Wilson loops and vertex operators in a matrix model

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We systematically construct wave functions and vertex operators in the type IIB (IKKT) matrix model by expanding a supersymmetric Wilson loop operator. They form a massless multiplet of the $\mathcal{N} = 2$ type IIB supergravity and automatically satisfy conservation laws.

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I. INTRODUCTION

Type IIB (IKKT) matrix model was proposed as a nonperturbative formulation of superstrings [1] and has been expected to be equivalent to the type IIB superstring. The Schwinger-Dyson equation of the Wilson lines are shown to describe the string field equation of motion of type IIB superstring in the light cone gauge [2]. Although there are still many issues to be resolved, the model has an advantage to other formulations of superstrings that we can discuss dynamics of space-time more directly [3]. The action of the model is given by

$$S_{\mathrm{IKKT}} = -\frac{1}{4}\mathrm{tr}[A_\mu, A_\nu]^2 - \frac{1}{2}\mathrm{tr}\bar{\psi}\Gamma^\mu[A_\mu, \psi], \tag{1.1}$$

where A^μ ($\mu = 0, \cdots, 9$) and ten-dimensional Majorana-Weyl fermion ψ are $N \times N$ bosonic and fermionic hermitian matrices. The action was originally derived from the Schild action for the type IIB superstring by regularizing the world sheet coordinates by matrices. It is interesting that the same action describes the effective action for N D-instantons [4]. This suggests a possibility that D-instantons [D(− 1)] can be considered as fundamental objects to generate both of the space-time and the dynamical fields (or strings) on the space-time. The bosonic matrices represent noncommutative coordinates of D(− 1)'s and the distribution of eigenvalues of A_μ is interpreted to form space-time.

If we take the above interpretation that the space-time is constructed by distribution of D-instantons, how can we interpret the SO(9, 1) rotational symmetry of the matrix model action? This symmetry can be interpreted in the sense of mean field. Namely we can consider that the system of N D-instantons are embedded in larger size $(N + M) \times (N + M)$ matrices as

$$\begin{pmatrix} ND(-1) & \\ & MD(-1)\ \text{as background for}\ ND(-1) \end{pmatrix}, \tag{1.2}$$

and consider the action (1.1) as an effective action in the background where the rest, M eigenvalues, distribute uniformly in ten dimensions. If the M eigenvalues distribute inhomogeneously, we may expect that the effective action for N D-instantons is modified so that they live in a curved space-time. This is analogous to a thermodynamic system. In a canonical ensemble, a subsystem in a heat bath is characterized by several thermodynamic quantities like temperature. Similarly a subsystem of N D-instantons in a "matrix bath" can be characterized by several thermodynamic quantities.

Since the matrix model has the $\mathcal{N} = 2$ type IIB supersymmetry

$$\begin{cases} \delta A_\mu = i\bar{\epsilon}\Gamma_\mu\psi, \\ \delta\psi = -\frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon + \epsilon' 1_N, \end{cases} \tag{1.3}$$

we expect that the configuration of the M D-instantons can describe condensation of massless fields of the type IIB supergravity and the thermodynamic quantities of the matrix bath are characterized by the values of the condensations.

In order to discuss which type of configurations for M D-instantons correspond to the condensation of the massless type IIB supergravity multiplet, we consider the supersymmetry transformations (1.3) in the system of $N + M$ D-instantons (1.2). In particular, we consider in this paper the simplest case that the background is represented by 1 D-instanton (namely $M = 1$). This simplification can be considered as a mean field approximation that the configuration of M D-instantons is represented by a mean field described by a single D-instanton. We call this extra D-instanton a **mean field D-instanton**. This kind of idea was first discussed by Yoneya in [5]. We hence embed $N \times N$ matrices into $(N + 1) \times (N + 1)$ matrices as

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$$A'_\mu = \begin{pmatrix} A_\mu & a_\mu \\ a^\dagger_\mu & y_\mu \end{pmatrix}, \qquad \psi' = \begin{pmatrix} \psi & \varphi \\ \varphi^\dagger & \xi \end{pmatrix}. \tag{1.4}$$

Here we use A'_μ, ψ' for $(N+1)\times(N+1)$ matrices and A_μ, ψ for $N\times N$ parts of the matrices. (y, ξ) is the coordinate of the mean field D-instanton and its configuration (or the wave function) $f(y, \xi)$ specifies a certain background of the massless type IIB supergravity multiplet. The supersymmetry transformation (1.3) for (A'_μ, ψ') can be rewritten in components as

$$\begin{cases} \delta A_\mu = i\bar{\epsilon}\Gamma_\mu\psi, \\ \delta y_\mu = i\bar{\epsilon}\Gamma_\mu\xi, \\ \delta a_\mu = i\bar{\epsilon}\Gamma_\mu\varphi, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \delta\psi = -\frac{i}{2}(F_{\mu\nu} + a_\mu a^\dagger_\nu - a_\nu a^\dagger_\mu)\Gamma^{\mu\nu}\epsilon + \epsilon' 1_N, \\ \delta\xi = -\frac{i}{2}(a^\dagger_\mu a_\nu - a^\dagger_\nu a_\mu)\Gamma^{\mu\nu}\epsilon + \epsilon', \\ \delta\varphi = -\frac{i}{2}\{(A_\mu - y_\mu)a_\nu - (A_\nu - y_\nu)a_\mu\}\Gamma^{\mu\nu}\epsilon, \end{cases} \tag{1.6}$$

where $F_{\mu\nu} = [A_\mu, A_\nu]$. We can obtain an effective action for the diagonal blocks by integrating the off-diagonal parts a_μ, ϕ. In the leading order of the perturbation, we can neglect terms depending on the off-diagonal fields and the supersymmetry (SUSY) transformations are given by

$$\begin{cases} \delta A_\mu = i\bar{\epsilon}\Gamma_\mu\psi, \\ \delta\psi = -\frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + \epsilon', \end{cases} \tag{1.7}$$

$$\begin{cases} \delta y_\mu = i\bar{\epsilon}\Gamma_\mu\xi, \\ \delta\xi = \epsilon'. \end{cases} \tag{1.8}$$

The first transformations for N D-instantons are the same as the original SUSY transformations, Eq. (1.3). The second ones are $\mathcal{N} = 2$ supersymmetry transformations for the single mean field D-instanton. The generators of the former are given by

$$\bar{\epsilon}Q_1 = i(\bar{\epsilon}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon\frac{\delta}{\delta\psi}, \tag{1.9}$$

$$\bar{\epsilon}Q_2 = \epsilon\frac{\delta}{\delta\psi}, \tag{1.10}$$

while those of the latter are given by

$$\bar{\epsilon}q_1 = i\bar{\epsilon}\Gamma_\mu\xi\frac{\partial}{\partial y_\mu}, \tag{1.11}$$

$$\bar{\epsilon}'q_2 = \epsilon'\frac{\partial}{\partial\xi}. \tag{1.12}$$

In order to obtain the correct wave functions $f(y, \xi)$ corresponding to the massless supergravity multiplet, we need to obtain the multiplet of wave functions that transform correctly under the supersymmetry transformation (1.8).

When the supersymmetry transformations (1.8) act on wave functions of the form $e^{-ik\cdot y}f(\xi)$, they become

$$\begin{aligned} \bar{\epsilon}q_1 f(\xi)e^{-ik\cdot y} &= (\bar{\epsilon}\not{k}\xi)f(\xi)e^{-ik\cdot y}, \\ \bar{\epsilon}'q_2 f(\xi)e^{-ik\cdot y} &= \epsilon'\frac{\partial}{\partial\xi}f(\xi)e^{-ik\cdot y}. \end{aligned} \tag{1.13}$$

In this paper, as a first step toward discussing dynamics of condensation, we derive a multiplet of wave functions for the single (mean field) D-instanton and corresponding vertex operators for the N D-instantons. Such vertex operators were partly obtained by Kitazawa [6] for the type IIB matrix model by using the supersymmetry transformations. We give a more systematic derivation of the vertex operators by expanding supersymmetric Wilson loop operators. They automatically form a supersymmetry multiplet and satisfy conservation laws. In the context of M(atrix) theory for D-particles or membrane theory, such vertex operators corresponding to the supergravity modes were also constructed by one-loop calculations in fermionic backgrounds [7] or by using supersymmetry transformations of the Wilson loop operator [8]. Vertex operators for matrix strings were also constructed in [9] by duality transformations of the above.

The content of the paper is as follows. In Sec. II, we construct a supersymmetric Wilson loop operator which is invariant under simultaneous supersymmetry transformations of N D-instantons and the remaining single D-instanton coordinate. In Sec. III, by using the supersymmetry transformations, we construct a multiplet of wave functions for the single D-instanton. In Sec. IV, we then expand the supersymmetric Wilson loop in terms of the wave functions derived in Sec. III and obtain a multiplet of vertex operators for N D-instantons. Section V is devoted to further discussions concerning dynamics of condensations. In the appendix, we summarize our notations and useful identities.

