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$D = 2$, $\mathcal{N} = 2$ supersymmetric σ models on non(anti)commutative superspace

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I extend the results of hep-th/0310137 to show that a general classical action for $D = 2$, $\mathcal{N} = 2$ sigma models on a non(anti)commutative superspace is not standard and contains infinite number of terms, which depend on the determinant of the non(anti)commutativity parameter $C^{\alpha\beta}$. I show that using Kähler normal coordinates the action can be written in a manifestly covariant manner. I introduce vector multiplets and obtain the $\mathcal{N} = 1/2$ supersymmetry transformations of the theory in the Wess-Zumino gauge. By explicitly deriving the expressions for vector and twisted superfields on non(anti)commutative superspace, I study the classical aspects of gauged linear sigma models.

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I. INTRODUCTION

Supersymmetric field theories defined on deformed superspaces have been studied for quite some time [1–6]. The recent interest in such theories is due to the realization that they arise naturally in certain limits of string theory in exactly the same way as noncommutative field theories arise in the Seiberg-Witten low energy limit [7].

In the context of Dijkgraaf-Vafa correspondence [8], it was shown that the deformation of the algebra of superspace coordinates allows the computation of nonperturbative contributions to the $\mathcal{N} = 1$ superpotential by summing over certain nonplanar diagrams on the matrix model side.

To be precise, using the pure spinor approach, the $D = 4$ sigma model action for D branes of type II superstring theory compactified on Calabi-Yau three-folds was considered in [9]. It was shown that, turning on a constant graviphoton background field in four dimensions (or more generally Ramond-Ramond (RR) two-forms in ten dimensions itself [10]), leads to a deformation of correlation functions of the superspace coordinates as:

$$\{\theta^\alpha, \theta^\beta\} = 2\alpha'^2 F^{\alpha\beta}. \quad (1.1)$$

Here α' is related to the inverse of string tension and $F^{\alpha\beta}$ is the self-dual graviphoton field strength.

Note that the anticommutation relations of the remaining superspace coordinates, $\bar{\theta}^{\dot{\alpha}}$, are not modified. This is, however, only possible in a Euclidean space, where setting the anti-self-dual part $F^{\dot{\alpha}\dot{\beta}}$ to zero, does not affect the string equations of motion. Further, it can be shown that this configuration of fluxes is stable and does not backreact on the metric, due to the vanishing of the energy-momentum tensor.

It was noted that the deformation in Eq. (1.1) does not survive the field theory limit $\alpha' \to 0$, as long as $F^{\alpha\beta}$ is a constant. But, the boundary term generated by the graviphoton vertex operator survives the field theory limit (as it turns out to be independent of α') and also breaks half of the supersymmetries. Nevertheless, it was shown in [9] that a suitable deformation of the gluino anticommutation relations cancels the boundary term and this restores the $\mathcal{N} = 1$ supersymmetry on the brane. This also restores the standard anticommutation relations of the fermionic coordinates θ^α and θ^β. Crucial to their analysis were the covariant quantization techniques developed in [11,12], for studying superstrings in Ramond-Ramond backgrounds in a manifestly superpoincaré invariant manner.

Mechanisms of supersymmetry breaking which come from superspace deformation and also survive in the field theory limit are very interesting. Hence, for the theory on the brane, the limit $\alpha' \to 0$, $F^{\alpha\beta} \to \infty$, $\alpha'^2 F^{\alpha\beta} = C^{\alpha\beta} =$ fixed was considered in [13,14], so as to preserve the nontrivial anticommutation relations (1.1). As mentioned before, an important consequence of the deformation in Eq. (1.1), is that half of the supersymmetry generators, due to their dependence on θ^α's, become nonlinear. As a result, they are no more the symmetries of the background. The surviving supertranslational symmetry along the $\bar{\theta}$ directions has been termed as the $\mathcal{N} = 1/2$ supersymmetry.

It is useful to note that one still continues to use the full superspace, but with the understanding that the translational symmetry in the θ directions is broken. Regardless of this aspect, it was shown in [13] that the classical action of four dimensional $\mathcal{N} = 1$ supersymmetric field theories with the superspace deformation as in Eq. (1.1), is still Lorentz invariant [in the sense that the non(anti)commutativity parameter $C^{\alpha\beta}$ appears only as (det$C$) in the action]. Further, the F terms were also shown to be invariant under the surviving $\mathcal{N} = 1/2$ supersymmetry transformations.

This was followed by a number of works on the classical and quantum aspects of the Wess-Zumino models in four dimensions, both perturbative and nonperturbative. Other interesting features [15–35] and generalizations to models with $\mathcal{N} = 2$ supersymmetry in four [36–40], as

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well as in two dimensions [41,42], have also been discussed.

In [42], we studied the consequences of the superspace deformation (1.1) for $D = 2$, $\mathcal{N} = 2$ supersymmetric theories characterized by a general Kähler potential and arbitrary superpotential. The classical action was shown to have a power series expansion in the determinant of the non(anti)commutativity parameter. The analysis was only for the case of a single chiral multiplet. Generalization to include several chiral multiplets, and hence to a sigma model, is of great importance.

Formulating sigma models in two dimensions is also interesting from the target space point of view. To be precise, the fact that the worldsheet is deformed by the relations (1.1), does not necessarily imply that the target space shares the same properties. Thus, it is of great interest to study $\mathcal{N} = 2$ theories on non(anti)commutative superspace, with Kähler manifolds as target spaces.

Further, it is well known that $\mathcal{N} = 2$ supersymmetric nonlinear sigma models [43–46] have provided invaluable insights in the physics of lower dimensional systems, dynamics of string theory in general curved backgrounds, mirror symmetry and Calabi-Yau geometries, topological field theories, etc. These theories are characterized by an underlying Kähler geometry which constrains the form of the classical action and, at the quantum level, puts severe restrictions on their ultraviolet structure [45,46]. Motivated by the above facts, in this paper, I first generalize the analysis of [42] to study $\mathcal{N} = 2$ supersymmetric nonlinear σ models characterized by a Kähler potential $\mathcal{K}(\Phi^i, \bar{\Phi}^j)$, where there are several chiral multiplets, Φ^i, $i = 1, \ldots, k$. This setup naturally leads to the formulation of sigma models with Kähler manifolds as target spaces.

On another front, chiral multiplets in the presence of gauge fields in two dimensions have been considered before, for providing interesting insights into various aspects of Mirror symmetry. Thus, we extend the analysis to include several chiral multiplets charged under a single vector multiplet and study gauged linear sigma models (GLSM) on non(anti)commutative superspace.

As discussed above, the motivation for studying GLSM's are many fold. First, a distinct feature that appears in two dimensions, compared to four dimensions, is that in addition to the chiral multiplets it is also possible to have twisted multiplets [47]. Sigma models having both kinds of multiplets are quite fascinating, as mirror symmetry interchanges the two. Thus, they allow a concrete understanding of the Landau-Ginzburg and Calabi-Yau phases of $\mathcal{N} = 2$ theories [48,49]. Further, mirror symmetry in the presence of fluxes is also being pursued. The fluxes coming from string theory can either be of Neveu-Schwarz–Neveu-Schwarz type or RR type. Since, the superspace deformation in Eq. (1.1) is coming from the study of superstrings in the RR backgrounds, it might be interesting to understand mirror symmetry in this set up.

Second, superstring compactifications on Calabi-Yau manifolds can generate nontrivial superpotentials in the effective four dimensional theory. It is of interest to get a better understanding of this superpotential, as it encodes important information about the vacuum structure of the theory. It has been known for a while that this superpotential can be studied by looking at the associated sigma model. But, for these sigma models to be useful, they have to be either conformally invariant or flow to conformally invariant theories in the IR limit. Nevertheless, it is still possible to deduce useful information from these models by twisting them to get topological theories. The observables and correlation functions in these topological sigma models do not depend on the metric and are also invariant under scale transformations.

Because of such varied applications, $\mathcal{N} = 2$ GLSM's have been studied by many authors. Further, one can add world sheets to the boundary by putting appropriate boundary conditions on the fields in the vector multiplet and study D branes via these models [50]. With this motivation, in this paper, I study the classical aspects of $D = 2$, $\mathcal{N} = 2$ sigma models defined on a non(anti)commutative superspace.

The rest of the paper is organized as follows. In Sec. II, I begin with the dimensional reduction of the relevant formulas from $D = 4$ to $D = 2$. In the following subsection, I discuss the various supersymmetry multiplets in the theory and also explicitly derive the supersymmetry and gauge transformations in the Wess-Zumino gauge. In Sec. III, I discuss the construction of classical action of the theory, while pointing out the emergence of a series expansion in (det$C$). Here, I use a certain normal coordinate expansion to write the action in a manifestly covariant fashion.

In Sec. IV, I present the classical action of the gauged linear sigma models, show the invariance under $\mathcal{N} = 1/2$ supersymmetry transformations, and also make some remarks about the superpotential of the theory. I present conclusions and discussion in Sec. V.

II. $\mathcal{N} = 2$ SUPERSPACE AND SUPERMULTIPLETS

In this section, I start by establishing our notations and conventions, while also reviewing certain general properties of non(anti)commutative superspace in two dimensions. Some relevant details can also be found in [42], but most of the results have been rederived so that the connection with four dimensional case [13] is more clear and also to ensure that the notations are compatible with the ones in [48]. In Sec. II B, I discuss the definitions of the matter and gauge multiplets, gauge transformation properties of the component fields, and explicitly construct the supersymmetry transformations in the Wess-Zumino gauge.

Before proceeding, it is useful to mention that we work in a Euclidean space, but continue to use Lorentzian signature for convenience [13]. The reason why the underlying space is Euclidean can be understood by going back to the four dimensional relations in Eq. (1.1). As discussed before, the deformation is imposed only over half of the fermionic coordinates, while the remaining half still satisfy the same old Grassmannian algebra. This is only possible in Euclidean space where the self-dual component $F_{\alpha\beta}$ can be turned on, while setting its anti-self-dual part $F_{\dot\alpha\dot\beta}$ to zero.

In a Minkowski spacetime, the self-dual and (anti)self-dual components of the graviphoton field strength are related by a complex conjugation. However, in a Euclidean space the two components transform independently under the two different SU(2) subgroups, which come from $SO(4) = SU(2)_L \times SU(2)_R$ [25,51].

Thus, compared to $\mathcal{N} = 1$ supersymmetric theories in Minkowski spacetime, the number of bosonic and fermionic fields of the theory are doubled and complexified in the corresponding Euclidean space. Now, in order to preserve reality conditions, one is forced to introduce the second supersymmetry. In other words, the only way to put consistent reality conditions on the fields of the theory is to extend the superspace to $D = 4$, $\mathcal{N} = 2$ theories. However, we continue to work with $\mathcal{N} = 1$ Euclidean superspace, given the understanding that all the fields of the theory are complex with no reality conditions on them.

A. $D = 4 \to D = 2$ dimensional reduction

We now start by discussing the dimensional reduction from $D = 4$ to $D = 2$. The superspace coordinates in $D = 4$ as given in [13] are: θ^α, $\bar\theta^{\dot\alpha}$, and y^μ, where α, $\dot\alpha$ represent the two chiralities of spinor indices. Raising and lowering of spinor indices is done as $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, where ϵ is the antisymmetric tensor whose nonzero components are given as $\epsilon^{01} = -\epsilon_{10} = 1$. y^μ denotes the chiral coordinates and is related to the standard IR^4 coordinates as:

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}. \tag{2.1}$$

The need for using chiral coordinates can be understood as follows. Once we introduce the deformation:

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \tag{2.2}$$

the standard IR^4 coordinates x^μ do not commute [13]. However, the coordinates y^μ can be taken to commute. In fact, all (anti)commutators of y^μ, θ^α, and $\bar\theta^{\dot\alpha}$ vanish, except (2.2).

It will be useful to obtain the $D = 2$, $\mathcal{N} = 2$ superspace by dimensional reduction of the above formulas, so that later on, the results obtained here can be directly compared to the ones in $D = 4$.

In making the reduction, we take the 2D fields to be independent of x^1 and x^2 and label the fermionic coordinates as $(\theta^0, \theta^1) = (\theta^-, \theta^+)$ and $(\theta_0, \theta_1) = (\theta_-, \theta_+)$. Here, the upper and lower components are further related as $\theta^- = \theta_+$, $\theta^+ = -\theta_-$. Similar identifications hold for the dotted indices as well. For the tensors $\sigma^\mu_{\alpha\dot\alpha}$ we use [52]:

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

After dimensional reduction, we find it convenient to use the following combination of chiral coordinates:

$$\xi^- = \frac{1}{2}(x^0 - x^3) - i\theta^-\bar\theta^-, \qquad \zeta^- = \frac{1}{2}(x^0 + x^3) - i\theta^+\bar\theta^+, \tag{2.4}$$

where $\xi^- = \frac{1}{2}(y^0 - y^3)$ and $\zeta^- = \frac{1}{2}(y^0 + y^3)$. Our non(-anti)commutative superspace can be obtained by translating the relations (2.2) to $D = 2$ as:

$$\{\theta^-, \theta^-\} = C^{00}, \qquad \{\theta^-, \theta^+\} = C^{01}, \qquad \{\theta^+, \theta^-\} = C^{10}, \qquad \{\theta^+, \theta^+\} = C^{11}. \tag{2.5}$$

Functions of θ^- and θ^+, say $f(\theta^-, \theta^+)$ and $g(\theta^-, \theta^+)$, are Weyl-ordered using the following definition of star product:

$$f * g = f \exp\left(-\frac{C^{00}}{2}\overleftarrow{\partial}_{\theta^-}\overrightarrow{\partial}_{\theta^-} - \frac{C^{01}}{2}\overleftarrow{\partial}_{\theta^-}\overrightarrow{\partial}_{\theta^+} - \frac{C^{10}}{2}\overleftarrow{\partial}_{\theta^+}\overrightarrow{\partial}_{\theta^-} - \frac{C^{11}}{2}\overleftarrow{\partial}_{\theta^+}\overrightarrow{\partial}_{\theta^+}\right) g. \tag{2.6}$$

The generators of supersymmetry transformations, written in the chiral basis (2.4) are,

$$Q_\pm = -\frac{\partial}{\partial\theta^\pm}, \qquad \bar Q_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - 2i\theta^\pm\left(\frac{\partial}{\partial y^0} \pm \frac{\partial}{\partial y^3}\right), \tag{2.7}$$

and these anticommute with the remaining set of derivatives, written in chiral basis as:

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - 2i\bar\theta^\pm\left(\frac{\partial}{\partial y^0} \pm \frac{\partial}{\partial y^3}\right), \qquad \bar D_\pm = -\frac{\partial}{\partial\bar\theta^\pm}. \tag{2.8}$$

In the chiral basis (2.4), the algebra of the supercovariant derivatives (2.8), does not get modified due to the deformation (2.5), as seen below:

$$\{\bar D_\pm, D_\pm\} = 2i\left(\frac{\partial}{\partial y^0} \pm \frac{\partial}{\partial y^3}\right) \qquad \text{and the rest all zero.} \tag{2.9}$$

However, the algebra of supercharges given in Eqs. (2.7) gets modified:

$$\begin{aligned}
\{Q_\pm, \bar{Q}_\pm\} &= -2i\Big(\frac{\partial}{\partial y^0} \pm \frac{\partial}{\partial y^3}\Big)\\
\{\bar{Q}_-, \bar{Q}_-\} &= -4C^{00}\Big(\frac{\partial}{\partial y^0} - \frac{\partial}{\partial y^3}\Big)^2\\
\{\bar{Q}_+, \bar{Q}_+\} &= -4C^{11}\Big(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^3}\Big)^2\\
\{\bar{Q}_-, \bar{Q}_+\} &= -4C^{01}\Big(\frac{\partial^2}{(\partial y^0)^2} - \frac{\partial^2}{(\partial y^3)^2}\Big),
\end{aligned} \tag{2.10}$$

and the rest all zero. As stated before, due to the dependence of $\bar{Q}$'s on the non(anti)commutative coordinates $\theta^\pm$, it is no more a symmetry of the theory. From the algebra (2.10), the only unbroken symmetry generators are $Q_\pm$. Hence, we only use these $\mathcal{N} = 1/2$ supersymmetry generators to study the theory.

