T^D + T_c^D)^{2/D-1}, \tag{6.2}$$

and

$$\tilde{S}^{(D)}(T) = 2\frac{(T^D + T_c^D)^{2/D-1}}{TT_c(T^\alpha + T_c^\alpha)}(T^D T_c^\alpha - T^\alpha T_c^D). \tag{6.3}$$

Note, in particular, that the relative sizes of the string corrections to the entropy are not small in this example unless $\alpha \geq D + 1$. This is the strengthened bound on α which emerges from Eq. (5.4) for this system.

These functions for $S^{(D)}(T)$ and $\tilde{S}^{(D)}(T)$ are plotted in Fig. 1 for the special cases with $D = 2$ and $D = 4$. We can immediately see the behavior of $\tilde{S}^{(D)}(T)$ as a function of T. At low temperatures $T \ll T_c$, we see that $\tilde{S}$ follows S quire closely; indeed the corrections to the traditional entropy [e.g., as measured by the ratio $(S - \tilde{S})/S$] remain small as long as $T \ll T_c$. At higher temperatures, however, $\tilde{S}$ is increasingly suppressed relative to S, and ultimately vanishes as $T \rightarrow T_c$. This is required by the fact that $\tilde{S}$ must be an odd function under $T \rightarrow T_c^2/T$ for all α.

For sufficiently small temperatures, our corrected entropies resemble the traditional entropy and grow with increasing temperature, with $d\tilde{S}^{(D)}/dT > 0$ all D and α. This conforms to our standard notions of entropy as a measure of disorder. However, as T approaches the critical temperature, we see that $d\tilde{S}^{(D)}/dT$ ultimately changes sign. At first glance, this might appear to signal an

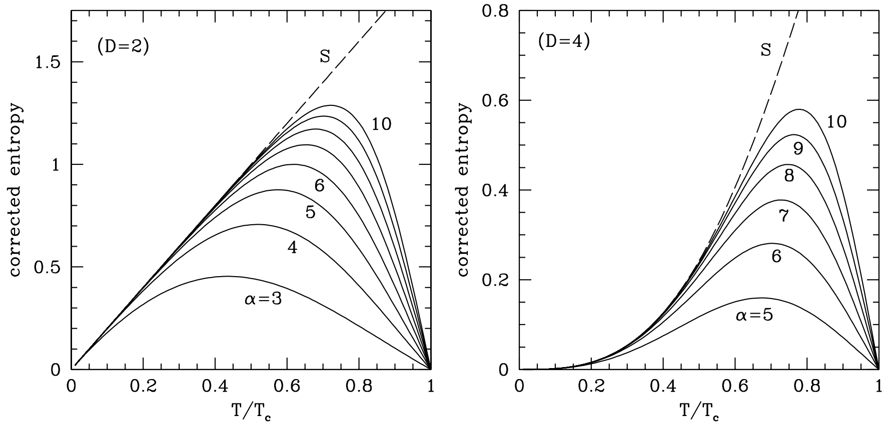

FIG. 1. The string-corrected entropies $\tilde{S}^{(D)}$ given in Eq. (6.3), plotted as functions of T for $D = 2$ (left plot) and $D = 4$ (right plot). In each case, we have plotted the string-corrected entropies for $D + 1 \leq \alpha \leq 10$, while the uncorrected entropy is indicated with a dashed line. In all cases, the corrected entropies are smaller than the traditional entropies, and vanish as $T \to T_c$. The parameter α governs the relative size of the string corrections and thus the rate with which the corrected entropy begins to separate from the uncorrected entropy.

inconsistency in our string-corrected thermodynamics. However, as is well known in string thermodynamics (see, e.g., Refs. [5,17–28]), we expect that a phase transition or other Hagedorn-related event should occur at large temperatures at or near T_c. Thus, rather than interpret $d\tilde{S}^{(D)}/dT < 0$ as a loss of disorder, it is tempting to interpret this sign change as the beginning of a possible phase transition and the conversion of the system into new degrees of freedom. Thus, as the temperature increases towards the critical temperature, fewer and fewer of the original degrees of freedom remain in the system, and thus the entropy associated with these original degrees of freedom begins to decrease.

Of course, verifying this speculation would require a more complete understanding of the nature of the string physics near the critical temperature. Our point here, however, is that a fully covariant treatment of entropy necessarily requires the introduction of corrections which, in this case, ultimately drive the corrected entropy to zero at the critical temperature.

B. String-corrected specific heat

We can also perform a similar analysis for the specific heat. Once again starting from Eq. (6.1), we obtain

$$c_V^{(D)}(T) = 2\frac{T^{D-1}}{T_c}(T^D + T_c^D)^{2/D-2}[T^D + (D-1)T_c^D], \tag{6.4}$$

and

$$\tilde{c}_V^{(D)}(T) = 2\frac{(T^D + T_c^D)^{2/D-2}}{TT_c(T^\alpha + T_c^\alpha)}[T^{2D}T_c^\alpha + T^\alpha T_c^{2D} + (D-1)(T^D T_c^{\alpha+D} + T^{\alpha+D}T_c^D)]. \tag{6.5}$$

These functions are plotted in Fig. 2 for the special cases with $D = 2$ and $D = 4$. Once again, we observe that $\tilde{c}_V^D \geq c_V^D$, with the bound saturating only at $T = 0$ and $T = T_c$. Note that $d\tilde{c}_V^{(D)}/dT = 0$ at $T = T_c$ for all α. This is a direct consequence of thermal duality, and indicates that the corrected specific heat loses all temperature sensitivity at T_c.

We conclude this section with two further comments. First, throughout this section, we have focused exclusively on the behavior for $T \leq T_c$. As mentioned above, we have done this in the expectation that a phase transition or other Hagedorn-related event should occur at large temperatures at or near T_c. However, from a purely mathematical perspective, we could easily have continued our analysis beyond T_c, since our string-corrected entropies and specific heats are (by construction) thermal duality invariant. For example, since the specific heat is necessarily an even (invariant) function under $T \to T_c^2/T$, we see that $\tilde{c}_V^{(D)}$ continues to remain positive for all T and ultimately declines beyond T_c. This is in sharp contrast to the uncorrected specific heat, which continues to rise indefinitely. On the other hand, the string-corrected entropy $\tilde{S}^{(D)}$ is necessarily an odd function under $T \to T_c^2/T$. Thus, $\tilde{S}^{(D)}$ becomes *negative* beyond T_c. This provides dramatic illustration of the fact that, as already anticipated from other considerations, new physics must intercede at or near the string scale.