II. SUPERSYMMETRIC WILSON LOOP

In order to construct wave functions $f_A(\xi)e^{-ik\cdot y}$ and vertex operators $V_A(A^\mu, \psi; k)$ that transform covariantly under supersymmetries (1.8) and (1.7) respectively (A denotes a field of a massless $\mathcal{N} = 2$ supergravity multiplet), we first consider a supersymmetric Wilson loop operator first introduced in [10] for the IIB matrix model:

$$w(C) = \mathrm{tr}\prod_{j=1} e^{\bar{\lambda}_j Q_1} e^{-i\epsilon k^\mu_j A_\mu} e^{-\bar{\lambda}_j Q_1}. \tag{2.1}$$

Since we are interested in the massless multiplet, we here consider the following simplest Wilson loop operator

$$\omega(\lambda, k) = e^{\bar{\lambda}Q_1}\mathrm{tr}e^{ik\cdot A}e^{-\bar{\lambda}Q_1}. \tag{2.2}$$

We will then show that by expanding the operator $\omega(\lambda, k)$ we can obtain a set of wave functions and vertex operators. Hereafter, we assume that the $N \times N$ matrices A_μ and ψ satisfy the equations of motion

$$[A^\nu, [A_\mu, A_\nu]] - \frac{1}{2}(\Gamma_0\Gamma_\mu)_{\alpha\beta}\{\psi_\alpha, \psi_\beta\} = 0, \tag{2.3}$$

$$\Gamma^\mu[A_\mu, \psi] = 0. \tag{2.4}$$

First we show that $\omega(\lambda, k)$ is invariant under simultaneous supersymmetry transformations for $N \times N$ matrices A^μ, ψ and the parameters (λ, k). When we act supersymmetry transformation $e^{\bar{\epsilon}Q_1}$ on $\omega(\lambda, k)$, it becomes

$$\begin{aligned} e^{\bar{\epsilon}Q_1}\omega(\lambda, k)e^{-\bar{\epsilon}Q_1} &= e^{\bar{\epsilon}Q_1}e^{\bar{\lambda}Q_1}\mathrm{tr}e^{ik\cdot A}e^{-\bar{\lambda}Q_1}e^{-\bar{\epsilon}Q_1} \\ &= e^{(A^\mu\bar{\epsilon}\Gamma_\mu\lambda)G}e^{(\bar{\epsilon}+\bar{\lambda})Q_1}\mathrm{tr}e^{ik\cdot A} \\ &\quad\times e^{-(\bar{\lambda}+\bar{\epsilon})Q_1}e^{(A^\nu\bar{\epsilon}\Gamma_\nu\lambda)G} \\ &= \omega(\epsilon + \lambda, k). \end{aligned} \tag{2.5}$$

Here G is the generator of U(N) transformation and we have used the commutation relation

$$\begin{aligned} [\bar{\epsilon}_1Q_1, \bar{\epsilon}_2Q_1] = 2A^\mu\bar{\epsilon}_1\Gamma_\mu\epsilon_2G + \bigg(&-\frac{7}{8}(\bar{\epsilon}_1\Gamma^\mu\epsilon_2)\Gamma_\mu + \frac{1}{16\cdot 5!} \\ &\times(\bar{\epsilon}_1\Gamma^{\mu_1\cdots\mu_5}\epsilon_2)\Gamma_{\mu_1\cdots\mu_5}\bigg)\Gamma_\lambda[A^\lambda, \psi]\frac{\delta}{\delta\psi}. \end{aligned} \tag{2.6}$$

The second term on the right-hand side vanishes due to the equation of motion (2.4). Similarly for the other supersymmetry transformation $e^{\bar{\epsilon}Q_2}$, the Wilson loop operator transforms as

$$\begin{aligned} e^{\bar{\epsilon}Q_2}\omega(\lambda, k)e^{-\bar{\epsilon}Q_2} &= e^{\bar{\epsilon}Q_2}e^{\bar{\lambda}Q_1}\mathrm{tr}e^{ik\cdot A}e^{-\bar{\lambda}Q_1}e^{-\bar{\epsilon}Q_2} \\ &= e^{\bar{\lambda}Q_1}e^{\bar{\epsilon}Q_2}e^{i(\bar{\lambda}\Gamma_\mu\epsilon)\delta/\delta A_\mu}\mathrm{tr}e^{ik\cdot A} \\ &\quad\times e^{-i(\bar{\lambda}\Gamma_\mu\epsilon)\delta/\delta A_\mu}e^{-\bar{\epsilon}Q_2}e^{-\bar{\lambda}Q_1} \\ &= e^{-(\bar{\lambda}\not{k}\epsilon)}\omega(\lambda, k), \end{aligned} \tag{2.7}$$

where we have used the commutation relation

$$[\bar{\epsilon}_1Q_1, \bar{\epsilon}_2Q_2] = -i(\bar{\epsilon}_1\Gamma^\mu\epsilon_2)\frac{\partial}{\partial A^\mu}. \tag{2.8}$$

From (2.5) and (2.7), the following two relations for the supersymmetric Wilson loop operator are obtained:

$$[\bar{\epsilon}Q_1, \omega(\lambda, k)] - \epsilon\frac{\partial}{\partial\lambda}\omega(\lambda, k) = 0, \tag{2.9}$$

$$[\bar{\epsilon}Q_2, \omega(\lambda, k)] + (\bar{\lambda}\not{k}\epsilon)\omega(\lambda, k) = 0. \tag{2.10}$$

These relations mean that the supersymmetric Wilson loop operator is invariant if we perform supersymmetry transformations (1.7) simultaneously with the supersymmetry transformations for (λ, k). By expanding $\omega(\lambda, k)$ in terms of an appropriate basis of wave functions for λ as

$$\omega(\lambda, k) = \sum_A f_A(\xi)V_A(A_\mu, \psi; k), \tag{2.11}$$

we can define supersymmetry transformations for the wave functions by

$$\delta^{(1)}f(\lambda, k) = \epsilon\frac{\partial}{\partial\lambda}f(\lambda, k), \tag{2.12}$$

$$\delta^{(2)}f(\lambda, k) = (\bar{\epsilon}\not{k}\lambda)f(\lambda, k). \tag{2.13}$$

These transformations are the same as (1.13) except that these two supersymmetries are interchanged. As we explain later, the interchanging can be realized by a charge conjugation operation.

The Majorana-Weyl fermion λ contains 16 degrees of freedom and there are 2^{16} independent wave functions for λ. To reduce the number, we impose a massless condition for the momenta k. Then since $\not{k}\lambda$ has only eight independent degrees of freedom the supersymmetry can generate only $2^8 = 256$ independent wave functions for λ. They form a massless type IIB supergravity multiplet containing a complex dilaton Φ, a complex dilatino $\tilde{\Phi}$, a complex antisymmetric tensor $B_{\mu\nu}$, a complex gravitino Ψ_μ, a real graviton $h_{\mu\nu}$, and a real 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$.

We now define a charge conjugation operation on the massless wave functions $f(\lambda, k)$. The charge conjugation is an operation to interchange a wave function with $p(\leq 8)$ λ's and that with $(8-p)$ λ's. It is defined by

$$(\hat{C}f)(\zeta, k) = f^c(\zeta, k) \equiv \int[d\lambda]e^{\bar{\zeta}\not{k}\lambda}f(\lambda, k), \tag{2.14}$$

where the integration of λ is performed with respect to eight λ's included in $\not{k}\lambda$. The integral measure is normalized so that $\hat{C}^2 = 1$. Acting $\hat{C}^2 = 1$ on a wave function, we get

$$\begin{aligned} (\hat{C}^2f)(\lambda', k) &= \int[d\zeta][d\lambda]e^{\bar{\zeta}\not{k}(\lambda-\lambda')}f(\lambda, k) \\ &= \int[d\zeta][d\lambda]e^{\bar{\zeta}\not{k}\lambda}f(\lambda + \lambda', k) \\ &= \int[d\zeta][d\lambda]\frac{1}{8!}(\bar{\zeta}\not{k}\lambda)^8f(\lambda + \lambda'). \end{aligned} \tag{2.15}$$

If we take a special momentum $k^\mu = (E, 0\cdots 0, E)$ and use the Gamma matrices given in the appendix, we have

$$\frac{1}{8!}(\bar{\zeta}\not{k}\lambda)^8 = (2E)^8(\zeta_9\zeta_{10}\cdots\zeta_{16})(\lambda_9\cdots\lambda_{16}), \tag{2.16}$$

and the normalization of the integration is given by

$$\int[d\zeta](2E)^4(\zeta_9\cdots\zeta_{16}) = 1. \tag{2.17}$$

It is easy to show that supersymmetry transformations for the charge conjugated fields are interchanged between $\delta^{(1)}$ and $\delta^{(2)}$:

$$(\delta^{(1)}f)^c(\zeta, k) = (\bar{\epsilon}\not{k}\zeta)f^c(\zeta) = \delta^{(2)}f^c(\zeta), \tag{2.18}$$

$$(\delta^{(2)}f)^c(\zeta, k) = \epsilon\frac{\partial}{\partial\zeta}f^c(\zeta) = \delta^{(1)}f^c(\zeta). \tag{2.19}$$

III. WAVE FUNCTIONS FOR THE IIB SUPERGRAVITY MULTIPLET

In this section we derive wave functions $f(\lambda, k)$ for a massless supergravity multiplet by using the transformations (2.12) and (2.13). It can be seen that these wave functions satisfy the SUSY transformations of the IIB supergravity.