B. $\mathcal{N} = 2$ multiplets

Let us start by discussing the $\mathcal{N} = 2$ matter and gauge multiplets in two dimensions. For the $C = 0$ case, the results are summarized in [48]. For the case with $C \neq 0$, the definition of the vector superfield and the subtleties in defining gauge transformations in $D = 4$ has been discussed in [13]. The discussion has been further extended to include chiral multiplets in [24]. Thus, the simplest way to obtain the vector and chiral multiplets in two dimensions is to do a dimensional reduction of the relevant formulas given in four dimensions.

As we will see, a naive dimensional reduction may not show some critical aspects associated with the definition of the multiplets. Thus, I choose to derive the proper definitions of vector and chiral superfields in $D = 2$ for the case $C \neq 0$. Later on, I compare these definitions with the ones obtained by a dimensional reduction and point out the differences. What we will see is that a direct reduction of the definition of vector superfields from $D = 4$ may give some additional terms, which can be ignored in $D = 2$.

1. Vector multiplet

Since, one of my interests is in formulating a gauge theory, I first introduce the vector superfield V. For simplicity, in this work I only consider Abelian gauge groups, in which case V is a single real function on the superspace. Towards the end, I comment on the generalization to the case of non-Abelian gauge groups.

Even after imposing the reality condition, there is a residual gauge invariance under which the vector superfield transforms infinitesimally as:

$$\delta e^V = -i\bar{\Lambda} * e^V + ie^V * \Lambda, \tag{2.11}$$

where $\Lambda = -\alpha(\xi^-, \zeta^-)$ and $\bar{\Lambda} = -\alpha(\xi^+, \zeta^+)$ are the gauge parameters with $\xi^+ = \xi^- + 2i\theta^-\bar{\theta}^-$ and $\zeta^+ = \zeta^- + 2i\theta^+\bar{\theta}^+$. This residual gauge invariance can be partially fixed by going to a Wess-Zumino gauge, in which case, V takes the form:

$$\begin{aligned}
V_{\rm wz} = &-\bar{\theta}^-\theta^-\nu_\xi - \bar{\theta}^+\theta^+\nu_\zeta + \sqrt{2}\bar{\theta}^+\theta^-\sigma + \sqrt{2}\bar{\theta}^-\theta^+\bar{\sigma}\\
&+ 2i\theta^-\theta^+(\bar{\theta}^+\bar{\lambda}_+ + \bar{\theta}^-\bar{\lambda}_-)\\
&- 2i\bar{\theta}^-\bar{\theta}^+(\theta^-\lambda_- + \theta^+\lambda_+)\\
&- 2\theta^-\theta^+\bar{\theta}^-\bar{\theta}^+\Big(D + \frac{i}{2}\partial_{\zeta^-}\nu_\xi + \frac{i}{2}\partial_{\xi^-}\nu_\zeta\Big).
\end{aligned} \tag{2.12}$$

In the above definition of the vector multiplet, for gauge fields ν_0, ν_1, we have introduced the notation $\nu_\xi = (\nu_0 - \nu_1)$ and $\nu_\zeta = (\nu_0 + \nu_1)$, as this combination will occur quite frequently in chiral basis. Further, in Eq. (2.12) $\sigma, \bar{\sigma}$ are complex scalars, $\lambda_\pm$, $\bar{\lambda}_\pm$ are the gauginos, and D is an auxiliary field.

To find out the gauge transformation properties of the component fields, we write $\bar{\Lambda}$ in terms of (ξ^-, ζ^-) coordinates as:

$$\begin{aligned}
\bar{\Lambda} = &-\alpha - 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\alpha - 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\alpha\\
&- 4\theta^-\bar{\theta}^-\theta^+\bar{\theta}^+\partial_{\xi^-}\partial_{\zeta^-}\alpha,
\end{aligned} \tag{2.13}$$

and calculate the right-hand side (RHS) of Eq. (2.11), where for V, we use the definition derived in Eq. (2.12). Some terms in the calculation, namely, the ones depending on C, are given below (identities used in the calculation are given in the appendix):

$$\begin{aligned}
-i\bar{\Lambda} * e^V + ie^V * \Lambda = \bar{\theta}^-\bar{\theta}^+[&i\partial_{\xi^-}\alpha(\theta^+C^{00} + \theta^-C^{01})\bar{\lambda}_+\\
&+ \partial_{\zeta^-}\alpha(\theta^-C^{11} + \theta^+C^{10})\bar{\lambda}_-\\
&+ 2(-\sqrt{2}C^{00}\sigma\partial_{\xi^-}\alpha + C^{01}\nu_\zeta\partial_{\xi^-}\alpha\\
&- C^{10}\nu_\xi\partial_{\zeta^-}\alpha + \sqrt{2}C^{11}\bar{\sigma}\partial_{\zeta^-}\alpha)].
\end{aligned} \tag{2.14}$$

Now, comparing the variation of the vector superfield and the result in Eq. (2.14), one can directly obtain the gauge transformations of the component fields of the vector multiplet, as given below:

$$\begin{aligned}
&\delta_g\nu_\xi = -2\partial_{\xi^-}\alpha \qquad \delta_g\nu_\zeta = -2\partial_{\zeta^-}\alpha\\
&\delta_g(\sigma, \bar{\sigma}) = 0 \qquad \delta_g D = 0 \qquad \delta_g\bar{\lambda}_\pm = 0\\
&\delta_g\lambda_- = -\frac{i}{2}(C^{01}\bar{\lambda}_+\partial_{\xi^-}\alpha + C^{11}\bar{\lambda}_-\partial_{\zeta^-}\alpha)\\
&\delta_g\lambda_+ = -\frac{i}{2}(C^{00}\bar{\lambda}_+\partial_{\xi^-}\alpha + C^{10}\bar{\lambda}_-\partial_{\zeta^-}\alpha).
\end{aligned} \tag{2.15}$$

These are not the standard gauge transformation properties of the component fields, due to the new C-dependent terms present in $\delta_g\lambda_\pm$. However, as suggested in [13], it is possible to cancel the new terms seen in $\delta_g\lambda_\pm$ by modifying the definition of $V_{\rm wz}$ to include certain new C-dependent terms. In fact from Eq. (2.11), it is possible to guess the kind of terms that need to be added to $V_{\rm wz}$.

The new terms to be added are of the following kind:

$$V_c = i\bar{\theta}^-\bar{\theta}^+[\theta^-(C^{01}\bar{\lambda}_+\nu_\xi + C^{11}\bar{\lambda}_-\nu_\zeta) + \theta^+(C^{00}\bar{\lambda}_+\nu_\xi + C^{10}\bar{\lambda}_-\nu_\zeta)]. \quad (2.16)$$

Below I argue that modifying the definition of vector superfield as in Eq. (2.18), has the effect of canceling the first two terms in the quantity given in Eq. (2.14). This in turn corresponds to restoring the standard gauge transformation property of the gauginos, i.e., $\delta_g\lambda_\pm = 0$.

The way to guess the new terms given in Eq. (2.16), is to note that $\partial_{\xi^-}\alpha$ and $\partial_{\zeta^-}\alpha$ appearing in $\delta_g\lambda_\pm$ are nothing but the gauge transformations of the gauge fields ν_ξ and ν_ζ. Thus, the terms in V_c have been chosen in such a way that $\delta_g V_c$ looks similar to the terms appearing in $\delta_g\lambda_\pm$. The rest is to adjust the coefficients by making this ansatz.

The remaining terms in the second line of Eq. (2.14), can also be understood to be coming from a modification of the gauge parameter as shown below:

$$\begin{aligned}\bar{\Lambda} = &-\alpha - 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\alpha - 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\alpha \\ &- 4\theta^-\bar{\theta}^-\theta^+\bar{\theta}^+\partial_{\xi^-}\partial_{\zeta^-}\alpha \\ &+ -i\bar{\theta}^-\bar{\theta}^+[-\sqrt{2}C^{00}\sigma\partial_{\xi^-}\alpha + C^{01}\nu_\zeta\partial_{\xi^-}\alpha \\ &- C^{10}\nu_\xi\partial_{\zeta^-}\alpha + \sqrt{2}C^{11}\bar{\sigma}\partial_{\zeta^-}\alpha].\end{aligned} \quad (2.17)$$

To summarize, choosing the final form of vector superfield in the C-deformed case to be

$$V^c_{\rm wz} = V_{\rm wz} + V_c, \quad (2.18)$$

with V_c given as in Eq. (2.16) and modifying the gauge parameter as in Eq. (2.17), the standard gauge transformation properties of the component fields are restored.

We note that the additional terms added to the definition of the vector superfield in four dimensions [13] are a bit different from the ones given in Eq. (2.16). If we dimensionally reduce the definitions given in [13], we get terms of the kind:

$$i\bar{\theta}^-\bar{\theta}^+\theta^-(C^{01}\bar{\lambda}_+\sigma + C^{11}\bar{\lambda}_-\bar{\sigma}), \quad (2.19)$$

which may contribute to Eq. (2.16). However, these terms contain 2D scalars (coming from 4D gauge fields) which do not vary under gauge transformations in $D = 2$ and, hence, do not affect the gauge transformation properties of any of the component fields. Thus, these terms do not play any role in the present analysis. Further, in [13], the vector superfield was *a priori* assumed to be matrix valued and the theory was non-Abelian. Since, for the present case, we only consider Abelian gauge groups, these terms do not occur. However, it is useful to note that if there are several vector multiplets, there is a restriction on the allowed gauge groups in the theory [25].

Before proceeding, it will be useful to write down the powers of the vector superfield (2.18), as shown below:

$$\begin{aligned}V^2_* = V * V = &\, 2\bar{\theta}^-\bar{\theta}^+[\theta^-\theta^+(-\nu_\zeta\nu_\xi + 2\sigma\bar{\sigma}) \\ &- (\det C)\bar{\lambda}_-\bar{\lambda}_+], \\ V^3_* = &\, 0.\end{aligned} \quad (2.20)$$

One can see that, as in the standard $C = 0$ case, star product of more than two vector superfields vanishes [13], and this will be needed while writing down the action.

2. Twisted multiplets

It has been known for quite some time that $\mathcal{N} = 2$ sigma models having both chiral and twisted chiral multiplets are helpful in understanding mirror symmetry. Hence, for the present case, we follow [48] and construct the twisted chiral superfield for an Abelian gauge theory as [48,53]:

$$\Sigma = \frac{1}{\sqrt{2}}\bar{D}_+D_-V, \quad (2.21)$$

where the modified vector superfield V is defined in Eq. (2.18). Using the algebra of the supercovariant derivatives given in Eq. (2.9), it is possible to show that the twisted chiral superfield satisfies the conditions, $D_-\Sigma = 0$, $\bar{D}_+\Sigma = 0$ and can be written in terms of its components as:

$$\begin{aligned}\Sigma = &\, \sigma + i\sqrt{2}\theta^+\bar{\lambda}_+ + i\sqrt{2}\bar{\theta}^-\Big[-\lambda_- + \frac{1}{2}C^{01}\nu_\xi(\bar{\lambda}_+ \\ &+ 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\bar{\lambda}_+) + \frac{1}{2}C^{11}\nu_\zeta(\bar{\lambda}_- + 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\bar{\lambda}_-)\Big] \\ &- \sqrt{2}\bar{\theta}^-\theta^+\Big(D - \frac{i}{2}\nu_{\xi\zeta}\Big) - 2i\bar{\theta}^-\theta^-\partial_{\xi^-}\sigma \\ &- 2\sqrt{2}\bar{\theta}^-\Big(\theta^+\theta^- - \frac{1}{2}C^{10}\Big)\partial_{\xi^-}\bar{\lambda}_+,\end{aligned} \quad (2.22)$$

where $\nu_{\xi\zeta} = \partial_{\xi^-}\nu_\zeta - \partial_{\zeta^-}\nu_\xi$ is the gauge field strength. Twisted antichiral superfield satisfying $D_+\bar{\Sigma} = 0$ and $\bar{D}_-\bar{\Sigma} = 0$ can be obtained in an analogous way from $\bar{\Sigma} = \frac{1}{\sqrt{2}}\bar{D}_-D_+V$, and is given below:

$$\begin{aligned}\bar{\Sigma} = &\, \bar{\sigma} - i\sqrt{2}\theta^-\bar{\lambda}_- + i\sqrt{2}\bar{\theta}^+\Big[\lambda_+ - \frac{1}{2}C^{00}\nu_\xi(\bar{\lambda}_+ \\ &+ 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\bar{\lambda}_+) - \frac{1}{2}C^{10}\nu_\zeta(\bar{\lambda}_- \\ &+ 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\bar{\lambda}_-)\Big] - \sqrt{2}\bar{\theta}^+\theta^-\Big(D + \frac{i}{2}\nu_{\xi\zeta}\Big) \\ &- 2i\bar{\theta}^+\theta^+\partial_{\zeta^-}\bar{\sigma} + 2\sqrt{2}\bar{\theta}^+\Big(\theta^-\theta^+ - \frac{1}{2}C^{01}\Big)\partial_{\zeta^-}\bar{\lambda}_-.\end{aligned} \quad (2.23)$$

All the component fields of twisted superfields are functions of (ξ^-, ζ^-). It is useful to compare the definitions of twisted superfields given in Eqs. (2.22) and (2.23) with the ones given in [48]. The only difference is the new

C-dependent terms, some of which arise from the additional terms added to the definition of vector superfield. These terms have also been expanded around (ξ^-, ζ^-) coordinates. Other C-dependent terms, for instance, the term in the fourth line of Eq. (2.22) can be obtained from the twisted chirality condition.

