Disappearance of the black hole singularity in loop quantum gravity

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We apply techniques recently introduced in quantum cosmology to the Schwarzschild metric inside the horizon and near the black hole singularity at $r = 0$. In particular, we use the quantization introduced by Husain and Winkler, which is suggested by Loop Quantum Gravity and is based on an alternative to the Schrödinger representation introduced by Halvorson. Using this quantization procedure, we show that the black hole singularity disappears and spacetime can be dynamically extended beyond the classical singularity.

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I. INTRODUCTION

A remarkable result of loop quantum cosmology [1] is the disappearance of the initial cosmological singularity present in the classical theory. The main results of loop quantum gravity [2], indeed, are the quantization of area and volume partial observables [3], which suggest that in the complete theory there cannot be spacetime points with infinity matter density. If this is correct, the quantum theory should control all classical singularities of general relativity. In this work, we apply techniques analogous to the ones used in loop quantum cosmology to study the $r = 0$ singularity in the interior of a Schwarzschild black hole.

In particular, we use the non-Schrödinger procedure of quantization introduced by Halvorson [4] and utilized in quantum cosmology by Husain and Winkler [5]. We focus on the Schwarzschild solution inside the horizon and near the singularity. We use the method introduced in [6] to express $1/r$, and therefore the curvature invariant $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = 48M^2G_N^2/r^6$, in terms of the volume operator. Following [6], we write the Hamiltonian constraint as well in terms of the volume. This allows us to express the quantum evolution equation as a difference equation for the coefficients for the physical states, and to completely control the singularity.

The paper is organized as follow. In Sec. II we briefly recall the properties of the Schwarzschild solution for $r < 2MG_N$, namely, inside the horizon. As well known, here the temporal and spatial (radial) coordinate exchange their role. In Sec. III we study the classical dynamics of a very simple model giving this solution. The Hamiltonian constraint depends on a single variable, and its classical solution yields the Schwarzschild metric inside the horizon, in the new temporal variable. In Sec. IV we quantize the system using the non-Schrödinger procedure of quantization of references [4,5]. In particular, we show that the singularity in $r = 0$ is resolved in quantum gravity and that the Hamiltonian constraint acts like a difference operator, as in loop quantum cosmology.

II. THE SCHWARZSCHILD SOLUTION INSIDE THE HORIZON

Consider the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2MG_N}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2MG_N}{r})} + r^2(\sin^2\theta d\phi^2 + d\theta^2) \tag{1}$$

for $r < 2MG_N$. This metric describe spacetime inside the horizon of a Schwarzschild black hole. The coordinate r is timelike and the coordinate t is spatial; for convenience we rename them as $r \equiv T$ and $t \equiv r$ with $T \in]0, 2MG_N[$ and $r \in]-\infty, +\infty[$. The metric reads then

$$ds^2 = -\frac{dT^2}{\left(\frac{2MG_N}{T} - 1\right)} + \left(\frac{2MG_N}{T} - 1\right)dr^2 + T^2(\sin^2\theta d\phi^2 + d\theta^2). \tag{2}$$

We eliminate the coefficient of dT^2 by defining a new temporal variable τ via

$$d\tau = \frac{dT}{\sqrt{\frac{2MG_N}{T} - 1}}. \tag{3}$$

The integration gives

$$\tau = -\sqrt{T(2MG_N - T)} + 2MG_N \arctan\left(\sqrt{\frac{T}{2MG_N - T}}\right) + \text{const.} \tag{4}$$

We take $\text{const} = 0$ because $\lim_{T\to 0}\tau(T) = \text{const}$. The function $T = T(\tau)$ is monotonic and convex, thus $\tau \in]0, 2MG_N\pi/2[$. In this new temporal variable the metric becomes

$$ds^2 = -d\tau^2 + \left(\frac{2MG_N}{T(\tau)} - 1\right)dr^2 + T(\tau)^2(\sin^2\theta d\phi^2 + d\theta^2). \tag{5}$$