A. Dilaton Φ and dilatino $\tilde{\Phi}$

We start with the simplest wave function which can be interpreted as a dilaton field Φ in the IIB supergravity multiplet:

$$\Phi(\lambda, k) = 1. \tag{3.1}$$

Dilatino wave function $\tilde{\Phi}$ can be generated from the dilaton wave function Φ by supersymmetry $\delta^{(2)}$ as

$$\delta^{(2)}\Phi(\lambda, k) = \bar{\epsilon}\not{k}\lambda \equiv \bar{\epsilon}\,\tilde{\Phi}(\lambda, k). \tag{3.2}$$

Hence the dilatino wave function is given by

$$\tilde{\Phi}(\lambda, k) = \not{k}\lambda. \tag{3.3}$$

The dilatino wave function automatically satisfies the equation of motion

$$\not{k}\tilde{\Phi} = 0, \tag{3.4}$$

because of the massless condition $k^2 = 0$. Then we can show the supersymmetry transformation between the dilaton and the dilatino:

$$\delta^{(1)}\tilde{\Phi} = \not{k}\epsilon = \Gamma^\mu\epsilon(-i\partial_\mu\Phi). \tag{3.5}$$

B. Antisymmetric tensor field $B_{\mu\nu}$

The wave function of the next field, an antisymmetric tensor field contains two λ's and can be generated from the dilatino wave function by $\delta^{(2)}$ transformation as

$$\delta^{(2)}\tilde{\Phi}(\lambda, k) = -\frac{1}{16}\Gamma^{\mu\nu\rho}\epsilon k_\mu(k^\sigma\bar{\lambda}\Gamma_{\nu\rho\sigma}\lambda) \equiv -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon H_{\mu\nu\rho}, \tag{3.6}$$

We identify $H_{\mu\nu\rho}$ as the field strength of the antisymmetric tensor $B_{\mu\nu}(\lambda, k)$,

$$H_{\mu\nu\rho} = i(k_\mu B_{\nu\rho} + k_\nu B_{\rho\mu} + k_\rho B_{\mu\nu}). \tag{3.7}$$

Then the wave function $B_{\mu\nu}$ is given by

$$B_{\mu\nu}(\lambda, k) = -\frac{1}{2}b_{\mu\nu} + (k_\mu v_\nu - k_\nu v_\mu) \equiv -\frac{1}{2}b_{\mu\nu} + k_{[\mu}v_{\nu]}, \tag{3.8}$$

where v_μ represents gauge degrees of freedom corresponding to the two form gauge field $B_{\mu\nu}$. Here we have defined an antisymmetric bilinear of λ by

$$b_{\mu\nu}(\lambda) \equiv k^\rho(\bar{\lambda}\Gamma_{\mu\nu\rho}\lambda). \tag{3.9}$$

They are the only independent bilinear forms constructed from eight independent massless spinors (namely nonzero components of $\not{k}\lambda$) and there are ${}_8C_2 = 28$ degrees of freedom. This number can be understood as follows. $b_{\mu\nu}$ satisfies two relations

$$k^\mu b_{\mu\nu} = 0, \tag{3.10}$$

$$b_{\mu\nu}\Gamma^{\mu\nu}\lambda = 0, \tag{3.11}$$

and an independent number of each relation is 9 and 8. Hence the number of independent $b_{\mu\nu}$ is ${}_{10}C_2 - 9 - 8 = 28$. The proof of the second relation (3.11) is given in the appendix.

For simplicity we fix the gauge degrees of freedom as $v_\mu = 0$. For the wave function (3.8), the equation of motion for the antisymmetric tensor is satisfied,

$$k^\mu H_{\mu\nu\rho} = 0, \tag{3.12}$$

because of $k^2 = 0$ and

$$k^\mu B_{\mu\nu} = 0. \tag{3.13}$$

A variation under the other supersymmetry $\delta^{(1)}$ of the wave function $B_{\mu\nu}(\lambda, k)$ is calculated as

$$\delta^{(1)}B_{\mu\nu} = -\bar{\epsilon}\Gamma_{\mu\nu}\tilde{\Phi}. \tag{3.14}$$

C. Gravitino Ψ_μ

A gravitino wave function contains three λ's and can be generated from $B_{\mu\nu}$ through $\delta^{(2)}$ supersymmetry transformation. It is defined through the SUSY transformation

$$\delta^{(2)}B_{\mu\nu} = 2i(\bar{\epsilon}\Gamma_{[\mu}\Psi_{\nu]} + k_{[\mu}\Lambda_{\nu]}). \tag{3.15}$$

Λ^μ is a gauge transformation parameter. Since the left-hand side of (3.15) becomes

$$\delta^{(2)}B_{\mu\nu} = (\bar{\epsilon}\not{k}\lambda)B_{\mu\nu}(\lambda, k) = -\frac{1}{2}(\bar{\epsilon}\not{k}\lambda)b_{\mu\nu}, \tag{3.16}$$

we can identify the wave function

$$\Psi_\mu(\lambda, k) = -\frac{i}{24}(k_\rho\Gamma^{\nu\rho}\lambda)b_{\mu\nu}, \tag{3.17}$$

and the gauge transformation parameter

$$\Lambda_\mu(\lambda, k) = -\frac{i}{12}(\bar\epsilon\Gamma^\nu\lambda)b_{\mu\nu}. \tag{3.18}$$

The wave function (3.17) automatically satisfies the equation of motion

$$k_\nu\Gamma^{\mu\nu\rho}\Psi_\rho = 0. \tag{3.19}$$

With the gauge choice in (3.17), this equation of motion is equivalent to

$$\not{k}\Psi_\mu = 0, \tag{3.20}$$

because of

$$\Gamma^\mu\Psi_\mu = k^\mu\Psi_\mu = 0. \tag{3.21}$$

The supersymmetry transformation $\delta^{(1)}$ for the gravitino wave function is given by

$$\begin{aligned}\delta^{(1)}\Psi_\mu(\lambda, k) &= -\frac{i}{24}[(\Gamma^\nu\not{k}\epsilon)b_{\mu\nu} + 2(\Gamma^\nu\not{k}\lambda)(\bar\epsilon\Gamma_{\mu\nu\rho}\lambda)k^\rho] \\ &= \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon H_{\mu\nu\rho} - \Gamma_{\mu\nu\rho\sigma}\epsilon H^{\nu\rho\sigma}] \\ &\quad + (\text{gauge tr.}).\end{aligned} \tag{3.22}$$

D. Graviton $h_{\mu\nu}$ and 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$

In the wave functions containing four λ's there are two fields, graviton $h_{\mu\nu}$ and 4-rank antisymmetric tensor field $A_{\mu\nu\rho\sigma}$. These wave functions can be read from the supersymmetry transformations of the gravitino field as

$$\delta^{(2)}\Psi_\mu(\lambda, k) = \frac{i}{2}\Gamma^{\lambda\rho}k_\rho h_{\mu\lambda}\epsilon + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon F_{\rho_1\cdots\rho_5} + (\text{gauge tr.}). \tag{3.23}$$

Here the field strength $F_{\mu\nu\rho\sigma\tau}(\lambda, k)$ is defined by

$$\begin{aligned}F_{\mu\nu\rho\sigma\tau} &= ik_\mu A_{\nu\rho\sigma\tau} + (\text{antisymmetrization}) \\ &= ik_{[\mu}A_{\nu\rho\sigma\tau]}.\end{aligned} \tag{3.24}$$

Since the left-hand side becomes

$$\delta^{(2)}\Psi_\mu(\lambda, k) = (\bar\epsilon\not{k}\lambda)\Psi_\mu \tag{3.25}$$

$$\begin{aligned}&= -\frac{i}{24}(\bar\epsilon\not{k}\lambda)(\Gamma^\nu\not{k}\lambda)b_{\mu\nu} \\ &= \frac{i}{12\cdot 16}\Gamma^{\nu\rho}\epsilon k_\rho b_\mu{}^\sigma b_{\sigma\nu} + \frac{i}{24\cdot 16} \\ &\quad\times\left[\frac{1}{5!}\Gamma^{\rho_1\rho_2\rho_3\rho_4}\epsilon k_{[\mu}b_{\rho_1\rho_2}b_{\rho_3\rho_4]} - (\text{gauge tr.})\right],\end{aligned} \tag{3.26}$$

we have the graviton wave function $h_{\mu\nu}$ as

$$h_{\mu\nu}(\lambda, k) = \frac{1}{96}b_\mu{}^\rho b_{\rho\nu}. \tag{3.27}$$

Because of the identity $b_{\mu\nu}b^{\mu\nu} = 0$, the graviton wave function is traceless. By using the self-duality of $F_{\mu\nu\rho\sigma\tau}$,

$$\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu F_{\rho_1\cdots\rho_5} = \Gamma^{\rho_1\cdots\rho_5}{}_\mu F_{\rho_1\cdots\rho_5} + 5\Gamma^{\rho_1\cdots\rho_4}F_{\rho_1\cdots\rho_4\mu} \tag{3.28}$$

$$= 10\Gamma^{\rho_1\cdots\rho_4}F_{\rho_1\cdots\rho_4\mu}, \tag{3.29}$$

we can also obtain the wave function for the field strength as

$$F_{\rho_1\cdots\rho_4\mu} = \frac{1}{32\cdot 4!}k_{[\mu}b_{\rho_1\rho_2}b_{\rho_3\rho_4]}. \tag{3.30}$$

and hence for the 4-rank antisymmetric tensor $A_{\rho_1\cdots\rho_4}$ as

$$A_{\rho_1\cdots\rho_4}(\lambda, k) = -\frac{i}{32(4!)^2}b_{[\rho_1\rho_2}b_{\rho_3\rho_4]}, \tag{3.31}$$

up to gauge transformations. It can be checked directly that the field strength $F_{\mu\nu\rho\sigma\tau}$ is self-dual with this wave function.

Under the other SUSY transformation $\delta^{(1)}$, these wave functions transform as follows:

$$\delta^{(1)}h_{\mu\nu} = -\frac{i}{2}\bar\epsilon\Gamma_{(\mu}\Psi_{\nu)} + (\text{gauge tr.}), \tag{3.32}$$

$$\delta^{(1)}A_{\mu\nu\rho\sigma} = -\frac{1}{32\cdot 4!}\bar\epsilon\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} + (\text{gauge tr.}), \tag{3.33}$$

where a round bracket for indices means symmetrization with a weight 1.

E. Charge conjugation and the other wave functions

The other wave functions in the massless supergravity multiplet can be similarly constructed by using the supersymmetry transformations. In the following we instead make use of the charge conjugation operation (2.14) to obtain the other wave functions.

