Scalar self-force on a static particle in Schwarzschild spacetime using the massive field approach

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(Received 5 October 2004; published 14 December 2004)

I use the recently developed massive field approach to calculate the scalar self-force on a static particle in a Schwarzschild spacetime. In this approach the scalar self-force is obtained from the difference between the (massless) scalar field, and an auxiliary massive scalar field combined with a certain limiting process. By applying this approach to a static particle in Schwarzschild I show that the scalar self-force vanishes in this case. This result conforms with a previous analysis [A. G. Wiseman, Phys. Rev. D **61**, 084014 (2000).].

DOI: 10.1103/PhysRevD.70.124016 PACS numbers: 04.25.–g

I. INTRODUCTION

The self-force is a force originating from the coupling between the charge of a particle and the field that this charge induces. In curved spacetime, one usually considers a fixed curved background spacetime and a given world line of a charged particle. The field induced by the point particle diverges at the particle's location, and therefore a regularization method is required to calculate the correct (and finite) self-force from this diverging field. For this problem, formal expressions for the self-force have been derived for various types of fields. Thus, DeWitt and Brehme derived an expression for the electromagnetic self-force [1]. More recently Mino, Sasaki, and Tanaka [2], and independently Quinn and Wald [3], have derived an expression for the gravitational self-force (here the electric charge is replaced by the mass of the particle, and the induced electric field is replaced by the linear gravitational perturbation induced by the particle's mass). Quinn [4] has recently derived the corresponding expression for the scalar self-force.

Although formal analytical expressions for the various types of self-forces are well-known, explicit analytical calculations of these self-forces are still a challenging task and were carried out only in a few cases. Here, one of the main difficulties is the nonlocal nature of the self-force in curved spacetime (i.e., the self-force value at each point on the particle's trajectory generically depends on entire past history of the particle). For a weakly curved spacetime the above mentioned explicit self-force expressions were derived by DeWitt and DeWitt [5], and by Pfenning and Poisson [6]. The self-force on a static particle was investigated analytically by several authors: Smith and Will have obtained a nonvanishing result for the electromagnetic self-force on a static particle in Schwarzschild [7]. Later Lohiya [8] derived the electromagnetic self-force on a static particle for other types of background spacetimes. Recently, Wiseman [9] has showed that the scalar self-force on a static particle in a Schwarzschild spacetime vanishes.

A practical method to calculate the self-force for generic orbits was devised by Barack and Ori [10] (this method was later improved [11,12]). In this method, one first calculates certain regularization parameters (usually analytically), and then uses these parameters to calculate the self-force (usually numerically). This method was implemented numerically in certain cases [13,14]. In particular, using this method (numerically) the scalar self-force on a static particle in Kerr-Newman background was calculated by Burko and Liu [15]. For other approaches to the self-force problem see [16–18].

Very recently, a new general method titled ''the massive field approach'' for the calculation of the scalar self-force was developed [19]. In this paper, I implement this new method to calculate analytically the scalar self-force on a particle, which is held static (by some external forces) in a Schwarzschild background spacetime (i.e., static with respect to Schwarzschild coordinates), and thereby I show that the self-force vanishes in this case. This result conforms with the above mentioned analysis by Wiseman (which used a different calculation method). The analysis given here therefore verifies Wiseman's result and furthermore demonstrates how the massive field approach can be implemented in practice.

In Ref. [19] it is shown that the scalar self-force can be obtained from the difference between the following retarded scalar fields: The first field is ϕ—the massless scalar field induced by the particle. This field satisfies the inhomogeneous massless scalar field equation, with a charge density ρ [see Eq. (4) below]. The second field is ϕ_Λ—an auxiliary massive scalar field satisfying the inhomogeneous massive field equation, with the *same* charge density ρ [see Eq. (14) below]. Reference [19] provides the following prescription for the calculation of the scalar self-force $f^{\rm self}_\mu$ in terms of these fields

$$f^{\rm self}_\mu(z_0) = q\lim_{\Lambda\to\infty}\Big\{\lim_{\delta\to 0}\Delta\phi_{,\mu}(x) + \frac{1}{2}q[\Lambda^2 n_\mu(z_0) + \Lambda a_\mu(z_0)]\Big\}, \tag{1}$$

