# Nonperturbative gravity and the spin of the lattice graviton

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The lattice formulation of quantum gravity provides a natural framework in which nonperturbative properties of the ground state can be studied in detail. In this paper we investigate how the lattice results relate to the continuum semiclassical expansion about smooth manifolds. As an example we give an explicit form for the lattice ground-state wave functional for semiclassical geometries. We then do a detailed comparison between the more recent predictions from the lattice regularized theory and results obtained in the continuum for the nontrivial ultraviolet fixed point of quantum gravity found using weak field and nonperturbative methods. In particular we focus on the derivative of the beta function at the fixed point and the related universal critical exponent $\nu$ for gravitation. Based on recently available lattice and continuum results we assess the evidence for the presence of a massless spin-two particle in the continuum limit of the strongly coupled lattice theory. Finally we compare the lattice prediction for the vacuum-polarization induced weak scale dependence of the gravitational coupling with recent calculations in the continuum, finding similar effects.

## I. INTRODUCTION

It is widely believed that an understanding of the properties of quantum gravitation would have important consequences in many areas of cosmology and high energy physics. Unfortunately approaches to quantum gravity based on linearized perturbation methods have had moderate success so far, as the underlying theory is known not to be perturbatively renormalizable [1,2]. A lack of perturbative renormalizability implies that an increasing number of counterterms needs to be added in order to make the theory finite order by order in perturbation theory. It has been recognized for some time though that the lack of perturbative renormalizability is not necessarily an obstacle in defining a consistent quantum theory [3], as several simpler field theory models suggest [4] (most notably the nonlinear sigma model above two dimensions) and recent rigorous results seem to support [5]. In the continuum nonperturbative renormalizability requires the existence of a nontrivial ultraviolet fixed point of the renormalization group. In the presence of a lattice momentum cutoff, the corresponding requirement is the existence of a phase transition with a divergent correlation length.

In the simplicial lattice formulation of quantum gravity one proceeds in a way similar to ordinary lattice gauge theories, and introduces a lattice ultraviolet regulator which in principle allows for controlled, systematic ana-

lytical [6,7] and numerical [8–10] nonperturbative calculations of ground-state properties, anomalous scaling dimensions and invariant correlations. Once the specific lattice action has been chosen, numerically exact results can be obtained on finite volume lattices which then need to be judiciously extrapolated to the infinite volume limit using the well-established methods of finite size scaling. In this paper we address the basic issue of the relationship between recent lattice results and a variety of approximate perturbative and nonperturbative results obtained in the continuum formulation, with the intent of establishing a set of connections between the two formulations that go beyond weak coupling perturbation theory and the perturbative expansion about smooth manifolds.

The starting point for a nonperturbative study of quantum gravity is usually a suitable definition of the discrete Feynman path integral. In the simplicial lattice approach one starts from the discretized Euclidean path integral for pure gravity, with the squared edge lengths taken as fundamental dynamical variables,

$$
Z_L = \int_0^\infty \prod_s [V_d(s)]^\sigma \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2] \exp\Big\{ -\sum_h (\lambda V_h
$$
$$
- k\delta_h A_h + a\delta_h^2 A_h^2/V_h + \cdots)\Big\} \qquad (1.1)
$$

(see Ref. [10] for notation). The above expression is supposed to represent a lattice discretization of the continuum Euclidean path integral for pure quantum gravity

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$$Z_C = \int \prod_x [\sqrt{g(x)}]^\sigma \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \exp\Bigg\{ - \int d^4x \sqrt{g}\Big(\lambda - \frac{k}{2}R$$

$$+ \frac{a}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots \Big) \Bigg\}, \qquad (1.2)$$

with $k^{-1} = 8\pi G$, $G$ Newton's constant, and the $\cdots$ represent higher order curvature invariant terms. The Regge lattice action propagates only spin-two degrees of freedom in the weak field limit, while the cosmological and measure terms contain only local volume contributions. The discrete gravitational measure in $Z_L$ can be considered as the lattice analog of the DeWitt [11,12] continuum functional measure [13–15]. A cosmological constant term with bare $\lambda > 0$ is needed for convergence of the path integral [8,16,17]. Without loss of generality one can rescale the metric and set $\lambda = 1$. The curvature squared terms ($a \to 0$) allow one to control the short distance fluctuations in the curvature, and as far as the functional measure parameter $\sigma$ is concerned most of the recent work has focused on the case $\sigma = 0$ (for more details on the choice of action and measure the reader is referred to the review [9], and further references therein).

The present numerical evidence from the discrete model of Eq. (1.1) suggests that quantum gravity in four dimensions exhibits a phase transition in the coupling $G$ between two physically distinct phases [10]: a strong coupling phase, in which the geometry becomes smooth at large scales,

$$\langle g_{\mu\nu} \rangle \approx c\eta_{\mu\nu} \qquad (G > G_c) \qquad (1.3)$$

with a vanishingly small average curvature in the vicinity of the critical point at $G_c$, and a weak coupling phase

$$\langle g_{\mu\nu} \rangle = 0 \qquad (G < G_c) \qquad (1.4)$$

in which the geometry becomes degenerate, bearing some resemblance to a dilute branched polymer. It is clear that based on its geometric properties, only the smooth phase is physically acceptable. The existence of a phase transition at finite coupling $G$ is usually associated with the appearance of an ultraviolet fixed point of the renormalization group, and implies in principle nontrivial scaling properties for the coupling constant and invariant correlations.

In the lattice theory the presence of a fixed point or phase transition is usually inferred (as in other lattice field theories) from the appearance of nonanalytic terms in invariant local averages, such as the average scalar curvature

$$\frac{\langle \int d^4x \sqrt{g}R(x) \rangle}{\langle \int d^4x \sqrt{g} \rangle} \underset{k \to k_c}{\sim} A_{\mathcal{R}}(k_c - k)^{4\nu - 1}. \qquad (1.5)$$

Without such singularities the lattice continuum limit cannot be taken, as one needs a divergent correlation length to define the lattice continuum limit. Indeed $k_c$

here is defined as the location of the nonanalyticity in the partition function and its averages, the latter often obtained by differentiation with respect to a source or some other parameter. A precise determination of $\nu$ then allows one to connect singularities in averages such as the one above to other long-distance properties of the theory. In particular the relation between the critical exponent $\nu$ and the derivative of the renormalization group beta function at the fixed point $\beta'(G_c) = -1/\nu$ implies a scale dependence of Newton's constant (due to gravitational vacuum-polarization effects) of the form

$$G(r) = G(0)\{1 + c(r/\xi)^{1/\nu} + O[(r/\xi)^{2/\nu}]\}, \qquad (1.6)$$

where $\xi$ is a renormalization group invariant scale parameter and $c$ a calculable numerical constant of order one. Detailed numerical studies of the Regge lattice gravity model give a value very close to $\nu^{-1} = 3$ [10]. Since one finds for the critical value $G_c \approx 0.626$ in units of the lattice spacing, one would conclude that the lattice theory is not weakly coupled in the vicinity of the fixed point. It seems natural to interpret the momentum scale $\xi^{-1}$ as arising due to a gravitational analog of dimensional transmutation, and it playing a role in gravitation similar to the universal scaling violation parameter $\Lambda_{\overline{MS}}$ of QCD [18]. Other lattice approaches to quantum gravity based on discrete dynamical triangulations with fixed edge lengths and which give rise to a rather different phase structure are reviewed in Ref. [19].

In this paper we examine a number of fundamental issues that have a bearing on the relationship between lattice and continuum models for quantum gravity. First we will consider the lattice analog of the semiclassical expansion for the ground-state wave functional of continuum gravity. Within the continuum formulation, the semiclassical expansion about smooth manifolds with bounded quantum fluctuations allows one to exhibit in a clear and direct way the transverse-traceless modes (or equivalently spin-two modes) as the only physical gravitational degrees of freedom. Such an expansion is most easily carried out with the Euclidean functional integral approach, wherein the gravitational action is expanded in the weak field metric, and the resulting Gaussian integrals are subsequently carried out.

In trying to construct the lattice analog of the ground-state functional for semiclassical gravity one has two options. The first procedure relies on constructing directly a lattice expression for the exponent of the ground-state functional, obtained by transcribing the continuum expression in terms of lattice variables. The first method we follow here is to proceed from the lattice expression for the contribution to the gravitational action from a fixed time slice, supplemented by the appropriate vacuum gauge conditions (the boundary action, which gives the ground-state wave function, is defined on a slice, the boundary slice). A crucial ingredient in this method is

the correct identification of the correspondence between continuum degrees of freedom (the metric) and the lattice variables (the edge lengths). This correspondence is fixed by the relationship between the lattice Regge action and the continuum Einstein action, at least in the weak field limit. The resulting lattice expression is then equivalent to the continuum one by construction.

The second procedure relies instead only on the expression for the lattice gravitational action, as computed in the weak field limit, and determines the explicit lattice form for the ground-state functional for linearized gravity by performing explicitly the necessary lattice Gaussian functional integrals. The resulting discrete expression can then be compared to the continuum one by reexpressing the edge lengths in terms of the metric. It is encouraging that the resulting lattice expression completely agrees with what is found by using the previous method.

It is advantageous in performing the above calculation to introduce spin projection operators, which separate out the spin-zero, spin-one and spin-two components of the gravitational action. As a by-product one can then show that the lattice gravitational action only propagates massless spin-two (or transverse-traceless) degrees of freedom in the weak field limit, as is the case in the continuum. Furthermore, as expected, the lattice ground-state functional for linearized gravity only contains those physical modes.