First the charge conjugation of the dilaton field is given by

$$\begin{aligned}\Phi^c(\zeta, k) &= \int[d\lambda]e^{\bar\zeta\not{k}\lambda} = (2E)^4(\zeta_9\cdots\zeta_{16}) \\ &= \frac{1}{8\cdot 8!}b_\mu{}^\nu b_\nu{}^\lambda b_\lambda{}^\sigma b_\sigma{}^\mu(\zeta).\end{aligned} \tag{3.34}$$

The determination of the coefficient is straightforward but not easy to obtain. We have determined the coefficient by using a computer and verified that it is consistent with the SUSY transformations of the wave functions.

The charge conjugated dilatino wave function becomes

$$\tilde\Phi^c(\zeta, k) = \int[d\lambda]e^{\bar\zeta\not{k}\lambda}\tilde\Phi(\lambda, k) = \frac{1}{8!}k_\alpha\Gamma^{\mu\nu\alpha}\lambda b_{\nu\rho}b^{\rho\sigma}b_{\sigma\mu}. \tag{3.35}$$

It also satisfies the same equation of motion as the dilatino field

$$\not{k}\tilde{\Phi}^c = 0. \quad (3.36)$$

By taking the charge conjugation of the transformation (3.2) and (3.5), we have

$$\delta^{(1)}\Phi^c(\zeta, k) = \bar{\epsilon}\tilde{\Phi}^c(\zeta, k), \quad (3.37)$$

$$\delta^{(2)}\tilde{\Phi}^c(\zeta, k) = \Gamma^\mu \epsilon(-i\partial_\mu \Phi^c). \quad (3.38)$$

The wave function for the charge conjugated antisymmetric tensor field is given by

$$B^c_{\mu\nu}(\zeta, k) = \int [d\lambda] e^{\bar{\zeta}\not{k}\lambda} B_{\mu\nu}(\lambda, k) = -\frac{1}{6!} b_{\mu\rho} b^{\rho\sigma} b_{\sigma\nu}. \quad (3.39)$$

From transformations (3.6) and (3.14), we have supersymmetry transformations for the charge conjugated field as

$$\delta^{(1)}\tilde{\Phi}^c(\zeta, k) = -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon(H_{\mu\nu\rho})^c, \quad (3.40)$$

$$\delta^{(2)} B^c_{\mu\nu} = -\bar{\epsilon}\Gamma_{\mu\nu}\tilde{\Phi}^c. \quad (3.41)$$

Finally the charge conjugated gravitino wave function becomes

$$\Psi^c_\mu(\zeta, k) = \int [d\lambda] e^{\bar{\zeta}\not{k}\lambda} \Psi_\mu(\lambda, k) = -\frac{i}{4\cdot 5!} k^\rho \Gamma_{\rho\lambda} \lambda b^{\lambda\sigma} b_{\mu\sigma}, \quad (3.42)$$

and its supersymmetry transformation is given by

$$\delta^{(1)} B^c_{\mu\nu} = 2i(\bar{\epsilon}\Gamma_{[\mu}\Psi^c_{\nu]} + k_{[\mu}\Lambda^c_{\nu]}), \quad (3.43)$$

$$\delta^{(2)}\Psi^c_\mu(\zeta, k) = \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon(H_{\mu\nu\rho})^c - \Gamma_{\mu\nu\rho\sigma}\epsilon(H^{\nu\rho\sigma})^c] + (\text{gauge tr.}). \quad (3.44)$$

Graviton and 4-rank antisymmetric tensor fields are invariant under the charge conjugation:

$$h^c_{\mu\nu} = h_{\mu\nu}, A^c_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma}. \quad (3.45)$$

Therefore we have the charge conjugated supersymmetry transformation as

$$\delta^{(1)}\Psi^c_\mu = \frac{i}{2}\Gamma^{\nu\rho} k_\rho h_{\mu\nu}\epsilon + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu \epsilon F_{\rho_1\cdots\rho_5} + (\text{gauge tr.}), \quad (3.46)$$

$$\delta^{(2)} h_{\mu\nu} = -\frac{i}{2}\bar{\epsilon}\Gamma_{(\mu}\Psi^c_{\nu)} + (\text{gauge tr.}), \quad (3.47)$$

$$\delta^{(2)} A_{\mu\nu\rho\sigma} = -\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_{[\mu\nu\rho}\Psi^c_{\sigma]} + (\text{gauge tr.}). \quad (3.48)$$

F. Wave functions and SUSY transformations

We here summarize the wave functions for the massless multiplet and their supersymmetry transformations.

(i) Wave functions

$$\begin{aligned}
\Phi(\lambda, k) &= 1, \qquad \tilde{\Phi}(\lambda, k) = \not{k}\lambda, \\
B_{\mu\nu}(\lambda, k) &= -\frac{1}{2} b_{\mu\nu}(\lambda), \\
\Psi_\mu(\lambda, k) &= -\frac{i}{24}(k_\sigma \Gamma^{\nu\sigma}\lambda) b_{\mu\nu}(\lambda), \\
h_{\mu\nu}(\lambda, k) &= \frac{1}{96} b_\mu{}^\rho b_{\rho\nu}(\lambda), \\
A_{\mu\nu\rho\sigma}(\lambda, k) &= -\frac{i}{32(4!)^2} b_{[\mu\nu} b_{\rho\sigma]}(\lambda), \\
\Psi^c_\mu(\lambda, k) &= -\frac{i}{4\cdot 5!} k^\rho \Gamma_{\rho\lambda}\lambda b^{\lambda\sigma} b_{\sigma\mu}(\lambda), \\
B^c_{\mu\nu}(\lambda, k) &= -\frac{1}{6!} b_{\mu\rho} b^{\rho\sigma} b_{\sigma\nu}(\lambda), \\
\tilde{\Phi}^c(\lambda, k) &= \frac{1}{8!} k_\alpha \Gamma^{\mu\nu\alpha}\lambda b_{\nu\rho} b^{\rho\sigma} b_{\sigma\mu}(\lambda), \\
\Phi^c(\lambda, k) &= \frac{1}{8\cdot 8!} b_\mu{}^\nu b_\nu{}^\lambda b_\lambda{}^\sigma b_\sigma{}^\mu(\lambda).
\end{aligned} \quad (3.49)$$

(ii) SUSY transformations

$$\begin{aligned}
\delta\Phi &= \bar{\epsilon}_2\tilde{\Phi}, \quad \delta\tilde{\Phi} = \not{k}\epsilon_1\Phi - \tfrac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_2 H_{\mu\nu\rho}, \\
\delta B_{\mu\nu} &= -\bar{\epsilon}_1\Gamma_{\mu\nu}\tilde{\Phi} + 2i(\bar{\epsilon}_2\Gamma_{[\mu}\Psi_{\nu]} + k_{[\mu}\Lambda_{\nu]}), \\
\delta\Psi_\mu &= \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon_1 H_{\mu\nu\rho} - \Gamma_{\mu\nu\rho\sigma}\epsilon_1 H^{\nu\rho\sigma}] \\
&\quad + \frac{i}{2}\Gamma^{\nu\rho} k_\rho h_{\mu\nu}\epsilon_2 \\
&\quad + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_2 F_{\rho_1\cdots\rho_5} + k_\mu\xi, \\
\delta h_{\mu\nu} &= -\frac{i}{2}\bar{\epsilon}_1\Gamma_{(\mu}\Psi_{\nu)} - \frac{i}{2}\bar{\epsilon}_2\Gamma_{(\mu}\Psi^c_{\nu)} + k_{(\mu}\xi_{\nu)}, \\
\delta A_{\mu\nu\rho\sigma} &= -\frac{1}{(4!)^2}\bar{\epsilon}_1\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} - \frac{1}{(4!)^2}\bar{\epsilon}_2\Gamma_{[\mu\nu\rho}\Psi^c_{\sigma]} \\
&\quad + k_{[\mu}\xi_{\nu\rho\sigma]}, \\
\delta\Psi^c_\mu &= \frac{i}{2}\Gamma^{\nu\rho} k_\rho h_{\mu\nu}\epsilon_1 + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_1 F_{\rho_1\cdots\rho_5} \\
&\quad + \frac{1}{24\cdot 4}[9\Gamma^{\nu\rho}\epsilon_2 H^c_{\mu\nu\rho} - \Gamma_\mu{}^{\nu\rho\sigma}\epsilon_2 H^c_{\nu\rho\sigma}] \\
&\quad + k_\mu\xi^c, \\
\delta B^c_{\mu\nu} &= 2i(\bar{\epsilon}_1\Gamma_{[\mu}\Psi^c_{\nu]} + k_{[\mu}\Lambda^c_{\nu]}) - \bar{\epsilon}_2\Gamma_{\mu\nu}\tilde{\Phi}^c, \\
\delta\tilde{\Phi}^c &= -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_1 H^c_{\mu\nu\rho} + \not{k}\epsilon_2\Phi^c, \\
\delta\Phi^c &= \bar{\epsilon}_1\tilde{\Phi}^c,
\end{aligned} \quad (3.50)$$

where ξ, ξ_μ, $\xi_{\mu\nu\rho}$, and Λ_μ are gauge parameters. This supersymmetry transformation is the same as that in [11] up to normalizations.

IV. VERTEX OPERATORS IN IIB MATRIX MODEL

In this section, we construct the vertex operators in the IIB matrix model. The construction can be done systematically by expanding the supersymmetric Wilson loop operator in terms of the wave functions $f_A(\lambda)$ constructed in the previous section.