where,

$$\Delta\phi(x) \equiv \phi(x) - \phi_\Lambda(x). \tag{2}$$

Here, Λ is the mass of the massive field, z_0 is the self-force evaluation point on the particle's world line, and x is a point near the world line defined as follows: at z_0 one constructs a unit spatial vector n^μ, which is perpendicular to the particle's world line but is otherwise arbitrary (i.e., at z_0 the following relations are satisfied: $n^\mu n_\mu = 1$, $n^\mu u_\mu = 0$). In the direction of n^μ one constructs a geodesic, which extends out an invariant length δ to the point $x(z_0, n^\mu, \delta)$. In the following section I shall calculate the scalar self-force on a static particle in Schwarzschild spacetime by implementing the mathematical prescription given by Eq. (1).

II. IMPLEMENTATION OF THE MASSIVE FIELD APPROACH

Here I use Eq. (1) to calculate the scalar self-force on a static particle in Schwarzschild spacetime. I use Schwarzschild coordinates throughout, where $x^\alpha = (t, r, \theta, \varphi)$, $g_{\mu\nu} = \mathrm{diag}(-f, f^{-1}, r^2, r^2\sin^2\theta)$, $f \equiv 1 - \frac{2M}{r}$. I denote the particle's world line with $z(\tau)$, where τ denotes the particle's proper time. More explicitly the particle world line is characterized by the spatial coordinates $(r_0, \theta_0, \varphi_0)$ which are constants, where $r_0 > 2M$.

By virtue of the spherical symmetry of the problem the two angular components of the self-force f_θ^{self} and $f_\varphi^{\mathrm{self}}$ vanish. To calculate the time component of the self-force f_t^{self} we use Eq. (1). For a static particle in Schwarzschild the scalar fields ϕ and ϕ_Λ are time independent, and therefore the term $\Delta\phi_{,t}$ in Eq. (1) vanishes. Noting that the vectors n_μ and a_μ are perpendicular to the particle's world line we find that f_t^{self} vanishes, by virtue of Eq. (1).

We now consider the radial component of the self-force f_r^{self}. Equation (1) implies that the calculation of f_r^{self} follows from the calculation of the field $\Delta\phi_{,r}$ near the particle's world line, combined with a limiting process. The first limit that has to be considered is $\delta \to 0$. Here, we take advantage of the arbitrariness in the definition of the unit spatial vector n_μ, and choose n_μ to be the radial unit vector $n_\mu = (0, f_0^{-1/2}, 0, 0)$, where $f_0 \equiv f(r_0)$. This choice defines the spatial geodesic which extends out from z_0 to x in the direction of n_μ to be a radial spatial geodesic (with $r \geq r_0$) denoted here with γ. The limit $\delta \to 0$ is now equivalent to the limit $r \to r_0$ along this geodesic. From Eq. (1) we now find that the radial component of the self-force is given by

$$f_r^{\mathrm{self}}(z_0) = q \lim_{\Lambda\to\infty}\left\{\lim_{r\to r_0}\Delta\phi_{,r}(r) + \frac{1}{2}q[\Lambda^2 n_r(z_0) + \Lambda a_r(z_0)]\right\}. \tag{3}$$

Here the limit $r \to r_0$ is considered along γ.

This section is organized as follows: First, in Sec. II A I calculate the field $\phi_{,r}$ along the radial geodesic γ, then in Sec. II B I calculate an expansion for the field $\phi_{\Lambda,r}$ in the vicinity of the particle's world line and along γ, and finally in Sec. II C I substitute $\phi_{,r}$ and $\phi_{\Lambda,r}$ in Eq. (3) and calculate f_r^{self}.