In subsequent sections of the paper we examine systematically the relationship between recent nonperturbative results obtained in the lattice theory and corresponding calculations performed in the continuum theory. The latter suggest the presence of a nontrivial ultraviolet fixed point in $G$, and in some cases have even led to definite predictions for the universal critical exponent of quantum gravity, which can therefore be compared quantitatively to the lattice results.

Besides relying on the recent lattice and continuum results for quantum gravity, one can also try to independently estimate the gravitational scaling dimensions using what is known based on exact and approximate renormalization group methods for spin zero (self-interacting scalar field in four dimensions) and spin one (Abelian noncompact pure gauge theories), for which a wealth of information is available on the critical indices. Based on this comparison, we will argue that these results too are consistent with what is known about the gravitational exponents in four dimensions. Finally we describe a simple geometric argument which interprets the value found for the gravitational exponent $\nu^{-1} = 3$.

## II. GROUND-STATE WAVE FUNCTIONAL OF LINEARIZED GRAVITY

According to the path integral prescription for Euclidean quantum gravity, the wave function of the state of minimum excitation for an asymptotically flat three-geometry with specified metric is

$$\Psi_0[{}^3g_{ij}, t] = \mathcal{N} \int_{{}^3g_{ij}} [dg_{\mu\nu}] e^{-I[g_{\mu\nu}]} \qquad (2.1)$$

where the integral is over all Euclidean four-geometries which are asymptotically flat and bounded at time $t$ by an asymptotically flat hypersurface with induced metric ${}^3g_{ij}$.

Kuchař [20] has given an expression for the ground-state wave functional of linearized gravity. In the vacuum or "Coulomb" gauge $h_{ik,k} = 0$, $h_{ii} = 0$, $h_{0i} = 0$ the ground-state functional is given by

$$\Psi_0[h_{ij}^{TT}, t] = \mathcal{N} \exp\left\{-\frac{1}{4l_P^2} \int d^3\mathbf{k}\, \omega_{\mathbf{k}} h_{ij}^{TT}(\mathbf{k}) \bar{h}_{ij}^{TT}(\mathbf{k})\right\}$$
$$(2.2)$$

where $h_{ij}^{TT}(\mathbf{k})$ is a Fourier component of the transverse-traceless part of the deviation of the three-metric from the flat three-metric in rectangular coordinates,

$$h_{ij}(\mathbf{x}, t) = {}^3g_{ij}(\mathbf{x}, t) - \delta_{ij}, \qquad (2.3)$$

$\omega_{\mathbf{k}} = |\mathbf{k}|$, $\mathcal{N}$ is a normalization factor, and $l_P = (16\pi G)^{1/2}$ is the Planck length in a system of units where $\hbar = c = 1$. Equivalently one can write, in real space and in terms of first derivatives of the fields, the expression

$$\Psi_0[h_{ij}^{TT}, t] = \mathcal{N} \exp\left[-\frac{1}{8\pi^2 l_P^2}\right.$$
$$\left. \times \int d^3\mathbf{x} \int d^3\mathbf{x}' \frac{h_{ij,k}^{TT}(\mathbf{x}) h_{ij,k}^{TT}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2}\right]. \quad (2.4)$$

The above ground-state wave function of linearized gravity was originally evaluated by Kuchař using canonical methods [20]. Later the same formula was obtained by Hartle using the Euclidean path integral prescription [21] (see Sec. III).

### A. Electromagnetic case

It is instructive, in view of the calculations to be done for the gravitational case in the next sections, to consider as an aside the much simpler case of electromagnetism. Indeed a completely analogous set of results holds for the ground-state functional in the electromagnetic case (as discussed in detail in the original references [20,21]), and brings out the relatively simple relationship between the original electromagnetic action and the quantity appearing in the exponent of the ground-state functional. Let us therefore first examine the structure of the ground-state functional in the electromagnetic case. In the Coulomb gauge $\partial_i A_i = 0$, $A_0 = 0$ the ground-state functional is given by Kuchař in terms of the transverse parts of the potentials only

$$\Psi_0[\mathbf{A}, t] = \mathcal{N} \exp\left[-\frac{1}{2}\int d^3\mathbf{k}\, k A_i^T(\mathbf{k})\bar{A}_i^T(\mathbf{k})\right] \quad (2.5)$$

or, equivalently, in terms of the **B** fields as

$$\Psi_0[\mathbf{B}, t] = \mathcal{N} \exp\left[-\frac{1}{4\pi^2}\int d^3\mathbf{x}\int d^3\mathbf{x}'\frac{\mathbf{B}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right]. \quad (2.6)$$

It will not be our purpose here to rederive the above expressions, and we refer to the original paper for further details on the functional integration procedure leading to Eq. (2.5). Instead we will point out that the expression in the exponent of Eq. (2.5) is related in a simple way to the original electromagnetic action. One has for the action appearing in the exponent of the Feynman path integral

$$I[A_\mu] = \frac{1}{4}\int d^4x\, F_{\mu\nu}(x)F^{\mu\nu}(x) \quad (2.7)$$

which for a single mode reduces to

$$I_k = \tfrac{1}{4}[k_\mu A_\nu(k) - k_\nu A_\mu(k)]^2. \quad (2.8)$$

The electromagnetic field can, as usual, be decomposed into transverse and longitudinal components. In terms of the transverse projection of the field

$$A_i^T = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)A_j \quad (2.9)$$

one has for $A_0 = 0$ (and $k_0 = 0$)

$$\mathbf{k}^2 A_i^T(\mathbf{k})\bar{A}_i^T(\mathbf{k}) = \tfrac{1}{2}[k_i A_j(\mathbf{k}) - k_j A_i(\mathbf{k})]^2 \quad (2.10)$$

and therefore, after summing over all modes,

$$\int d^3\mathbf{k}\,\omega_{\mathbf{k}} A_i^T(\mathbf{k})\bar{A}_i^T(\mathbf{k}) = \int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k}}}[k_i A_j(\mathbf{k}) - k_j A_i(\mathbf{k})]^2 \quad (2.11)$$

with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$. The only point we want to make here is that, up to a numerical factor, the expression appearing in the exponent of Eq. (2.5) is the same as the contribution written in Eq. (2.11), which follows from the continuum action Eq. (2.7) plus the gauge choice. In the next section we will see that a similar relationship holds in the gravitational case as well, at least in the linearized case. This will provide us with a first way of writing down the *exponent* of the vacuum functional, starting from the original action and following a set of steps analogous to the ones followed here for the much simpler case of electromagnetism, relying of course on the continuum result derived by the authors of Refs. [20,21], and, in particular, the main result of Eq. (2.2).

### B. Spin projections

Returning to the case of the gravitational field one can follow a procedure similar to the one just outlined, first in the continuum and then on the lattice, and eventually obtain a lattice expression for the exponent of the vacuum functional for linearized gravity, starting from Kuchař's original expression, Eq. (2.2). It will not be our purpose here to rederive the gravitational vacuum functional of Eq. (2.2) from the original path integral by performing the required functional integrals for a prescribed boundary metric (we refer to the original reference [21] for details on this procedure), which we will assume here to be correct. We will show here instead that in the continuum the exponent of the ground-state functional for linearized gravity in Eq. (2.2) can be obtained from the continuum action by suitably expanding the gravitational action in the weak field limit, and then imposing a set of appropriate gauge conditions.

By later following a similar procedure on the lattice, a corresponding discrete expression can easily be obtained. The resulting expression will provide a first way of discretizing the exponent of the vacuum functional of Eq. (2.2). In Sec. III, the same result will be obtained by following a significantly more elaborate but also more satisfactory procedure, in which one starts from the original lattice gravitational actions, and explicitly performs the required Gaussian functional integrals in the presence of appropriate boundaries to obtain the vacuum functional.

The first step in the above-mentioned procedure involves the expansion of the continuum Lagrangian density $-\sqrt{g}R$ in the weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad (2.12)$$

with $\kappa = \sqrt{8\pi G}$ and $|h_{\mu\nu}|$ small. The quadratic part [2,22] is then given by

$$\begin{aligned}\mathcal{L}_{\text{sym}} = &-\tfrac{1}{2}\partial_\mu h_{\mu\nu}\partial_\nu h_{\rho\rho} + \tfrac{1}{2}\partial_\mu h_{\mu\nu}\partial_\rho h_{\rho\nu}\\&-\tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial_\mu h_{\nu\rho} + \tfrac{1}{4}\partial_\mu h_{\nu\nu}\partial_\mu h_{\rho\rho}\end{aligned} \quad (2.13)$$

with residual gauge invariance

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \quad (2.14)$$

and $\xi_\nu$ an arbitrary gauge function. For one mode with wave vector $k$ one has

$$\begin{aligned}\mathcal{L}_{\text{sym}} = &\tfrac{1}{2}k_\mu h_{\mu\nu}k_\nu h_{\rho\rho} - \tfrac{1}{2}k_\mu h_{\mu\nu}k_\rho h_{\rho\nu} + \tfrac{1}{4}k_\mu h_{\nu\rho}k_\mu h_{\nu\rho}\\&-\tfrac{1}{4}k_\mu h_{\nu\nu}k_\mu h_{\rho\rho}\end{aligned} \quad (2.15)$$

$$\begin{aligned}= &\tfrac{1}{2}k_1^2(-h_{23}^2 - h_{24}^2 - h_{34}^2 + h_{22}h_{33} + h_{22}h_{44} + h_{33}h_{44})\\&+\cdots.\end{aligned} \quad (2.16)$$

A gauge-fixing term can be added of the form

$$\mathcal{L}_{\text{fix}} = -\tfrac{1}{2}(k_\nu h_{\mu\nu} - \tfrac{1}{2}k_\mu h_{\nu\nu})(k_\rho h_{\mu\rho} - \tfrac{1}{2}k_\mu h_{\rho\rho}) \quad (2.17)$$

giving for the combined gauge-fixed weak field action

$$\begin{aligned}\mathcal{L}_{\text{tot}} &= \mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{fix}}\\&= -\tfrac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\mu h_{\alpha\beta} + \tfrac{1}{8}\partial_\mu h_{\alpha\alpha}\partial_\mu h_{\beta\beta}.\end{aligned} \quad (2.18)$$

However in the following we shall rely instead on the vacuum (or Coulomb) gauge fixing which gives the ground-state functional for linearized gravity discussed previously.