A vector superfield by itself is not a gauge invariant object and, hence, is not directly used to construct the action for the gauge fields. Rather, the twisted superfields derived from V are used in writing down a gauge invariant action for gauge fields. In other words, twisted superfields play the role of gauge invariant field strength for the superspace U(1) gauge fields.

We now write down the supersymmetry transformations of the component fields of the vector multiplet. It is easier to derive them from the twisted multiplets as follows:

$$\delta\Sigma = (\epsilon^+ Q_+ + \epsilon^- Q_-)\Sigma, \tag{2.24}$$

with similar relations for the twisted antichiral multiplet. Comparing the RHS of the above equation with the variation of the component fields in the definition of Σ given in Eq. (2.22), we get:

$$\delta\sigma = i\sqrt{2}\epsilon^+\bar{\lambda}_+ \qquad \delta\bar{\sigma} = -i\sqrt{2}\epsilon^-\bar{\lambda}_- \qquad \delta\bar{\lambda}_+ = 0$$
$$\delta\bar{\lambda}_- = 0 \qquad \delta\nu_\zeta = 2i\epsilon^-\bar{\lambda}_+ \qquad \delta\nu_\xi = -2i\epsilon^+\bar{\lambda}_-. \tag{2.25}$$

The above transformations are the same even for the $C = 0$ theory. However, the transformation properties of the remaining component fields get modified by certain new terms, as seen below:

$$\begin{aligned}\delta\lambda_+ &= -\sqrt{2}\epsilon^+\partial_{\zeta^-}\bar{\sigma} + i\epsilon^-\Big(D + \frac{i}{2}\nu_{\xi\zeta}\Big) \\ &\quad + i(C^{00}\epsilon^+ + C^{10}\epsilon^-)\bar{\lambda}_+\bar{\lambda}_- \\ \delta\lambda_- &= \sqrt{2}\epsilon^+\partial_{\xi^-}\sigma - i\epsilon^+\Big(D - \frac{i}{2}\nu_{\xi\zeta}\Big) + i(C^{01}\epsilon^+ \\ &\quad + C^{11}\epsilon^-)\bar{\lambda}_+\bar{\lambda}_-.\end{aligned} \tag{2.26}$$

It is useful to compare the above results with the ones obtained by dimensional reduction from [13].

3. Chiral multiplets

Now, the chiral and antichiral superfields satisfying $\bar{D}_\pm\Phi = 0$ and $D_\pm\bar{\Phi}_o = 0$, respectively, can be written in a Weyl-ordered form, as shown below [13,42]:

$$\Phi = \phi + \sqrt{2}\theta^-\psi_- + \sqrt{2}\theta^+\psi_+ - 2\theta^-\theta^+ F, \tag{2.27}$$

$$\begin{aligned}\bar{\Phi}_o &= \bar{\phi} - \sqrt{2}\bar{\theta}^-\bar{\psi}_- - \sqrt{2}\bar{\theta}^+\bar{\psi}_+ + 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\bar{\phi} \\ &\quad + 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\bar{\phi} + \bar{\theta}^-\bar{\theta}^+(2\bar{F} \\ &\quad - 2\sqrt{2}i\theta^-\partial_{\xi^-}\bar{\psi}_+ + 2\sqrt{2}i\theta^+\partial_{\zeta^-}\bar{\psi}_- \\ &\quad + 4\theta^-\theta^+\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}).\end{aligned} \tag{2.28}$$

Note that we have used $C^{01} = C^{10}$ in Weyl-ordering the above expressions. Also, the definitions given in Eq. (2.4) have been used in writing the antichiral superfield. All the component fields are taken to be functions of ξ^- and ζ^-, unless specified otherwise.

The $\mathcal{N} = 1/2$ supersymmetry transformations of the component fields in the chiral and antichiral multiplet are standard and were also derived in [42]. We give them below for later use:

$$\delta\phi = \sqrt{2}\epsilon^+\psi_+ + \sqrt{2}\epsilon^-\psi_- \qquad \delta\psi_- = -\sqrt{2}\epsilon^+ F$$
$$\delta\psi_+ = \sqrt{2}\epsilon^- F \qquad \delta F = 0 \qquad \delta\bar{\phi} = 0$$
$$\delta\bar{\psi}_+ = i\sqrt{2}\epsilon^+\partial_{\zeta^-}\bar{\phi} \qquad \delta\bar{\psi}_- = i\sqrt{2}\epsilon^-\partial_{\xi^-}\bar{\phi}$$
$$\delta\bar{F} = i\sqrt{2}\epsilon^+\partial_{\zeta^-}\bar{\psi}_- - i\sqrt{2}\epsilon^-\partial_{\xi^-}\bar{\psi}_+. \tag{2.29}$$

Note that the above transformations do not take into account the coupling with the vector multiplet. Supersymmetry transformations for the matter multiplet coupled to the gauge multiplet will be derived explicitly in the following subsection.

Now we couple the matter and the vector multiplets by making the chiral and antichiral superfields transform in a certain representation of the gauge group. Thus, under a gauge transformation, the matter superfields transform as $\Phi' = e^{-i\Lambda} * \Phi$, $\bar{\Phi}'_o = \bar{\Phi}_o * e^{i\bar{\Lambda}}$ or infinitesimally as:

$$\delta\Phi = -i\Lambda * \Phi, \qquad \delta\bar{\Phi}_o = i\bar{\Phi}_o * \bar{\Lambda}, \tag{2.30}$$

where $\Lambda = -\alpha$ and $\bar{\Lambda}$ gets modified due to the additional C-dependent terms added to the vector superfield as given in Eq. (2.17).

Following the discussion in the case of the vector multiplet, one can compare the left-hand side (LHS) and RHS of each of the equations in (2.30), to get the gauge transformation properties of the (anti)chiral multiplet. As it turns out, and is also pointed out in [24], the component fields of the matter multiplet do not have standard transformation properties, due to certain additional C-dependent terms. For instance, the transformation of the auxiliary field takes the form:

$$\begin{aligned}\delta_g\bar{F} &= -i\alpha\bar{F} + \frac{\bar{\alpha}}{2}[C^{10}\nu_\zeta\partial_{\xi^-}\alpha - \sqrt{2}C^{00}\sigma\partial_{\xi^-}\alpha \\ &\quad + \sqrt{2}C^{11}\bar{\sigma}\partial_{\zeta^-}\alpha - C^{01}\nu_\xi\partial_{\zeta^-}\alpha] + iC^{01}\partial_{\xi^-}\bar{\phi}\partial_{\zeta^-}\alpha \\ &\quad - iC^{10}\partial_{\zeta^-}\bar{\phi}\partial_{\xi^-}\alpha.\end{aligned} \tag{2.31}$$

However, as discussed in the case of the vector multiplet above, it is possible to guess the terms that should be added to the chiral superfields, so that the component fields have the standard gauge transformation properties. Thus, we modify the definition of the antichiral superfield by adding certain C-dependent terms as $\bar{\Phi} = \bar{\Phi}_o + \bar{\Phi}_c$, where $\bar{\Phi}_o$ is defined in Eq. (2.28) and $\bar{\Phi}_c$ is given as:

$$\bar{\Phi}_c = -i\bar{\theta}^- \bar{\theta}^+ [\sqrt{2}C^{00}\partial_{\xi^-}(\sigma\bar{\phi}) - \sqrt{2}C^{11}\partial_{\zeta^-}(\bar{\sigma}\,\bar{\phi}) - C^{10}\partial_{\zeta^-}(\nu_{\xi}\bar{\phi}) + C^{01}\partial_{\xi^-}(\nu_{\zeta}\bar{\phi})]. \tag{2.32}$$

One can again check that the new terms $\bar{\Phi}_c$ are such that the C-dependent terms appearing in $\delta_g \bar{F}$ are canceled. It turns out that the definition of the chiral superfield need not be modified. With the modified definitions of the matter superfields, we write down the gauge transformations of the component fields as shown below:

$$\delta_g \phi = +i\alpha\phi, \qquad \delta_g \psi_{\pm} = +i\alpha\psi_{\pm}, \qquad \delta_g F = +i\alpha F, \tag{2.33}$$

$$\delta_g \bar{\phi} = -i\alpha\bar{\phi}, \qquad \delta_g \bar{\psi}_{\pm} = -i\alpha\bar{\psi}_{\pm}, \qquad \delta_g \bar{F} = -i\alpha\bar{F}. \tag{2.34}$$

It is important to note that the additional C-dependent terms that have been added to the antichiral superfield do not spoil the chirality conditions. Thus, the new field $\bar{\Phi}$ still satisfies $D_{\pm}\bar{\Phi} = 0$.

C. Wess-Zumino gauge

Supersymmetry transformations for the component fields of the vector multiplet, in the Wess-Zumino (WZ) gauge were derived in Sec. II. Here, I discuss the supersymmetry transformation properties of the chiral multiplet.

It is well known that the WZ gauge breaks supersymmetry. In other words, the supersymmetry transformations do not leave the gauge-fixing conditions invariant. For this reason, in the WZ gauge, every supersymmetry transformation has to be supplemented by an appropriate gauge transformation. The supersymmetry transformations of the chiral and antichiral multiplets are already given in Eqs. (2.29).

Before doing anything, one can guess that the $\mathcal{N} = 1/2$ supersymmetry transformations of the chiral multiplet remain unchanged even after the coupling with vector multiplet. This can be understood by noting that the modification for the chiral multiplet comes from variations under $\bar{\epsilon}\bar{Q}$. However, as discussed earlier, $\bar{Q}$'s are no more the symmetries of the theory and, hence, the supersymmetry transformations of chiral multiplet do not change and are the same as the ones given in Eqs. (2.29). However, the supersymmetry transformations of the antichiral multiplet get modified in the WZ gauge and we derive them below.

There are various ways to realize the supersymmetry transformations in the WZ gauge. The straightforward way to derive the transformations is to note that, in the presence of gauge fields, the antichiral superfield takes the form:

$$\bar{\Phi}' = \bar{\Phi} * e^{V}. \tag{2.35}$$

Since in this work I only consider a single vector multiplet, the superfield $\bar{\Phi}'$ transforms under a U(1) gauge group and satisfies the condition $\mathcal{D}_{\pm}\bar{\Phi}' = 0$, where $\mathcal{D}_{\pm}$ denotes a gauge covariant derivative (the explicit form of which I introduce later). The RHS of Eq. (2.35) can be evaluated straightforwardly. Then, one can calculate $\delta\bar{\Phi}' = (\epsilon^+ Q_+ + \epsilon^- Q_-)\bar{\Phi}'$ and compare it with the variation of the RHS of Eq. (2.35).

The above procedure will give the combined supersymmetry and gauge transformations of the component fields. We will, however, resort to another method by which one can calculate the appropriate gauge transformation corresponding to every supersymmetry transformation. Since we have already calculated the supersymmetry transformations of the antichiral multiplet in Eqs. (2.29), all we need to do is to determine the appropriate gauge transformations. We follow the method discussed in [54].

A general vector superfield on a non(anti)commutative superspace can be written as:

$$V = V^c_{\rm wz} + i(\bar{\tilde{\Lambda}} - \tilde{\Lambda}), \tag{2.36}$$

where the fields which survive in the WZ gauge and the other fields which can be set equal to zero have been separated out in Eq. (2.36). Here, $V^c_{\rm wz}$ is the vector superfield in the WZ gauge, as given in Eq. (2.18) and $(\bar{\tilde{\Lambda}})$ $\tilde{\Lambda}$ is the (anti)chiral superfield containing other fields, as shown below:

$$\tilde{\Lambda} = \tilde{\phi} + \sqrt{2}\theta^- \tilde{\psi}_- + \sqrt{2}\theta^+ \tilde{\psi}_+ - 2\theta^- \theta^+ \tilde{F}, \tag{2.37}$$

$$\bar{\tilde{\Lambda}} = \bar{\tilde{\phi}} - \sqrt{2}\bar{\theta}^- \bar{\tilde{\psi}}_- - \sqrt{2}\bar{\theta}^+ \bar{\tilde{\psi}}_+ + 2\bar{\theta}^- \bar{\theta}^+ \bar{\tilde{F}}. \tag{2.38}$$

For the rest of the analysis, we set all the component fields of $\tilde{\Lambda}$ in Eq. (2.37) to zero. This is consistent with the WZ gauge choice due to the reasons already discussed above. Now, if one naively sets all the component fields appearing in Eq. (2.38) to zero, then that is not enough to preserve the gauge choice. This is due to the fact that some of the component fields may transform under $\mathcal{N} = 1/2$ supersymmetry transformations. As a result, the fields which have been set equal to zero can be recovered back by a supersymmetry transformation.

Thus, for the antichiral multiplet, one can make a choice for the component fields of $\bar{\tilde{\Lambda}}$ appearing in Eq. (2.38). For some of the fields, the choice does not involve any C-dependent pieces and they are already known in the standard literature. For instance, for some of the fields one can guess the terms by looking at the analogous expressions given in [54], for the $C = 0$ case in four dimensions. Thus, we choose:

$$\bar{\bar{\phi}} = 0, \qquad \bar{\bar{\psi}}_- = \sqrt{2}\epsilon^- \nu_\xi \bar{\phi} - 2\epsilon^+ \bar{\sigma}\,\bar{\phi}, \qquad \bar{\bar{\psi}}_+ = \sqrt{2}\epsilon^+ \nu_\zeta \bar{\phi} - 2\epsilon^- \sigma \bar{\phi}. \tag{2.39}$$

For the auxiliary field $\bar{\bar{F}}$, the choice involves adding certain C-dependent pieces apart from the usual pieces. There is a way to guess the terms, but what I will do is to give the relevant terms below and then at the end, it will be clear as to why this particular choice has been made:

$$\bar{\bar{F}} = 2C^{01}\epsilon^- \bar{\lambda}_+ \nu_\xi + 2C^{11}\epsilon^- \bar{\lambda}_- \nu_\zeta + 2C^{00}\epsilon^+ \bar{\lambda}_+ \nu_\xi + 2C^{10}\epsilon^+ \bar{\lambda}_- \nu_\zeta. \tag{2.40}$$

Note that $\bar{\bar{F}}$ will have some $C = 0$ pieces as well. Further, the gauge parameter has been chosen in such a way that the sum of a supersymmetry and a gauge transformation vanishes, i.e., $(\delta_s + \delta_g)\bar{\bar{\Lambda}} = 0$.