First we rewrite the Wilson loop operator (2.2) in terms of the supersymmetry transformations of $(ik\cdot A)$ as follows:

$$\omega(\lambda, k) = e^{\bar\lambda Q_1}\mathrm{tr}e^{ik\cdot A}e^{-\bar\lambda Q_1} = \mathrm{tr}e^{G}, \tag{4.1}$$

where G is given as a finite sum

$$\begin{aligned}G &= ik\cdot A + [\bar\lambda Q_1, ik\cdot A] + \frac{1}{2}[\bar\lambda Q_1,[\bar\lambda Q_1, ik\cdot A]] + \cdots \\ &\quad + \frac{1}{n!}[\bar\lambda Q_1,\cdots,[\bar\lambda Q_1, e^{ik\cdot A}]] + \cdots \\ &= \sum_{i=0}^{8} G_i.\end{aligned} \tag{4.2}$$

Note that the sum terminates at $i = 8$ because there are only eight independent λ's for the on shell ($k^2 = 0$) Wilson loop operator. Each term can be evaluated as follows:

$$G_0 = ik\cdot A, \tag{4.3}$$

$$G_1 = -(\bar\lambda \not k\psi), \tag{4.4}$$

$$G_2 = \frac{i}{4}b^{\mu\nu}[A_\mu, A_\nu], \tag{4.5}$$

$$G_3 = -\frac{1}{3!}b^{\mu\nu}[\bar\lambda\Gamma_\mu\psi, A_\nu], \tag{4.6}$$

$$\begin{aligned}G_4 = \frac{1}{4!}\Big\{&\frac{i}{2}b^{\mu\nu}(\bar\lambda\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], A_\nu] \\ &- ib^{\mu\nu}[\bar\lambda\Gamma_\mu\psi, \bar\lambda\Gamma_\nu\psi]\Big\},\end{aligned} \tag{4.7}$$

$$\begin{aligned}G_5 = -\frac{1}{5!}\Big\{&b^{\mu\nu}(\bar\lambda\Gamma_{\mu\rho\sigma}\lambda)[[\bar\lambda\Gamma^\rho\psi, A^\sigma], A_\nu] \\ &+ \frac{3}{2}b^{\mu\nu}(\bar\lambda\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], \bar\lambda\Gamma_\nu\psi]\Big\}.\end{aligned} \tag{4.8}$$

$$\vdots$$

Note that G_n contains n λ's. In order to obtain a vertex operator of each field, we need to expand $\omega(\lambda, k)$ and collect all terms with the same number of λ as

$$\begin{aligned}\omega(\lambda, k) &= \mathrm{tr}(e^{ik\cdot A+\sum_{i=1}^{8}G_i}) \\ &= \mathrm{Str}e^{ik\cdot A}\Big\{1 + G_1 + \Big(\frac{1}{2}G_1\cdot G_1 + G_2\Big) + \Big(\frac{(G_1^3).}{3!} + G_1\cdot G_2 + G_3\Big) + \Big[\frac{(G_1^4).}{4!} + \frac{1}{2}(G_1^2).\cdot G_2 \\ &\quad + \frac{1}{2}(G_2^2). + G_1\cdot G_3 + G_4\Big] + \Big[\frac{(G_1^5).}{5!} + \frac{1}{3!}(G_1^3).\cdot G_2 + \frac{1}{2}G_1\cdot(G_2^2). \\ &\quad + \frac{1}{2}(G_1^2).\cdot G_3 + G_2\cdot G_3 + G_1\cdot G_4 + G_5\Big] + \cdots\Big\}.\end{aligned} \tag{4.9}$$

Here "Str" means a symmetrized trace which is defined by

$$\begin{aligned}\mathrm{Str}e^{ik\cdot A}B_1\cdot B_2\cdots B_n = \times\int_0^1 dt_1\int_{t_1}^1 dt_2\cdots\int_{t_{n-2}}^1 dt_{n-1}\mathrm{tr}e^{ik\cdot At_1}B_1e^{ik\cdot A(t_2-t_1)}B_2\cdots e^{ik\cdot A(t_{n-1}-t_{n-2})}B_{n-1}e^{ik\cdot A(1-t_{n-1})}B_n \\ + [\text{permutations of } \mathrm{B}_\mathrm{i}'\mathrm{s}(i = 2, 3, \cdots, n)].\end{aligned} \tag{4.10}$$

The center dot on the left-hand side means that the operators B_i are symmetrized. We denoted $\underbrace{G_k\cdot G_k\cdots G_k}_{n}$ as (G_k^n).. Various properties of the symmetrized trace are given in the appendix. For notational simplicity we sometimes use Str with a single operator like $\mathrm{Str}(e^{ik\cdot A}B)$ which is equivalent to an ordinary trace. If we set $k = 0$, the symmetrized trace becomes

$$\mathrm{S\,tr}(B_1\cdot B_2\cdots B_n) = \frac{1}{n!}\sum_{\text{perm.}}\mathrm{tr}(B_{i_1}B_{i_2}\cdots B_{i_n}). \tag{4.11}$$

A. Dilaton Φ and dilatino $\tilde\Phi$

Dilaton vertex operator V^Φ is given by the leading order of λ, namely, a term without λ,

$$V^{\Phi} = \mathrm{tr} e^{ik\cdot A}. \qquad (4.12)$$

Dilatino vertex operator $V^{\tilde{\Phi}}$ is read from the term with a single λ. This is also easily obtained as

$$\mathrm{tr} e^{ik\cdot A} G_1 = \mathrm{tr} e^{ik\cdot A}(-\bar{\lambda}\not{k}\psi) = (\mathrm{tr} e^{ik\cdot A}\bar{\psi})\cdot(\not{k}\lambda), \qquad (4.13)$$

$$V^{\tilde{\Phi}} = \mathrm{tr} e^{ik\cdot A}\bar{\psi}. \qquad (4.14)$$

B. Antisymmetric tensor field $B_{\mu\nu}$

The vertex operator for the antisymmetric tensor $B_{\mu\nu}$ can be obtained from the terms with two λ's:

$$\begin{aligned}\mathrm{Str} e^{ik\cdot A}\left(\frac{1}{2}G_1\cdot G_1 + G_2\right) &= \mathrm{Str} e^{ik\cdot A}\left(\frac{1}{2}(\bar{\lambda}k\psi)\cdot(\bar{\lambda}k\psi) + \frac{i}{4}b^{\mu\nu}[A_\mu, A_\nu]\right) \\ &= \mathrm{Str} e^{ik\cdot A}\left(-\frac{1}{32}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) + \frac{i}{4}[A_\mu, A_\nu]\right)b^{\mu\nu}. \qquad (4.15)\end{aligned}$$

Hence the vertex operator for the antisymmetric tensor field is given by

$$V^B_{\mu\nu} = \mathrm{Str} e^{ik\cdot A}\left(\frac{1}{16}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) - \frac{i}{2}[A_\mu, A_\nu]\right). \qquad (4.16)$$

This vertex operator satisfies

$$k^\mu V^B_{\mu\nu} = 0, \qquad (4.17)$$

which assures the gauge invariance of the coupling with the wave function obtained in the previous section, $B^{\mu\nu}(\lambda)V^B_{\mu\nu}$.

C. Gravitino Ψ_μ

The 3rd order terms give the gravitino Ψ_μ vertex operator as

$$\begin{aligned}&\mathrm{Str} e^{ik\cdot A}\left(\frac{1}{3!}G_1\cdot G_1\cdot G_1 + G_1\cdot G_2 + G_3\right) \\ &= \mathrm{Str} e^{ik\cdot A}\left[-\frac{1}{6}(\bar{\lambda}\not{k}\psi)^3 - \frac{i}{4}(\bar{\lambda}\not{k}\psi)b^{\mu\nu}\cdot[A_\mu, A_\nu] - \frac{1}{6}b^{\mu\nu}[A_\mu, \bar{\lambda}\Gamma_\nu\psi]\right].\end{aligned} \qquad (4.18)$$

Here the following relation is useful:

$$\begin{aligned}b_{\mu\nu}(\lambda)(\bar{\lambda}\not{k}\psi) = \frac{1}{4}\{&b_\mu{}^\sigma k^\rho(\bar{\lambda}\Gamma_{\sigma\nu\rho}\psi) - b_\nu{}^\sigma k^\rho(\bar{\lambda}\Gamma_{\sigma\mu\rho}\psi) \\ &- k_\mu b_\nu{}^\sigma(\bar{\lambda}\Gamma_\sigma\psi) + k_\nu b_\mu{}^\sigma(\bar{\lambda}\Gamma_\sigma\psi)\}.\end{aligned} \qquad (4.19)$$

Using this relation, the first term on the right-hand side of (4.18) becomes

$$\left[-\frac{i}{12}\mathrm{Str} e^{ik\cdot A}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi)\cdot\bar{\psi}\Gamma^\nu\right]\Psi^\mu(\lambda), \qquad (4.20)$$

where $\Psi^\mu(\lambda)$ is the wave function of the gravitino (3.17). Similarly the second term on the right-hand side of (4.18) is rewritten as

$$\mathrm{Str} e^{ik\cdot A}\left[-\frac{i}{12}k^\rho b_\mu{}^\sigma(\bar{\psi}\Gamma_\nu\Gamma_{\rho\sigma}\lambda)\cdot[A_\mu, A_\nu] - \frac{1}{6}b^{\mu\nu}(\bar{\lambda}\Gamma_\nu\psi)\cdot[A_\mu, ik\cdot A]\right]. \qquad (4.21)$$

By using the relation (A33) in the appendix, it is easily understood that the last term cancels the third term of (4.18). Therefore the terms with three λ's become

$$\begin{aligned}&\mathrm{Str} e^{ik\cdot A}\left(\frac{1}{3!}(G_1^3). + G_1\cdot G_2 + G_3\right) \\ &= \mathrm{Str} e^{ik\cdot A}\left(-\frac{i}{12}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) - 2[A_\mu, A_\nu]\right)\cdot\bar{\psi}\Gamma^\nu\Psi^\mu(\lambda),\end{aligned} \qquad (4.22)$$

and thus we have the vertex operator for the gravitino

$$V^\Psi_\mu = \mathrm{Str} e^{ik\cdot A}\left(-\frac{i}{12}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) - 2[A_\mu, A_\nu]\right)\cdot\bar{\psi}\Gamma^\nu. \qquad (4.23)$$

The second term is a matrix regularization of the supercurrent $\bar{J}_\mu = \{X_\mu, X_\nu\}\bar{\psi}\gamma^\nu$ associated with the supersymmetry $\delta\psi = \epsilon'$ of the Schild action. Here {} is the Poisson bracket on the world sheet.