It will be advantageous to define three independent spin projection operators, which show explicitly the unique decomposition of the continuum gravitational action for linearized gravity into spin-two (transverse-traceless) and spin-zero (conformal mode) parts [23,24]. The spin-two projection operator is defined as

$$
P^{(2)}_{\mu\nu\alpha\beta} = \frac{1}{3k^2}(k_\mu k_\nu \delta_{\alpha\beta} + k_\alpha k_\beta \delta_{\mu\nu}) - \frac{1}{2k^2}(k_\mu k_\alpha \delta_{\nu\beta}
$$
$$
+ k_\mu k_\beta \delta_{\nu\alpha} + k_\nu k_\alpha \delta_{\mu\beta} + k_\nu k_\beta \delta_{\mu\alpha})
$$
$$
+ \frac{2}{3k^4} k_\mu k_\nu k_\alpha k_\beta + \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})
$$
$$
- \frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}, \tag{2.19}
$$

the spin-one projection operator as

$$
P^{(1)}_{\mu\nu\alpha\beta} = \frac{1}{2k^2}(k_\mu k_\alpha \delta_{\nu\beta} + k_\mu k_\beta \delta_{\nu\alpha} + k_\nu k_\alpha \delta_{\mu\beta}
$$
$$
+ k_\nu k_\beta \delta_{\mu\alpha}) - \frac{1}{k^4} k_\mu k_\nu k_\alpha k_\beta \tag{2.20}
$$

and the spin-zero projection operator as

$$
P^{(0)}_{\mu\nu\alpha\beta} = -\frac{1}{3k^2}(k_\mu k_\nu \delta_{\alpha\beta} + k_\alpha k_\beta \delta_{\mu\nu}) + \frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}
$$
$$
+ \frac{1}{3k^4} k_\mu k_\nu k_\alpha k_\beta. \tag{2.21}
$$

The sum of the three spin projection operators is then equal to unity

$$
P^{(2)}_{\mu\nu\alpha\beta} + P^{(1)}_{\mu\nu\alpha\beta} + P^{(0)}_{\mu\nu\alpha\beta} = \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}). \tag{2.22}
$$

As a result one can define for the metric three orthogonal fields of definite spin, the transverse-traceless (spin-two) part

$$
h^{TT}_{\mu\nu} = P_{\mu\alpha}h_{\alpha\beta}P_{\beta\nu} - \frac{1}{3}P_{\mu\nu}P_{\alpha\beta}h_{\beta\alpha} \tag{2.23}
$$

the longitudinal (spin-one) part

$$
h^{L}_{\mu\nu} = h_{\mu\nu} - P_{\mu\alpha}h_{\alpha\beta}P_{\beta\nu} \tag{2.24}
$$

and the trace (spin-zero) part

$$
h^{T}_{\mu\nu} = \frac{1}{3}P_{\mu\nu}P_{\alpha\beta}h_{\alpha\beta} \tag{2.25}
$$

such that their sum gives $h$

$$
h = h^{TT} + h^L + h^T. \tag{2.26}
$$

Here we have defined the projection operator

$$
P_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{\Delta}\partial_\mu\partial_\nu \tag{2.27}
$$

or equivalently in momentum space

$$
P_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \tag{2.28}
$$

Using the three spin projection operators, the action for linearized gravity can then be rewritten (for one mode) simply as

$$
\mathcal{L}_{\mathrm{sym}} = -\frac{1}{4}h_{\mu\nu}[P^{(2)} - 2P^{(0)}]_{\mu\nu\alpha\beta}k^2 h_{\alpha\beta}. \tag{2.29}
$$

Imposing the gauge condition $h_{i0} = h_{00} = h_{ik,k} = 0$ (and setting $k_0 = 0$ again, as in the electromagnetic case, to eliminate some unwanted terms) one obtains

$$
-\frac{1}{4}k^2 \bar{h}^{TT}_{ij}h^{TT}_{ij} + \frac{1}{2}k^2 \bar{h}^{T}_{ij}h^{T}_{ij} \tag{2.30}
$$

with the second (spin zero) vanishing after further imposing the trace condition $h_{ii} = 0$. The resulting expression is then identical, up to a factor, to the expression appearing in the exponent of the ground-state functional of linearized gravity of Eq. (2.2). We should stress here again that the above steps are *not* intended as a derivation of the Kuchař and Hartle result of Eq. (2.2), and are presented here instead to bring out the relationship between the original gravitational action density and the expression appearing in the exponent of the ground-state wave functional, and as a prescription for obtaining one from the other.

### C. Lattice transverse-traceless modes

In this section an expression for the lattice analog of the exponent of the Kuchař and Hartle expression for the vacuum functional [Eq. (2.2)] will be obtained from the Regge lattice gravitational action, which will then give a lattice candidate for the gravitational wave functional. The procedure will be very similar, and parallel to, the one followed in the previous section dealing purely with the continuum case. As a first step one needs to perform the weak field expansion for the Regge action

$$
I_R = \sum_h A_h \delta_h \tag{2.31}
$$

where $A_h$ is the area of the hinge $h$, and $\delta_h$ is the deficit angle at the same hinge. Following Ref. [7] each hypercube in a hypercubic lattice is divided up into 24 four-simplices, with vertices at $(0,0,0,0)$, $(0,0,0,1)$ ... $(1,1,1,1)$ (without loss of generality one can take the lattice spacing to be one). The lengths of the 15 edges connecting the vertices $i$ and $j$ are denoted by $l_{ij}$, where $i$ and $j$ range from 1 to 15, with the coordinates of the end points interpreted as binary numbers (for more details, see Sec. III). Next each link length is allowed to fluctuate by an amount $1 + e$ around the hypercubic lattice value. To lowest order in the edge fluctuation, the lattice action is given by a quadratic form

$$I_R = \frac{1}{2} \sum_{ij} e_i M_{ij} e_j \qquad (2.32)$$

with $M$ a local matrix connecting only nearest-neighbor points. In Fourier space one can write for each of the 15 displacements $e_i^{a,b,c,d}$, defined at the vertex of the cube with labels $(a, b, c, d)$,

$$e_i^{(a,b,c,d)} = (\omega_1)^a (\omega_2)^b (\omega_4)^c (\omega_8)^d e_i^0 \qquad (2.33)$$

with $\omega_1 = e^{ik_1}$, $\omega_2 = e^{ik_2}$, $\omega_4 = e^{ik_3}$, $\omega_8 = e^{ik_4}$ (we use the binary notation for $\omega$ and $e$, but the regular notation for $k_i$). For one mode (one set of $\omega$'s) one obtains therefore (see Appendix B in Ref. [7])

$$6e_1^2 + 16e_3^2 + 18e_7^2 + (\omega_1\omega_4 + \omega_2\omega_4 + \omega_1\omega_8 + \omega_2\omega_8 + \bar{\omega}_1\bar{\omega}_4 + \bar{\omega}_2\bar{\omega}_4 + \bar{\omega}_1\bar{\omega}_8 + \bar{\omega}_2\bar{\omega}_8)e_1e_2 - (8 + 4\omega_2$$
$$+ 4\bar{\omega}_2)e_1e_3 - (2\omega_1 + 2\bar{\omega}_1 + 2\omega_2\omega_4 + 2\bar{\omega}_2\bar{\omega}_4)e_1e_6 - (12 + 6\omega_4 + 6\bar{\omega}_4)e_3e_7 + \cdots. \qquad (2.34)$$

Each coefficient is real, as expected from the reality of the action. Thus, for example, in the above expression, we have

$$\omega_1\omega_4 + \omega_2\omega_4 + \omega_1\omega_8 + \omega_2\omega_8 + \bar{\omega}_1\bar{\omega}_4 + \bar{\omega}_2\bar{\omega}_4 + \bar{\omega}_1\bar{\omega}_8 + \bar{\omega}_2\bar{\omega}_8$$
$$= 4\cos[\tfrac{1}{2}(k_3 - k_4)]\{\cos[\tfrac{1}{2}(2k_1 + k_3 + k_4)] + \cos[\tfrac{1}{2}(2k_2 + k_3 + k_4)]\} \sim 8 - 2k_1^2 - 2k_2^2 - 2k_1k_3 - 2k_2k_3 - 2k_3^2$$
$$- 2k_1k_4 - 2k_2k_4 - 2k_4^2 + O(k^4). \qquad (2.35)$$