Hence, the sum of supersymmetry and gauge transformations for the component fields of the antichiral multiplet in the WZ gauge can now be calculated. The ones which remain the same as in the $C = 0$ theory are given below:

$$\begin{aligned}(\delta_s + \delta_g)\bar{\phi} &= 0,\\ (\delta_s + \delta_g)\bar{\psi}_- &= i\sqrt{2}\epsilon^- \bar{\mathcal{D}}_{\xi^-}\bar{\phi} - 2Q\epsilon^+ \bar{\sigma}\,\bar{\phi},\\ (\delta_s + \delta_g)\bar{\psi}_+ &= i\sqrt{2}\epsilon^+ \bar{\mathcal{D}}_{\zeta^-}\bar{\phi} - 2Q\epsilon^- \sigma \bar{\phi}.\end{aligned} \tag{2.41}$$

It is understood that the supersymmetry transformation for the auxiliary field will be modified, and is given as:

$$\begin{aligned}(\delta_s + \delta_g)\bar{F} = &-i\sqrt{2}\epsilon^- \bar{\mathcal{D}}_{\xi^-}\bar{\psi}_+ + i\sqrt{2}\epsilon^+ \bar{\mathcal{D}}_{\zeta^-}\bar{\psi}_- \\ &+ 2Q(\epsilon^+ \bar{\psi}_+ \bar{\sigma} - \epsilon^- \bar{\psi}_- \sigma) - 2iQ\bar{\phi}(\epsilon^+ \lambda_+ \\ &- \epsilon^- \lambda_-) - 2QC^{00}\epsilon^+ \bar{\mathcal{D}}_{\xi^-}(\bar{\lambda}_+ \bar{\phi}) \\ &- 2QC^{11}\epsilon^- \bar{\mathcal{D}}_{\zeta^-}(\bar{\lambda}_- \bar{\phi}) \\ &- 2QC^{10}\epsilon^- \bar{\mathcal{D}}_{\xi^-}(\bar{\lambda}_+ \bar{\phi}) \\ &- 2QC^{01}\epsilon^+ \bar{\mathcal{D}}_{\zeta^-}(\bar{\lambda}_- \bar{\phi}).\end{aligned} \tag{2.42}$$

Now, one can justify the choice of the terms given in Eq. (2.40). The first thing to note is that the only modification one expects for the supersymmetry variation of $\bar{F}$ is from additional terms added to the definition of the antichiral superfield which are proportional to $\bar{\theta}^- \bar{\theta}^+$. These are precisely C-dependent terms given in Eq. (2.32). Under supersymmetry variation, the terms in Eq. (2.32) transform as:

$$\begin{aligned}\delta\bar{\Phi}_c = 2\bar{\theta}^- \bar{\theta}^+ [&C^{00}\epsilon^+ \partial_{\xi^-}(\bar{\lambda}_+ \bar{\phi}) + C^{11}\epsilon^- \partial_{\zeta^-}(\bar{\lambda}_- \bar{\phi}) \\ &+ C^{10}\epsilon^+ \partial_{\zeta^-}(\bar{\lambda}_- \bar{\phi}) + C^{01}\epsilon^- \partial_{\xi^-}(\bar{\lambda}_+ \bar{\phi})].\end{aligned} \tag{2.43}$$

From Eq. (2.42), one can understand that the unique choice of terms in Eq. (2.40) is such that they add to the terms in Eq. (2.43) and form a gauge covariant derivative. This, as we know, is the ultimate aim of writing supersymmetry transformations in the WZ gauge. Thus, the choice of C-dependent terms made in Eq. (2.40) is correct and unique.

Thus, Eqs. (2.41) and (2.42) summarize the $\mathcal{N} = 1/2$ supersymmetry transformations of the antichiral multiplet, and the corresponding transformations of the fields in the chiral multiplet are given by the first four equations in Eqs. (2.29). Now, as an explicit check, one can directly calculate these supersymmetry transformations from Eq. (2.35) and show that they are indeed correct.

These supersymmetry transformations will be used in Sec. IV to check the invariance of the gauged linear sigma model action.

III. SIGMA MODELS WITH ARBITRARY KÄHLER POTENTIAL

In previous work [42], I studied $\mathcal{N} = 2$ supersymmetric theories in two dimensions, characterized by an arbitrary Kähler potential and superpotential with the superspace deformation as in Eqs. (2.5). The discussion was limited to the case of a single chiral multiplet. It is interesting to generalize the discussion to include several multiplets, as this generalization leads to the construction of a sigma model and is also useful in analyzing the target space geometry.

Thus, in this section, I first generalize the results of [42] and study sigma models characterized by an arbitrary Kähler potential. I show that the classical action admits a series expansion in the determinant of the non(-anti)commutativity parameter. In fact, it is possible to write terms in this series expansion, at an arbitrary order, in a closed form. In the later part, I use a normal coordinate expansion to write the action in a covariant fashion. In [55], a specific Kähler potential was considered, and CP^n models were analyzed in four dimensions.

A. Expansion of the Kähler potential

Let us start by giving the most general form of the classical action for supersymmetric sigma models on general Kähler manifolds:

$$I = \int d^2y d^4\theta \mathcal{K}(\Phi^i, \bar{\Phi}^j), \tag{3.1}$$

where $\mathcal{K}(\Phi^i, \bar{\Phi}^j)$ is the Kähler potential with Φ^i, $\bar{\Phi}^j$ denoting N chiral and antichiral superfields, respectively.

To obtain the action in terms of the component fields, the Kähler potential is Taylor-expanded around the bosonic fields ϕ, $\bar{\phi}$ as:

$$\mathcal{K}(\Phi, \bar{\Phi}) = \mathcal{K}(\phi^i, \bar{\phi}^j) + L^i\mathcal{K}^i + R^i\mathcal{K}_{,\bar{j}} + \frac{1}{2!}L^i * L^j\mathcal{K}_{,ij} + \frac{1}{2!}R^i * R^j\mathcal{K}_{,\bar{i}\bar{j}} + \frac{1}{2!}[L^i * R^j]\mathcal{K}_{,i\bar{j}} + \frac{1}{3!}[L^i * L^j * R^k]\mathcal{K}_{,ij\bar{k}} + \frac{1}{3!}[L^i * R^j * R^k]\mathcal{K}_{,i\bar{j}\bar{k}} + \cdots + \frac{1}{n!}L^n_*\mathcal{K}_{,i_1i_2\cdots i_n} + \frac{1}{m!}R^m_*\mathcal{K}_{,\bar{j}_1\bar{j}_2\cdots\bar{j}_m} + \cdots + \frac{1}{(n+m)!}[L^n_* * R^m_*]\mathcal{K}_{,i_1i_2\cdots i_n\bar{j}_1\bar{j}_2\cdots\bar{j}_m} + \cdots. \tag{3.2}$$

A few remarks are in order regarding the expansion of the Kähler potential given above. First, in Eq. (3.2), n, m are integers and we use the shorthand notation:

$$\mathcal{K}_{,i_1i_2\cdots i_n\bar{j}_1\bar{j}_2\cdots\bar{j}_m} = \frac{\partial^{(n+m)}\mathcal{K}}{\partial\Phi^{i_1}\partial\Phi^{i_2}\cdots\partial\Phi^{i_n}\partial\bar{\Phi}^{j_1}\partial\bar{\Phi}^{j_2}\cdots\partial\bar{\Phi}^{j_m}}\Bigg|_{\Phi^i=\phi^i,\bar{\Phi}^i=\bar{\phi}^i}, \tag{3.3}$$

for the derivatives of the Kähler potential with respect to the chiral and antichiral superfields evaluated at $\Phi^i = \phi^i$ and $\bar{\Phi}^i = \bar{\phi}^i$. Note that the order of taking derivatives of the Kähler potential with respect to the chiral or antichiral superfields does not matter. In other words, $\mathcal{K}_{,i_1i_2\cdots i_n\bar{j}_1\bar{j}_2\cdots\bar{j}_m}$ is symmetric under any interchange of i indices or j indices or an i index with a j index. This symmetry will be useful while writing down the action. Further, in Eq. (3.2), the square brackets $[\cdots]$ stand for all possible combinations of star product of L^n with R^m, where $L^n_* = L^{i_1} * L^{i_2} * \ldots\ldots * L^{i_n}$ and $R^m_* = R^{i_1} * R^{i_2} * \ldots\ldots * R^{i_m}$. Explicitly,[1]

$$L^i = \Phi^i - \phi^i = +\sqrt{2}\theta^-\psi^i_- + \sqrt{2}\theta^+\psi^i_+ - 2\theta^-\theta^+F^i, \tag{3.4}$$

$$\begin{aligned}R^i &= \bar{\Phi}^i - \bar{\phi}^i \\ &= -\sqrt{2}\bar{\theta}^-\bar{\psi}^i_- - \sqrt{2}\bar{\theta}^+\bar{\psi}^i_+ + 2i\theta^-\bar{\theta}^-\partial_{\xi^-}\bar{\phi}^i \\ &\quad + 2i\theta^+\bar{\theta}^+\partial_{\zeta^-}\bar{\phi}^i + \bar{\theta}^-\bar{\theta}^+(i2\sqrt{2}\theta^+\partial_{\zeta^-}\bar{\psi}^i_- \\ &\quad - i2\sqrt{2}\theta^-\partial_{\xi^-}\bar{\psi}^i_+ + 2\bar{F}^i + 4\theta^-\theta^+\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}^i),\end{aligned} \tag{3.5}$$

where we have suppressed the functional dependence of the component fields on (ξ^-, ζ^-). The need for considering all possible combinations (square brackets) in Eq. (3.2) has been explained in great detail in [42], and I do not repeat it here. However, in the present case, there is an additional permutational symmetry which I illustrate below.

Consider for instance, a term of the form $L^i * L^j$ in the expansion of the Kähler potential in Eq. (3.2). If there was only one chiral multiplet, this term would just be $L * L$. However, if there are many chiral multiplets, then $L^i * L^j$ is not the same as $L^j * L^i$ due to the additional C-dependent terms coming from the star product. This can be seen by explicitly calculating the two terms as shown below:

$$\begin{aligned}L^i * L^j &= -C^{00}\psi^i_-\psi^j_- - C^{11}\psi^i_+\psi^j_+ - 2\Big(\theta^-\theta^+ \\ &\quad - \frac{1}{2}C^{01}\Big)\psi^i_-\psi^j_+ + 2\Big(\theta^-\theta^+ + \frac{1}{2}C^{10}\Big)\psi^i_+\psi^j_- \\ &\quad - \sqrt{2}(C^{00}\theta^+ + C^{01}\theta^-)\bar{\psi}^i_-F^j + \sqrt{2}(C^{00}\theta^+ \\ &\quad + C^{01}\theta^-)\bar{\psi}^j_-F^i + \sqrt{2}(C^{10}\theta^+ + C^{11}\theta^-)\bar{\psi}^i_+F^j \\ &\quad - \sqrt{2}(C^{10}\theta^+ + C^{11}\theta^-)\bar{\psi}^j_+F^i - (\det C)F^iF^j,\end{aligned} \tag{3.6}$$

and $L^j * L^i$ can be obtained by interchanging the indices i and j in the above equation. Now, one can check that, considering the permutation $[L^i * L^j] = L^i * L^j + L^j * L^i$, there are a lot of cancellations and only a few terms survive, as seen below:

$$[L^i * L^j] = -2\{2\theta^-\theta^+(\psi^i_-\psi^j_+ + \psi^j_-\psi^i_+) + (\det C)F^iF^j\}. \tag{3.7}$$

Thus, it is useful to repeat that, apart from all possible combinations of L's and R's considered in the expansion of the Kähler potential in Eq. (3.2), one has additional symmetry factors coming from the permutation of indices in either L's or R's. This will be discussed further later on, while writing down the action.

Hence, in what follows, we consider such permutations as in Eq. (3.7) to write down the action. First, using the definitions of L given in Eq. (3.4), one can generalize the result in Eq. (3.7) to calculate $[L^i * L^j * L^k]$. One can again show that there are many cancellations by considering all possible permutations. Proceeding in this manner, one can check that the result in Eq. (3.7) can be generalized to derive a general formula for the star product of arbitrary number of L's as shown below [42]:

$$[L^{2n}_*] = (-1)^n(\det C)^{n-1}[4n\theta^-\theta^+\{F^{i_1}F^{i_2}\cdots F^{i_{2n-2}}\psi^{i_{2n-1}}_-\psi^{i_{2n}}_+ + \text{perm.}\} + (2n)!(\det C)F^{i_1}F^{i_2}\cdots F^{i_{2n}}], \tag{3.8}$$

[1] The notations used in [42] to write down similar expressions are a bit different. One can use the following coordinate changes to recover the results in [42] : $A \to \phi, \bar{A} \to \bar{\phi}$; $\bar{\psi}_L \to i\sqrt{2}\bar{\psi}_-, \bar{\psi}_R \to i\sqrt{2}\bar{\psi}_+$; $\psi_L \to i\sqrt{2}\psi_-, \psi_R \to i\sqrt{2}\psi_+$; $F \to 2iF, \bar{F} \to -2i\bar{F}$. For the Grassmannian coordinates, the map is $\theta \to -\theta^-, \chi \to \theta^+, \bar{\theta} \to -\bar{\theta}^-, \bar{\chi} \to \bar{\theta}^+$. Further, one also has to take $\partial_{\xi^-} \to 2\partial_{\xi^-}$ and $\partial_{\zeta^-} \to 2\partial_{\zeta^-}$.

$$[L_*^{2n+1}] = (-1)^n (\det C)^n [2n\{F^{i_1}F^{i_2}\cdots F^{i_{2n-1}}\psi_-^{i_{2n}}\psi_+^{i_{2n+1}} + \text{perm.}\} + \{F^{i_1}F^{i_2}\cdots F^{i_{2n}}L^{i_{2n+1}} + \text{perm.}\}]. \tag{3.9}$$

Note that in Eqs. (3.8) and (3.9) and in what follows, the permutations are understood to be among the $i_1 \cdots i_n$ indices. In obtaining the identities given in Eqs. (3.8) and (3.9), we have also made use of the fact that certain terms involving fermions are antisymmetric under the interchange of two indices where as the derivatives of the Kähler potential are symmetric under such interchange of indices.