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We introduce two function $a^2(\tau) \equiv \frac{2MG_N}{T(\tau)} - 1$ and $b^2(\tau) \equiv T^2(\tau)$ and redefine $\tau \equiv t$. The metric reads

$$ds^2 = -dt^2 + \left(\frac{2MG_N}{b(t)} - 1\right)dr^2 + b(t)^2(\sin^2\theta d\phi^2 + d\theta^2). \tag{6}$$

Notice that a metric written in terms of two functions $a(t)$ and $b(t)$ with the form

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)(\sin^2\theta d\phi^2 + d\theta^2) \tag{7}$$

is the metric of an homogeneous, anisotropic space with spatial section of topology $\mathbf{R} \times \mathbf{S}^2$. In our case, $a(t)$ is a function of $b(t)$, $a = a[b(t)]$.

III. CLASSICAL THEORY

The complete action for gravity can be written in the form

$$S = \frac{1}{16\pi G_N}\int d^3x dt N h^{1/2}[K_{ij}K^{ij} - K^2 + ^{(3)}\mathrm{R}] \tag{8}$$

If we specialize this for metrics of the form (7), the action becomes [7]

$$\begin{aligned} S = &-\frac{1}{16\pi G_N}\int dt \int_0^R dr \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \sin\theta b^2 a\Big[2\frac{\dot{b}^2}{b^2} + 4\frac{\dot{a}\,\dot{b}}{ab} - \frac{2}{b^2}\Big] = \\ &= -\frac{R}{2G_N}\int dt[a\dot{b}^2 + 2\dot{a}\,\dot{b}\,b - a], \end{aligned} \tag{9}$$

where R is a cutoff on the space radial coordinate. We can work also with radial densities because the model is homogeneous and all the following results remain identical. In another way, the spatial homogeneity enable us to fix a linear radial cell $\mathcal{L}_r$ and restrict all integrations to this cell [1]. Recalling from (6) that the two functions $a(t)$ and $b(t)$ are not independent, and satisfy

$$a^2(t) = \frac{2MG_N}{b(t)} - 1, \tag{10}$$

we can write the action in terms of a single function

$$S = \frac{R}{2G_N}\int dt\Big[\frac{\sqrt{b}}{\sqrt{2MG_N}}\Big(1 - \frac{b}{2MG_N}\Big)^{-1/2}\dot{b}^2 + \frac{\sqrt{2MG_N}}{\sqrt{b}}\Big(1 - \frac{b}{2MG_N}\Big)^{1/2}\Big]. \tag{11}$$

Now we calculate the Hamiltonian, which is also the Hamiltonian constraint (see Appendix B). The momentum is

$$p = \frac{R\sqrt{b}}{G_N\sqrt{2MG_N}}\Big(1 - \frac{b}{2MG_N}\Big)^{-1/2}\dot{b}, \tag{12}$$

and so

$$\begin{aligned} H &= p\dot{b} - L \\ &= \Big(\frac{G_N p^2}{2R} - \frac{R}{2G_N}\Big)\Big[\frac{\sqrt{2MG_N}}{\sqrt{b}}\Big(1 - \frac{b}{2MG_N}\Big)^{1/2}\Big]. \end{aligned} \tag{13}$$

We can now show that the Hamiltonian constraint produce the correct classical dynamics. We express the Hamiltonian constraint in terms of $\dot{b}$

$$H = \frac{R}{G_N}\left[\frac{\dot{b}^2}{\sqrt{\frac{2MG_N}{b} - 1}} - \sqrt{\frac{2MG_N}{b} - 1}\right] = 0. \tag{14}$$

The solution is

$$\dot{b}^2 = \Big(\frac{2MG_N}{b} - 1\Big) \tag{15}$$

and this is exactly the Eq. (3) with solution (4) that reproduces the Schwarzschild metric.