A. Massless scalar field

Here we calculate $\phi_{,r}$ along the spatial radial geodesic γ. The massless scalar field ϕ satisfies

$$\Box\phi = -4\pi\rho. \tag{4}$$

Here $\Box\phi \equiv \phi_{;\mu}{}^{\mu}$, and $\rho(x)$ is the scalar charge density. For a point particle this charge density is given by

$$\rho(x) = q\int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}}\delta^4[x - z(\tau)]d\tau, \tag{5}$$

where g denotes the determinant of the background metric. For a static world line Eq. (5) gives

$$\rho = \frac{q\sqrt{f_0}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0)}{r^2\sin\theta}. \tag{6}$$

Next, we solve Eq. (4) for the charge density given by Eq. (6); for this purpose we decompose ρ and ϕ into spherical harmonics:

$$\rho = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\rho^{lm}(r)Y^{lm}(\theta, \varphi), \tag{7}$$

$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\phi^{lm}(r)Y^{lm}(\theta, \varphi). \tag{8}$$

The coefficients in the charge density decomposition are given by

$$\rho^{lm} = q\frac{\delta(r - r_0)\sqrt{f_0}}{r^2}Y^{lm*}(\theta_0, \varphi_0). \tag{9}$$

The above mode decompositions together with Eq. (4) give the following set of decoupled ordinary differential equations (for a static source)

$$(r^2 f\phi^{lm}_{,r})_{,r} - l(l+1)\phi^{lm} = -4\pi r^2\rho^{lm}. \tag{10}$$

These equations have the following analytical solutions

$$\phi^{lm} = \frac{4\pi q\sqrt{f_0}}{M}Y^{lm*}(\theta_0, \varphi_0)P_l(z_<)Q_l(z_>). \tag{11}$$

Here $z \equiv \frac{r}{M} - 1$, $z_0 \equiv z(r_0)$; $z_>$ and $z_<$ denote the larger and smaller terms from the pair $\{z_0, z\}$, respectively; and P_l, Q_l denote Legendre functions of the first and second kind, respectively. We now substitute Eq. (11) into Eq. (8) and sum over the multipole number m which gives

$$\phi = \frac{q\sqrt{f_0}}{M}\sum_{l=0}^{\infty}(2l+1)P_l(z_<)Q_l(z_>)P_l(\cos\alpha). \tag{12}$$

Here,

$$\cos\alpha \equiv \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0).$$

Recall that here we are interested only in the solution along the radial geodesic γ. Along γ we have $\cos\alpha = 1$, which considerably simplifies the summation in Eq. (12), and we find that along this geodesic

$$\phi = q\frac{\sqrt{f_0}}{r - r_0}.$$

Finally, we differentiate this expression with respect to r and obtain

$$\phi_{,r} = -q\frac{\sqrt{f_0}}{(r - r_0)^2}. \tag{13}$$

B. Massive scalar field

We now calculate an approximate expression for $\phi_{\Lambda,r}$ along the radial geodesic γ. Note that for the implementation of the prescription summarized by Eq. (3), it is sufficient to have an approximate expression for $\phi_{\Lambda,r}$ in the vicinity of the particle's world line, and as $\Lambda \to \infty$. We therefore expand $\phi_{\Lambda,r}$ along γ in powers of $(r - r_0)$ and in powers of Λ^{-1}; and keep only terms that do not vanish as $r \to r_0$, and as $\Lambda \to \infty$.

The massive scalar field ϕ_Λ satisfies

$$(\Box - \Lambda^2)\phi_\Lambda = -4\pi\rho. \tag{14}$$

Here, the charge density ρ is given by Eq. (6)—the same charge density as in the massless field equation. Decomposing ϕ_Λ and ρ into spherical harmonics gives

$$\phi_\Lambda = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{r}Y^{lm}(\theta,\varphi)\phi_\Lambda^{lm}(r), \tag{15}$$

$$\rho = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{r}Y^{lm}(\theta,\varphi)\tilde{\rho}^{lm}(r), \tag{16}$$

where $\tilde{\rho}^{lm} = r\rho^{lm}$. From the massive field Eq. (14) together with the spherical harmonics decompositions (15) and (16), we obtain the following infinite set of decoupled ordinary differential equations for the spherical harmonics coefficients ϕ_Λ^{lm}:

$$\phi^{lm}_{\Lambda,r^*r^*} - [V^l(r) + \Lambda^2 f]\phi_\Lambda^{lm} = -4\pi\tilde{\rho}^{lm}f. \tag{17}$$