For the star product of R's, we derive the following results using Eq. (3.5):

$$[R_*^2] = -4\bar\theta^-\bar\theta^+[2\bar\psi_-^i\bar\psi_+^j - i\sqrt{2}\theta^+\bar\psi_-^i\partial_{\zeta^-}\bar\phi^j + i\sqrt{2}\theta^-\bar\psi_+^i\partial_{\xi^-}\bar\phi^j - 2\theta^-\theta^+\partial_{\xi^-}\bar\phi^i\partial_{\zeta^-}\bar\phi^j + \text{perm.}], \tag{3.10}$$

$$R_*^m = 0, \quad \text{for } m > 2. \tag{3.11}$$

The other terms appearing in the expansion of the Kähler potential correspond to the star product of arbitrary powers of L's and R's. It is convenient to calculate the star product of even and odd powers of L with R and R_*^2 separately. First we have:

$$[L^i * R^j]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = -8(i\psi_-^i\partial_{\zeta^-}\bar\psi_-^j + i\psi_+^i\partial_{\xi^-}\bar\psi_+^j + F^i\bar F^j), \tag{3.12}$$

$$[L^i * R^j * R^k]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = -24i\{iF^i(\bar\psi_-^j\bar\psi_+^k + \text{perm.}) + \psi_-^i(\bar\psi_-^j\partial_{\zeta^-}\bar\phi^k + \text{perm.}) + \psi_+^i(\bar\psi_+^j\partial_{\xi^-}\bar\phi^k + \text{perm.})\}. \tag{3.13}$$

Note that in Eqs. (3.12) and (3.13), we have only written the terms which are proportional to $\bar\theta^-\bar\theta^+\theta^-\theta^+$, as only these terms contribute to the action after integration over the Grassmannian coordinates. Now, the identities in Eqs. (3.12) and (3.13) can be generalized to[2]:

$$[L_*^{2n+1} * R_l]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = 4(2n+2)(-1)^n(\det C)^n[2n\{F^{i_1}F^{i_2}\cdots F^{i_{2n-1}}\psi_-^{i_{2n}}\psi_+^{i_{2n+1}} + \text{perm.}\}\partial_{\xi^-}\partial_{\zeta^-}\bar\phi_l + i\{F^{i_1}F^{i_2}\cdots F^{i_{2n}}\psi_-^{i_{2n+1}} + \text{perm.}\}\partial_{\zeta^-}\bar\psi_- + i\{F^{i_1}F^{i_2}\cdots F^{i_{2n}}\psi_+^{i_{2n+1}} + \text{perm.}\}\partial_{\xi^-}\bar\psi_+ + (2n+1)!F^{i_1}F^{i_2}\cdots F^{i_{2n+1}}\bar F_l], \tag{3.14}$$

$$[L_*^{2n+1} * R^j * R^k]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = -4(2n+3)(2n+2)(-1)^n i(\det C)^n[-2ni(F^{i_1}F^{i_2}\cdots F^{i_{2n-1}}\psi_-^{i_{2n}}\psi_+^{i_{2n+1}} + \text{perm.}) \times (\partial_{\xi^-}\bar\phi^j\partial_{\zeta^-}\bar\phi^k + \text{perm.}) - i(F^{i_1}F^{i_2}\cdots F^{i_{2n}}\psi_-^{i_{2n-1}} + \text{perm.})(\bar\psi_-^j i\partial_{\zeta^-}\bar\phi^k + \text{perm.}) - i(F^{i_1}F^{i_2}\cdots F^{i_{2n}}\psi_+^{i_{2n+1}} + \text{perm.})(\bar\psi_+^j\partial_{\xi^-}\bar\phi^k + \text{perm.}) + (2n+1)!F^{i_1}F^{i_2}\cdots F^{i_{2n+1}}(\bar\psi_-^j\bar\psi_+^k + \text{perm.})]. \tag{3.15}$$

Similarly, the star product of even powers of L with R and R_*^2 can be shown to be:

$$[L_*^{2n} * R^k]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = 4(2n+1)(-1)^n(\det C)^{n-1}[2n\bar F^k(F^{i_1}F^{i_2}\cdots F^{i_{2n-2}}\psi_-^{i_{2n-1}}\psi_{+,i_{2n}} + \text{perm.}) + (2n)!(\det C)F^{i_1}F^{i_2}\cdots F^{i_{2n}}\partial_{\xi^-}\partial_{\zeta^-}\bar\phi^k], \tag{3.16}$$

$$[L_*^{2n} * R^j * R^k]|_{\bar\theta^-\bar\theta^+\theta^-\theta^+} = -4(2n+2)(2n+1)(-1)^n(\det C)^{n-1}[2n(F^{i_1}F^{i_2}\cdots F^{i_{2n-2}}\psi_-^{i_{2n-1}}\psi_{+,i_{2n}} + \text{perm.}) \times (\bar\psi_-^j\bar\psi_+^k + \text{perm.}) - (2n)!(\det C)F^{i_1}F^{i_2}\cdots F^{i_{2n}}(\partial_{\xi^-}\bar\phi^j\partial_{\zeta^-}\bar\phi^k + \text{perm.})]. \tag{3.17}$$

One can check that all the identities derived in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) go over to the ones derived in [42] for the case of a single chiral and antichiral supermultiplet, apart from some permutational factors.

Now, substituting the results given in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) and performing integration over the Grassmannian coordinates in the usual way, it is possible to derive the full classical action for the $\mathcal{N} = 2$ supersymmetric sigma model on a non(anti)commutative superspace.

[2]Note that when we calculate terms of the kind $[L_*^{2n+1} * R_l]$ and the ones to follow, we are actually writing down $[(L_*^{2n+1}) * R_l]$, where the additional square bracket corresponds to permutations of indices of L. However, in what follows we do not write this additional square bracket explicitly.

Before proceeding, we note that the sole effect of the permutations seen in the identities in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) is to contribute an overall symmetry factor which cancels in the action. We illustrate this aspect for a couple of terms in the action and it will be clear that the argument can be generalized to all the terms in the action. A term in the expansion of the Kähler potential (3.2) of the form $[L^i * R^j * R^k]|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+}\mathcal{K}_{,i\bar{j}\bar{k}}$ can be rewritten as $\frac{1}{2!}[L^i * (R^j * R^k)]|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+}\mathcal{K}_{,i\bar{j}\bar{k}}$. This can in turn be written as:

$$\frac{1}{2!}24i[F^i(\bar{\psi}^j_-\bar{\psi}^k_+ + \bar{\psi}^k_-\bar{\psi}^j_+) + \psi^i_-(\bar{\psi}^j_-\partial_{\zeta^-}\bar{\phi}^k + \bar{\psi}^k_-\partial_{\zeta^-}\bar{\phi}^j) + \psi^i_+(\bar{\psi}^j_+\partial_{\xi^-}\bar{\phi}^k + \bar{\psi}^k_+\partial_{\xi^-}\bar{\phi}^j)]\mathcal{K}_{,i\bar{j}\bar{k}}, \tag{3.18}$$

where in writing the above equation, we have used the result in Eq. (3.13) and permuted terms have been explicitly written down. However, as discussed before, one can use the symmetry of the Kähler potential under the interchange of j and k indices, i.e., $\mathcal{K}_{,i\bar{j}\bar{k}} = \mathcal{K}_{,i\bar{k}\bar{j}}$. Using this symmetry, the result in Eq. (3.18) can be rewritten as:

$$24i[F^i(\bar{\psi}^j_-\bar{\psi}^k_+) + \psi^i_-(\bar{\psi}^j_-\partial_{\zeta^-}\bar{\phi}^k) + \psi^i_+(\bar{\psi}^j_+\partial_{\xi^-}\bar{\phi}^k)]\mathcal{K}_{,i\bar{j}\bar{k}}. \tag{3.19}$$

One can notice that the permutations in Eq. (3.18) contributed an overall symmetry factor of 2 which canceled with 2! in the denominator in Eq. (3.19). Since there are only two possible permutations of the terms of the kind $\bar{\psi}^j_-\bar{\psi}^k_+$, etc. in Eq. (3.18), the symmetry factor one gets is 2.

The above arguments can be easily generalized to get rid of the permutations appearing in all the identities in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17). In fact, after using this symmetry of the Kähler potential, the identities in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) will then go over to the ones derived in [42] for the case of a single chiral and antichiral supermultiplet, multiplied by appropriate overall symmetry factors.

One can illustrate the above discussion by considering a more general term in the action. After using the symmetry of the Kähler potential as discussed above, we have:

$$\begin{aligned}[L_*^{2n+1} * R_l]|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+}\mathcal{K}_{,i_1i_2\cdots i_{2n+1}\bar{j}} = {}& 4(2n+1)!(2n+2)(-1)^n(\det C)^n F^{i_1}F^{i_2}\cdots F^{i_{2n-1}}[2n\psi^{i_{2n}}_-\psi^{i_{2n+1}}_+\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}_l \\ &+ iF^{i_{2n}}\psi^{i_{2n+1}}_-\partial_{\zeta^-}\bar{\psi}_- + iF^{i_{2n}}\psi^{i_{2n+1}}_+\partial_{\xi^-}\bar{\psi}_+ + F^{i_{2n}}F^{i_{2n+1}}\bar{F}_l]\mathcal{K}_{,i_1i_2\cdots i_{2n+1}\bar{j}},\end{aligned} \tag{3.20}$$

where $(2n+1)!$ is the symmetry factor obtained after eliminating the permutations. However, this symmetry factor will cancel after writing the LHS of Eq. (3.20) as $\frac{1}{(2n+1)!}[(L_*^{2n+1}) * R_l]$. This is nothing but the identity given in Eq. (3.14). Similarly, various terms in the action can be rearranged and the rest of the identities given in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17) can be used in an analogous fashion, while writing down the action.

We further note that, in writing down the general identities given in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17), we have used the fact that for terms in the action proportional to $\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+$ it is possible to push all the L's to one side and all the R's to the other side. The proof for the case of a single chiral multiplet has been given in [42] and can also be rigorously shown to be valid in the case of several chiral multiplets as well.

B. Classical action in normal coordinates

Following the discussion in the previous subsection and collecting all the results derived in Eqs. (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), and (3.17), and substituting them in the expansion of the Kähler potential given in Eq. (3.2), we find that the action can be divided into two parts as $I = I_0 + I_C$, with I_0 and I_C corresponding to the C-independent and C-dependent parts, respectively. First I_0 can be deduced to be:

$$I_0 = \int d^2x\Big[\Big(\frac{1}{2}\partial_{\xi^-}\phi^i\partial_{\zeta^-}\bar{\phi}^j + \frac{1}{2}\partial_{\zeta^-}\phi^i\partial_{\xi^-}\bar{\phi}^j + i\psi^i_-\partial_{\zeta^-}\bar{\psi}^j_- + i\psi^i_+\partial_{\xi^-}\bar{\psi}^j_+ + F^i\bar{F}^j\Big)\mathcal{K}_{,i\bar{j}} + \psi^i_-\psi^j_+\bar{F}^k\mathcal{K}_{,ij\bar{k}} + (i\psi^i_-\bar{\psi}^j_-\partial_{\zeta^-}\bar{\phi}^k + i\psi^i_+\bar{\psi}^j_+\partial_{\xi^-}\bar{\phi}^k - F^i\bar{\psi}^j_-\bar{\psi}^k_+)\mathcal{K}_{,i\bar{j}\bar{k}} + (\psi^i_-\bar{\psi}^k_-\psi^j_+\bar{\psi}^l_+)\mathcal{K}_{,ij\bar{k}\bar{l}}\Big], \quad (3.21)$$

where the derivatives of the Kähler potential are defined in Eq. (3.3). This action should be compared to the one in standard literature [56]. I_C can be derived in a similar fashion and is given as [42]:

$$I_C = -\sum_{n=2}^{\infty}(-1)^n(\det C)^{n-1}\int d^2x\frac{F^{i_1}F^{i_2}\cdots F^{i_{2n-2}}}{(2n-1)!}[\psi^{i_{2n-1}}_-\psi^{i_{2n}}_+\bar{F}^j\mathcal{K}_{,i_1i_2\cdots i_{2n}\bar{j}} + \psi^{i_{2n-1}}_-\bar{\psi}^j_-\psi^{i_{2n}}_+\bar{\psi}^k_+\mathcal{K}_{,i_1i_2\cdots i_{2n}\bar{j}\bar{k}}] + \sum_{n=1}^{\infty}(\det C)^n(-1)^n\int d^2xF^{i_1}F^{i_2}\cdots F^{i_{2n-1}}\Big[\frac{1}{(2n)!}F^{i_{2n}}\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}^j\mathcal{K}_{,i_1i_2\cdots i_{2n}\bar{j}} + \frac{1}{(2n+1)!}(2n\psi^{i_{2n}}_-\psi^{i_{2n+1}}_+\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}^j + iF^{i_{2n}}\psi^{i_{2n+1}}_-\partial_{\zeta^-}\bar{\psi}^j_- + iF^{i_{2n}}\psi^{i_{2n+1}}_+\partial_{\xi^-}\bar{\psi}_+ + F^{i_{2n}}F^{i_{2n+1}}\bar{F}^j)\mathcal{K}_{,i_1i_2\cdots i_{2n+1}\bar{j}} + \frac{1}{(2n)!}F^{i_{2n}}\partial_{\xi^-}\bar{\phi}^j\partial_{\zeta^-}\bar{\phi}^k\mathcal{K}_{,i_1i_2\cdots i_{2n}\bar{j}\bar{k}} + \frac{1}{(2n+1)!}(2n\psi^{i_{2n}}_-\psi^{i_{2n+1}}_+\partial_{\xi^-}\bar{\phi}^j\partial_{\zeta^-}\bar{\phi}^k - F^{i_{2n}}F^{i_{2n+1}}\bar{\psi}^j_-\bar{\psi}^k_+ - F^{i_{2n}}\bar{\psi}^j_-\psi^{i_{2n+1}}_-\partial_{\zeta^-}\bar{\phi}^k - F^{i_{2n}}\bar{\psi}^j_+\psi^{i_{2n+1}}_+\partial_{\xi^-}\bar{\phi}^k)\mathcal{K}_{,i_1i_2\cdots i_{2n+1}\bar{j}\bar{k}}\Big]. \quad (3.22)$$

The full action for the $\mathcal{N}=2$ supersymmetric sigma model on a non(anti)commutative superspace is thus given by Eqs. (3.21) and (3.22). Note that certain overall factors have been taken out in writing the actions given above. Further, the actions given above differ from the ones in [42] by some overall factors and also by signs of some terms. One can explicitly see the correspondence, by using the map given in the footnote above, between the variables used in [42] and ones used here. Once these notational differences are taken into account, the action given by Eqs. (3.21) and (3.22) is the same as the one given in [42].