This gravitino vertex operator is shown to satisfy

$$k^\mu V^\Psi_\mu = 0. \qquad (4.24)$$

The first term of V^Ψ_μ trivially satisfies this relation and the second term is calculated as follows:

$$\begin{aligned}k^\mu(\text{the 2nd term of } V^\Psi_\mu) &= -2\mathrm{tr}\int_0^1 dt e^{ik\cdot At}[k\cdot A, A_\mu] \times e^{ik\cdot A(1-t)}\bar{\psi}\Gamma^\mu \\ &= 2i\mathrm{tr}[e^{ik\cdot A}, A_\mu]\bar{\psi}\Gamma^\mu \\ &= 2i\mathrm{tr} e^{ik\cdot A}[A_\mu, \bar{\psi}]\Gamma^\mu \\ &= 0.\end{aligned} \qquad (4.25)$$

In the last line, we used the equation of motion for the fermion, $\Gamma^\lambda[A_\lambda, \psi] = 0$. (4.24) assures the gauge invariance of the coupling with the gravitino wave function

$$V^\Psi_\mu\Psi^\mu. \qquad (4.26)$$

D. Graviton and 4-rank antisymmetric tensor field

The next terms with four λ's give the vertex operators for the graviton $h_{\mu\nu}$ and the 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$. The calculation becomes more complicated and we need to use various identities involving fermions. Here we only write down the final results:

$$
\begin{aligned}
&\mathrm{Str}e^{ik\cdot A}\left(\frac{(G_1^4).}{4!}+\frac{1}{2}G_1\cdot G_1\cdot G_2+\frac{1}{2}G_2\cdot G_2+G_1\cdot G_3+G_4\right)\\
&=\mathrm{Str}e^{ik\cdot A}\left(\frac{1}{4!}(\bar\lambda k\psi)^4+\frac{i}{8}(\bar\lambda k\psi)^2 b^{\mu\nu}\cdot[A_\mu,A_\nu]-\frac{1}{32}b^{\mu\nu}b^{\alpha\beta}[A_\mu,A_\nu]\cdot[A_\alpha,A_\beta]+\frac{1}{6}(\bar\lambda k\psi)b^{\mu\nu}\cdot[\bar\lambda\Gamma_\mu\psi,A_\nu]\right.\\
&\left.\quad-\frac{i}{24}b^{\mu\nu}[\bar\lambda\Gamma_\mu\psi,\bar\lambda\Gamma_\nu\psi]+\frac{i}{48}b_{\mu\nu}(\bar\lambda\Gamma^{\alpha\beta\mu}\lambda)[[A_\alpha,A_\beta],A_\nu]\right)\\
&=\frac{1}{48}b^\mu{}_a b^{a\nu}\mathrm{Str}e^{ik\cdot A}\left\{[A_\mu,A^\rho]\cdot[A_\nu,A_\rho]+\frac{1}{2}\bar\psi\cdot\Gamma_\mu[A_\nu,\psi]+\frac{i}{4}k^\lambda(\bar\psi\cdot\Gamma_{\mu\lambda\sigma}\psi)\cdot[A^\sigma,A_\nu]-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\lambda}{}^\sigma\psi)\right.\\
&\left.\quad\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\right\}+\frac{1}{3}\cdot\left(-\frac{1}{32}\right)(b^{\mu\nu}b^{\rho\sigma}+b^{\mu\rho}b^{\sigma\nu}+b^{\mu\sigma}b^{\nu\rho})\mathrm{Str}e^{ik\cdot A}\left\{[A_\mu,A_\nu]\cdot[A_\rho,A_\sigma]+C\bar\psi\cdot\Gamma_{\mu\nu\rho}[A_\sigma,\psi]\right.\\
&\left.\quad-\frac{3i}{4}Ck^\lambda(\bar\psi\cdot\Gamma_{\mu\nu\lambda}\psi)\cdot[A_\rho,A_\sigma]-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\nu\lambda}\psi)\cdot(\bar\psi\cdot\Gamma_{\rho\sigma\tau}\psi)\right\},
\end{aligned}
\tag{4.27}
$$

where C is a numerical constant which we could not determine in this approach of the calculation. But we can instead make use of other information of the block-block interaction briefly explained in the next subsection and determine it to be $C = -1/3$. Therefore we have the vertex operators for the graviton and the 4-rank antisymmetric tensor field, respectively,

$$
\begin{aligned}
V^h_{\mu\nu}=2\mathrm{Str}e^{ik\cdot A}&\left\{[A_\mu,A^\rho]\cdot[A_\nu,A_\rho]+\frac{1}{4}\bar\psi\cdot\Gamma_{(\mu}[A_{\nu)},\psi]\right.\\
&-\frac{i}{8}k^\rho\bar\psi\cdot\Gamma_{\rho\sigma(\mu}\psi\cdot[A_{\nu)},A^\sigma]-\frac{1}{8\cdot 4!}\\
&\left.\times k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\lambda}{}^\sigma\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\right\},
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
V^A_{\mu\nu\rho\sigma}=-i\mathrm{Str}e^{ik\cdot A}&\left\{F_{[\mu\nu}\cdot F_{\rho\sigma]}+C\bar\psi\cdot\Gamma_{[\mu\nu\rho}[A_{\sigma]},\psi]\right.\\
&-\frac{3i}{4}Ck^\lambda\bar\psi\cdot\Gamma_{\lambda[\mu\nu}\psi\cdot F_{\rho\sigma]}-\frac{1}{8\cdot 4!}\\
&\left.\times k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\lambda[\mu\nu}\psi)\cdot(\bar\psi\cdot\Gamma_{\rho\sigma]\tau}\psi)\right\},
\end{aligned}
\tag{4.29}
$$

where $F_{\mu\nu} = [A_\mu, A_\nu]$. These vertex operators satisfy the conservation laws by similar calculations as (4.25),

$$
k^\nu V^h_{\mu\nu}=0,\qquad k^\sigma V^A_{\mu\nu\rho\sigma}=0,
\tag{4.30}
$$

if we use the equations of motion (2.3) and (2.4). In the vertex operator of the graviton, while the fourth term trivially satisfies this equation, the first three terms multiplied by k_μ are combined to become a term proportional to the equations of motion. In the case of the 4-rank antisymmetric tensor, by multiplying k_μ, the forth term trivially vanishes and so does the first term due to the Jacobi identity. The second and the third terms satisfy the conservation law because of properties of the symmetrized trace.

Thus the couplings with the graviton and the 4-rank antisymmetric tensor wave functions

$$
h^{\mu\nu}V^h_{\mu\nu},\qquad A^{\mu\nu\rho\sigma}V^A_{\mu\nu\rho\sigma},
\tag{4.31}
$$

are, respectively, gauge invariant.

E. Other vertex operators

The other vertex operators are obtained from the terms containing more λ's and the calculations of them become exponentially more difficult. Therefore we do not proceed with this calculation here and instead give a part of the vertex operators by using other approaches.

The IIB matrix model can be regarded as a matrix regularization of the Schild type action for the IIB superstring. The supercurrent of the Schild action associated with the homogeneous supersymmetry (1.3) is given by

$$
J^{(2)}_\mu=\{X_\mu,X_\nu\}\{X_\rho,X_\sigma\}\Gamma^{\rho\sigma}\Gamma^\nu\psi-\frac{2i}{3}(\bar\psi\Gamma^\nu\{X_\mu,\psi\})\Gamma_\nu\psi.
\tag{4.32}
$$

It is then expected that the vertex operator for the charge conjugation of the gravitino includes a term which is a matrix regularization of the above supercurrent of the Schild action. Hence we have

$$
\begin{aligned}
V^{\Psi^c}_\mu=\mathrm{Str}e^{ik\cdot A}&\Big([A_\mu,A_\nu]\cdot[A_\rho,A_\sigma]\cdot\Gamma^{\rho\sigma}\Gamma^\nu\psi\\
&+\frac{2}{3}\bar\psi\cdot\Gamma_\nu[A_\mu,\psi]\cdot\Gamma^\nu\psi\Big).
\end{aligned}
\tag{4.33}
$$

This satisfies the relation $k^\mu V^{\Psi^c}_\mu = 0$ up to the equations of motion. Of course, the vertex operator will also contain other terms which include more fermions and momentum k_μ.

In the IIB matrix model, the interactions between supergravity modes can be obtained from the one-loop calculation by integrating out off-diagonal components of the matrices. These interaction terms are interpreted as exchange of massless supergravity particles between vertex operators for the diagonal blocks of the matrices. Exchange of the graviton, dilaton, and antisymmetric tensor field is identified in [1] by calculation of one-loop effective action without fermionic backgrounds. With fermionic backgrounds we can also identify exchange of the fermionic fields such as gravitinos and dilatinos. Moreover we can also read off other terms of the bosonic vertex operators containing even number of fermion fields such as the second term of the graviton vertex operator (4.28) or the coefficient C in the 4th rank antisymmetric tensor field. The one-loop effective action expanded with respect to the inverse powers of the relative distance between two blocks was given in [7,12,13]:

$$W^{(i,j)} = -3S\mathcal{T}r^{(i,j)}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\nu\sigma}\mathcal{F}_{\sigma\tau}\mathcal{F}_{\tau\mu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\mathcal{F}_{\tau\sigma}\mathcal{F}_{\tau\sigma}\Big)\frac{1}{(d^{(i)}-d^{(j)})^8} - 3S\mathcal{T}r^{(i,j)}(\bar{\Psi}\Gamma^{\mu}\Gamma^{\nu}\Gamma^{\rho}\mathcal{F}_{\sigma\mu}\mathcal{F}_{\nu\rho}[\mathcal{A}_{\sigma},\Psi]) \times\frac{1}{(d^{(i)}-d^{(j)})^8} + W^{(i,j)}_{\Psi^4} + O\Big[\frac{1}{(d^{(i)}-d^{(j)})^9}\Big]. \quad (4.34)$$

$W^{(i,j)}$ expresses the interaction between the ith block and jth block and $(d^{(i)} - d^{(j)})$ is the distance between the center of mass coordinate of the ith block and that of the jth block. $\mathcal{T}r$ is the trace of the adjoint operators and $\mathcal{F}$, $\mathcal{A}$, and Ψ are adjoint operators which act as $\mathcal{O}S = [O, S]$. W_{Ψ^4} denotes terms including four Ψ's. The terms up to $O(r^{-7})$ cancel each other when backgrounds are restricted to satisfy the matrix model equations of motion. From the above result we can identify some terms in the vertex operators.