Here we introduced the tortoise coordinate r^* (see, e.g., [20]) defined by $\frac{dr*}{dr} = \frac{1}{f}$; we also defined $V^l(r) \equiv f(\frac{f'}{r} + \frac{l(l+1)}{r^2})$ and $f' \equiv \frac{df}{dr}$. We now express the solution of the inhomogeneous Eq. (17) in terms of two linearly independent solutions of the corresponding homogeneous equation. Denoting these homogeneous solutions with $\chi^l_\pm$: such that χ^l_+ vanishes as $r^* \to \infty$, and χ^l_- is regular as $r^* \to -\infty$, we find that

$$\phi_\Lambda^{lm} = \frac{-4\pi q\sqrt{f_0}Y^{lm*}(\theta_0,\phi_0)}{r_0}\frac{\chi^l_-(r_<)\chi^l_+(r_>)}{W[\chi^l_-,\chi^l_+]_{r_0}}. \tag{18}$$

Here $W[\chi^l_-,\chi^l_+] = \chi^l_{+,r^*}\chi^l_- - \chi^l_{-,r^*}\chi^l_+$ denotes the Wronskian, and $r_>$ and $r_<$ denote the larger and smaller terms from the pair $\{r_0, r\}$, respectively; the subscript r_0 indicates that the term in the square brackets is evaluated at $r = r_0$.

Next, we calculate the sum over the multipole number m in Eq. (15). Introducing $\phi^l_\Lambda \equiv \sum_{m=-l}^{l}\frac{1}{r}\phi_\Lambda^{lm}Y^{lm}$, and using Eq. (18) we find that along the spatial radial geodesic γ the multipoles ϕ^l_Λ are given by

$$\phi^l_\Lambda = \frac{-2qL\sqrt{f_0}}{rr_0}\frac{\chi^l_-(r_0)\chi^l_+(r)}{W[\chi^l_-,\chi^l_+]_{r_0}}, \tag{19}$$

where $L \equiv l + \frac{1}{2}$.

Next, we calculate an asymptotic expansion for $\phi_{\Lambda,r}$ as $\Lambda \to \infty$. This requires summation of the terms $\phi^l_{\Lambda,r}$, followed by an asymptotic expansion in powers of Λ^{-1}. First we consider the summation over the multipole number l

$$\phi_{\Lambda,r} = \sum_{l=0}^{\infty}\phi^l_{\Lambda,r}. \tag{20}$$

Recall that for our purposes it is sufficient to calculate $\phi_{\Lambda,r}$ along the radial geodesic γ, and up to $O[(r - r_0)^0]$ (inclusive). As discussed in Ref. [19] the field $\phi_{\Lambda,r}$ diverges on the particle's world line. However, the individual multipole terms $\phi^l_{\Lambda,r}(r_0)$, are finite on this world line,[1] only their sum over the multipole number l diverges there. To calculate this sum (off the world line) it is useful to split this sum into two parts: a sum which diverges at the limit $r \to r_0$, and a sum which is finite at this limit. This splitting simplifies the calculations, and allows us to use different calculation methods for the two sums. We therefore express $\phi_{\Lambda,r}$ as

$$\phi_{\Lambda,r} = \sum_{l=0}^{\infty}h^l + \sum_{l=0}^{\infty}(\phi^l_{\Lambda,r} - h^l). \tag{21}$$

This splitting is considered on the radial geodesic γ, and the functions $h_l(r)$ will be defined below, such that the second sum in Eq. (21) remains finite as $r \to r_0$ [see Eq. (31) below]. This requirement implies that the values of $h_l(r)$ in the vicinity of the world line must reflect the leading asymptotic expansion of $\phi^l_{\Lambda,r}(r)$ as $L \to \infty$. We comment here that a similar method for calculating a mode sum was devised in Ref. [11] (for a somewhat different purpose).

We shall now derive the required expressions for the functions $h^l(r)$. This requires an analysis of the asymptotic behavior of $\phi^l_{\Lambda,r}(r)$ as $L \to \infty$ in the vicinity of the world line. For this purpose we use the WKB approximation, which enables us to calculate an asymptotic expansion of the solutions of Eq. (17) in inverse powers of L (or in

[1]For an approximated expression for these multipoles see Eq. (25) below.