In [42], the action has been shown to preserve the $\mathcal{N}=1/2$ supersymmetry of the theory in great detail. I do not repeat the calculations here. However, I have checked that the actions given in Eqs. (3.21) and (3.22) are invariant under the $\mathcal{N}=1/2$ supersymmetry transformations given in Eqs. (2.29).

The power series expansion we see in the C-dependent part of the action given in Eq. (3.22) is because of the arbitrariness of the Kähler potential. The fact that the series can be summed and written in a closed form is important. In fact, terms to an arbitrary order in $(\det C)$ can be easily deduced from Eq. (3.22). However, at this stage, it is not clear whether the full action can be written in terms of covariant quantities. This will have to be taken care while studying the quantum aspects of the theory. In other words, the question is, whether it is possible to see that the quantities like $\mathcal{K}_{,i_1i_2\cdots i_{2n}\bar{j}\bar{k}}$, etc. can be written in terms of proper geometric tensors.

In the $C=0$ case, one can eliminate the auxiliary fields by their equations of motion and see that the action can be written in terms of proper geometric quantities. To be precise, from Eq. (3.21) one can deduce that $F^i = -\psi^j_-\psi^k_+\Gamma^i_{jk}$ with similar relation for $\bar{F}$. Substituting these back in Eq. (3.21), one arrives at:

$$I_0 = \int d^2x[(\partial_{\xi^-}\phi^i\partial_{\zeta^-}\bar{\phi}^j + i\psi^i_-\mathbf{D}_{\zeta^-}\bar{\psi}^j_- + i\psi^i_+\mathbf{D}_{\xi^-}\bar{\psi}^j_+)g_{i\bar{j}} - \psi^i_-\bar{\psi}^k_-\psi^j_+\bar{\psi}^l_+\mathcal{R}_{j\bar{k}i\bar{l}}], \quad (3.23)$$

with covariant derivative defined as $\mathbf{D}_{\xi^-}\bar{\psi}^i_+ = \partial_{\xi^-}\bar{\psi}^i_+ + \Gamma^{\bar{i}}_{\bar{j}k}\bar{\psi}^j_+\partial_{\xi^-}\bar{\phi}^k$ and with a similar relation for the covariant derivative of $\bar{\psi}^i_-$. Note that we have done a partial integration in Eq. (3.23). Further, we have used the fact that, for Kähler manifolds, the metric can be obtained from the Kähler potential as:

$$g_{i\bar{j}} = \frac{\partial}{\partial\Phi^i}\frac{\partial}{\partial\bar{\Phi}^j}\mathcal{K}(\Phi,\bar{\Phi}). \quad (3.24)$$

And further, many components of the curvature tensor are zero. The only nonzero components are of the kind $\mathcal{R}_{j\bar{k}i\bar{l}}$ or $\mathcal{R}_{\bar{j}k\bar{i}l}$. There are many further simplifications, and a brief collection of relevant formulas has been given in Sec. 2.2 of [42].

In the $C\neq 0$ case, it is not clear whether the auxiliary fields can still be eliminated. The equation of motion of F and $\bar{F}$ which was found for the $C=0$ case may not be valid when the full action in Eq. (3.22) is considered because of the infinite number of terms in the action. Since it is difficult to work with the nth order action, below we first analyze the action only to order $(\det C)$. From Eq. (3.21) and (3.22), one can write down the full action $I = I_0 + I_C$, to order $(\det C)$ as:

$$I = \int d^2x\bigg[\bigg(\frac{1}{2}\partial_{\xi^-}\phi^i\partial_{\zeta^-}\bar{\phi}^j + \frac{1}{2}\partial_{\zeta^-}\phi^i\partial_{\xi^-}\bar{\phi}^j + i\psi_-^i\partial_{\zeta^-}\bar{\psi}_-^j + i\psi_+^i\partial_{\xi^-}\bar{\psi}_+^j + F^i\bar{F}^j\bigg)g_{i\bar{j}} + \psi_-^i\psi_+^j\bar{F}^k\Gamma_{ij\bar{k}} + (i\psi_-^i\bar{\psi}_-^j\partial_{\zeta^-}\bar{\phi}^k + i\psi_+^i\bar{\psi}_+^j\partial_{\xi^-}\bar{\phi}^k - F^i\bar{\psi}_-^j\bar{\psi}_+^k)\Gamma_{i\bar{j}\bar{k}} + (\psi_-^i\bar{\psi}_-^k\psi_+^j\bar{\psi}_+^l)\partial_i\Gamma_{j\bar{k}\bar{l}} + (\det C)F^p\bigg[-\frac{F^q}{6}\{\psi_-^l\psi_+^m\bar{F}^j\partial_p\partial_q\Gamma_{lm\bar{j}} + \psi_-^l\bar{\psi}_-^j\psi_+^m\bar{\psi}_+^k\partial_p\partial_q\partial_l\Gamma_{m\bar{j}\bar{k}}\} - \frac{1}{2}F^q\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}^j\Gamma_{pq\bar{j}} - \frac{1}{6}(2\psi_-^q\psi_+^l\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi}^j + iF^q\psi_-^l\partial_{\zeta^-}\bar{\psi}_-^j + iF^q\psi_+^l\partial_{\xi^-}\bar{\psi}_+ + F^qF^l\bar{F}^j)\partial_p\Gamma_{ql\bar{j}} - \frac{1}{2}F^q\partial_{\xi^-}\bar{\phi}^j\partial_{\zeta^-}\bar{\phi}^k\partial_p\Gamma_{q\bar{j}\bar{k}} - \frac{1}{6}(2\psi_-^q\psi_+^l\partial_{\xi^-}\bar{\phi}^j\partial_{\zeta^-}\bar{\phi}^k - F^qF^l\bar{\psi}_-^j\bar{\psi}_+^k - F^q\bar{\psi}_-^j\psi_-^l\partial_{\zeta^-}\bar{\phi}^k - F^q\bar{\psi}_+^j\psi_+^l\partial_{\xi^-}\bar{\phi}^k)\partial_p\partial_q\Gamma_{l\bar{j}\bar{k}}\bigg]\bigg], \tag{3.25}$$

with the notation $\Gamma_{ij\bar{k}} = g_{l\bar{k}}\Gamma^l_{ij}$. Further, the affine connections are obtained from the Kähler potential as: $\Gamma_{ij\bar{k}} = \frac{\partial}{\partial\Phi^i}\frac{\partial}{\partial\Phi^j}\frac{\partial}{\partial\bar{\Phi}^k}\mathcal{K}$.

One can notice that various terms in the action in Eq. (3.25) are noncovariant and the equation of motion of auxiliary fields may be hard to find. Thus, it is not possible to eliminate the auxiliary fields even from the action (3.25) to first order in $(\det C)$, due to various noncovariant terms. In other words, the action (3.22) to all orders in $(\det C)$ has to be expressed in terms of proper geometric quantities. Here, we show that it is possible to employ a normal coordinate expansion and express the action in a manifestly covariant way.

In [55], a specific Kähler potential was considered and the case of CP^n models was discussed. After gauging the sigma model by introducing vector multiplets, it was possible to write the action in a closed form and the number of terms in the action was also finite. Hence, the auxiliary field equations of motion were simple to obtain. Thus, in Sec. IV we consider a simple Kähler potential and show that the auxiliary fields appearing in the chiral 2D superfields can be eliminated by their equations of motion. However, we do not treat the fields of the vector multiplet as auxiliary and, hence, they are not eliminated, unlike the case in [55].

To ensure the covariance of various terms coming from the background field expansion of the sigma model action, Riemann normal coordinates were used in [57,58]. This analysis was further used in the study of ultraviolet structure of the bosonic and supersymmetric nonlinear sigma models [46]. The need for using normal coordinates was at the quantum level in doing a background field expansion. Further, the discussion was explicitly for the case of $\mathcal{N} = 1$ sigma models. Although the method can be applied to $\mathcal{N} = 2$ sigma models, the results were not manifestly covariant under $\mathcal{N} = 2$ supersymmetry. Recently, progress has been made in this direction and the analysis has been generalized to $\mathcal{N} = 2$ supersymmetric sigma models on Kähler manifolds [59]. In [59], new normal coordinates were introduced and it was shown that the background field expansion of the action can be written in a covariant manner. However, it was pointed out that a manifestly supersymmetric expansion in these new coordinates is still not possible. The new coordinates are holomorphic and hence are, nevertheless, more suitable for Kähler manifolds.

Thus, below we show that using the Kähler normal coordinates introduced in [59] the action can be written in terms of covariant geometric quantities. Since the origin of these noncovariant terms is in the expansion of the Kähler potential, we try to identify the terms in this expansion which give us the action (3.25). Thus, we find that the sigma model action to first order in $(\det C)$ given in Eq. (3.25), can be obtained from the following terms in the expansion of the Kähler potential:

$$I = \int d^2yd^4\theta\bigg[R^i\mathcal{K}_{,\bar{i}} + \frac{1}{2!}(R^i * R^j)\mathcal{K}_{,\bar{i}\bar{j}} + \frac{1}{2!}(L^i * R^j)\mathcal{K}_{,i\bar{j}} + \frac{1}{3!}(L_*^2 * R^k)\mathcal{K}_{,ij\bar{k}} + \frac{1}{3!}(L^i * R_*^2)\mathcal{K}_{,i\bar{j}\bar{k}} + \frac{1}{4!}(L_*^3 * R^k)\mathcal{K}_{,ijl\bar{k}} + \frac{1}{4!}(L_*^2 * R_*^2)\mathcal{K}_{,ik\bar{j}\bar{l}} + \frac{1}{5!}(L_*^4 * R^k)\mathcal{K}_{,ijlm\bar{k}} + \frac{1}{5!}(L_*^3 * R_*^2)\mathcal{K}_{,imk\bar{j}\bar{l}} + \frac{1}{6!}(L_*^4 * R_*^2)\mathcal{K}_{,ikmp\bar{j}\bar{l}}\bigg] + \cdots, \tag{3.26}$$

where $(\cdots)$ in the above Eq. (3.26) corresponds to other higher order terms in $(\det C)$ in the expansion of the Kähler potential. Now, as we discussed, the sigma model action (3.25), obtained from Eq. (3.26) above, is not invariant under general coordinate transformations. Thus, to write the Eq. (3.26) in a covariant form, we rearrange the expansion of the Kähler potential and claim that the action can be written as:

$$I = \int d^2yd^4\theta[\bar{f}(\bar{\pi}) + g_{i\bar{j}}(\pi,\bar{\pi})\pi^i * \bar{\pi}^j + \mathcal{R}_{i\bar{j}k\bar{l}}\pi^i * \bar{\pi}^j * \pi^k * \bar{\pi}^l + \cdots], \tag{3.27}$$

where we have introduced new superfields π^i and $\bar{\pi}^j$, which are given in terms of the old superfields as:

$$\pi^i = L^i + \frac{1}{2!} L^l * L^m g^{i\bar{k}} \mathcal{K}_{,lm\bar{k}} + \frac{1}{3!} L^l * L^m * L^n g^{i\bar{k}} \mathcal{K}_{,lmn\bar{k}} + \frac{1}{4!} L^l * L^m * L^n * L^p g^{i\bar{k}} \mathcal{K}_{,lmnp\bar{k}} + \cdots, \quad (3.28)$$

$$\bar{\pi}^j = R^j + \frac{1}{2!} R^k * R^l g^{m\bar{j}} \mathcal{K}_{,\bar{k}\bar{l}m}. \quad (3.29)$$

Note that, in terms of these new coordinates, one does not have to consider various permutation and combination of indices. However, in new coordinates, one still continues to use the star product as given in Eq. (2.6). As a consequence, from Eq. (3.29), one can show that the star product of more than three $\bar{\pi}$'s vanishes. Further, in Eq. (3.27), the function $\bar{f}$ is completely antiholomorphic and is given in terms of the old variables as [first two terms on the right hand side of Eq. (3.26)]:

$$\bar{f} = R^i \mathcal{K}_{,\bar{i}} + \frac{1}{2!} [R^i * R^j] \mathcal{K}_{,\bar{i}\bar{j}}. \quad (3.30)$$

To write the above function in terms of the new variables, one has to invert the relations given in Eq. (3.29) as shown:

$$R^j = \bar{\pi}^j - \frac{1}{2!} \bar{\pi}^k * \bar{\pi}^l g^{m\bar{j}} \mathcal{K}_{,\bar{k}\bar{l}m}. \quad (3.31)$$

Now, using the above relations, the function $\bar{f}$ can be written in terms of the new superfields. Generally speaking, what one can actually do is to rearrange the expansion of the Kähler potential given in Eq. (3.2) as:

$$\mathcal{K}(\Phi, \bar{\Phi}) = \mathcal{K}(\pi, \bar{\pi}) + f(\pi) + \bar{f}(\bar{\pi}) + g_{i\bar{j}} \pi^i * \bar{\pi}^j + \mathcal{R}_{i\bar{j}k\bar{l}} \pi^i * \bar{\pi}^j * \pi^k * \bar{\pi}^l + \cdots, \quad (3.32)$$

where the functions f and $\bar{f}$ are holomorphic and antiholomorphic, respectively. The function $\bar{f}$ is defined in Eq. (3.30) above and f is given as:

$$f(\pi) = L^i \mathcal{K}_{,i} + \frac{1}{2!} [L^l * L^m] \mathcal{K}_{,lm} + \frac{1}{3!} [L^l * L^m * L^n] \mathcal{K}_{,lmn} + \frac{1}{4!} [L^l * L^m * L^n * L^p] \mathcal{K}_{,lmnp} + \cdots + [L^n_*] \mathcal{K}_{,i_1 i_2 \cdots i_n} + \cdots. \quad (3.33)$$

Now, one can write the function f in terms of new superfields π by inverting the relations given in Eq. (3.28) as:

$$L^i = \pi^i - \frac{1}{2!} \pi^l * \pi^m g^{i\bar{k}} \mathcal{K}_{,lm\bar{k}} + \cdots, \quad (3.34)$$

and using this relation in Eq. (3.33). Notice that the functions f and $\bar{f}$ explicitly contain many noncovariant quantities. However, these functions can be absorbed into a redefinition of the Kähler potential by a Kähler gauge transformation:

$$\mathcal{K}'(\pi, \bar{\pi}) = \mathcal{K}(\pi, \bar{\pi}) + f(\pi) + f(\bar{\pi}). \quad (3.35)$$

Now, one can check that the expansion of the Kähler potential given in Eq. (3.27) generates all the terms in Eq. (3.26) to order (det$C$). In addition, the term $\pi^i * \bar{\pi}^j * \pi^k * \bar{\pi}^l$ in Eq. (3.27) will also give terms proportional to (det$C$)^2, etc. The proof that an expansion of the kind given in Eq. (3.27) generates all the terms in the action has been discussed in detail in [59] and can be checked in our case as well by explicit calculation. The only difference compared to the case given in [59] is the presence of star products instead of the ordinary product. The expansion in new variables has the advantage that all the terms coming from it are covariant. This can be explicitly checked by writing the transformations of various component fields under holomorphic coordinate transformations and using the fact that the coordinates (3.28) transform as holomorphic tangent vectors on target space [59].