Our second comment concerns the duality weights of the entropy and specific heat. As already mentioned at the beginning of this section (and as explained more fully in Ref. [1]), the *uncorrected* entropy $S^{(D)}$ in this example is actually already covariant, with duality weight D and sign $+1$. Thus, it may seem that no further corrections are necessary in this case. However, as we have seen in Sec. V, the proper duality weight and sign for the entropy are $k = 0$ and $\gamma = -1$, respectively. Thus, the net effort

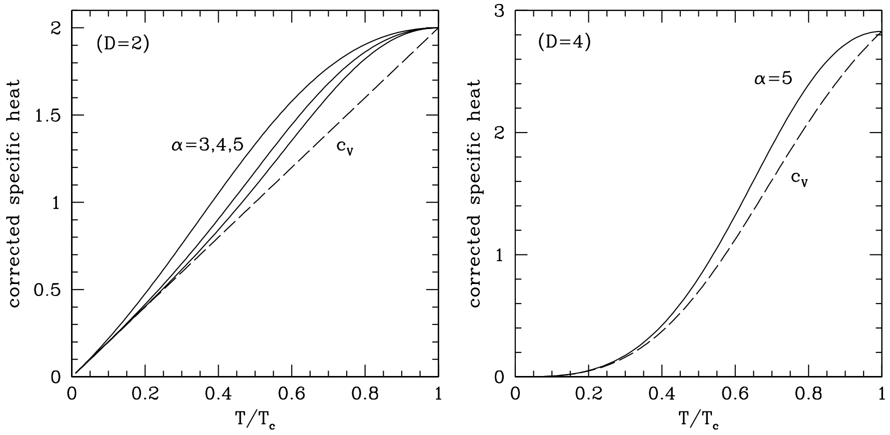

FIG. 2. The string-corrected specific heats $\tilde{c}_V^{(D)}$ given in Eq. (6.5), plotted as functions of T for $D = 2$ (left plot) and $D = 4$ (right plot). For the sake of clarity, we have illustrated only the cases with the largest relative corrections: We have taken $\alpha = 3, 4, 5$ for $D = 2$, and $\alpha = 5$ for $D = 4$. Note that, in all cases, the corrected specific heats exceed the traditional specific heat, agreeing with the traditional heat only at $T = 0$ and $T = T_c$. Moreover, as a consequence of thermal duality, we have $d\tilde{c}_V^{(D)}/dT = 0$ at $T = T_c$ for all α.

or our corrections in this case is to "convert" an even entropy function of weight D into an odd entropy function of weight zero. Of course, these corrections also simultaneously restore duality invariance to the specific heat, where it was otherwise lacking.

C. Effective dimensionalities and holography

Finally, we now investigate the *scaling* behavior of our corrected thermodynamic quantities as functions of temperature. As we shall see, this will enable us to provide a possible physical interpretation to our string-theoretic corrections.

In ordinary quantum field theory, the free energy $F(T)$ at large temperatures typically scales as T^D, where D is the spacetime dimension. This in turn implies that the entropy S should scale as T^{D-1}. However, in string theory we have $F(T) \sim T^2$ as $T \to \infty$, implying that $S(T) \sim T$ as $T \to \infty$. Thus, string theory behaves asymptotically as though it has an effective dimensionality $D_{\text{eff}} = 2$.

At first glance, these two sets of results might not appear to be in conflict since they apply to different theories. However, the field-theory limit of string theory is expected to occur for $T \ll T_c$, and thus the field-theory behavior must be embedded within the larger string-theory behavior. It is therefore interesting to examine the effective dimensionality (i.e., the effective scaling exponent) of our thermodynamic quantities as a function of temperature. As discussed in Ref. [1], it is easiest to define this effective dimensionality $D_{\text{eff}}(T)$ by considering the entropy: Since $S(T)$ is a monotonically increasing function of T, we can define $D_{\text{eff}}(T)$ as the effective scaling exponent at temperature T, setting $S(T) \sim T^{D_{\text{eff}}-1}$. We thus obtain

$$D_{\text{eff}} \equiv 1 + \frac{d\ln S}{d\ln T} = 1 + \frac{T}{S}\frac{dS}{dT} = 1 + \frac{c_V}{S}, \qquad (6.6)$$

where the last equality follows from the thermodynamic identity $c_V = T dS/dT$.

Given the entropy $S^{(D)}$ in Eq. (6.2) and the specific heat $c_V^{(D)}$ in Eq. (6.4), it is straightforward to calculate $D_{\text{eff}}(T)$ as a function of temperature T. This calculation was originally performed in Ref. [1], where a plot of $D_{\text{eff}}(T)$ is given. In each case, it is found that D_{eff} *interpolates* between $D_{\text{eff}} = D$ for $T \ll T_c$ and $D_{\text{eff}} = 2$ for $T \gg T_c$. It is, of course, easy to interpret this result. At small temperatures $T \ll T_c$, the entropy behaves as we expect on the basis of field theory, growing according to the power law $S^{(D)}(T) \sim T^{D-1}$. Indeed, this low-temperature limit of string theory can be identified as the high-temperature limit of the low-energy effective field theory. However, as T approaches the string scale T_c, we see that this scaling behavior begins to change, with the T^{D-1} growth in the entropy ultimately becoming the expected *linear* growth for $T \gg T_c$. This is then the asymptotic *string* limit.