inverse powers of Λ). Here there is a problem though, since the WKB approximation is invalid in the vicinity of the black-hole horizon. This can easily be inferred from Eq. (17) which has a "turning point" at the event horizon [i.e., the term in the brackets in Eq. (17) vanishes at $r = 2M$]. Therefore, we cannot impose a boundary condition at $r = 2M$ on the (WKB approximated) solution χ^l_-. We shall therefore consider a general approximated solution for χ^l_- in the region where the WKB approximation is valid, without imposing any boundary condition. We shall eventually deal with this boundary condition problem in the appendix, by considering a different approximation method. For the accuracy required by the calculations below it will be sufficient to keep the first three leading terms in the WKB approximation, which reads

$$\chi^l_+ \approx e^{-S_0+S_1-S_2}, \tag{22}$$

$$\chi^l_- \approx c_1 e^{+S_0+S_1+S_2} + c_2 e^{-S_0+S_1-S_2}. \tag{23}$$

Here c_1 and c_2 are independent of r^*, and the functions S_i, $(i = 0, 1, 2)$ are given by (see, e.g., [21])

$$S_0 = \int_{2M}^{r} f^{-1}\sqrt{U}dr', \qquad S_1 = -\frac{1}{4}\ln U,$$
$$S_2 = \int_{2M}^{r} \frac{1}{f}\left(\frac{f\partial_{r'}(fU')}{8U^{3/2}} - \frac{5(fU')^2}{32U^{5/2}}\right)dr', \tag{24}$$

where $U(r') \equiv V^l(r') + \Lambda^2 f(r')$, and $U' \equiv \frac{dU}{dr'}$. Examining the asymptotic properties of the functions $\partial_r S_i$, we find that as $L \to \infty$ (for a fixed Λ) the functions $\partial_r S_i$ are $O(L^{1-i})$; and as $\Lambda \to \infty$ (for a fixed L) the functions $\partial_r S_i$ are $O(\Lambda^{1-i})$. We now substitute Eqs. (22) and (23) into Eq. (19), and differentiate with respect to r, this gives

$$\phi^l_{\Lambda,r} \approx \frac{q}{\sqrt{f_0}rr_0}\frac{Le^S}{[S_{0,r} + S_{2,r}]_{r_0}}\left(S_{,r} - \frac{1}{r}\right), \tag{25}$$

where,

$$S \equiv -[S_0(r) - S_0(r_0)] + [S_1(r) - S_1(r_0)] - [S_2(r) - S_2(r_0)]. \tag{26}$$

This WKB approximation is accurate up to $O(L^{-1})$ [and up to $O(\Lambda^{-1})$]. At the limit where $L \to \infty$ (or $\Lambda \to \infty$) the contributions to Eq. (25) from the term which contains the coefficients c_1 and c_2 [originating in Eq. (23)] vanishes faster then any negative power of L (or Λ), and therefore this contribution was neglected here (see the appendix for details).

To find the asymptotic expansion for $\phi^l_{\Lambda,r}$, as $L \to \infty$, we first calculate the leading asymptotic expansion of S. Expanding S gives

$$S = -\alpha L + O(L^0). \tag{27}$$

Employing Eqs. (24), (26), and (27) we find that α is approximated by

$$\alpha = \frac{r - r_0}{\sqrt{f_0}r_0} - \frac{(r - r_0)^2(r_0 - M)}{2r_0^3 f_0^{3/2}} + O[(r - r_0)^3]. \tag{28}$$

Equation (27) implies that as a function of L the term e^S in Eq. (25) has an essential singularity at infinity. To deal with this singularity we simply multiply Eq. (25) by $e^{-\alpha L}e^{\alpha L}$, and obtain

$$\phi^l_{\Lambda,r} \approx e^{-\alpha L}\left[\frac{q}{\sqrt{f_0}rr_0}\frac{Le^{S+\alpha L}}{[S_{0,r} + S_{2,r}]_{r_0}}\left(S_{,r} - \frac{1}{r}\right)\right]. \tag{29}$$

Note that no further approximation was made here, since we merely multiplied Eq. (25) by unity. However, in this form the term in the square brackets does not contain an essential singularity as $L \to \infty$, and therefore this term can be expanded in powers of L without difficulties. Note that higher orders in the expansion of S do not give rise to a similar essential singularity, and therefore do not require special attention. Moreover, higher orders in the expansion of α in Eq. (28) will have a vanishing contribution at the limit $r \to r_0$ (see below), and are not required here. An expansion of the term in the square brackets in Eq. (29) reads

$$\frac{q}{\sqrt{f_0}rr_0}\frac{Le^{S+\alpha L}}{[S_{0,r} + S_{2,r}]_{r_0}}\left(S_{,r} - \frac{1}{r}\right) = A(r)L + B(r) + \frac{C(r)}{L} + O(L^{-2}). \tag{30}$$