In the case of the vertex operator for the charge conjugation of the antisymmetric tensor $V^{B^c}_{\mu\nu}$, the leading term with the least number of fermion fields can be read from the calculations of the block-block interaction as

$$\mathrm{Str}e^{ik\cdot A}\Big([A_\mu, A_\rho]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\nu] - \frac{1}{4}[A_\mu, A_\nu] \cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\rho]\Big). \quad (4.35)$$

Requiring the current conservation, $k^\mu V^{B^c}_{\mu\nu} = 0$, it can be understood that the vertex operator should include the following terms:

$$V^{B^c}_{\mu\nu} = \mathrm{Str}e^{ik\cdot A}\Big([A_\mu, A_\rho]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\nu] - \frac{1}{4}[A_\mu, A_\nu]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\rho] - \frac{1}{4}\bar{\psi}\cdot\Gamma_{(\mu}[A_{\rho)}, \psi]\cdot[A^\rho, A_\nu] + \frac{1}{4}\bar{\psi}\cdot\gamma_{(\nu}[A_{\rho)}, \psi] \cdot[A^\rho, A_\mu] + \frac{1}{16}\bar{\psi}\cdot\Gamma_{\rho\sigma[\mu}\psi\cdot[A_{\nu]}, [A^\rho, A^\sigma]] - \frac{i}{8}k_\lambda\bar{\psi}\cdot\Gamma^{\lambda\rho\sigma}\psi\cdot[A_\mu, A_\rho]\cdot[A_\nu, A_\sigma]\Big). \quad (4.36)$$

For the charge conjugations of the dilaton, the leading terms of the vertex operators can be similarly read from block-block interactions as

$$V^{\Phi^c} = \mathrm{Str}e^{ik\cdot A}\Big\{[A_\mu, A_\nu]\cdot[A^\nu, A^\rho]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\mu] - \frac{1}{4}[A_\mu, A_\nu]\cdot[A^\nu, A^\mu]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\rho] + [A_\sigma, A_\mu]\cdot[A_\nu, A_\rho]\cdot\bar{\psi}\Gamma^\mu\Gamma^{\nu\rho}\cdot[A_\sigma, \psi]\Big\}. \quad (4.37)$$

The charge conjugated dilatino vertex operator can be obtained from this charge conjugated dilaton vertex operator by supersymmetry transformations. The leading order term is proportional to

$$V^{\tilde{\Phi}^c} = \mathrm{Str}e^{ik\cdot A}\Big\{\Big([A_\mu, A_\rho]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\nu] - \frac{1}{4}[A_\mu, A_\nu]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\rho]\Big)\cdot\Gamma^{\mu\nu}\psi + \frac{1}{24}[A_\mu, A_\nu]\cdot[A_\rho, A_\sigma]\cdot[A_\lambda, A_\tau]\cdot\Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi\Big\}. \quad (4.38)$$

In order to obtain complete forms of the vertex operators, we need to accomplish the calculation which we performed in the previous section. The calculation is very complicated and tough. As we briefly explained above, we can instead determine the leading order terms of the vertex operators from the calculations of block-block interactions with bosonic and fermionic backgrounds.

V. CONCLUSIONS AND DISCUSSIONS

In this paper we have constructed a set of wave functions and vertex operators in the IIB matrix model by expanding the supersymmetric Wilson loop operator. They form a massless multiplet of the type IIB supergravity. The vertex operators satisfy conservation laws, for instance Eq. (4.24) or (4.30) by using equations of motion for A^μ and ψ.

When we couple these vertex operators to background fields such as a graviton $h^{\mu\nu}$ field and integrate out the matrices, we can obtain the effective action for the background fields. Schematically it is written as

$$e^{-S_{\rm eff}[h^{\mu\nu}]} = \int dA d\psi e^{-S+\sum_k h^{\mu\nu} V^h_{\mu\nu}}. \tag{5.1}$$

Since vertex operators satisfy the conservation laws, the effective action $S_{\rm eff}[h^{\mu\nu}]$ has a gauge symmetry and it may be written as a sum of gauge invariant terms:

$$S_{\rm eff}[h^{\mu\nu}] = \int d^{10}x[c_1(N)\sqrt{g} + c_2(N)\sqrt{g}R + \cdots]. \tag{5.2}$$

Here $c_i(N)$ are N dependent coefficients. This is reminiscent of the induced gravity. Of course in order to show that the gravity theory indeed appears as above, we need to show that the graviton is formed as a bound state and calculate the coefficients as functions of the matrix size N. Both of them are very difficult to perform but we can instead make use of the large N renormalization group as we will discuss below.

Here we discuss the origin of the conservation laws. We have used the supersymmetric Wilson loops and the supersymmetry transformations in order to obtain the vertex operators. We did not use the explicit form of the action. Nonetheless the vertex operators satisfy the conservation laws by using the equations of motion derived from the action (1.1). This is due to the commutation relation of the supersymmetry generators (2.6). Recall that the supersymmetric Wilson loop is invariant under simultaneous supersymmetry transformations of the matrices and wave functions as Eqs. (2.9) and (2.10) only by using the equations of motion. On the other hand, the supersymmetry transformations of the wave functions (3.50) contain gauge transformations. Because of it, the vertex operators satisfy conservation laws by using the equations of motion. In this sense, the conservation laws for the vertex operators follow from the supersymmetries. In string theories, conformal invariance guarantees the gauge invariance and the decoupling of unphysical modes from the S-matrix elements. It would be interesting to search for such a hidden symmetry in matrix models.

Another interesting issue is to obtain the equation of motion for the background field of the matrix models. In string theories, conformal invariance plays an important role in deriving an equation of motion for the background. In the matrix model, we expect that a large N renormalization group will play such a role. Matrix models are believed to describe string theories in the large N limit. As we discussed in the introduction, we implicitly assume that there are background D(− 1)'s other than the N D(− 1)'s and a modification of the configurations of the background D(− 1)'s leads to a modification of the background field for the N D(− 1)'s. Hence stability of the background must be related to the stability of the background configurations under integrations of the background D(− 1)'s. More concretely, we start from the matrix model for $(N+1)\times(N+1)$ Hermitian matrices A'_μ with a graviton coupling

$$S_{\rm IKKT}[A'_\mu] + \int dk h^{\mu\nu}(k) V^h_{\mu\nu}[A'_\mu], \tag{5.3}$$

and integrate 1 D(− 1) [which we call a mean field D(− 1)]. Then we arrive at a matrix model action for $N\times N$ Hermitian matrices A_μ with a modified graviton coupling

$$S_{\rm IKKT}[A_\mu] + \int dk h'^{\mu\nu}(k) V^h_{\mu\nu}[A_\mu], \tag{5.4}$$

and we can obtain a renormalization group flow for the coupling constant

$$h^{\mu\nu}(k) \to h^{\mu\nu}(k) + \delta h^{\mu\nu}(k). \tag{5.5}$$

Fixed points of this renormalization group flow will give the equations of motion for the background fields. Though the calculation itself is very difficult, we can also in principle obtain renormalization group flow for the coefficients $c_i(N)$ of the effective action (5.2). We want to investigate these issues in future publications.

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APPENDIX

1. MAJORANA-WEYL REPRESENTATION

We use the following Majorana-Weyl representation,

$$\begin{aligned}
&\Gamma_0 = i\sigma_1\otimes 1\otimes 1\otimes 1\otimes 1, && \Gamma_1 = i\epsilon\otimes\epsilon\otimes\epsilon\otimes\epsilon\otimes\epsilon,\\
&\Gamma_2 = i\epsilon\otimes\epsilon\otimes 1\otimes\sigma_1\otimes\epsilon, && \Gamma_3 = i\epsilon\otimes\epsilon\otimes 1\otimes\sigma_3\otimes\epsilon,\\
&\Gamma_4 = i\epsilon\otimes\epsilon\otimes\sigma_1\otimes\epsilon\otimes 1, && \Gamma_5 = i\epsilon\otimes\epsilon\otimes\sigma_3\otimes\epsilon\otimes 1,\\
&\Gamma_6 = i\epsilon\otimes\epsilon\otimes\epsilon\otimes 1\otimes\sigma_1, && \Gamma_7 = i\epsilon\otimes\epsilon\otimes\epsilon\otimes 1\otimes\sigma_3,\\
&\Gamma_8 = i\epsilon\otimes\sigma_1\otimes 1\otimes 1\otimes 1, && \Gamma_9 = i\epsilon\otimes\sigma_3\otimes 1\otimes 1\otimes 1,\\
&\Gamma_{11} = \sigma_3\otimes 1\otimes 1\otimes 1\otimes 1,
\end{aligned} \tag{A1}$$

where $\epsilon = i\sigma_2$.