These observations originally appeared in Ref. [1]. However, given these observations, let us now proceed to determine the effective dimensionalities $\tilde{D}_{\text{eff}}$ of our *string-corrected* entropies. In complete analogy with D_{eff}, these corrected effective dimensionalities $\tilde{D}_{\text{eff}}$ may be defined as

$$\tilde{D}_{\text{eff}} \equiv 1 + \frac{d\ln\tilde{S}}{d\ln T} = 1 + \frac{T}{\tilde{S}}\frac{d\tilde{S}}{dT}. \qquad (6.7)$$

Note that, since $\tilde{c}_V \neq T d\tilde{S}/dT$, we cannot write Eq. (6.7) easily in terms of $\tilde{c}_V$.

The results are plotted in Fig. 3 for $D = 4$. As expected, all of our corrected entropies exhibit an initial scaling with $\tilde{D}_{\text{eff}} = D = 4$ as $T/T_c \to 0$; this is guaranteed by our original requirement that $\alpha \geq D + 1$. This implies that none of our string corrections disturb the

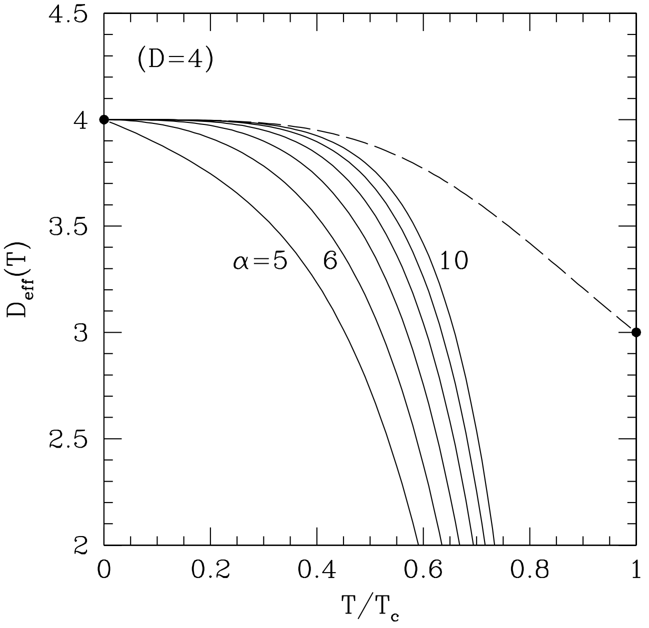

FIG. 3. The effective dimensionalities $\tilde{D}_{\rm eff}$ of the four-dimensional string-corrected entropies $\tilde{S}^{(4)}$, plotted as functions of T for $5 \le \alpha \le 10$. The effective dimensionality of the uncorrected entropy $S^{(4)}$ is also shown (dashed line).

expected field-theoretic behavior at low temperature. However, as T becomes larger and approaches the string scale, we see that the net effect of our string correction is to reduce the effective scaling dimensionality of the entropy even more rapidly than in the uncorrected case.

Thus, combining our results from Figs. 1 and 3, we see that our string corrections have two net effects on the entropy as $T \to T_c$: They reduce its overall magnitude, and they also reduce its scaling exponent (effective dimensionality) as a function of temperature. It is important to stress that these are, in principle, uncorrelated effects: The first relates to the overall size of a function, while the second has to do with its rate of growth. As a stark example of this point observe that, if the scaling behavior of the corrected entropy had been $(T/T_c)^3$ rather than $(T/T_c)^4$ for all $T \le T_c$, this *decrease* in the scaling exponent would have resulted in an *increase* in the entropy, not a decrease. This would have been interpreted as the appearance of more degrees of freedom at low temperatures, not fewer.

Of course, there is a natural interpretation for an effect which simultaneously decreases not only the entropy but also the effective dimensionality that governs its scaling: Such an effect is holographic. Thus, we see that our duality-inspired corrections to the laws of thermodynamics are holographic in nature, enhancing the tendency towards holography that already exists in traditional string thermodynamics. Indeed, as originally observed in Ref. [1], we see from Fig. 3 that the *uncorrected* effective dimensionality already shows a holographic decline from $D_{\rm eff} = 4$ at $T/T_c \ll 1$ to $D_{\rm eff} = 3$ at $T \to T_c$. Our corrections thus enhance this effect, introducing this holographic behavior even more strikingly and at lower temperatures.

Of course, as discussed more fully in Ref. [1], there are a number of outstanding issues that need to be addressed before we can truly identify this phenomenon with holography. In particular, an analysis formulated in flat space (such as ours) cannot address questions pertaining to the geometry of holography, and thus cannot determine whether the modified scaling behavior and the implied reduction in the number of associated degrees of freedom are really to be associated with a lower-dimensional subspace (or boundary) of the original geometry. Indeed, such an analysis is beyond the scope of this paper, and would require reformulating the predictions of thermal duality for string theories in nontrivial D-dimensional backgrounds, and then developing a map between degrees of freedom in the bulk of the D-dimensional volume and those on a lower-dimensional section of this volume. Thus, as indicated in Ref. [1], the possible connection between thermal duality and holography remains to be explored further.

VII. ALTERNATIVE FORMULATIONS FOR A DUALITY-COVARIANT THERMODYNAMICS

In this section, we shall investigate other possible formulations for a duality-covariant thermodynamics. As we shall see, a wide set of possibilities exists: Some of these lead to drastically different phenomenologies, while others have drastically different theoretical underpinnings.

A. Alternative covariant derivatives

First, as stressed in Sec. IV, our thermal duality-covariant derivative in Eq. (4.8) is not unique: *Any* function of $g(T)$ satisfying Eq. (4.4) can be exploited in the construction of a covariant derivative as in Eq. (4.2). As an example, let us again remain within the class of covariant functions $g(T)$ given in Eq. (4.6) and consider the physics that results if we choose $\gamma_g = -1$ rather than $\gamma_g = +1$. Our covariant derivative in Eq. (4.8) then becomes

$$D_T^{(-)} = \frac{d}{dT} + \frac{k}{T}\frac{T^\alpha}{T_c^\alpha - T^\alpha}, \tag{7.1}$$

leading to the definitions

$$\begin{aligned} \tilde{S} &\equiv -D_T^{(-)} F = S - \frac{2T^{\alpha-1}F}{T_c^\alpha - T^\alpha} = S - \frac{2T^\alpha}{T_c^\alpha - T^\alpha}\mathcal{V}, \\ \tilde{c}_V &\equiv D_T^{(-)} U = c_V + \frac{2T^{\alpha-1}U}{T_c^\alpha - T^\alpha} = c_V - \frac{2T^{\alpha+1}}{T_c^\alpha - T^\alpha}\frac{d\mathcal{V}}{dT}. \end{aligned} \tag{7.2}$$

Note that, unlike the case with $\gamma_g = +1$, we now have

$\tilde S \geq S$ and $\tilde c_V \leq c_V$. However, as required, we still find that our string-corrected quantities $\tilde S$ and $\tilde c_V$ are duality covariant with weight $k = 0$ and signs ∓ 1, respectively. Moreover, in the case of the covariant example given in Eq. (6.1), a proper low-temperature (field theory) limit is guaranteed for all $\alpha \geq D$.