Here the coefficients $A(r)$,$B(r)$, and $C(r)$ are independent of L. We now define the functions $h^l(r)$ by multiplying the asymptotic expansion in Eq. (30) by $e^{-\alpha L}$, which gives

$$h^l(r) \equiv e^{-\alpha(r)L}\left[A(r)L + B(r) + \frac{C(r)}{L}\right]. \tag{31}$$

Having this definition, the functions $h^l(r)$ coincide with the leading asymptotic expansion of $\phi^l_{\Lambda,r}(r)$ in the vicinity of the world line, as required. Moreover, the difference $(\phi^l_{\Lambda,r} - h^l)$, when evaluated on the world line is $O(L^{-2})$. Therefore, the second sum in Eq. (21) converges on the particle's world line, as required. We comment that at the limit $r \to r_0$ the coefficients $A(r)$, $B(r)$, $C(r)$ coincide with the first three mode-sum regularization parameters introduced by Barack and Ori in Ref. [11] (for the particular problem which is considered here).

Next, we calculate the first sum in Eq. (21). Employing Eq. (31) we obtain

$$\sum_{l=0}^{\infty} h^l = \sum_{l=0}^{\infty} e^{-\alpha L}\left[AL + B + \frac{C}{L}\right]. \tag{32}$$

Note that for $r > r_0$ this sum converges due to the factor $e^{-\alpha L}$. Summing separately over the various terms (without the coefficients) in the square brackets in Eq. (32) gives

$$\sum_{l=0}^{\infty} Le^{-\alpha L} = \frac{\cosh(\alpha/2)}{4\sinh^2(\alpha/2)}, \qquad \sum_{l=0}^{\infty} e^{-\alpha L} = \frac{2}{\sinh(\alpha/2)},$$
$$\sum_{l=0}^{\infty} L^{-1}e^{-\alpha L} = 2\mathrm{arctanh}(e^{-\alpha/2}). \tag{33}$$

Equation (28) implies that as $r \to r_0$ the first sum in Eq. (33) is $O[(r-r_0)^{-2}]$, the second sum is $O[(r-r_0)^{-1}]$, and the third sum is $O\{\log[(r-r_0)^{-1}]\}$. Our calculation has to be accurate up to $O[(r-r_0)^0]$, and therefore the coefficients $A(r)$, $B(r)$, $C(r)$ in Eq. (32), and the function $\alpha(r)$ have to be accurate up to $[O(r-r_0)^2]$. We now substitute Eq. (33) in Eq. (32), and calculate the coefficients A, B, and C to the required order. For this calculation we use Eqs. (24), (26), (28), and (30). We find that the first sum in Eq. (21) is given by [22]

$$\sum_{l=0}^{\infty} h^l = -\frac{q\sqrt{f_0}}{(r-r_0)^2} + \frac{q}{2\sqrt{f_0}}\Lambda^2 + O(r-r_0). \tag{34}$$

Next, we consider the second sum in Eq. (21) in the vicinity of the world line:

$$\sum_{l=0}^{\infty}[\phi^l_{\Lambda,r}(r) - h^l(r)] = \sum_{l=0}^{\infty}[\phi^l_{\Lambda,r}(r_0) - h^l(r_0)] + O(r-r_0). \tag{35}$$

Recall that we only need the asymptotic form (as $\Lambda \to \infty$) of this expression. Introducing the notations $y^l \equiv l/\Lambda$, $y^{\bar{l}} \equiv L/\Lambda$, and $\phi_{\Lambda,r}(L,r) \equiv \phi^l_{\Lambda,r}(r)$, $h(L,r) \equiv h^l(r)$. We find that as $\Lambda \to \infty$ the sum in Eq. (35) can be approximated with a Riemann integral

$$\Lambda \sum_{l=0}^{\infty}[\phi_{\Lambda,r}(y^{\bar{l}}\Lambda, r_0) - h(y^{\bar{l}}\Lambda, r_0)](y^{l+1} - y^l)$$
$$\approx \Lambda \int_0^{\infty}[\phi_{\Lambda,r}(\Lambda y, r_0) - h^l(\Lambda y, r_0)]dy. \tag{36}$$