It is important to note that the Riemann normal coordinates introduced in [46] are inherently nonchiral. In other words, at a time, only one of the old or new coordinates can be made chiral superfields. This problem carries over to the case of Kähler normal coordinates as well, although various quantities are evaluated with respect to the bosonic background. For instance, our old superfield L is evaluated with respect to the bosonic background as it is given by $\Phi - \phi$ and satisfies $\bar{D}_\pm L = 0$. However, the expansion in new coordinates $(\pi, \bar{\pi})$ will not preserve chirality as various geometric quantities will have both holomorphic and antiholomorphic indices [59].

The coordinate transformations given in Eq. (3.30) include all the terms in the expansion of the Kähler potential. Hence, the full nth order action given in Eq. (3.22) can be written in a covariant manner. Further, it is possible to use the expansion of the Kähler potential in terms of the new variables given in Eq. (3.27) to write the action in terms of the component fields. This action (3.22), written in terms of the new coordinates, will be useful while employing background field methods to study the quantum structure of the theory. For this purpose, one needs to calculate covariant expressions for the expansion of various geometric quantities. Explicit expressions to a certain order are given in [59] for the $C = 0$ case. It should be interesting to find out similar expressions in our case as well. Further, once the component form of the covariant action is calculated from Eq. (3.27), it might be possible to find the equations of motion of the auxiliary fields F and $\bar{F}$.

Further, one can do a background field expansion for the simple case of a constant background, $\partial \phi_0 = 0$. This background field expansion of the action can be argued to be manifestly invariant under general holomorphic coordinate transformations [59]. However, in the present case, it may not be manifestly invariant under $\mathcal{N} = 1/2$ supersymmetry transformations [59]. In this section, we have outlined how the $\mathcal{N} = 2$ sigma model action on a non(anti)commutative superspace can be written in a covariant manner by transforming to the new normal coordinates. It would be interesting to pursue these issues further.

IV. GAUGED LINEAR SIGMA MODELS

In this section, I derive the classical action for gauged linear sigma models. I show that the action is invariant under $\mathcal{N} = 1/2$ supersymmetry transformations. The matter content of the theory is as follows. We have k chiral superfields S^i which transform with charges Q_i^a under the s vector multiplets V^a. As stated before, we only consider Abelian gauge groups, which for our purposes will be $U(1)^s$.

The superspace action corresponding to above multiplets can be written as a gauged linear sigma model and consists of four parts [48]:

$$I = I_{\text{kin}} + I_W + I_{\text{gauge}} + I_{r,\theta}, \tag{4.1}$$

where the terms are, respectively, the kinetic term of the chiral superfields, the superpotential interaction, the kinetic term of the gauge fields, and the Fayet-Illiopoulos and theta terms. The construction of all these terms is discussed below. In writing the formulas, at some places I suppress the indices corresponding to the number of multiplets for convenience.

A. Chiral superfield action

The gauge invariant kinetic term for the Chiral superfields takes the form:

$$I_{\text{kin}} = \int d^2y d^4\theta \bar{\Phi} * e^V * \Phi, \tag{4.2}$$

where $d^2y = d\xi^- d\zeta^-$, $d^4\theta = d\bar{\theta}^- d\bar{\theta}^+ d\theta^- d\theta^+$ and the integrand can be evaluated owing to the results in Eq. (2.20), as shown below:

$$\bar{\Phi} * e^V * \Phi = \bar{\Phi} * \Phi + \bar{\Phi} * V * \Phi + \frac{1}{2}\bar{\Phi} * V_*^2 * \Phi. \tag{4.3}$$

I calculate each of the terms in Eq. (4.3) separately and use them in the action (4.2). The calculations are given in Appendix B. Thus, using the formulas derived in Eqs. (B1) and (B2) in Eq. (4.2), and performing integration over the Grassmannian coordinates in the usual way, we find that the action can be written as:

$$\begin{aligned} I_{\text{kin}} = \sum_i \int d\xi^- d\zeta^- \Big[&\frac{1}{2}\bar{\mathcal{D}}_{\xi^-}\phi_i \mathcal{D}_{\zeta^-}\bar{\phi}_i + \frac{1}{2}\bar{\mathcal{D}}_{\zeta^-}\phi_i \mathcal{D}_{\xi^-}\bar{\phi}_i + i\bar{\psi}_{-,i}\mathcal{D}_{\zeta^-}\psi_{-,i} + i\bar{\psi}_{+,i}\mathcal{D}_{\xi^-}\psi_{+,i} + F_i\bar{F}_i \\ &- 2\sum_a \bar{\sigma}_a\sigma_a Q_{i,a}{}^2\bar{\phi}_i\phi_i - \sqrt{2}\sum_a Q_{i,a}(\bar{\sigma}_a\bar{\psi}_{+i}\psi_{-i} + \sigma_a\bar{\psi}_{-i}\psi_{+i}) + \sum_a D_a Q_{i,a}\bar{\phi}_i\phi_i - i\sqrt{2}\sum_a Q_{i,a}\bar{\phi}_i(\psi_{-,i}\lambda_{+,a} \\ &- \psi_{+,i}\lambda_{-,a}) - i\sqrt{2}\sum_a Q_{i,a}\phi_i(\bar{\lambda}_{-,a}\bar{\psi}_{+,i} - \bar{\lambda}_{+,a}\bar{\psi}_{-,i}) - \sum_a Q_{i,a}[\sqrt{2}C^{00}(iF\partial_{\xi^-}\sigma_a\bar{\phi}_i - \bar{\lambda}_{+,a}\psi_{-,i}\bar{\mathcal{D}}_{\xi^-}\bar{\phi}_i) \\ &- \sqrt{2}C^{11}(-iF\partial_{\zeta^-}\bar{\sigma}_a\bar{\phi}_i + \bar{\lambda}_{-,a}\psi_{+,i}\bar{\mathcal{D}}_{\zeta^-}\bar{\phi}_i) - C^{01}(iF_i\bar{\phi}_i\nu_{\xi\zeta} + \sqrt{2}\bar{\lambda}_{+,a}\psi_{+,i}\bar{\mathcal{D}}_{\xi^-}\bar{\phi}_i - \sqrt{2}\bar{\lambda}_{-,a}\psi_{-,i}\bar{\mathcal{D}}_{\zeta^-}\bar{\phi}_i)] \\ &- 2\sum_a Q_{i,a}{}^2(\det C)F_i\bar{\phi}_i\bar{\lambda}_{-,a}\bar{\lambda}_{+,a}\Big], \end{aligned} \tag{4.4}$$

where $\mathcal{D}_{\xi^-} = \partial_{\xi^-} + \frac{i}{2}\nu_\xi$, $\mathcal{D}_{\zeta^-} = \partial_{\zeta^-} + \frac{i}{2}\nu_\zeta$ are the gauge covariant derivatives and $\bar{\mathcal{D}}_{\xi^-}$, $\bar{\mathcal{D}}_{\zeta^-}$ denote the corresponding complex conjugates, respectively. The $C = 0$ part of the kinetic action for the chiral superfields is seen to be equivalent to the standard action given in [48]. By using the transformation properties of the component fields, it can be shown that the full kinetic action for the chiral superfields is gauge invariant.

Now, since the $C = 0$ part of action (4.4) is the same as the one given in [48], one need not explicitly show that this part is invariant under the $\mathcal{N} = 1/2$ supersymmetry transformations. One can still check this by using the supersymmetry transformations given in Eq. (2.41) and also the one obtained by putting $C = 0$ in Eq. (2.42).

For the case with $C \neq 0$, we know from Eq. (2.42) and (2.26) that only the supersymmetry variation of $\bar{F}$ and $\lambda_\pm$ have C-dependent terms. Thus, these terms are expected to cancel the variation of all the C-dependent terms of the action (4.4). This is exactly what we show below.

Let us name the terms in the action depending on C as I_c. Then, the supersymmetry variation of these terms is:

$$\begin{aligned} \delta I_c = (C^{00}\epsilon^+ + C^{10}\epsilon^-)[i\bar{\lambda}_+\bar{\lambda}_- + 2\bar{\mathcal{D}}_{\xi^-}(\bar{\lambda}_+\bar{\phi})] \\ + (C^{11}\epsilon^- + C^{01}\epsilon^+)[i\bar{\lambda}_+\bar{\lambda}_- + 2\bar{\mathcal{D}}_{\zeta^-}(\bar{\lambda}_-\bar{\phi})]. \end{aligned} \tag{4.5}$$

Note that we have ignored the pieces whose variations are trivially zero. For instance, the variation of the term proportional to (det$C$) in the action (4.4) is zero identically. Now, one can guess that the terms obtained in Eq. (4.5) are exactly canceled by the C-dependent terms coming from the following terms of the action (4.4):

$$-i\sqrt{2}\bar{\phi}[\psi_-(\delta\lambda_+) - \psi_+(\delta\lambda_-)] + F(\delta\bar{F}). \tag{4.6}$$

Using the supersymmetry transformations given in Eq. (2.42) and (2.26) in the above equation, one can explicitly show that the terms in Eq. (4.5) are exactly canceled. Thus, the chiral superfield action (4.4) is invariant under the $\mathcal{N} = 1/2$ supersymmetry of the theory.

B. Gauge kinetic part

A gauge invariant action for the vector superfields can be constructed from the twisted chiral superfields as shown below [48]:

$$I_{\text{gauge}} = -\sum_a \frac{1}{4e_a^2} \int d^2y d^4\theta \bar{\Sigma}_a \Sigma_a, \tag{4.7}$$

where e_a $a = 1, \dots, s$ are the gauge coupling constants in case one has several vector multiplets. Using the definitions of the twisted superfields given in Eqs. (2.22) and (2.23), the action can be written in the component form as:

$$\begin{aligned} I_{\text{gauge}} = -\sum_a \frac{1}{e_a^2} \int d^2y\Big(&\frac{1}{2}{\nu_{\xi\zeta,a}}^2 + \frac{1}{2}D_a^2 + i\bar{\lambda}_{+,a}\partial_{\xi^-}\lambda_{+,a} \\ &+ i\bar{\lambda}_{-,a}\partial_{\zeta^-}\lambda_{-,a} - \partial_{\xi^-}\sigma_a\partial_{\zeta^-}\bar{\sigma}_a \\ &- \frac{i}{2}C^{01}\nu_{\xi\zeta,a}\bar{\lambda}_{+,a}\bar{\lambda}_{-,a}\Big). \end{aligned} \tag{4.8}$$

The kinetic energy for the gauge fields given above is, apart from some new C-dependent terms, the same as the one given in [48]. The C-dependent term involving the gauge field strength is gauge invariant on its own. It is useful to compare this action with the dimensional reduction of the action in [13]. Thus, the $C = 0$ part can be taken to be invariant under the $\mathcal{N} = 1/2$ supersymmetry of the theory. This can as well be explicitly checked by using the transformations given in Eq. (2.41).

The $C \neq 0$ part of the action can also be shown to be $\mathcal{N} = 1/2$ supersymmetric as follows. We note that only the variations of $\lambda_\pm$ contain certain C-dependent terms. Thus in Eq. (4.8), the C-dependent terms obtained from $i\bar{\lambda}_{+,a}\partial_{\xi^-}(\delta\lambda)_{+,a} + i\bar{\lambda}_{-,a}\partial_{\zeta^-}(\delta\lambda)_{-,a}$ are:

$$\begin{aligned} -\bar{\lambda}_+\partial_{\xi^-}[(C^{00}\epsilon^+ + C^{10}\epsilon^-)\bar{\lambda}_+\bar{\lambda}_-] - \bar{\lambda}_-\partial_{\zeta^-}[(C^{01}\epsilon^+ \\ + C^{11}\epsilon^-)\bar{\lambda}_+\bar{\lambda}_-]. \end{aligned} \tag{4.9}$$

One can see that the terms given in the above are identically zero. Thus, one can guess that the $C \neq 0$ part of the action has to be supersymmetric on its own. It is straightforward to check that the $C \neq 0$ part of the action is invariant under $\mathcal{N} = 1/2$ supersymmetry transformations as $\delta\bar{\lambda}_\pm = 0$ and the other term also vanishes. Thus, the gauge kinetic part of the action is invariant under the $\mathcal{N} = 1/2$ supersymmetry of the theory. Further, the gauge invariance of the action can also be explicitly checked.

It should be interesting to take the $e^2 \to \infty$ limit where the fields of the vector multiplet become auxiliary and can be eliminated by their equations of motion. The auxiliary field equations of motion might have many C-dependent pieces [55], which might effect the target space metric. In this manner, it would be possible to study the consequences of various new C-dependent terms in the classical action on the sigma model metric in UV and IR (see Appendix B of [60]).

C. r and θ terms

The Fayet-Iliopoulos (FI) and the theta angle terms can be obtained from the twisted superfields as in [48]. The FI term is the vector superfield integrated over the whole of superspace. This term should still be the same, since we have not added any C-dependent term proportional to $\theta^2\bar{\theta}^2$ to the definition of vector superfield given in Eq. (2.18). Thus, we have:

$$I_{r,\theta} = -r_a \int d^2y D^a + \frac{\theta_a}{2\pi} \int d^2y \nu_{\xi\zeta,a}, \tag{4.10}$$

where, as defined before, $\nu_{\xi\zeta} = \partial_{\xi^-}\nu_\zeta - \partial_{\zeta^-}\nu_\xi$.