We now introduce an approximation. In the quantum theory, we will be interest in the region of the scale the Planck length l_p around the singularity. We assume that the Schwarzschild radius $r_s \equiv 2MG_N$ is much larger than this scale, and that $b(t) = T(t)$. In this approximation we can write

$$1 - \frac{b}{2MG_N} \sim 1 \tag{16}$$

and H becomes

$$H = \Big(\frac{G_N p^2}{2R} - \frac{R}{2G_N}\Big)\frac{\sqrt{2MG_N}}{\sqrt{b}}. \tag{17}$$

The volume is

$$\begin{aligned} V &= \int dr d\phi d\theta h^{1/2} = 4\pi R a b^2 \\ &= 4\pi R\sqrt{2MG_N}b^{3/2}\sqrt{1 - \frac{b}{2MG_N}}; \end{aligned} \tag{18}$$

in the previously approximation

$$V = 4\pi R\sqrt{2MG_N}b^{3/2} \equiv l_o b^{3/2}. \tag{19}$$

The canonical pair is given by $b \equiv x$ and p, with Poisson brackets $\{x, p\} = 1$.

We now assume that $x \in \mathbb{R}$ (and introduce the absolute value where appropriate). This choice is not correct classically, because for $b \equiv x = 0$ we have the singularity. But it allows us to open the possibility that the situation be different in the quantum theory. We introduce an algebra of classical observables, and we write the quantities of physical interest in terms of those variables. We are motivated by loop quantum gravity to use the fundamental variables x and

$$U_\gamma(p) \equiv \exp\left(\frac{8\pi G_N \gamma}{L} ip\right) \tag{20}$$

where γ is a real parameter and L fixes the unit of length. The parameter γ is necessary to separate the momentum point in the phase space. (Choosing $8\pi G_N \gamma/L = 1$ we obtain the some value of U for p and $p + 2\pi n$). This variable can be seen as the analog of the holonomy variable of loop quantum gravity.

A straightforward calculation gives

$$\begin{aligned} \{x, U_\gamma(p)\} &= 8\pi G_N \frac{i\gamma}{L} U_\gamma(p), \\ U_\gamma^{-1}\{V^n, U_\gamma\} &= l_0^n U_\gamma^{-1}\{|x|^{3n/2}, U_\gamma\} \\ &= i8\pi G_N l_0^n \frac{\gamma}{L}\frac{3n}{2}\mathrm{sgn}(x)|x|^{(3n/2)-1}. \end{aligned} \tag{21}$$

These formulas allow us to express inverse powers of x in terms of a Poisson bracket, following Thiemann's trick [6]. As we will see below, the volume operator has zero as an eigenvalue, therefore so we must take $n \geqslant 0$ for the second equation to be well define din the quantum theory. On the other hand, if we want that the power of x on the right hand side be negative we need $n \leqslant 2/3$. The choice $n = 1/3$ gives

$$\frac{\mathrm{sgn}(x)}{\sqrt{|x|}} = -\frac{2Li}{(8\pi G_N) l_0^{1/3}\gamma} U_\gamma^{-1}\{V^{1/3}, U_\gamma\}. \tag{22}$$

We use this relation in the next section to write physical operators. We are interested to the quantity $\frac{1}{|x|}$ because classically this quantity diverge for $|x| \to 0$ and produce the singularity. We are also interested to the Hamiltonian constraint and the dynamics and we will use (22) for writing the Hamiltonian.

IV. QUANTUM THEORY

We construct the quantum theory proceeding in analogy with the procedure used in loop quantum gravity. The first step is the choice of an algebra of classical functions to be represented as quantum configuration operators. We choose the algebra generated by the functions

$$W(\lambda) = e^{i\lambda x/L}, \tag{23}$$

where $\lambda \in \mathbb{R}$. The algebra consists of all function of the form

$$f(x) = \sum_{j=1}^{n} c_j e^{i\lambda_j x/L}, \tag{24}$$

where $c_j \in \mathbb{C}$, and their limits with respect to the sup norm. This is the algebra $AP(\mathbb{R})$ of the *almost periodic functions* over $\mathbb{R}$. The algebra $AP(\mathbb{R})$ is isomorphic to $C(\bar{\mathbb{R}}_{\mathrm{Bohr}})$, the algebra of continuous functions on the Bohr-compactification of $\mathbb{R}$. This suggests to take the Hilbert space $L_2(\bar{\mathbb{R}}_{\mathrm{Bohr}}, d\mu_0)$, where $d\mu_0$ is the Haar measure on $\bar{\mathbb{R}}_{Bohr}$. With this choice the basis states in the Hilbert space are