We now substitute the WKB approximation given by Eq. (25) into the integral in Eq. (36). Note that contributions to $\phi_{\Lambda,r}(y^{\bar{l}}\Lambda, r_0)$ from the functions S_i with $i \geq 3$ are convergent upon summation over the multipole number l. However, these sums vanish as $\Lambda \to \infty$, and therefore it is sufficient to keep only the first three leading terms in the WKB approximation. We now substitute Eq. (25) into Eq. (36) and expand the result in inverse powers of Λ [technically it is useful to substitute $U(r) = U_0 + \Delta U(r)$ into Eq. (25), where $U_0 \equiv U(r_0)$, and formally expand this expression with respect to U_0, and keep only terms that will eventually give a nonvanishing contribution as $\Lambda \to \infty$] we find that in the vicinity of the particle's world line the second sum in Eq. (21) is given by [22]

$$\sum_{l=0}^{\infty}(\phi^l_{\Lambda,r} - h^l) = \frac{qM}{2r_0^2 f_0}\Lambda + O(\Lambda^{-1}) + O(r-r_0). \tag{37}$$

Substituting Eqs. (34) and (37) in Eq. (21) gives

$$\phi_{\Lambda,r} = -\frac{q\sqrt{f_0}}{(r-r_0)^2} + \frac{q}{2}(\Lambda a_r + \Lambda^2 n_r) + O(r-r_0) + O(\Lambda^{-1}). \tag{38}$$

Here $n_r \equiv f_0^{-1/2}$ is the radial component of the unit spatial vector n_μ, and $a_r \equiv \frac{M}{r_0^2 f_0}$ is the radial component of the four-acceleration of the static particle in Schwarzschild.

C. The radial self-force

We now calculate the radial self-force by implementing the prescription given by Eq. (3). First we calculate the field $\Delta\phi_{,r}(r)$ in the vicinity of the particle's world line. From Eqs. (13) and (38) we obtain

$$\Delta\phi_{,r} = -\frac{q}{2}(\Lambda a_r + \Lambda^2 n_r) + O(r-r_0) + O(\Lambda^{-1}). \tag{39}$$

Note that the field $\Delta\phi_{,r}(r)$ remains finite as $r \to r_0$ (while the fields $\phi_{,r}$ and $\phi_{\Lambda,r}$ diverge at this limit)—this is a general property of $\Delta\phi_{,r}(r)$ (see Ref. [19]). The cancellation of the divergent terms allows us to take the limit $r \to r_0$ of $\Delta\phi_{,r}(r)$. Following Eq. (3) we add the term $\frac{q}{2} \times [\Lambda a_r(z_0) + \Lambda^2 n_r(z_0)]$ to this limit. This added term exactly annihilates the $O(\Lambda)$ and $O(\Lambda^2)$ terms in our expression—this is a general property as well (see Ref. [19]). We now complete this calculation by taking the limit $\Lambda \to \infty$ in Eq. (3), which gives

$$f_r^{\mathrm{self}}(z_0) = q\lim_{\Lambda\to\infty}\left\{\lim_{r\to r_0}\Delta\phi_{,r}(r) + \frac{1}{2}q[\Lambda^2 n_r(z_0) + \Lambda a_r(z_0)]\right\} = 0. \tag{40}$$

Combining this result with the results for the other components of the self-force, we conclude that the scalar self-force on a static particle in Schwarzschild vanishes.

ACKNOWLEDGMENTS

I am grateful to Amos Ori for valuable discussions.