Normally to show the equivalence of the Regge action to the continuum Einstein-Hilbert action one needs to replace the $e$ fields with metric components (or alternatively, as done in Ref. [7], use trace reversed metric components), with body principals expanded as

$$e_1 = -1 + [1 + \omega_1 h_{11}]^{1/2}, \qquad e_2 = -1 + [1 + \omega_2 h_{22}]^{1/2}, \qquad e_4 = -1 + [1 + \omega_4 h_{33}]^{1/2},$$
$$e_8 = -1 + [1 + \omega_8 h_{44}]^{1/2}, \qquad (2.36)$$

face diagonals as

$$e_3 = -1 + [1 + \tfrac{1}{2}\omega_1\omega_2(h_{11} + h_{22}) + h_{12}]^{1/2}, \qquad e_5 = -1 + [1 + \tfrac{1}{2}\omega_1\omega_4(h_{11} + h_{33}) + h_{13}]^{1/2},$$
$$e_9 = -1 + [1 + \tfrac{1}{2}\omega_1\omega_8(h_{11} + h_{44}) + h_{14}]^{1/2}, \qquad e_6 = -1 + [1 + \tfrac{1}{2}\omega_2\omega_4(h_{22} + h_{33}) + h_{23}]^{1/2},$$
$$e_{10} = -1 + [1 + \tfrac{1}{2}\omega_2\omega_8(h_{22} + h_{44}) + h_{24}]^{1/2}, \qquad e_{12} = -1 + [1 + \tfrac{1}{2}\omega_4\omega_8(h_{33} + h_{44}) + h_{34}]^{1/2}, \qquad (2.37)$$

body diagonals as

$$e_7 = -1 + \{1 + \tfrac{1}{3}\omega_1\omega_2\omega_4(h_{11} + h_{22} + h_{33}) + \tfrac{1}{3}[(1 + \omega_4)h_{12} + (1 + \omega_1)h_{23} + (1 + \omega_2)h_{13}]\}^{1/2},$$
$$e_{11} = -1 + \{1 + \tfrac{1}{3}\omega_1\omega_2\omega_8(h_{11} + h_{22} + h_{44}) + \tfrac{1}{3}[(1 + \omega_8)h_{12} + (1 + \omega_1)h_{24} + (1 + \omega_2)h_{14}]\}^{1/2},$$
$$e_{13} = -1 + \{1 + \tfrac{1}{3}\omega_1\omega_2\omega_4(h_{11} + h_{33} + h_{44}) + \tfrac{1}{3}[(1 + \omega_8)h_{13} + (1 + \omega_1)h_{34} + (1 + \omega_4)h_{14}]\}^{1/2},$$
$$e_{14} = -1 + \{1 + \tfrac{1}{3}\omega_2\omega_4\omega_8(h_{22} + h_{33} + h_{44}) + \tfrac{1}{3}[(1 + \omega_8)h_{23} + (1 + \omega_2)h_{34} + (1 + \omega_4)h_{24}]\}^{1/2}, \qquad (2.38)$$

and finally hyperbody diagonals as

$$e_{15} = -1 + [1 + \tfrac{1}{4}\omega_1\omega_2\omega_4\omega_8(h_{11} + h_{22} + h_{33} + h_{44}) + \tfrac{3}{4}(h_{12} + h_{13} + h_{14} + h_{23} + h_{24} + h_{34})]^{1/2}, \qquad (2.39)$$

although the latter quantity is not needed, as it does not appear in the Regge action to lowest order in the weak field expansion. Each expression is then expanded out for weak $h$, giving for example

$$e_1 = \tfrac{1}{2}\omega_1 h_{11} + O(h^2), \qquad e_3 = \tfrac{1}{2}h_{12} + \tfrac{1}{4}\omega_1\omega_2(h_{11} + h_{22}) + O(h^2),$$
$$e_7 = \tfrac{1}{6}(h_{12} + h_{13} + h_{23}) + \tfrac{1}{6}(\omega_1 h_{23} + \omega_2 h_{13} + \omega_4 h_{12}) + \tfrac{1}{6}\omega_1\omega_2\omega_4(h_{11} + h_{22} + h_{33}) + O(h^2) \qquad (2.40)$$

and so on for the other edges by permuting indices. Setting then $\omega_1 = e^{ik_1} \ldots \omega_8 = e^{ik_4}$ (we switch here from the binary notation for the $\omega$'s to a normal notation for the $k$'s), the resulting answer is finally expanded out in $k$ to give exactly the weak field expansion of the continuum Einstein action as given in Eqs. (2.13) and (2.15), and completely parallels the procedure for recovering the continuum limit of the lattice action as described originally in Refs. [7,25].

To obtain a suitable lattice expression for the ground-state functional of linearized gravity one needs to compute the contribution to the lattice gravitational action from a given time slice, and subsequently impose the appropriate discrete vacuum gauge conditions. This will then give, as in the continuum, the contribution appearing in the exponent of the ground-state functional for linearized gravity as it appears in Eq. (2.2).

The first step involves therefore the imposition of the vacuum gauge conditions $h_{ik,k} = 0$, $h_{00} = h_{0i} = 0$ which gives

$$e_8 = 0, \quad e_9 = \tfrac{1}{2}\omega_8 e_1, \quad e_{10} = \tfrac{1}{2}\omega_8 e_2, \quad e_{12} = \tfrac{1}{2}\omega_8 e_4,$$
$$e_{11} = \tfrac{1}{3}(1 + \omega_8)e_3 - \tfrac{1}{6}(1 - \omega_8)(\omega_2 e_1 + \omega_1 e_2),$$
$$e_{13} = \tfrac{1}{3}(1 + \omega_8)e_5 - \tfrac{1}{6}(1 - \omega_8)(\omega_4 e_1 + \omega_1 e_4), \quad (2.41)$$
$$e_{14} = \tfrac{1}{3}(1 + \omega_8)e_6 - \tfrac{1}{6}(1 - \omega_8)(\omega_2 e_4 + \omega_4 e_2)$$

and results in an action contribution of the form

$$2e_1^2 + 8e_3^2 + (\omega_1\omega_4 + \omega_2\omega_4 + \bar{\omega}_1\bar{\omega}_4 + \bar{\omega}_2 + \bar{\omega}_4)e_1 e_2$$
$$- 2(\omega_2 + \bar{\omega}_2 + 2)e_1 e_3 - 2(\omega_1\omega_2 + \bar{\omega}_1\bar{\omega}_2 + \omega_4$$
$$+ \bar{\omega}_4)e_3 e_4 + 4(\omega_2 + \bar{\omega}_2 + \omega_4 + \bar{\omega}_4)e_3 e_5 + \cdots$$
$$(2.42)$$

where the dots indicate again additional terms obtainable by permutation of indices.

To verify that this is indeed the correct expression one can use the expansion of the $e_i$'s in terms of the $h_{ij}$'s, as given in Eq. (2.39), and then expand out the $\omega$'s in powers of $k$. After setting $k_4 = 0$ (the analog of setting $k_0 = 0$ in the previous section), one obtains

$$\tfrac{1}{2}(k_1^2 h_{23}^2 - k_1^2 h_{22}h_{33} - 2k_1 k_2 h_{13} h_{23} + 2k_1 k_2 h_{12} h_{33}$$
$$+ k_2^2 h_{13}^2 - k_2^2 h_{11} h_{33} + 2k_1 k_3 h_{13} h_{22} - 2k_1 k_3 h_{12} h_{23}$$
$$- 2k_2 k_3 h_{12} h_{13} + 2k_2 k_3 h_{11} h_{23} + k_3^2 h_{12}^2 - k_3^2 h_{11} h_{22})$$
$$(2.43)$$

which can in turn be rewritten as the sum of two parts, the first part being the transverse-traceless contribution

$$\tfrac{1}{4}\mathbf{k}^2 \mathrm{Tr}\{^3 h[P^3 h P - \tfrac{1}{2}P\mathrm{Tr}(P^3 h)]\} = \tfrac{1}{4}\mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k})h_{ij}^{TT}(\mathbf{k})$$
$$(2.44)$$

$$\bar{h}_{ij}^{TT} h_{ij}^{TT} = \mathrm{Tr}\{^3 h[P^3 h P - \tfrac{1}{2}P\mathrm{Tr}(P^3 h)]\} \quad (2.45)$$

with $P_{ij} = \delta_{ij} - k_i k_j / \mathbf{k}^2$, and the second part arising due to the trace component of the metric

$$-\tfrac{1}{4}\mathbf{k}^2 \mathrm{Tr}[P\mathrm{Tr}(P^3 h)P\mathrm{Tr}(P^3 h)] = \mathbf{k}^2 \bar{h}_{ij}^{T}(\mathbf{k})h_{ij}^{T}(\mathbf{k}) \quad (2.46)$$

with $h^T = \tfrac{1}{2}P\mathrm{Tr}(P^3 h)$. In the vacuum gauge $h_{ik,k} = 0$, $h_{ii} = 0$, $h_{0i} = 0$ one needs to further solve for the metric components $h_{12}$, $h_{13}$, $h_{23}$ and $h_{33}$ in terms of the two remaining degrees of freedom, $h_{11}$ and $h_{22}$,

$$h_{12} = -\frac{1}{2k_1 k_2}(h_{11}k_1^2 + h_{22}k_2^2 + h_{11}k_3^2 + h_{22}k_3^2),$$
$$h_{13} = -\frac{1}{2k_1 k_3}(h_{11}k_1^2 - h_{22}k_2^2 - h_{11}k_3^2 - h_{22}k_3^2),$$
$$h_{23} = -\frac{1}{2k_2 k_3}(-h_{11}k_1^2 + h_{22}k_2^2 - h_{11}k_3^2 - h_{22}k_3^2),$$
$$h_{33} = -h_{11} - h_{22},$$
$$(2.47)$$

and show that the second (trace) part vanishes. For example, in terms of the $e$ variables the vacuum gauge condition $h_{ik,k} = 0$ then reads

$$[2(1 - \bar{\omega}_1) + \omega_2(1 - \omega_2) + \omega_4(1 - \omega_4)]e_1$$
$$+ \omega_1(1 - \omega_2)e_2 + \omega_1(1 - \omega_4)e_4 - 2(1 - \omega_2)e_3$$
$$- 2(1 - \omega_4)e_5 = 0 \quad (2.48)$$

and permutations. The above manipulations then show that the expression given in Eq. (2.42) is indeed the sought-after lattice analog for the continuum expression $\mathbf{k}^2 h_{ij}^{TT} \bar{h}_{ij}^{TT}$ appearing in the exponent of the ground-state functional of linearized gravity.