At first glance, the definitions in Eq. (7.2) might appear to be unacceptable because of the apparent divergences in $\tilde S$ and $\tilde c_V$ as $T \to T_c$. For example, since the corrections to the entropy are positive and the corrections to the specific heat are negative, we might worry that the definitions in Eq. (7.2) would result in the asymptotic behavior $\tilde S \to \infty$ and $\tilde c_V \to -\infty$ as $T \to T_c$. While a positively divergent entropy leads to no specific difficulty (and might be interpreted as a Hagedorn-like phenomenon), a negative specific heat necessarily results in an inconsistent thermodynamics in which thermal fluctuations grow without bound and ultimately destabilize the system.

However, these concerns are ultimately spurious. Because the internal energy U vanishes at $T = T_c$ as a result of thermal duality, the specific heat actually remains finite and positive as $T \to T_c$. Indeed, the divergence in the definition of the covariant derivative cancels against the vanishing of the internal energy, resulting in a string-corrected specific heat which takes the finite asymptotic value

$$\tilde c_V^{(D)} \to 2^{2/D-1} D(1 - 2/\alpha) \quad \text{as } T \to T_c. \qquad (7.3)$$

Note that this quantity is positive for all $\alpha > 2$.

The resulting string-corrected entropies and specific heats are plotted in Fig. 4 for $D = 4$. As expected, we see that $\tilde c_V^{(4)}$ remains positive in all cases, while $\tilde S^{(4)}$ is now monotonically increasing as a function of temperature for all $T \leq T_c$. Clearly, the effect of these corrections is no longer "holographic" as it was for $\gamma_g = 1$. However, this possibility also results in a fully consistent, duality-covariant string thermodynamics.

At present, we have no physical basis on which to prefer one version of the covariant thermodynamics over another. Even though they lead to drastically different phenomenologies, they are each internally self-consistent and have the same low-temperature (field-theoretic) limits. However, our main point in this paper is that *some* string-theoretic correction is necessary in order to restore thermal duality covariance to the usual rules of thermodynamics, and that it is possible to introduce such corrections without disturbing the usual low-temperature physics associated with traditional thermodynamics. The decision as to the preferred specific form of the covariant derivative awaits a full string calculation, perhaps along the lines discussed in Sec. III.

B. Alternative thermodynamic structures

Changing the specific form of the covariant derivative is not the only way in which we might approach the construction of an alternative thermodynamics. Indeed, even within the context of a fixed covariant derivative, there are other structural options that can be explored.

In order to understand these other options, let us first recall the structure of the traditional thermodynamics. This structure is defined through the definitions in Eq. (2.3), and is illustrated in Fig. 5(a). Note that the thermodynamic quantities are related to each other through temperature multiplications and differentiations, forming a closed self-consistent set of definitions. Of course, the temperature derivatives involved in these definitions do not respect thermal duality, which is why we were motivated to construct a thermal duality-covariant temperature derivative. Using this, we then developed a manifestly duality-covariant thermodynamics by replacing all ordinary temperature derivatives with duality-

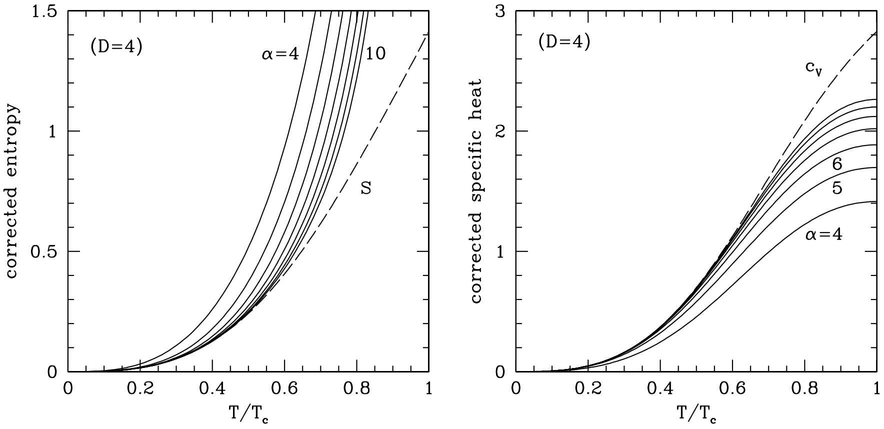

FIG. 4. The string-corrected four-dimensional entropies of $\tilde S^{(4)}$ and specific heats $\tilde c_V^{(4)}$ given in Eq. (7.2), plotted as functions of T. In each case, we have plotted these thermodynamic quantities for $4 \leq \alpha \leq 10$, while the uncorrected quantities are indicated with a dashed line. Note that $\tilde c_V^{(D)}$ remains positive in all cases, while $\tilde S^{(D)}$ is now monotonically increasing as a function of temperature for all $T \leq T_c$.