APPENDIX: GREEN-LIOUVILLE APPROXIMATION

Here I shall derive Eq. (25). In particular I shall provide justification for neglecting the term which depends on c_1 and c_2 in the expression for $\phi^l_{\Lambda,r}$. As discussed in Sec. II B, the WKB approximation is invalid in the vicinity of the event horizon. This situation prevents us from patching the WKB approximation for the homogeneous solution χ^l_- to a regular boundary condition at the event horizon. In the region where the WKB approximation is valid, the solution χ^l_- is given by Eq. (23). By substituting Eqs. (22) and (23) into Eq. (19), and differentiating with respect to r, we find that

$$\phi^l_{\Lambda,r} \approx \frac{q}{\sqrt{f_0} r r_0} \frac{L e^S}{[S_{0,r} + S_{2,r}]_{r_0}} \left(S_{,r} - \frac{1}{r} \right) \left(1 + \frac{c_2}{c_1} e^{-2S_0 - 2S_2} \right). \tag{A1}$$

We now focus on the contribution from the last term in the last brackets, and explain why this term can be neglected here. For this we follow an analysis by Rowan and Stephenson [23], who showed that by using the Green-Liouville approximation method, one can obtain an approximate solution to Eq. (17), which is valid in the entire $r \geq 2M$ region. First, we express the homogeneous wave equation for $\chi^l_\pm$ as

$$[r^2 f \chi^l_{,r}]_{,r} - [l(l+1) + \Lambda^2 r^2]\chi^l = 0. \tag{A2}$$

Next we introduce $x = (r/M)f$, $N = M\Lambda$ and make the following transformations from x to ξ:

$$\xi'^2 = \beta^2 \left(\frac{2+x}{x} \right) + \frac{\alpha^2}{x(x+2)}. \tag{A3}$$

Here, $\xi' = (\frac{d\xi}{dx})$, $\beta^2 = N^2/k^2$, $\alpha^2 = [l(l+1)]/k^2$, and $k^2 = N^2 + l(l+1)$. Note that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. The new variable ξ is defined by

$$\xi = \int_0^x |\xi'| dx. \tag{A4}$$

Next, we define ψ^l such that

$$\psi^l = \sqrt{\xi' x(x+2)} \chi^l. \tag{A5}$$

Equation (A2) now reads

$$\psi^l_{,\xi\xi} - [k^2 - (2\xi)^{-2} + g_1(\xi)]\psi^l = 0. \tag{A6}$$

Here $g_1(\xi)$ is a slowly varying function of ξ, which is bounded everywhere. This function is $O(1)$ near the event horizon, and it is $O(\xi^{-2})$ as $\xi \to \infty$. The full expression for $g_1(\xi)$ (see [23]) is not required here.

For large values of k (which correspond to large values of N and/or large values of l) Eq. (A6) can be solved using perturbation analysis. The leading order in this approximation is obtained by completely neglecting the small contribution from the function $g_1(\xi)$ in this equation. The remaining equation can be solved exactly, giving two linearly independent solutions, from which the corresponding $\chi^l_\pm$ can be calculated. These approximate homogeneous solutions are given by

$$\chi^l_+ \approx \sqrt{\xi}[\xi' x(x+2)]^{-1/2} K_0(k\xi),$$
$$\chi^l_- \approx \sqrt{\xi}[\xi' x(x+2)]^{-1/2} I_0(k\xi).$$

Here I_0 and K_0 are the modified Bessel functions of the first and second kind, respectively. By continuing the perturbation analysis to higher orders we find that the approximate solution χ^l_- at any (fixed) order has the following asymptotic form (as $k \to \infty$)

$$\chi^l_- \approx F_-(k,\xi) e^{k\xi} + F_+(k,\xi) e^{-k\xi}. \tag{A7}$$

For $\xi \neq 0$ the functions $F_\pm(k,\xi)$ vanish at the limit $k \to \infty$. At the domain where both WKB and Green-Liouville approximations are valid, we equate Eq. (A7) with the corresponding WKB approximation [i.e., with Eq. (23), extended to the required order]. Using this equation (for an arbitrary fixed order) we find that the term $\frac{c_2}{c_1} e^{-2S_0 - 2S_2}$ approaches zero faster then any negative power of k, as $k \to \infty$. We therefore conclude that the last term in the last brackets in Eq. (A1) can be neglected in our calculation. Note that the above mentioned equation can only provide us with a bound (which is sufficient for our purpose). This equation, however, cannot be used to determine the value of the coefficient c_2 (though it can be used to determine c_1 up to a given order). The difficulty with the coefficient c_2 is that it multiplies a subdominant term—a term which vanishes exponentially.

I comment here that the entire perturbation analysis can be done with Green-Liouville approximation; this method has the advantage of being valid in the entire $r \geq 2M$ region. However, we find that the WKB approximation is simpler, especially for the higher orders which are required by this analysis.

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