We conclude this section by outlining an example of a potentially useful application for the above results. The explicit construction of the ground-state wave functional of linearized lattice gravity in terms of lattice transverse-traceless modes makes it possible at least in principle to compare the lattice and continuum results in the limit of small curvatures, such as would be obtained, for example, from lattice simulations by imposing flat boundary conditions at spatial infinity. After imposing the boundary conditions by suitably restricting the values for the edge lengths on the lattice boundary such that the deficit angle is zero there, one would then have to further enforce the lattice vacuum gauge conditions of Eq. (2.48) so as to finally make contact with the semi-classical lattice functional of Eq. (2.42). But no gauge fixing is required for determining invariant averages obtained via the partition function of Eq. (1.1), so in practice the gauge conditions would have to be imposed configuration by configuration, by progressively applying local gauge transformations (such as the ones described in Ref. [14]) so as to gradually transform the edge lengths for each configuration to the lattice analog of the vacuum gauge. It is expected that after such a transformation the edge distributions on a fixed time slice should follow closely the distribution of Eq. (2.42), if indeed as expected the only surviving physical modes are transverse traceless.

## III. GROUND-STATE WAVE FUNCTIONAL FOR LINEARIZED REGGE CALCULUS

In the previous section a lattice expression for the exponent of the vacuum functional for linearized gravity was obtained, following a procedure which closely parallels the one used in the continuum to go from the original gravitational action density to the exponent of Eq. (2.2). In this section the ground-state functional will instead be derived by performing directly the discrete functional integration over the interior metric perturbations for a lattice with boundary. This will be one of the

main results of this paper, and is the analog of Hartle's continuum calculation of the ground-state wave functional of linearized gravity, using Regge calculus. We shall now briefly outline Hartle's calculation since our calculation later is (intended to be) a discrete version of his. For further details on the method we refer to the original paper.

In linearized gravity, the Einstein action is expanded to quadratic order in deviations of the metric from its flat-space value. On a surface which becomes the flat surface $t = \text{const}$ when the metric perturbations are zero, we write the three-metric as

$$^3 g_{ij} = \delta_{ij} + h_{ij}, \tag{3.1}$$

and $h_{ij}$ can be decomposed into a transverse-traceless part, a longitudinal part and the trace. Since the physical degrees of freedom are the two independent components of $h_{ij}^{TT}$, the transverse-traceless part, the wave function on the surface can be written as

$$\Psi_0 = \Psi_0[h_{ij}^{TT}(\mathbf{x}), t]. \tag{3.2}$$

The Euclidean Einstein action is given by

$$l_P^2 I(g) = - \int_M d^4 x \sqrt{g} R - 2 \int_{\partial M} d^3 x \sqrt{^3 g} K, \tag{3.3}$$

and for linearized gravity, the Euclidean four-metric in the functional integral is written as

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + h_{\alpha\beta}(x) \tag{3.4}$$

and the action is expanded to quadratic order in $h_{\alpha\beta}$. The boundary $\partial M$ is taken to be a flat slice in flat Euclidean space, and $M$ is the region of flat Euclidean space to the past of this. The $h_{\alpha\beta}$ are required to vanish at infinity so that $g_{\alpha\beta}$ is asymptotically flat.

The action is required to be gauge invariant, and gauge-fixing terms in the four-volume and on the surface are included in the functional integral. The metric perturbations are divided into conformal equivalence classes [26] by writing

$$h_{\alpha\beta}(x) = \phi_{\alpha\beta}(x) + 2\delta_{\alpha\beta}\chi(x). \tag{3.5}$$

The integration contour for $\chi$ is rotated to purely imaginary values to make the integral over $\chi$ converge. Then the field $\phi_{\alpha\beta}$ is decomposed as

$$\phi_{\alpha\beta} = \hat{\phi}_{\alpha\beta} + f_{\alpha\beta}, \tag{3.6}$$

where $\hat{\phi}_{\alpha\beta}$ is a solution of the linearized field equations which satisfies the gauge and boundary conditions. The unique solution is essentially that the spatial components are the $h_{ij}^{TT}$ and the other components vanish. The integral over the $f_{\alpha\beta}$ contributes only to the normalization factor, as does that over $\chi$. The result for the ground-state wave function of linearized gravity is

$$\Psi_0[h_{ij}^{TT}, t] = \mathcal{N} \exp\left[ -\frac{1}{4l_P^2} \int d^3\mathbf{k}\, \omega_k h_{ij}^{TT}(\mathbf{k}) \bar{h}_{ij}^{TT}(\mathbf{k}) \right], \tag{3.7}$$

where $h_{ij}^{TT}(\mathbf{k})$ is the Fourier transform of $h_{ij}^{TT}(\mathbf{x})$, $\omega_k = |\mathbf{k}|$, $\mathcal{N}$ is a normalization factor and $l_P = \sqrt{16\pi G}$ is the Planck length. This is exactly the same formula as that obtained by Kuchař using canonical methods.

Linearized Regge calculus can be implemented as the theory of the small fluctuations of edge lengths away from their flat-space values, in a tessellation of four-dimensional space using rectangular hypercubes subdivided into four-simplices. The methods of subdivision and the notation are described in detail in earlier work on linearized Regge calculus [7], the difference in this case being that we have a four-dimensional Euclidean space with a flat boundary. The binary notation in Ref. [7], which we shall also use here, comes from interpreting lattice vectors $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, $(0, 1, 0, 0)$ and $(1, 0, 0, 0)$ as binary numbers, giving the $x_1$, $x_2$, $x_4$, and $x_8$ directions. For ease of notation here, we shall take the boundary surface to be $x_8 = 0$, and the Euclidean four-space to be $x_8 \geq 0$ (to avoid lots of minus signs). Unlike the continuum case, we shall take periodic boundary conditions in the 1-, 2- and 4-directions, while the space will be asymptotically flat in the 8-direction. With unit lattice spacing, the flat-space edge lengths will be 1, $\sqrt{2}$, $\sqrt{3}$ and 2, and the perturbed edge lengths will be written as in Sec. II, as

$$l_i^j = L_i^j(1 + e_i^j), \tag{3.8}$$

where $L$ is the flat-space edge length, $e$ is a small perturbation, and in each case, the upper index $j$ denotes the lattice point at which the edge is based and the lower index $i$ denotes the direction of that edge (all in binary notation). (Note a small change in notation from Ref. [7] where the small perturbations were called $\delta$.) Thus, for instance, the $e$'s based at the origin and lying in the boundary hypersurface will be $e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0, e_7^0$.

In brief, our method is to write down the action for the semi-infinite four-dimensional space and to perform a functional integral over the internal perturbations, leaving an expression in terms of the $e$'s on the boundary. The aim is then to identify the quadratic expression in the surface $e$'s with the discrete version of the integral of $h_{ij}^{TT}(\mathbf{k}) \bar{h}^{TTij}(\mathbf{k})$. The calculation is long and tedious so we shall give relatively little detail, but enough for the reader to reproduce it if required.

### A. Interior terms

Consider an interior vertex, which for simplicity we shall label as if it were the origin. The $e$'s for the edges

based at this vertex will be $e_i^0$, $i = 1, 2, \ldots, 15$, and we first write the total quadratic contribution (the first non-vanishing order) to the action, which involves any of these $e$'s. This will arise from the Regge action $\Sigma A_h \delta_h$, for the hypercube based at the origin and from neighboring hypercubes. This is given explicitly in Appendix B in Ref. [7].

The next step is to differentiate the action with respect to each of the $e_i^0$ in turn to obtain their classical equations of motion. Below we give an example of an equation of motion of each type, for $e_1^0$, $e_3^0$ and $e_7^0$ respectively:

$$
\begin{aligned}
0 =\, & 6e_1^0 - 4(e_3^0 + e_5^0 + e_6^0) - 2(e_6^1 + e_{10}^1 + e_{12}^1) \\
& - 4(e_3^{-2} + e_5^{-4} + e_9^{-8}) + 3(e_7^{-2} + e_{11}^{-2} + e_7^{-4} + e_{13}^{-4} \\
& + e_{11}^{-8} + e_{13}^{-8}) + e_4^3 + e_8^3 + e_2^5 + e_8^5 + e_2^9 + e_4^9 + e_2^{-6} \\
& + e_4^{-6} - 2e_6^{-6} + e_2^{-10} + e_8^{-10} - 2e_{10}^{-10} + e_4^{-12} \\
& + e_8^{-12} - 2e_{12}^{-12}
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
0 =\, & 8e_3^0 - 2(e_1^0 + e_2^0) - 3(e_7^0 + e_{11}^0) + 2(-e_2^1 + e_6^1 \\
& + e_{10}^1) + 2(-e_1^2 + e_5^2 + e_9^2) - e_4^3 - e_8^4 - e_4^{-4} - e_8^{-8} \\
& + 2(e_5^{-4} + e_6^{-4} + e_9^{-8} + e_{10}^{-8}) - 3(e_7^{-4} + e_{11}^{-8})
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
0 =\, & 6e_7^0 - 2e_3^0 - 2e_5^0 - 2e_6^0 + e_2^1 + e_4^1 - 2e_6^1 + e_1^2 + e_4^2 \\
& - 2e_5^2 + e_1^4 + e_2^4 - 2e_3^4.
\end{aligned} \tag{3.11}
$$

All other equations of motion may be obtained by cyclic permutations of the indices.