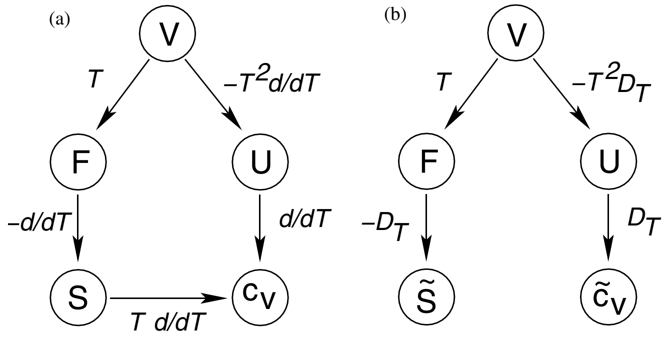

FIG. 5. Relations between thermodynamic quantities. (a) Traditional thermodynamics: All thermodynamic quantities are related to each other through temperature multiplications and differentiations. (b) Our string-corrected thermodynamics: We replace the usual temperature derivatives by duality-covariant derivatives, maintaining the definitions $\tilde{S}$ and $\tilde{c}_V$ in terms of their respective thermodynamic potentials F and U. However, $\tilde{c}_V$ is no longer related to $\tilde{S}$ through either type of temperature derivative.

covariant derivatives. This resulted in a version of thermodynamics whose structure is illustrated in Fig. 5(b). Indeed, as evident in Eq. (5.1), our new quantities $\tilde{S}$ and $\tilde{c}_V$ are defined as covariant derivatives of their respective thermodynamic potentials F and U.

However, this replacement of d/dT by D_T does not preserve the entire structure of the traditional thermodynamics: The final direct "link" between the entropy and specific heat is broken. In the traditional thermodynamics, these two quantities are related by the identity

$$c_V = T\frac{d}{dT}S, \tag{7.4}$$

yet $\tilde{c}_V$ and $\tilde{S}$ are not related in this way through either Td/dT or TD_T. (Note that, since $\tilde{S}$ has zero weight, $d/dT$and D_T are actually the same operator when acting on $\tilde{S}$.) Indeed, the fact that $\tilde{c}_V \neq Td\tilde{S}/dT$ is immediately apparent upon comparing Figs. 1, 2, and 4.

Of course, one might argue that preserving Eq. (7.4) is not as critical as preserving the identifications of the entropy and specific heat as derivatives of their respective potentials. However, in traditional thermodynamics, the identity (7.4) is critical for interpreting entropy in terms of heat transfer,

$$dS = \frac{dQ}{T}. \tag{7.5}$$

To see this, recall that a heat transfer dQ induces a change in internal energy $dU = dQ$ (where we are not distinguishing between exact and inexact differentials and where we have set $dW = 0$). However, since $U = c_V dT$, we see that Eq. (7.5) cannot hold unless S and c_V are related through Eq. (7.4).

There are various ways in which this situation can be addressed. One option, of course, is to regard the relation (7.4) as more fundamental than the separate relations between either the entropy or specific heat and their respective thermodynamic potentials. We could then establish a covariant thermodynamics by replacing our previous definition for the corrected specific heat with a new definition stemming directly from the corrected entropy

$$\tilde{S} = -D_T F, \qquad \tilde{c}'_V = TD_T\tilde{S}. \tag{7.6}$$

This option is illustrated in Fig. 6(a). Alternatively, we could retain the previous corrected specific heat $\tilde{c}_V$, and implicitly define a new corrected entropy (up to an overall additive constant) relative to this specific heat:

$$\tilde{c}_V = D_T U, \qquad \tilde{c}_V = TD_T\tilde{S}'. \tag{7.7}$$

This option is illustrated in Fig. 6(b).

Despite their differences, each of these options results in a fully consistent, duality-covariant thermodynamics. The primary difference between them, of course, is in the interpretation given to their corrected entropies. The corrected entropy $\tilde{S}$, which appears in Eqs. (5.1) and (7.6), is derived from the free energy which in turn is derived directly from the partition function $\mathcal{V}$. This entropy should thus retain its interpretation as a counting of states (i.e., as a measure of disorder). The corrected entropy $\tilde{S}'$, by contrast, is defined implicitly through Eq. (7.7). This entropy should thus retain its interpretation pertaining to heat transfer.

Given these observations, the question then arises as to whether there exist any special covariant derivatives D_T for which *all* of the "links" in these diagrams are generalized and continue to hold. As we shall now prove, only one such derivative exists.

To see this, we first observe that the diagram in Fig. 5(a) "closes" for the usual thermodynamics as a result of the operator identity

$$T\frac{d^2}{dT^2}T = \frac{d}{dT}T^2\frac{d}{dT}. \tag{7.8}$$

This in turn holds as a result of the commutation relation

$$[T, d/dT] = -1. \tag{7.9}$$

Indeed, when acting on $\mathcal{V}$, each side of Eq. (7.8) provides a different route to the second derivative c_V: The left side passes through F and S, while the right side passes through U.

We now seek to duplicate this success for our covariant derivative D_T. It is straightforward to demonstrate that

$$[T, D_T^{(k)}] = -1 \quad \text{for all } k, \tag{7.10}$$

where k is the weight coefficient within D_T; indeed, Eq. (7.9) is nothing but the $k = 0$ special case of Eq. (7.10). However, in order to have our diagrams "close" for arbitrary vacuum amplitudes $\mathcal{V}$, our covariant derivatives must now satisfy the generalized relation

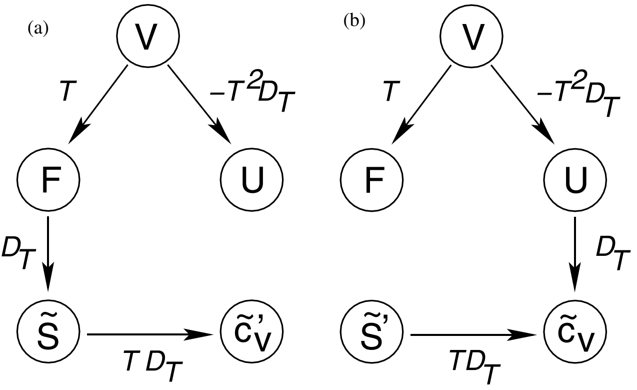

FIG. 6. Relations between thermodynamic quantities in alternative formulations of duality-covariant thermodynamics. (a) In this version based on Eq. (7.6), the corrected entropy $\tilde{S}$ is defined through the free energy, but the corrected specific heat $\tilde{c}_V$ is defined through the corrected entropy. (b) In this version based on Eq. (7.7), the corrected specific heat is defined through the internal energy, and the corrected entropy is defined implicitly through the corrected specific heat.