D. Superpotential terms

If we assume an arbitrary superpotential, then the interaction terms in the action turn out to have the form:

$$I_W = \int d^2x d^2\theta W(\Phi) + \int d^2x d^2\bar{\theta}\, \bar{W}(\bar{\Phi}). \tag{4.11}$$

As was shown in [42], the component form of the superpotential can be obtained by expanding around the bosonic fields ϕ and $\bar{\phi}$ as:

$$\begin{aligned} W(\Phi)|_{\theta^-\theta^+} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\det C)^n F^{i_1}F^{i_2}\cdots F^{i_{2n}}(F^{i_{2n+1}} \\ \times W_{,i_1\cdots i_{2n+1}} + \psi_-^{i_{2n+1}}\psi_+^{i_{2n+2}}W_{,i_1\cdots i_{2n+2}}), \end{aligned} \tag{4.12}$$

where as before, we use the notation $W_{,i} = \frac{\partial W}{\partial \Phi^i}$ evaluated at $\Phi = \phi$. In the above equations, we have only written down the terms proportional to $\theta^+\theta^-$ and $\bar{\theta}^+\bar{\theta}^-$, respectively. It is important to note that the hermiticity of the theory is spoiled due to the asymmetry of the holomorphic and the antiholomorphic parts of the superpotential [15]. This can in fact be noted by looking at the asymmetric way in which F and $\bar{F}$ terms appear in the kinetic action. Following the examples in four dimensions [15], in the present case also, it may be possible consider supersymmetric vacua which come from $\bar{W}(\bar{\Phi})$ only, as $W(\Phi)$ may not be stable due to radiative corrections.

The antiholomorphic part of the superpotential in the component form is:

$$\bar{W}(\bar{\Phi})|_{\bar{\theta}^-\bar{\theta}^+} = -\bar{F}^i\bar{W}_{,i} + \bar{\psi}_-^i\bar{\psi}_+^j\bar{W}_{,ij}, \tag{4.13}$$

where $\bar{W}_{,i} = \frac{\partial \bar{W}}{\partial \bar{\Phi}^i}|_{\bar{\Phi}=\bar{\phi}}$.

It is possible to eliminate the auxiliary fields and write down the F term constraints as follow from their equations of motion:

$$F_i = \bar{W}_{,i},$$
$$\bar{F}_i = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\det C)^n F^{i_1} F^{i_2} \cdots F^{i_{2n-1}} [(2n+1) F^{i_{2n}} W_{,i_1 \cdots i_{2n+1}} + (2n) \psi_-^{i_{2n+1}} \psi_+^{i_{2n+2}} W_{,i_1 \cdots i_{2n+2}}]$$
$$+ i \sum_a Q_{i,a} (\sqrt{2} C^{00} \partial_{\xi^-} \sigma_a \bar{\phi}_i + \sqrt{2} C^{11} \partial_{\zeta^-} \bar{\sigma}_a \bar{\phi}_i - C^{01} \bar{\phi}_i \nu_{\xi\zeta}) + 2 \sum_a Q_{i,a}{}^2 (\det C) \bar{\phi}_i \bar{\lambda}_{-,a} \bar{\lambda}_{+,a}. \quad (4.14)$$

The $\bar{F}$ constraint contains new C-dependent pieces compared to the standard case. In the present case, it was possible to solve for the auxiliary field, unlike the case in the previous section, where the Kähler potential was arbitrary.

We eliminate the auxiliary fields from the action and write down the potential energy for the bosonic fields of the theory as:

$$U = \sum_{n=0}^{\infty} \frac{(-1)^n (\det C)^n}{(2n+1)!} [(2n+1)(\bar{W}_{,i})^{2n+1} W_{,i_1 \cdots i_{2n+1}} + (2n) \times (\bar{W}_{,i})^{2n} \psi_-^{i_{2n+1}} \psi_+^{i_{2n+2}} W_{,i_1 \cdots i_{2n+2}}] + \left[\frac{D_a^2}{2e^2} + 2\sigma_a \bar{\sigma}_a Q_{i,a}{}^2 \phi^i \bar{\phi}^i \right]. \quad (4.15)$$

The potential seen above is again an expansion in powers of ($\det C$). Further, there are also many higher powers of the $\frac{\partial \bar{W}}{\partial \bar{\phi}^i}$. So, if $\bar{W}(\bar{\phi})$ is chosen to be zero, then no matter what $W(\phi)$ is, the F terms are zero and one is only left with D terms in the potential.

To draw more conclusions, let us look at the potential given in Eq. (4.15) to first order in ($\det C$):

$$U = \frac{\partial \bar{W}}{\partial \bar{\phi}^i} \frac{\partial W}{\partial \phi^i} - (\det C) \frac{\partial \bar{W}}{\partial \bar{\phi}^i} \frac{\partial \bar{W}}{\partial \bar{\phi}^j} \left(\frac{1}{2} \frac{\partial \bar{W}}{\partial \bar{\phi}^k} \frac{\partial^3 W}{\partial \phi^i \partial \phi^j \partial \phi^k} + \frac{1}{3} \psi_-^k \psi_+^l \frac{\partial^4 W}{\partial \phi^i \partial \phi^j \partial \phi^k \partial \phi^l} \right) + \left(\frac{D_a^2}{2e^2} + 2\sigma_a \bar{\sigma}_a Q_{i,a}^2 \phi^i \bar{\phi}^i \right). \quad (4.16)$$

First, we see that the potential for the scalar fields also contains some fermionic pieces. In the $C = 0$ case, these fermionic pieces are absent and one can independently look at the F-flatness and D-flatness conditions. However, in the present case, the pieces depending on C come with a negative sign and, hence, it is important to understand their role while looking for supersymmetric vacua. In some simple cases, by suitable choice of W and $\bar{W}$, the fermionic pieces can be dropped.

Before proceeding, we note that in Eq. (4.16), the D term can be set to zero independently, as there are no other terms which depend on the gauge coupling:

$$D = e^2 \left(\sum Q \phi \bar{\phi} - r \right) = 0. \quad (4.17)$$

Notice that the D-flatness condition is the same as in the $C = 0$ theory. It is the analogue of this condition in [48] that gives the target space as CP^{n-1}.

Now, one can make an appropriate choice for the superpotentials and impose further conditions on the target space geometry. For instance, following [48], one can take the matter content to be, say, two chiral superfields Φ^1, Φ^2 of charge 1 each and one chiral superfield P of charge -2, such that the superpotential $W(\Phi) = P * G(\Phi^1, \Phi^2)$ is gauge invariant, quasihomogeneous, and satisfies the constraints coming from R-symmetry invariance. Further, we can also choose $\bar{W}(\bar{\Phi}) = \bar{P} * \bar{G}(\bar{\Phi}^1, \bar{\Phi}^2)$. In this case, one can show that the fermionic terms drop out.

Once again, the D term can be set equal to zero independently. The analogue of this condition in terms of [48] would give the target space to be CP^1. Further, with the above choice of the superpotentials, evaluating Eq. (4.16), one can have new terms in the potential which depend on ($\det C$). For instance, the potential can have terms of the kind ($\det C$) ($\bar{p}^2 \bar{G} \frac{\partial G}{\partial \phi^1} \frac{\partial G}{\partial \phi^2}$), apart from the standard terms which one normally gets [48]. Here, $\bar{p}$, ϕ^1, ϕ^2 are the lowest components of corresponding superfields. It should be interesting to vary the Fayet-Illiopoulos parameter r and study the phases of the above theory. In particular, to study the additional restrictions put by the terms depending on C on the target space geometry.

V. DISCUSSION

To conclude, in this paper, I have extended the results of my previous work [42] to write down the action for $D = 2$, $\mathcal{N} = 2$ sigma models characterized by an arbitrary Kähler potential on a non(anti)commutative superspace to include several chiral multiplets. Despite the fact that there are an infinite number of terms, a general term in the action can be written down in a closed form. This is due to the fact that the action turns out to be a series expansion in ($\det C$)F.

It was shown that the action can be written in a manifestly covariant manner by using the Kähler normal coordinates. This will be needed while analyzing the quantum structure of the theory. It would be interesting to apply the background field methods to study the action using normal coordinates. Since the Kähler normal coordinates transform as holomorphic tangent vectors on the target manifold, one expects that the background field expansion will also be manifestly covariant. However, the background field expansion may not preserve chirality and in this process invariance under the $\mathcal{N} = 1/2$ supersymmetry transformations may also be lost. It is important to further study these features so as to address the question of renormalizability of the theory.

In the second part, the analysis was extended to include vector multiplets as well. I wrote down the classical action for gauged linear sigma models on non(anti)commutative spaces. The gauge transformations and the supersymmetry transformations for the vector and chiral multiplets were derived explicitly in the Wess-Zumino gauge. To ensure the correctness of component calculations, the action was explicitly shown to be invariant under the $\mathcal{N} = 1/2$ supersymmetry transformations. The bosonic potential of the theory was shown to contain various higher powers of the derivatives of the superpotential. The D-term constraint is still the same as in the $C = 0$ theory. It would be interesting to turn on superpotentials considering various number of chiral multiplets and see what kinds of restrictions can be put on the target space geometry. This would be the first step to study the phases of this model, in parallel to [48].

It is known [13] that supersymmetric theories defined on non(anti)commutative superspace do not have a chiral ring structure due to the absence of the $\bar{Q}$ supersymmetry. This can also be inferred from the fact that the product of an arbitrary number of the chiral superfields does not vanish, in general. This has some straightforward implications for topological field theories. In the $C = 0$ theory, it is known that if the left and (nonanamolous) right R symmetries are unbroken, then it is possible to have A and B twists. Many important properties of the untwisted models and several aspects of mirror symmetry have been studied from the topological A and B models. For the present case, the absence of the chiral ring suggests that it may not be possible to have the standard B twist. However, it is possible to have the A twist, where the operators are in $Q = Q_+ + Q_-$ cohomology. It should be interesting to study these topological models.

There are other avenues one can explore. Taking the $e^2 \to \infty$ limit, one can look at the boundary terms generated from the gauged linear sigma model action. Since, there are new C-dependent terms in the GLSM action, one expects new terms to be generated at the boundaries. These terms will play a crucial role while studying sigma models with boundaries and, hence, will be relevant in the study of D branes using GLSM's [50]. On another front, one can look to solve the D-flatness conditions and study the sigma model metric in the UV and IR. Further, it should also be interesting to study closed string tachyon condensation [61] in this setting. I hope to come back to these issues in future.

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APPENDIX A: SOME IDENTITIES USED IN THE TEXT

Some of the identities used in the text are given below. They can be derived using the definition of star product given in Eq. (2.6):

$$\theta^- * \theta^- = \frac{1}{2}C^{00}, \tag{A1}$$

$$\theta^+ * \theta^+ = \frac{1}{2}C^{11}, \tag{A2}$$

$$\theta^- * \theta^+ = \theta^-\theta^+ - \frac{1}{2}C^{01}, \tag{A3}$$

$$\theta^+ * \theta^- = \theta^+\theta^- - \frac{1}{2}C^{10}, \tag{A4}$$

$$\theta^- * (\theta^-\theta^+) = -\theta^- * (\theta^+\theta^-) = \frac{1}{2}(C^{00}\theta^+ + C^{01}\theta^-), \tag{A5}$$

$$\theta^+ * (\theta^+\theta^-) = -\theta^+ * (\theta^-\theta^+) = \frac{1}{2}(C^{11}\theta^- + C^{10}\theta^+), \tag{A6}$$

$$(\theta^-\theta^+) * (\theta^-\theta^+) = -(\theta^+\theta^-) * (\theta^-\theta^+) = -\frac{1}{4}(\det C). \tag{A7}$$

APPENDIX B: DETAILS OF GLSM ACTION

Below are given some details of the calculation corresponding to the chiral superfield action. We write down the $\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+$ terms coming from each of the pieces appearing on the RHS of Eq. (4.3):

$$\bar{\Phi} * \Phi|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+} = 4\phi\partial_{\xi^-}\partial_{\zeta^-}\bar{\phi} - 4i\psi_+\partial_{\xi^-}\bar{\psi}_+ - 4i\psi_-\partial_{\zeta^-}\bar{\psi}_- - 4F\bar{F} + 2iF[\sqrt{2}C^{00} - \partial_{\xi^-}(\sigma\bar{\phi})\sqrt{2}C^{11}\partial_{\zeta^-}(\bar{\sigma}\,\bar{\phi}) + C^{01}\partial_{\xi^-}(\nu_\zeta\bar{\phi}) - C^{10}\partial_{\zeta^-}(\nu_\xi\bar{\phi})], \tag{B1}$$

$$\begin{aligned}\bar{\Phi} * V * \Phi|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+} &= \bar{\phi}\{-\phi(2D + i\partial_\zeta\nu_\xi + i\partial_\xi\nu_\zeta) + 2\sqrt{2}i\lambda_-\psi_+ - 2\sqrt{2}i\lambda_+\psi_- - 2\sqrt{2}i(C^{01}\bar{\lambda}_+\nu_\xi + C^{11}\bar{\lambda}_-\nu_\zeta)\psi_+ \\ &\quad + 2\sqrt{2}i(C^{00}\bar{\lambda}_+\nu_\xi + C^{10}\bar{\lambda}_-\nu_\zeta)\psi_-\} + \sqrt{2}\bar{\psi}_-\{\sqrt{2}\nu_\zeta\psi_- + 2\sigma\psi_+ + 2i\phi\bar{\lambda}_+\} \\ &\quad + \sqrt{2}\bar{\psi}_+\{\sqrt{2}\nu_\xi\psi_+ + 2\bar{\sigma}\psi_- - 2i\phi\bar{\lambda}_-\} - 2i\partial_{\xi^-}\bar{\phi}\{\phi\nu_\zeta + \nu_\zeta FC^{01} + \sqrt{2}\sigma FC^{00} - i\sqrt{2}\bar{\lambda}_+\psi_-C^{00} \\ &\quad + i\sqrt{2}\bar{\lambda}_+\psi_+C^{01}\} - 2i\partial_{\zeta^-}\bar{\phi}\{\phi\nu_\xi - \nu_\xi FC^{10} - \sqrt{2}\bar{\sigma}FC^{11} + i\sqrt{2}\bar{\lambda}_-\psi_+C^{11} - i\sqrt{2}\bar{\lambda}_-\psi_-C^{10}\} \\ \bar{\Phi} * V_*^2 * \Phi|_{\bar{\theta}^-\bar{\theta}^+\theta^-\theta^+} &= 2\phi\bar{\phi}(-\nu_\zeta\nu_\xi + 2\sigma\bar{\sigma}) + 4\bar{\phi}(\det C)\bar{\lambda}_-\bar{\lambda}_+F.\end{aligned} \tag{B2}$$

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