We introduce new integration variables $f_i^j$ by

$$
e_i^j = \hat{e}_i^j + f_i^j, \tag{3.12}
$$

where the $\hat{e}_i^j$ satisfy the equations of motion above. By subtracting each of the $\hat{e}_i^j$ times the corresponding classical equation of motion from the contribution to the action based at the origin, the $\hat{e}_i^j$ are completely eliminated, leaving only Gaussian integrals over the $f_i^j$, which contribute only to the normalization. The same feature appears in the continuum. (Note that as in Ref. [7], cross terms of the form $e_i^0 e_k^j$, where $j$ is a neighboring lattice point of the origin, are assigned half to each of the lattice points involved.) This elimination of the interior terms would seem to hold independently of whether we impose periodic boundary conditions or asymptotic flatness.

## B. Boundary terms

Having integrated over all contributions to the action from interior vertices, we are now left with the contributions assigned to vertices on the $x_8 = 0$ boundary. These will consist not only of terms involving $e$'s based at vertices on the boundary hypersurface, but also of contributions from vertices one layer in but assigned (as explained above) partly to the boundary layer.

Suppose now that the origin is back on the boundary hypersurface. The total contribution to the action involving $e_i^0$ has more than 200 terms so will not be reproduced here. Clearly there are no terms involving $e_i^{-j}$ with $j = 8, 9, \ldots, 14$ since the boundary is at $x_8 = 0$. All $e_i^j$ terms with $i$ or $j = 8, 9, \ldots, 14$ are as for interior vertices, and most of the other terms are half their interior values.

Recall that the $e$'s in the boundary three-surface ($e_i^0$, $i = 1, 2, \ldots, 7$) are fixed but $e_i^0$ with $i = 8, 9, \ldots, 14$ are to be varied and have exactly the same classical equations of motion as before. We again simplify the action by subtracting $e_i^0$ times its equation of motion, for each of these, and then eliminate $e_{11}^0$, $e_{13}^0$ and $e_{14}^0$ using their equations of motion. At the same time, we eliminate $e_7^0$ using a constraint identical to its equation of motion. This looks somewhat suspect, but the motivation and justification are as follows. In three-dimensional linearized Regge calculus performed in a manner completely analogous to the four-dimensional case in Ref. [7], the $e_7^0$ mode is not dynamical and satisfies a constraint which turns out to be identical to its equation of motion; this reduces the number of variables to six, the correct number. We apply this result to our three-dimensional boundary hypersurface.

As a first step in linking our position-space representation of the action to the momentum representation in the Kuchař-Hartle formula [20,21], we take the Fourier transform in the 1-, 2- and 4-directions, which are those in which there are periodic boundary conditions. (In the 8-direction, we have contributions from only one other layer, the first interior one). The details are explained in Ref. [7] but there the complex nature of the Fourier transforms was not taken into account. Here our convention is that $e_i^0 e_j^a$ transforms to $\omega_a \tilde{e}_i^0 \bar{\tilde{e}}_j^0$, where $a = 1, 2, 4$ and $\omega_a = e^{ik_a}$. For consistency with this, the linear expressions in the $e$'s in the equations of motion transform slightly differently from in Ref. [7], with $e_i^a$ transforming to $\bar{\omega}_a \tilde{e}_i^0$. To simplify the notation, we immediately drop the tildes from the Fourier transforms and the superscripts 0.

We have not yet eliminated $e_8$, $e_9$, $e_{10}$ and $e_{12}$; this is not straightforward in the way it was for $e_{11}$, $e_{13}$ and $e_{14}$ as their equations of motion are simultaneous equations for the four $e$'s. For example, the Fourier transforms of those for $e_8$ and $e_9$, with $\alpha_i$ defined to be $1 - \omega_i$, are

$$e_8[2(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_4|^2) - \alpha_1\bar{\alpha}_2 - \alpha_2\bar{\alpha}_1 - \alpha_1\bar{\alpha}_4 - \alpha_4\bar{\alpha}_1 - \alpha_2\bar{\alpha}_4 - \alpha_4\bar{\alpha}_2] + 2e_9(-|\alpha_2|^2 - |\alpha_4|^2 + \alpha_1\bar{\alpha}_2 + \alpha_1\bar{\alpha}_4)$$

$$+ 2e_{10}(-|\alpha_1|^2 - |\alpha_4|^2 + \alpha_2\bar{\alpha}_1 + \alpha_2\bar{\alpha}_4) + 2e_{12}(-|\alpha_1|^2 - |\alpha_2|^2 + \alpha_4\bar{\alpha}_1 + \alpha_4\bar{\alpha}_2) = e_1(2\alpha_1 + 2\alpha_2 + 2\alpha_4 - \bar{\alpha}_2$$

$$- \bar{\alpha}_4 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_4 + \alpha_1\bar{\alpha}_2 + \alpha_1\bar{\alpha}_4) + e_2(2\alpha_1 + 2\alpha_2 + 2\alpha_4 - \bar{\alpha}_1 - \bar{\alpha}_4 - 2\alpha_1\alpha_2 - 2\alpha_2\alpha_4 + \alpha_2\bar{\alpha}_1 + \alpha_2\bar{\alpha}_4)$$

$$+ e_4(2\alpha_1 + 2\alpha_2 + 2\alpha_4 - \bar{\alpha}_1 - \bar{\alpha}_2 - 2\alpha_1\alpha_4 - 2\alpha_2\alpha_4 + \alpha_4\bar{\alpha}_1 + \alpha_4\bar{\alpha}_2) + 2e_3(2\alpha_1\alpha_2 - \alpha_1 - \alpha_2) + 2e_5(2\alpha_1\alpha_4$$

$$- \alpha_1 - \alpha_4) + 2e_6(2\alpha_2\alpha_4 - \alpha_2 - \alpha_4) + e_1^8(-2\alpha_1 - \alpha_2 - \alpha_4 + 2\bar{\alpha}_2 + 2\bar{\alpha}_4) + e_2^8(-2\alpha_2 - \alpha_1 - \alpha_4 + 2\bar{\alpha}_1 + 2\bar{\alpha}_4)$$

$$+ e_4^8(-2\alpha_4 - \alpha_1 - \alpha_2 + 2\bar{\alpha}_1 + 2\bar{\alpha}_2) + 2e_3^8(\alpha_1 + \alpha_2) + 2e_5^8(\alpha_1 + \alpha_4) + 2e_6^8(\alpha_2 + \alpha_4), \tag{3.13}$$

$$e_8(-|\alpha_2|^2 - |\alpha_4|^2 + \alpha_2\bar{\alpha}_1 + \alpha_4\bar{\alpha}_1) + 2e_9(|\alpha_2|^2 + |\alpha_4|^2) - 2e_{10}\alpha_2\bar{\alpha}_1 - 2e_{12}\alpha_4\bar{\alpha}_1 = e_1(\bar{\alpha}_2 + \bar{\alpha}_4)$$

$$+ e_2(2\bar{\alpha}_1 - \alpha_2 - \alpha_2\bar{\alpha}_1) + e_4(2\bar{\alpha}_1 - \alpha_4 - \alpha_1\bar{\alpha}_1) + 2e_3\alpha_2 + 2e_5\alpha_4 + e_1^8(\alpha_2 + \alpha_4) + e_2^8(\alpha_2 - 2\bar{\alpha}_1)$$

$$+ e_4^8(\alpha_4 - 2\bar{\alpha}_1) - 2e_3^8\alpha_2 - 2e_5^8\alpha_4, \tag{3.14}$$

and those for $e_{10}$ and $e_{12}$ by cyclic permutations of the indices.

These equations are not all independent. Adding the left-hand sides gives zero, while adding the right-hand sides gives

$$0 = 2[e_1(|\alpha_2|^2 + |\alpha_4|^2 - \alpha_1\alpha_2 - \alpha_1\alpha_4) + e_2(|\alpha_1|^2 + |\alpha_4|^2 - \alpha_1\alpha_2 - \alpha_2\alpha_4) + e_4(|\alpha_1|^2 + |\alpha_2|^2 - \alpha_1\alpha_4 - \alpha_2\alpha_4)$$

$$+ 2e_3\alpha_1\alpha_2 + 2e_5\alpha_1\alpha_4 + 2e_6\alpha_2\alpha_4]. \tag{3.15}$$

This expression is precisely the three-dimensional scalar curvature at the origin [27] and is constrained to be zero. [The constraint on a three-dimensional hypersurface also includes $(trK)^2$ and $trK^2$ terms but these terms are of higher order in the $e$'s.] We can also show that $\alpha_1$ times the $e_9$ equation plus $\alpha_2$ times the $e_{10}$ equation plus $\alpha_4$ times the $e_{12}$ equation gives zero on the left-hand side but on the right-hand side, it gives half the difference between the three-dimensional curvature scalars at the origin and at vertex 8, and so again is zero. ($^3R$ is also zero at the interior vertex 8 as the constraint is propagated into the bulk.)