$$TD_T^{(0)}D_T^{(2)}T = D_T^{(2)}T^2D_T^{(0)}. \tag{7.11}$$

Without loss of generality, let us write $D_T^{(0)} = d/dT$ and $D_T^{(2)} = d/dT + g(T)/T$, as in Eq. (4.2). We then find

$$[D_T^{(2)}, D_T^{(0)}] = -\frac{d}{dT}\left(\frac{g}{T}\right), \tag{7.12}$$

which, along with the commutation relation in Eq. (7.10), enables us to reduce Eq. (7.11) to the differential equation $dh/dT = -h/T$, where $h \equiv g/T$. The only solution to this equation has $h \sim T^{-1}$, or g equal to a constant.[1] However, according to Eq. (4.4), this constant must be equal to $-k/2$. We thus find that $g(T) = -k/2$ is the unique solution which preserves all of our thermodynamic identities, resulting in the unique covariant derivative

$$D_T = \frac{d}{dT} - \frac{k}{2T}. \tag{7.13}$$

Remarkably, this is precisely the derivative that we already found in Eq. (4.9). However, we see that we would now have to take this as our covariant derivative for *all* values of T in order to preserve all of the links in our covariant thermodynamics. In other words, following Eq. (5.3), we would have to define $\tilde{S} \equiv U/T$ for all T.

It is easy to see how this corrected entropy manages to retain both of its interpretations pertaining to heat transfer and state counting. By defining $\tilde{S} \equiv U/T$, we are providing a direct relation between the corrected entropy and the internal energy, which in turn can be directly related to heat. A similar argument applies to counting states. Recall that the traditional entropy S is special in that it depends on the temperature only through the normalized Boltzmann probabilities $P_i \equiv p_i/Z$, where $p_i \equiv \exp(-E_i/T)$ and $Z \equiv \sum_i p_i$:

$$S = -\sum_i P_i \ln P_i. \tag{7.14}$$

It is this expression which enables us to associate the emergence of order with the vanishing of S: As $T \to 0$, we find $P_i = 0$ for all excited states and $P_i = 1$ for the ground state. However, if we now take $\tilde{S} \equiv U/T = S + F/T$. We find

$$\begin{aligned}\tilde{S} &= S + F/T = -\sum_i P_i \ln P_i - \ln Z \\ &= -\sum_i P_i(\ln P_i - \ln Z) - \ln Z = -\sum_i P_i \ln p_i,\end{aligned} \tag{7.15}$$

where in the first line we have identified $F = -T\ln Z$ (as appropriate for the usual canonical ensemble). We thus see that $\tilde{S} \equiv U/T$ is given by an expression which is similar to Eq. (7.14) but in which the final normalized Boltzmann probability P_i is simply replaced by the *unnormalized* Boltzmann probability p_i. Indeed, with this definition, $\tilde{S}$ is sensitive to the distribution of *individual* Boltzmann probabilities in precisely the same way as the usual entropy S, and differs only in its dependence on their combined sum. Thus, $\tilde{S}$ can also be taken as a direct measure of disorder.

Unfortunately, these definitions fail a crucial test: They do not have a smooth limit as $T/T_c \to 0$ in which traditional thermodynamics is restored. Rather, this solution for D_T exists only as a special point, a unique alternative thermodynamics which does not connect smoothly back to the traditional case. We are therefore forced to disregard this possibility. We see, then, that it is not generally possible to construct a thermal duality-covariant thermodynamics which simultaneously preserves *all* of the traditional relations between our thermodynamic quantities.

VIII. DISCUSSION

In the previous sections, we have advanced a proposal to modify traditional thermodynamics. Needless to say, this raises a number of important issues. In this section, we shall address some of these issues and provide some possible interpretations and resolutions.

First, one might wonder whether it is truly necessary to take the drastic step of modifying the rules of thermodynamics. After all, even in condensed-matter physics, there are systems (such as arrays of Josephson junctions) which exhibit temperature-inversion symmetries which

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[1] This conclusion can also be reached directly by observing that $\tilde{c}_V$ and $\tilde{S}$ are related through the modified identity $\tilde{c}_V = Td\tilde{S}/dT + (dg/dT)F$. This reduces to the traditional identity only if $g(T)$ is a constant. However, the above derivation involving the commutation relations of our covariant derivative exposes the underlying algebraic structure behind the failure of the traditional identity when $g(T)$ is not a constant.

are analogues of thermal duality. One does not modify the laws of thermodynamics when analyzing these systems; one merely accepts the fact that their free energies and their entropies may exhibit the underlying symmetry to different degrees. It is natural to wonder, then, whether the laws of thermodynamics should really vary with the system. Should not the laws of thermodynamics *transcend* the system under study?

The critical difference, however, is that this proposal is not about a particular system or configuration of matter. Rather, this is a proposal whose inspiration is string theory, a purported theory of matter itself at the most fundamental energy scales. Thus, the proposed string-theoretic corrections are to be interpreted as universal, valid for all systems regardless of their underlying symmetries. The analogy with gauge invariance is apt. Classical electromagnetism is the theory underlying all electromagnetic phenomena, and it exhibits gauge invariance at its most fundamental level. Within the framework of this theory, regardless of the particular system or charge distribution under study, we do not ascribe physical reality to quantities which are not gauge invariant; likewise we would not tolerate a calculational methodology which explicitly breaks gauge invariance in a way that does not lead to gauge-invariant results. If thermal duality is truly a fundamental string symmetry, then the same should be true here. Just as gauge invariance is used as a guide when performing calculations and extending our models into new domains, thermal duality is similarly being exploited to determine the forms of possible string-theoretic corrections to the laws of thermodynamics. We know that such corrections are necessary because the traditional laws explicitly break a symmetry which we are holding to be fundamental.