$$|\lambda\rangle \equiv |e^{i\lambda x/L}\rangle, \qquad \langle\mu|\lambda\rangle = \delta_{\mu,\lambda}. \tag{25}$$

The action of the configuration operators $\hat{W}(\lambda)$ on the basis is defined by

$$\hat{W}(\lambda)|\mu\rangle = e^{i\lambda\hat{x}/L}|\mu\rangle = e^{i\lambda\mu}|\mu\rangle. \tag{26}$$

These operators are weakly continuous in λ. This implies the existence of a self-adjoint operator $\hat{x}$, acting on the basis states according to

$$\hat{x}|\mu\rangle = L\mu|\mu\rangle. \tag{27}$$

Next, we introduce the operator corresponding to the classical momentum function $U_\gamma = e^{i8\pi G_N \gamma p/L}$. We define the action of $\hat{U}_\gamma$ on the basis states using the definition (27) and using a quantum analog of the Poisson bracket between x and U_γ

$$\hat{U}_\gamma|\mu\rangle = |\mu - \gamma\rangle, \qquad [\hat{x}, \hat{U}_\gamma] = -\gamma L \hat{U}_\gamma. \tag{28}$$

Using the standard quantization procedure $[,] \to i\hbar\{,\}$, and using (21) we obtain

$$-\gamma L = i\hbar(8\pi G_N)\frac{i\gamma}{L}, \qquad L = \sqrt{8\pi} l_p. \tag{29}$$

A. Volume operator and disappearance of the singularity

Near the singularity we can use the approximation (19). The action of the volume operator on the basis states is

$$\hat{V}|\mu\rangle = l_0|x|^{3/2}|\mu\rangle = l_0|L\mu|^{3/2}|\mu\rangle. \tag{30}$$

Recall that the dynamics is all in the function $b(t)$, which is equal to the radial Schwarzschild coordinate inside the horizon $b(t) = T(t)$, because inside the horizon we can change coordinates from t to τ of Eq. (4) (we remember the redefinition $\tau \to t$). The function $b(t)$ generated by the dynamics is monotonic and convex. The important point is that $b(t = 0) = 0$ and this is the Schwarzschild singularity. We now show that the term $(2MG_N)/b(t)$ does not diverge in the quantum theory and therefore there is no singularity in the quantum theory.

We use the relation (22) and we promote the Poisson brackets to commutators. In this way we obtain (for $\gamma = 1$) the operator

$$\widehat{\frac{1}{|x|}} = \frac{1}{2\pi l_p^2 l_0^{2/3}}(\hat{U}^{-1}[\hat{V}^{1/3}, \hat{U}])^2. \tag{31}$$

The action of this operator on the basis states is

$$\widehat{\frac{1}{|x|}}|\mu\rangle = \sqrt{\frac{2}{\pi l_p^2}}(|\mu|^{1/2} - |\mu - 1|^{1/2})^2|\mu\rangle. \tag{32}$$

We can now see that the spectrum is bounded from below and so we have not singularity in the quantum theory. In fact, for example, the curvature invariant

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = \frac{48M^2G_N^2}{r^6} \equiv \frac{48M^2G_N^2}{T^6} = \frac{48M^2G_N^2}{T(t)^6} \equiv \frac{48M^2G_N^2}{b(t)^6} \tag{33}$$

is finite in quantum mechanics in fact the eigenvalue of $1/|x|$ for the state $|0\rangle$ corresponds to the classical singularity and in the quantum case it is $\sqrt{2/\pi l_p^2}$, which is the largest possible eigenvalue. For this particular value the curvature invariant it is not infinity

$$\mathcal{R}_{\mu\nu\rho\sigma}\widehat{\mathcal{R}^{\mu\nu\rho\sigma}}|0\rangle = \widehat{\frac{48M^2G_N^2}{|x|^6}}|0\rangle = \frac{384M^2G_N^2}{\pi^3 l_P^6}|0\rangle. \tag{34}$$

On the other hand, for $|\mu| \to \infty$ the eigenvalues go to zero, which is the expected behavior of $1/|x|$ for large $|x|$.