Thus we have a situation where only two of the equations are independent, and where the consistency is guaranteed by the constraint. To solve the equations symmetrically, we take

$$\lambda = e_8, \tag{3.16}$$

$$\mu = \alpha_1 e_9 + \alpha_2 e_{10} + \alpha_4 e_{12} \tag{3.17}$$

to be arbitrary parameters. We then obtain

$$e_9 = \frac{1}{2\Sigma}(B + \lambda X_{24} + 2\bar{\alpha}_1\mu), \tag{3.18}$$

$$e_{10} = \frac{1}{2\Sigma}(C + \lambda X_{14} + 2\bar{\alpha}_2\mu), \tag{3.19}$$

$$e_{12} = \frac{1}{2\Sigma}(D + \lambda X_{12} + 2\bar{\alpha}_4\mu), \tag{3.20}$$

where $\Sigma = |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_4|^2$ and $X_{ij} = |\alpha_i|^2 + |\alpha_j|^2 - \alpha_i\bar{\alpha}_k - \alpha_j\bar{\alpha}_k$ with $i, j, k = 1, 2, 4$ and $k \neq i, j$. $B$, $C$ and $D$ are the expressions on the right-hand sides of the $e_9$, $e_{10}$ and $e_{12}$ equations. We now substitute for $\bar{e}_9$, $\bar{e}_{10}$ and $\bar{e}_{12}$ in the boundary action. The total coefficients obtained for $\bar{\lambda}$ and $\bar{\mu}$ are both multiples of $^3R$ at the origin and so vanish.

What remains is an extremely long expression. We write the coefficient of $e_i$ in this as $W_i + U_i$ (for "wanted" and "unwanted"). We will now give the expressions for $i = 1, 3$, all others being obtainable by cyclic permutations of the indices.

$$4\Sigma W_1 = [\bar{e}_1(|\alpha_2|^2 + |\alpha_4|^2 - \alpha_2\bar{\alpha}_4 - \alpha_4\bar{\alpha}_2) + \bar{e}_2(-|\alpha_4|^2 + 2\alpha_1\alpha_4 + 2\bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\bar{\alpha}_2 - \alpha_1\bar{\alpha}_4 - \alpha_4\bar{\alpha}_2) + \bar{e}_4(-|\alpha_2|^2$$

$$+ 2\alpha_1\alpha_2 + 2\bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\bar{\alpha}_2 - \alpha_1\bar{\alpha}_4 - \alpha_2\bar{\alpha}_4) + 2\bar{e}_3(\alpha_4\bar{\alpha}_2 - |\alpha_4|^2) + 2\bar{e}_5(\alpha_2\bar{\alpha}_4 - |\alpha_2|^2) + 2\bar{e}_6(\alpha_1\bar{\alpha}_2 + \alpha_1\bar{\alpha}_4$$

$$- 2\bar{\alpha}_2\bar{\alpha}_4)](2 + \Sigma) - 2\bar{e}_1^8(|\alpha_2|^2 + |\alpha_4|^2 - \alpha_2\bar{\alpha}_4 - \alpha_4\bar{\alpha}_2) - 2\bar{e}_2^8(-|\alpha_4|^2 + 2\alpha_1\alpha_4 + 2\bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\bar{\alpha}_2 - \alpha_1\bar{\alpha}_4$$

$$- \alpha_4\bar{\alpha}_2) - 2\bar{e}_4^8(-|\alpha_2|^2 + 2\alpha_1\alpha_2 + 2\bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\bar{\alpha}_2 - \alpha_1\bar{\alpha}_4 - \alpha_2\bar{\alpha}_4) - 4\bar{e}_3^8(\alpha_4\bar{\alpha}_2 - |\alpha_4|^2) - 4\bar{e}_5^8(\alpha_2\bar{\alpha}_4$$

$$- |\alpha_2|^2) - 4\bar{e}_6^8(\alpha_1\bar{\alpha}_2 + \alpha_1\bar{\alpha}_4 - 2\bar{\alpha}_2\bar{\alpha}_4), \tag{3.21}$$

$$2\Sigma W_3 = [\bar{e}_1(\alpha_2\bar{\alpha}_4 - |\alpha_4|^2) + \bar{e}_2(\alpha_1\bar{\alpha}_4 - |\alpha_4|^2) + \bar{e}_4(\alpha_1\bar{\alpha}_4 + \alpha_2\bar{\alpha}_4 - 2\alpha_1\alpha_2) + 2\bar{e}_3|\alpha_4|^2 - 2\bar{e}_5\alpha_2\bar{\alpha}_4$$

$$- 2\bar{e}_6\alpha_1\bar{\alpha}_4](2 + \Sigma) - 2\bar{e}_1^8(\alpha_2\bar{\alpha}_4 - |\alpha_4|^2) - 2\bar{e}_2^8(\alpha_1\bar{\alpha}_4 - |\alpha_4|^2) - 2\bar{e}_4^8(\alpha_1\bar{\alpha}_4 + \alpha_2\bar{\alpha}_4 - 2\alpha_1\alpha_2)$$

$$- 4\bar{e}_3^8|\alpha_4|^2 + 4\bar{e}_5^8\alpha_2\bar{\alpha}_4 + 4\bar{e}_6^8\alpha_1\bar{\alpha}_4, \tag{3.22}$$

$$\Sigma U_1 = \bar{e}_2(|\alpha_1|^2 - \bar{\alpha}_2|\alpha_1|^2 + \alpha_1\alpha_4 + \alpha_4\bar{\alpha}_2 - \alpha_1\alpha_4\bar{\alpha}_2 + \bar{\alpha}_2^2) + \bar{e}_4(|\alpha_1|^2 - \bar{\alpha}_4|\alpha_1|^2 + \alpha_1\alpha_2 + \alpha_2\bar{\alpha}_4 - \alpha_1\alpha_2\bar{\alpha}_4 + \bar{\alpha}_4^2)$$
$$+ 2\bar{e}_3\bar{\alpha}_1\bar{\alpha}_2 + 2\bar{e}_5\bar{\alpha}_1\bar{\alpha}_4 + 2\bar{e}_6\bar{\alpha}_2\bar{\alpha}_4 - \bar{e}_1^8(\alpha_1\alpha_2 + \alpha_1\alpha_4 - \bar{\alpha}_1\bar{\alpha}_2 - \bar{\alpha}_1\bar{\alpha}_4) + \bar{e}_2^8(-|\alpha_1|^2 + |\alpha_2|^2 - \alpha_1\alpha_2 + \bar{\alpha}_1\bar{\alpha}_2$$
$$- \alpha_1\alpha_4 + \bar{\alpha}_2\bar{\alpha}_4) + \bar{e}_4^8(-|\alpha_1|^2 + |\alpha_4|^2 - \alpha_1\alpha_4 + \bar{\alpha}_1\bar{\alpha}_4 - \alpha_1\alpha_2 + \bar{\alpha}_2\bar{\alpha}_4) - 2\bar{e}_3^8\bar{\alpha}_1\bar{\alpha}_2 - 2\bar{e}_5^8\bar{\alpha}_1\bar{\alpha}_4 - 2\bar{e}_6^8\bar{\alpha}_2\bar{\alpha}_4,$$
$$\tag{3.23}$$

$$\Sigma U_3 = 2[\bar{e}_1\alpha_2\bar{\alpha}_1 + \bar{e}_2\alpha_1\bar{\alpha}_2 - \bar{e}_4\alpha_1\alpha_2(1 - \bar{\alpha}_4) + \bar{e}_1^8\alpha_1\alpha_2 + \bar{e}_2^8\alpha_1\alpha_2 + \bar{e}_4^8\alpha_1\alpha_2]. \tag{3.24}$$

We see that

$$\Sigma U_1 = [{}^3R(0) - {}^3R(8)]/2 - \bar{e}_1(|\alpha_2|^2 + |\alpha_4|^2 - \bar{\alpha}_1\bar{\alpha}_2 - \bar{\alpha}_1\bar{\alpha}_2 - \bar{\alpha}_1\bar{\alpha}_4) + \bar{e}_2(\bar{\alpha}_2^2 - |\alpha_4|^2 - \alpha_1\bar{\alpha}_2 + \alpha_1\alpha_4 + \alpha_4\bar{\alpha}_2$$
$$+ \bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\alpha_4\bar{\alpha}_2) + \bar{e}_4(\bar{\alpha}_4^2 - |\alpha_2|^2 - \alpha_1\bar{\alpha}_4 + \alpha_1\alpha_2 + \alpha_2\bar{\alpha}_4 - \bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\alpha_2\bar{\alpha}_4) + (\bar{e}_1^8 + \bar{e}_2^8 + \bar{e}_4^8)(|\alpha_2|^2$$
$$+ |\alpha_4|^2 - \alpha_1\alpha_2 - \alpha_1\alpha_4). \tag{3.25}$$