2. PROPERTIES OF GAMMA MATRICES

(i) Metric

$$g_{\mu\nu} = {\rm diag}(-1, +1, \ldots, +1) \quad (D=10) \tag{A2}$$

(ii) Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \tag{A3}$$

$$\Gamma_{11} \equiv \Gamma_0\Gamma_1\cdots\Gamma_9, \qquad (\Gamma_{11})^2 = 1 \tag{A4}$$

(iii) Hermiticity

$$(\Gamma_\mu)^\dagger = \Gamma^\mu = \Gamma_0\Gamma_\mu\Gamma_0 \tag{A5}$$

$$(\Gamma_0)^\dagger = -\Gamma_0, (\Gamma_i)^\dagger = \Gamma_i \ \ (i = 1, 2, \ldots, 9) \tag{A6}$$

$$(\Gamma_{11})^\dagger = \Gamma_{11} \tag{A7}$$

Under our representations,

$$(\Gamma_0)^{\mathrm{T}} = \Gamma_0, \qquad (\Gamma_i)^{\mathrm{T}} = -\Gamma_i, \tag{A8}$$

and

$$\Gamma_0\Gamma_\mu\Gamma_0 = -(\Gamma_\mu)^{\mathrm{T}}. \tag{A9}$$

(iv) $\bar\psi \equiv \psi^\dagger\Gamma_0$

$$(i\bar\psi\psi)^* = -i\psi^\dagger(\Gamma_0)^\dagger\psi = i\bar\psi\psi \tag{A10}$$

$$(\mathrm{tr}\bar\psi\Gamma_\mu[A^\mu, \psi])^* = \mathrm{tr}\bar\psi\Gamma_\mu[A^\mu, \psi] \tag{A11}$$

(v) Charge conjugation

$$\psi^c = C\bar\psi^{\mathrm{T}} = \psi^*, \qquad C = \Gamma_0 \tag{A12}$$

(vi) Weyl spinor

$$\psi = \Gamma_{11}\psi \tag{A13}$$

$$\begin{aligned}\bar\psi_1\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi_2 &= \psi_1^\dagger\Gamma_0\Gamma_{\mu_1\mu_2\cdots\mu_n}\Gamma_{11}\psi_2 \\ &= (-1)^{n+1}\bar\psi_1\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi_2\end{aligned} \tag{A14}$$

Therefore bilinear forms of spinors vanish unless n is odd.

(vii) Majorana spinor

$$\psi^c = \psi \quad \longrightarrow \psi = \psi^* \tag{A15}$$

Under our representations,

$$\begin{aligned}\bar\psi_1\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi_2 &= -\psi_2^{\mathrm{T}}(\Gamma_{\mu_1\mu_2\cdots\mu_n})^{\mathrm{T}}(\Gamma_0)^{\mathrm{T}}\psi_1^* \\ &= -(-1)^{n(n-1)/2}\bar\psi_2\Gamma_{\mu_1\mu_2\cdots\mu_n}\psi_1.\end{aligned} \tag{A16}$$

When $\psi_1 = \psi_2$ is a Majorana-Weyl spinor, therefore, bilinear forms of spinors vanish unless $n =$ 3 or 7.

3. FIERZ IDENTITY

The Fierz identity is given by [14]

$$\begin{aligned}(\bar\psi_1 M\psi_2)(\bar\psi_3 N\psi_4) = -\frac{1}{32}\sum_{n=0}^{5} C_n(\bar\psi_1\Gamma_{A_n}\psi_4) \\ \times (\bar\psi_3 N\Gamma_{A_n}M\psi_2),\end{aligned} \tag{A17}$$

$$\begin{aligned}&C_0 = 2, \qquad C_1 = 2, \qquad C_2 = -1, \\ &C_3 = -\frac{1}{3}, \qquad C_4 = \frac{1}{12}, \qquad C_5 = \frac{1}{120}.\end{aligned} \tag{A18}$$

where A_n is indexes for n-rank Gamma matrix.

We here note some useful relations related to the Fierz identity. We set

$$A = f^{\alpha\beta\gamma}(\bar\xi\Gamma_{\alpha\beta\gamma}\xi)\bar\xi, \tag{A19}$$

$$B = f^{\alpha\beta\gamma}(\bar\xi\Gamma_{\nu\beta\gamma}\xi)\bar\xi\Gamma_\alpha{}^\nu, \tag{A20}$$

$$C = f^{\alpha\beta\gamma}(\bar\xi\Gamma_{\mu\nu\alpha}\xi)\bar\xi\Gamma^{\mu\nu}{}_{\beta\gamma}, \tag{A21}$$

where $f^{\alpha\beta\gamma}$ is an arbitrary antisymmetric tensor. Performing the Fierz transformation, we find

$$A = -\frac{1}{32}(2A + 6B - 3C), \tag{A22}$$

$$B = -\frac{1}{32}(14A + 10B + 3C). \tag{A23}$$

From these relations we obtain

$$(\bar\xi\Gamma_{\alpha\beta\gamma}\xi)\bar\xi\Gamma^{\alpha\beta} = 0. \tag{A24}$$

The following relation holds:

$$2X + Y - Z = 0, \tag{A25}$$

where

$$X = f^{\mu\nu}k^\rho(\bar\epsilon k\!\!\!/\xi)(\bar\xi\Gamma_{\mu\nu\rho}\xi), \tag{A26}$$

$$Y = f^{\mu\nu}k^\rho k_\nu(\bar\epsilon\Gamma^a\xi)(\bar\xi\Gamma_{a\mu\rho}\xi), \tag{A27}$$

$$Z = f^{\mu\nu}k^\rho k^\sigma(\bar\epsilon\Gamma^a{}_{\nu\sigma}\xi)(\bar\xi\Gamma_{a\mu\rho}\xi). \tag{A28}$$

Here $f^{\mu\nu}$ is an arbitrary antisymmetric tensor and $k^2 = 0$.

We can derive the following identity from the Fierz transformation:

$$\begin{aligned}b_{\mu\nu}b_{\rho\sigma} = \frac{1}{3}(b_{\mu\nu}b_{\rho\sigma} + b_{\sigma\nu}b_{\mu\rho} - b_{\sigma\mu}b_{\nu\rho}) + \frac{1}{6} \\ \times (g_{\sigma\mu}b_\nu{}^\alpha b_{\alpha\rho} - g_{\sigma\nu}b_\mu{}^\alpha b_{\alpha\rho} + g_{\rho\nu}b_\mu{}^\alpha b_{\alpha\sigma} \\ - g_{\rho\mu}b_\nu{}^\alpha b_{\alpha\sigma}) + \frac{1}{6}(k_\nu b_\mu{}^\alpha - k_\mu b_\nu{}^\alpha)(\bar\lambda\Gamma_{\rho\sigma\alpha}\lambda) \\ + \frac{1}{6}(k_\sigma b_\rho{}^\alpha - k_\rho b_\sigma{}^\alpha)(\bar\lambda\Gamma_{\mu\nu\alpha}\lambda),\end{aligned} \tag{A29}$$

where $b_{\mu\nu} = k^\rho(\bar\lambda\Gamma_{\mu\nu\rho}\lambda)$.

The following relations among the gamma matrices hold:

$$\begin{aligned} \Gamma^\mu \Gamma_{A_n} \Gamma_\mu &= (-1)^n (10-2n)\Gamma_{A_n}, \\ \Gamma_{\alpha\beta\gamma}\Gamma_\mu \Gamma^{\alpha\beta\gamma} &= 288\Gamma_\mu, \\ \Gamma_{\alpha\beta\gamma}\Gamma_{\mu\nu\rho}\Gamma^{\alpha\beta\gamma} &= -48\Gamma_{\mu\nu\rho}, \\ \Gamma_{\alpha\beta\gamma}\Gamma_{\mu\nu\rho\sigma\lambda}\Gamma^{\alpha\beta\gamma} &= 0. \end{aligned} \tag{A30}$$

3. SYMMETRIZED TRACE

The symmetrized trace is defined in (4.10). In particular, explicit forms for two and three operators are written as

$$\mathrm{S\,tr}(e^{ik\cdot A}B\cdot C) = \mathrm{tr}\int_0^1 dt e^{ik\cdot At}Be^{ik\cdot A(1-t)}C, \tag{A31}$$

$$\mathrm{Str}(e^{ik\cdot A}B\cdot C\cdot D) = \mathrm{tr}\int_0^1 dt_1 \int_{t_1}^1 dt_2 e^{ik\cdot At_1}Be^{ik\cdot A(t_2-t_1)} \times Ce^{ik\cdot A(1-t_2)}D + (C\longleftrightarrow D) \tag{A32}$$

where all matrices are bosonic. The definitions for fermionic matrices can be obtained by replacing the bosonic matrices on the above equations with the fermionic matrices multiplied by Grassmann odd numbers. The center dot on the left-hand side means that matrices are inserted at different places. We note useful equations related to the symmetrized trace,

$$\mathrm{S\,tr}(e^{ik\cdot A}[ik\cdot A, A_\alpha]\cdot \psi_\beta) = \mathrm{tr}[e^{ik\cdot A}, A_\alpha]\psi_\beta, \tag{A33}$$

$$\begin{aligned} &\mathrm{S\,tr}(e^{ik\cdot A}\bar{\psi}\cdot \Gamma_{\mu\nu\lambda}\psi\cdot [ik\cdot A, A_\rho]) \\ &\quad = 2\mathrm{Str}(e^{ik\cdot A}\bar{\psi}\cdot \Gamma_{\mu\nu\lambda}[A_\rho, \psi]), \end{aligned} \tag{A34}$$

where the following relation is used,

$$[e^{ik\cdot A}, B] = \int_0^1 dt e^{ik\cdot At}[ik\cdot A, B]e^{ik\cdot A(1-t)}. \tag{A35}$$

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