B. Hamiltonian constraint

We now study the quantization of the Hamiltonian constraint near the singularity, in the approximation (17). There is no operator p in quantum representation that we have chosen, hence we choose the following alternative representation for p^2. Consider the classical expression

$$p^2 = \frac{L^2}{(8\pi G_N)^2}\lim_{\gamma\to 0}\left(\frac{2 - U_\gamma - U_\gamma^{-1}}{\gamma^2}\right). \tag{35}$$

We have can give a physical interpretation to γ as $\gamma = l_p/L_{phys}$, where L_{Phys} is the characteristic size of the system. Using this, we write the Hamiltonian constraint as

$$\hat{H} = \frac{A_1}{l_0^{1/3}}[\hat{U}_\gamma + \hat{U}_\gamma^{-1} - (2 - A_2)1]\mathrm{sgn}(x)(\hat{U}^{-1}[\hat{V}^{1/3}, \hat{U}]) \tag{36}$$

where $A_1 = L^3G_N/(8\pi G_N)^{5/2}\gamma^3 R l_0^{1/3}\hbar$ and $A_2 = 8\pi R^2\gamma^2/l_P^2$. The action of $\hat{H}$ on the basis states is

$$\hat{H}|\mu\rangle = C\mathcal{V}_{\frac{1}{2}}(\mu)[|\mu - \gamma\rangle + |\mu + \gamma\rangle - (2 - C')|\mu\rangle], \tag{37}$$

where $C = A_1 L^{1/2}$ and $C' \equiv A_2$, and

$$\mathcal{V}_{1/2}(\mu) = \begin{cases} -||\mu - \gamma|^{1/2} - |\mu|^{1/2}| & \text{for} \quad \mu \neq 0 \\ |\gamma|^{1/2} & \text{for} \quad \mu = 0 \end{cases} \tag{38}$$

If we calculate the action of $\hat{H}$ and $1/|x|$ on the state of zero volume eigenvalue we obtain

$$\hat{H}|0\rangle = C|\gamma|^{1/2}[| - \gamma\rangle + |\gamma\rangle - (2 - C')|0\rangle], \quad \widehat{\frac{1}{|x|}}|0\rangle = \sqrt{\frac{2}{\pi l_P^2}}|0\rangle. \tag{39}$$

This finite value of $\frac{1}{|x|}$ can be interpreted as the effect of the quantization on the classical singularity.

We now study the solution of the Hamiltonian constraint. The solutions are in the $C^\star$ space that is the dual of the dense subspace C of the kinematical space $\mathcal{H}$. A generic element of this space is

$$\langle\psi| = \sum_\mu \psi(\mu)\langle\mu|. \tag{40}$$

The constraint equation $\hat{H}|\psi\rangle = 0$ is now interpreted as an equation in the dual space $\langle\psi|\hat{H}^\dagger$; from this equation we can derive a relation for the coefficients $\psi(\mu)$

$$\mathcal{V}_{1/2}(\mu + \gamma)\psi(\mu + \gamma) + \mathcal{V}_{1/2}(\mu - \gamma)\psi(\mu - \gamma) - (2 - C')\mathcal{V}_{1/2}(\mu)\psi(\mu) = 0. \tag{41}$$

This relation determines the coefficients for the physical dual state. We can interpret this states as describing the *quantum spacetime* near the singularity. From the difference Eq. (41) we obtain physical states as combinations of a countable number of components of the form $\psi(\mu + n\gamma)|\mu + n\gamma\rangle$ ($\gamma \sim l_P/L_{\mathrm{Phys}} \sim 1$); any component corresponds to a particular value of volume, so we can interpret $\psi(\mu + \gamma)$ as the wave function describing the black hole near the singularity at the time $\mu + \gamma$. A solution of the Hamiltonian constraint corresponds to a linear combination of black hole states for particular values of the volume or equivalently particular values of the time.