Likewise, one might wonder whether we must take thermal duality as a fundamental symmetry. After all, thermal duality might simply be an accident of certain compactifications. However, thermal duality is intimately related to T duality and Lorentz invariance, and both of these are certainly fundamental symmetries in string theory. Indeed, T duality is often taken as evidence that strings "feel" the spacetime in which they propagate in a way that does not distinguish between large and small. Symmetries such as these are not considered accidents; rather, they are taken as clues, evidence for the need to reinterpret the nature of time and space at the string scale. Since the roots of thermal duality are firmly embedded in T duality, it would seem that the implications of thermal duality should be taken just as seriously. Thus, if T duality tells us that our understanding of space itself may require modification at the string scale, the correspondence between compactified zero-temperature theories and uncompactified finite-temperature theories suggests that the same must be true of our understanding of thermodynamics. It is then completely natural that the laws of thermodynamics would require modification.

There are, of course, closed string compactifications which fail to exhibit thermal duality, just as there are closed string compactifications which fail to be self-dual under T-duality transformations. Indeed, the analogy is exact at a mathematical level: Such compactifications have certain orbifold twists which mix into the compactification and spontaneously break the underlying symmetry. However, the important point is that these are only *spontaneous* breakings of the fundamental symmetry; as with all Scherk-Schwarz breakings, the symmetry-breaking effects scale with the inverse volume of the compactification and disappear in the infinite-volume limit. The existence of compactifications in which these symmetries are spontaneously broken does not alter the primary point that these are still fundamental symmetries in string theory, and we should not expect the *rules* of the theory itself to violate them. As stated in the introduction, it is acceptable if the entropy $S(T)$ turns out to be noncovariant because the underlying vacuum amplitude $\mathcal{V}(T)$ is noncovariant for a particular twisted string ground state. It is not acceptable, however, if the covariance of $S(T)$ is lost only because this quantity is defined in a way that fails to respect the underlying symmetry.

Another important issue concerns the possible interpretations of the string-corrected quantities $\tilde{S}$ and $\tilde{c}_V$. For example, a conservative interpretation would be to assert that these new quantities are merely the proper eigenquantities with respect to thermal duality transformations, and that the entropy and specific heat are not eigenquantities but rather linear combinations of these eigenquantities. In this way, one would not need to impose the further interpretation that $\tilde{S}$ is itself the actual entropy, that $\tilde{c}_V$ is itself the actual specific heat. In other words, one would then avoid the need to interpret the extra string-suppressed terms in the definitions of $\tilde{S}$ and $\tilde{c}_V$ as *corrections*.

However, while such an interpretation is logically consistent, we would then be placed in the somewhat awkward position of associating physical observables such as entropy and specific heat with mathematical quantities that fail to exhibit our fundamental symmetries. If we believe fully that entropy and specific heat are physical observables, we are motivated to associate them with mathematical quantities such as $\tilde{S}$ and $\tilde{c}_V$ which are consistent with these symmetries. While this is indeed a more ambitious interpretation of the results, such an interpretation seems especially natural in light of the fact that the extra terms involved are, as noted, suppressed by powers of the string scale and hence are unobservable at low temperatures.

But this in turn raises another important issue. Given this stronger interpretation, a quantity such as entropy now has an extra contribution in its definition, one which depends on an energy scale T_c which is in turn related to the string scale. How can this be justified, given our expectation that entropy is merely a counting of states? Indeed, it seems that entropy should be a pure number without reference to any physical scale.

There are two potential answers to this question. First, the covariant derivative with $\alpha = 0$ actually does not introduce any new scale T_c. Moreover, this is the unique derivative which restores thermal duality while simultaneously managing to close all of the links in the thermodynamics diagrams in Sec. VII. Of course, this derivative does not admit traditional thermodynamics as a low-temperature limit, thus requiring that it be interpreted only as strongly as the above "conservative" approach would permit.

The second answer, however, is the more relevant one. It is certainly true that one is, in general, introducing a physical scale into the definition of entropy; this was hardly to be avoided, since the symmetry one is attempting to restore by doing so also contains a physical scale. However, this is not just any scale: This is the fundamental scale of string theory, the scale which one expects to govern the relative sizes of string-related phenomena associated with quantum gravity and a possible breakdown of our usual notions of spacetime geometry. It is not too much to imagine that this profound alteration should also affect the very meaning of degrees of freedom and counting of states. Indeed, it is natural to suspect a connection with holography in this regard.

Of course, such a state of affairs would seem completely natural if the number of degrees of freedom in the theory were to change as one approaches the string scale due to some hitherto unknown gravitational or string-induced effect. This would indeed be in the spirit of holography. However, at first glance, it might appear that our proposed modification to the laws of thermodynamics does not appear to be changing physical quantities such as the degrees of freedom of the theory; by redefining entropy, it may instead appear that our proposal merely changes the probabilistic rules by which they are counted.

At a deeper level, however, it is not readily evident how to distinguish between the two situations. Even with the usual definition of entropy, we count all states equally because we assume that each microstate of the system is equally likely to occur, that a given system explores all of its energetically allowed states with equal probability. This assumption is ultimately the bedrock of standard thermodynamics, but it is possible that this assumption is violated at the string scale. After all, we already know that this assumption is violated in purely classical (deterministic) systems, which must obey the Poincaré recurrence theorem and hence cannot truly explore the space of states completely randomly. In such systems, the validity of such an assumption becomes a question of time scales, and these ultimately depend on the relevant physical parameters of the system. Even in a quantum-mechanical system, this assumption is justified only in a rough statistical sense, thanks to quantum-mechanical uncertainties in specifying our states; once again, the validity of the assumption depends on the physical parameters of the system. It is therefore not too much to expect that, near the string scale, new quantum-gravitational or string-induced effects may also ultimately distort the manner in which the system explores all of its energetically allowed states. If so, the string-corrected entropy may be precisely what accounts for this phenomenon, providing a recipe for computing an "effective" number of degrees of freedom after all gravitational or string-induced effects are included. Indeed, as long as the final corrected entropy exhibits thermal duality along with the other thermodynamic quantities, it may not be possible to determine whether the true change is in the number of degrees of freedom or in the manner by which they are counted. Only the final count is important.

Clearly, such discussions ultimately tend in a philosophical direction and do not lead to simple answers. However, the important point is that the proposed thermodynamics differs from the standard thermodynamics only through effects which are unmeasurably small at temperatures much below the string scale. Given that physics is an experimental science, we cannot prove or disprove this proposal except through recourse to aesthetics. In this case, aesthetics means symmetry. The proposed modifications to thermodynamics restore one symmetry, namely, thermal duality, but imply profound changes to our understanding of entropy. Thus, it is natural that our understanding of quantities such as entropy would require profound alteration as we approach the fundamental scale of quantum gravity and string theory.

IX. CONCLUSIONS AND OPEN QUESTIONS

In this paper, we have addressed a fundamental issue: Is it possible to construct a thermodynamics which is manifestly covariant with respect to the thermal duality symmetry of string theory?

In one sense, this approach was successful. We were able to construct a manifestly covariant derivative, and through this derivative we were able to construct a manifestly covariant thermodynamics which not only reduces to the standard thermodynamics at low temperatures, but which leads to corrections that become significant only near the string scale. This alone guarantees that such a theory is experimentally viable as an extension to the standard rules of thermodynamics. Given that this theory restores a fundamental duality symmetry where it was otherwise lacing, we believe that such extensions to the rules of thermodynamics are worthy of further exploration.

Adopting this attitude, we are then led to a number of outstanding questions. First, of course, there are several theoretical issues. Most importantly, we needed to make an assumption for the form of the function $g(T)$ in our covariant derivative. While many of our main conclusions are independent of the specific form of $g(T)$, it still remains to calculate this function from first principles through a string calculation analogous to that discussed in Sec. III. This would, we believe, place our proposal on firmer theoretical footing. Another theoretical issue concerns the possible relation, if any, between our results and holography. Given that we are changing the rules by which entropy is to be calculated—indeed changing the very definition of entropy itself—it is important to study whether and how the effects of these string corrections can be interpreted in a holographic context. We have already seen, for example, that in many cases these string corrections tend to profoundly alter the scaling behavior of the entropy with temperature, thereby decreasing the effective spacetime dimensionality associated with the entropy. However, as discussed earlier, interpreting this effect as truly holographic would also require a geometric understanding of how the degrees of freedom contributing to $\tilde{S}$ may be mapped from a volume to the boundary of a volume. This issue cannot be addressed in our formulation which is thus far based on strings in flat (infinite-volume) backgrounds.

There are also many phenomenological issues that are prompted by our approach. For example, how do our results extend to theories in which thermal duality is spontaneously broken (see, e.g., Refs. [4,5,29–33]), as well as to open strings and branes? The answers to these questions could have important implications for recent brane-world scenarios. Likewise, it is interesting to consider the possible applications of our results to early-universe cosmology, particularly regarding the issues of Hagedorn-like phase transitions and entropy generation.

In another sense, however, our investigations have perhaps raised more questions than they have answered, and in this regard we are quite sympathetic to the discomfort of conservative readers. The structure of thermodynamics is so tightly constrained, and the underpinnings of thermodynamics rest on such elementary axioms of probability and state counting, that it would seem to be an extremely risky undertaking to attempt any alteration or generalization of these principles. We have already seen in Sec. VII, for example, that there are several possible generalizations of the traditional rules of thermodynamics, yet none of these approaches simultaneously preserves all of the different shades of interpretation that are normally ascribed to quantities such as entropy.

Many of these theoretical issues could perhaps be resolved (or at least placed on firmer footing) if we were to develop a formulation of our generalized thermodynamics based on the *microcanonical* ensemble. Yet we can immediately see the difficulties in doing so. By its very nature, thermal duality is a symmetry with respect to transformations in temperature; clearly temperature is the independent variable. In order to develop an equivalent microcanonical formulation, however, we require the internal energy U to be the independent variable. We would thus need to express thermal duality as a symmetry under transformations of U. We would then attempt to take our string-corrected entropy $\tilde{S}$ as the fundamental quantity (i.e., the string-corrected counting of states), and demonstrate that $d\tilde{S}/dU$ (or even a covariant derivative $D_U\tilde{S}$) is equivalent to the inverse of our original temperature $1/T$. However, it is easy to verify that this microcanonical approach does not generally lead to results which are consistent with those of the canonical ensemble. Indeed, we believe that the fundamental difficulty in this approach rests on the need to find a microcanonical ensemble equivalent of thermal duality—i.e., a formulation of this symmetry which does not take T as the independent parameter. As long as our approach to string thermodynamics rests on the canonical ensemble and string partition functions, this formulation is likely to elude us. Similar issues concerning the relation between the microcanonical and canonical ensembles are well known to exist in attempting to understand the Hagedorn transition, and may also play a role in generic problems concerning the interplay between gravity and thermodynamics, such as the Jeans instability.

Thus, what are we to make of these results? On the one hand, we could be content with the observation that there exist special solutions for $\mathcal{V}(T)$, as discussed in Ref. [1], for which the traditional entropy S (and occasionally even the specific heat c_V) turn out to be duality covariant. Indeed, in Ref. [1], we conjectured that these special solutions $\mathcal{V}(T)$ may represent the exact results of actual string calculations when the contributions from all orders in string perturbation theory are included. However, we continue to remain sympathetic to the original motivation of this paper, namely, that the rules of thermodynamics should themselves respect this symmetry in a manifest fashion. Indeed, it is by thrashing out how this can occur that we continue to hope to gain insight into the possible nature of temperature, state counting, and thermodynamics near the string scale. After all, if thermal effects can truly be associated with spacetime compactification through the Matsubara/Kaluza-Klein correspondence, then our expectations of an unusual "quantum geometry" near the string scale—one which does not distinguish between "large" and "small"—should simultaneously lead to expectations of an equally unusual thermodynamics near the string scale which does not distinguish between "hot" and "cold" in the traditional sense. Thermal duality should then serve as a tool towards deducing the nature of these new effects. We thus consider the investigation in this paper to be an initial, and hopefully provocative, attempt in this direction.

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