V. CONCLUSIONS

We have applied the quantization procedure of [5] to the case of the Schwarzschild singularity. This procedure is alternative to the Schrödinger quantization and it is suggested by loop quantum cosmology. The main results are:

(1) The classical black hole singularity near $r \sim 0$, which in our coordinate is $b(t) \equiv T(t) \sim 0$, disappears from the quantum theory. Classical divergent quantities are bounded in the quantum theory. For instance:

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = \frac{48M^2G_N^2}{b(t)^6} \to \widehat{\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}}|0\rangle = \frac{48\widehat{M^2G_N^2}}{|x|^6}|0\rangle = \frac{384M^2G_N^2}{\pi^3 l_P^6}|0\rangle.$$

(2) The quantum Hamiltonian constraint gives a discrete difference equation for the coefficients of the physical states.

It is interesting to observe that beyond the classical singularity the function $b \equiv x$ is negative. One can speculate, extrapolating the form of the metric that "on the other side" of the singularity there is no horizon: a black hole and a white hole are connected [8].

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APPENDIX A

We give here the explicit form of some tensors used in the paper. The spatial diagonal metric tensor is

$$h_{ij} = \begin{pmatrix} a^2(t) & 0 & 0 \\ 0 & b^2(t)\sin^2\theta & 0 \\ 0 & 0 & b^2(t) \end{pmatrix}. \tag{A1}$$

The inverse spatial metric tensor is

$$h^{ij} = \begin{pmatrix} a^{-2}(t) & 0 & 0 \\ 0 & b^{-2}(t)\sin^{-2}\theta & 0 \\ 0 & 0 & b^{-2}(t) \end{pmatrix}. \tag{A2}$$

The extrinsic curvature is $K_{ij} = -\frac{1}{2}\frac{\partial h_{ij}}{\partial t}$, and so

$$K_{ij} = \begin{pmatrix} -a\dot{a} & 0 & 0 \\ 0 & -b\dot{b}\sin^2\theta & 0 \\ 0 & 0 & -b\dot{b} \end{pmatrix}. \tag{A3}$$

$$K \equiv K_{ij}h^{ij} = -\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) \qquad K_{ij}K^{ij} = \frac{\dot{a}^2}{a^2} + 2\frac{\dot{b}^2}{b^2}$$
$$K_{ij}K^{ij} - K^2 = -\left(\frac{2\dot{b}^2}{b^2} + 4\frac{\dot{a}\,\dot{b}}{ab}\right) \tag{A4}$$

The Ricci curvature for the space section is

$$^{(3)}\mathrm{R} = \frac{2}{b^2} \tag{A5}$$

APPENDIX B

In this appendix we report the Hamiltonian for our system and we show that reproduces the correct equation of motion. We can start from the Hamiltonian

$$H = \left(\frac{p^2}{2M_P^2R} - \frac{M_P^2R}{2}\right)\left[\frac{\sqrt{2MG_N}}{\sqrt{b}}\left(1 - \frac{b}{2MG_N}\right)^{1/2}\right], \tag{B1}$$

and calculate the Hamilton equation for b ($\dot{b} = \frac{\partial H}{\partial p}$)

$$\dot{b} = \frac{p}{M_P^2R}\sqrt{\frac{2MG_N}{b} - 1}. \tag{B2}$$

At this point using the constraint $H = 0$ with (B2), we obtain

$$\dot{b}^{\,2} = \left(\frac{2MG_N}{b} - 1\right), \tag{B3}$$

that is the equation of motion for $b(t)$ that reproduce the Schwarzschild solution.

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