Repeatedly using ${}^3R(0) = {}^3R(8) = 0$, we then have the following expression for the boundary action:

$$\Sigma(e_1 U_1 + e_2 U_2 + e_3 U_3 + e_4 U_4 + e_5 U_5 + e_6 U_6) = [\bar{e}_1^8 + \bar{e}_2^8 + \bar{e}_4^8 - (\bar{e}_1 + \bar{e}_2 + \bar{e}_4)]^3 R(0)/2 + U, \tag{3.26}$$

where the remainder $U$ is given by

$$U = \{e_1[\bar{e}_1(\bar{\alpha}_1\bar{\alpha}_2 - \alpha_1\alpha_2 + \bar{\alpha}_1\bar{\alpha}_4 - \alpha_1\alpha_4) + \bar{e}_2(|\alpha_2|^2 + \bar{\alpha}_2^2 - \alpha_1\alpha_2 - \alpha_1\bar{\alpha}_2 + \alpha_4\bar{\alpha}_2 + \bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\alpha_4\bar{\alpha}_2) + \bar{e}_4(|\alpha_4|^2$$
$$+ \bar{\alpha}_4^2 - \alpha_1\alpha_4 - \alpha_1\bar{\alpha}_4 + \alpha_2\bar{\alpha}_4 + \bar{\alpha}_2\bar{\alpha}_4 - \alpha_1\alpha_2\bar{\alpha}_4)] + e_2[\bar{e}_1(|\alpha_1|^2 + \bar{\alpha}_1^2 - \alpha_2\bar{\alpha}_1 - \alpha_1\alpha_2 + \alpha_4\bar{\alpha}_1 + \bar{\alpha}_1\bar{\alpha}_4$$
$$- \alpha_2\alpha_4\bar{\alpha}_1) + \bar{e}_2(\bar{\alpha}_1\bar{\alpha}_2 - \alpha_1\alpha_2 + \bar{\alpha}_2\bar{\alpha}_4 - \alpha_2\alpha_4) + \bar{e}_4(|\alpha_4|^2 + \bar{\alpha}_4^2 - \alpha_2\bar{\alpha}_4 - \alpha_2\alpha_4 + \alpha_1\bar{\alpha}_4 + \bar{\alpha}_1\bar{\alpha}_4 - \alpha_1\alpha_2\bar{\alpha}_4)]$$
$$+ e_4[\bar{e}_1(|\alpha_1|^2 + \bar{\alpha}_1^2 - \alpha_4\bar{\alpha}_1 - \alpha_1\alpha_4 + \alpha_2\bar{\alpha}_1 + \bar{\alpha}_1\bar{\alpha}_2 - \alpha_2\alpha_4\bar{\alpha}_1) + \bar{e}_2(|\alpha_2|^2 + \bar{\alpha}_2^2 - \alpha_4\bar{\alpha}_2 - \alpha_2\alpha_4 + \alpha_1\bar{\alpha}_2$$
$$+ \bar{\alpha}_1\bar{\alpha}_2 - \alpha_1\alpha_4\bar{\alpha}_2) + \bar{e}_4(\bar{\alpha}_1\bar{\alpha}_4 - \alpha_1\alpha_4 + \bar{\alpha}_2\bar{\alpha}_4 - \alpha_2\alpha_4)] + 2e_3[\bar{e}_1(\alpha_2\bar{\alpha}_1 + \alpha_1\alpha_2) + \bar{e}_2(\alpha_1\bar{\alpha}_2 + \alpha_1\alpha_2)$$
$$+ \bar{e}_4(\alpha_1\alpha_2\bar{\alpha}_4)] + 2e_5[\bar{e}_1(\alpha_4\bar{\alpha}_1 + \alpha_1\alpha_4) + \bar{e}_2\alpha_1\alpha_4\bar{\alpha}_2 + \bar{e}_4(\alpha_1\bar{\alpha}_4 + \alpha_1\alpha_4)] + 2e_6[\bar{e}_1\alpha_2\alpha_4\bar{\alpha}_1 + \bar{e}_2(\alpha_4\bar{\alpha}_2 + \alpha_2\alpha_4)$$
$$+ \bar{e}_4(\alpha_2\bar{\alpha}_4 + \alpha_2\alpha_4)]\}. \tag{3.27}$$

The next step is to expand in powers of $k$, using $\omega_i = e^{ik_i}$, $\alpha_i = 1 - \omega_i$ (in contrast to the previous section, here we keep the binary notation for the $k_i$'s). For the remainder $U$ above, we obtain

$$U = 2i(k_1\bar{e}_1 + k_2\bar{e}_2 + k_4\bar{e}_4)^3 R(0) + O(k^4). \tag{3.28}$$

For $W_1$ and $W_3$, we have

$$2(k_1^2 + k_2^2 + k_4^2)W_1 = (\bar{e}_1 - \bar{e}_1^8)(k_2^2 + k_4^2 - 2k_2k_4) - (\bar{e}_2 - \bar{e}_2^8)(k_4^2 + 3k_1k_4 + 3k_2k_4 + k_1k_2) - (\bar{e}_4 - \bar{e}_4^8)(k_2^2 + 3k_1k_2$$
$$+ 3k_2k_4 + k_1k_4) + 2(\bar{e}_3 - \bar{e}_3^8)(k_2k_4 - k_4^2) + 2(\bar{e}_5 - \bar{e}_5^8)(k_2k_4 - k_2^2) + 2(\bar{e}_6 - \bar{e}_6^8)(k_1k_2 + k_1k_4$$
$$+ 2k_2k_4) + O(k^3), \tag{3.29}$$

$$(k_1^2 + k_2^2 + k_4^2)W_3 = (\bar{e}_1 - \bar{e}_1^8)(k_2k_4 - k_4^2) + (\bar{e}_2 - \bar{e}_2^8)(k_1k_4 - k_4^2) + (\bar{e}_4 - \bar{e}_4^8)(k_1k_4 + k_2k_4 + 2k_1k_2) + 2(\bar{e}_3 - \bar{e}_3^8)k_4^2$$
$$- 2(\bar{e}_5 - \bar{e}_5^8)k_2k_4 - 2(\bar{e}_6 - \bar{e}_6^8)k_1k_4 + O(k^3). \tag{3.30}$$

Thus our final expression for the boundary action is

$$\frac{1}{2(k_1^2 + k_2^2 + k_4^2)}\{e_1[(\bar{e}_1 - \bar{e}_1^8)(k_2^2 + k_4^2 - 2k_2k_4) - (\bar{e}_2 - \bar{e}_2^8)(k_4^2 + 3k_1k_4 + 3k_2k_4 + k_1k_2) - (\bar{e}_4 - \bar{e}_4^8)(k_2^2 + 3k_1k_2$$
$$+ 3k_2k_4 + k_1k_4) + 2(\bar{e}_3 - \bar{e}_3^8)(k_2k_4 - k_4^2) + 2(\bar{e}_5 - \bar{e}_5^8)(k_2k_4 - k_2^2) + 2(\bar{e}_6 - \bar{e}_6^8)(k_1k_2 + k_1k_4 + 2k_2k_4)$$
$$+ O(k^3)] + e_2(\ldots) + e_4(\ldots) + 2e_3[(\bar{e}_1 - \bar{e}_1^8)(k_2k_4 - k_4^2) + (\bar{e}_2 - \bar{e}_2^8)(k_1k_4 - k_4^2) + (\bar{e}_4 - \bar{e}_4^8)(k_1k_4$$
$$+ k_2k_4 + 2k_1k_2) + 2(\bar{e}_3 - \bar{e}_3^8)k_4^2 - 2(\bar{e}_5 - \bar{e}_5^8)k_2k_4 - 2(\bar{e}_6 - \bar{e}_6^8)k_1k_4 + O(k^3)] + 2e_5(\ldots) + 2e_6(\ldots)\}. \tag{3.31}$$

The coefficients of $e_2$ and $e_4$ can be obtained from those of $e_1$, and those of $e_5$ and $e_6$ from those of $e_3$, by cyclic permutation of indices.

This is to be compared with the expression for $h_{ij}^{TT}\bar{h}^{TTij}$ stated earlier, Eq. (2.42) supplemented by the gauge conditions of Eq. (2.48). Note that that expression does not distinguish between $\bar{e}_i e_j$ and $\bar{e}_j e_i$, and once this is taken into account, the expressions are identical (apart from an overall minus sign which arises from the fact that

the Regge calculus expressions are calculated from the changes in dihedral angles rather than the deficit angles). Thus we can write our expression for the action as

$$I = \int d^3\mathbf{k}\, h_{ij}^{TT}[\bar{h}^{TTij}(0) - \bar{h}^{TTij}(8)] \qquad (3.32)$$

which is the lattice equivalent of the continuum expression

$$I = \int d^3\mathbf{k}\, h_{ij}^{TT}\left(-\frac{\partial}{\partial x_8}\bar{h}^{TTij}\right) = \int d^3\mathbf{k}\,\omega_k h_{ij}^{TT}\bar{h}^{TTij} \qquad (3.33)$$

where $x_8$ is the coordinate normal to the boundary and $\omega_k = k_8 = \sqrt{k_1^2 + k_2^2 + k_4^2}$. In the last step we have made use of the usual replacement of a finite difference by a derivative, which is correct here up to higher order terms in $k$, and we do not need to distinguish here between the lattice and continuum version of $\omega_k$. Also we have used the behavior of $h(k)$ in a space with Euclidean signature, where, if the expansion is periodic in the 1-, 2-, and 4-directions, we shall have

$$h(k) = C\exp[i(k_1 x_1 + k_2 x_2 + k_4 x_4) - k_8 x_8], \qquad (3.34)$$

with $k_8^2 = k_1^2 + k_2^2 + k_4^2$ to satisfy the wave equation $\Box h = 0$. The ground-state wave function obtained is thus identical to the continuum result of Kuchař and Hartle.

## IV. THE EVIDENCE FOR SPIN TWO

In the weak field limit the lattice theory described by the partition function Eq. (1.1) is known to be equivalent to the continuum theory of a massless spin-two particle, as embodied in Einstein's general relativity with a cosmological constant term. One would hope that the local gauge invariance (continuous lattice diffeomorphism invariance) of the discrete gravitational action under metric deformations—taken sufficiently small so as not to violate the triangle inequalities [14]—would be powerful enough to ensure that the lattice theory still describes a regularized model for quantum gravity, even away from smooth manifolds. In this section we shall examine the evidence in support of the argument that the lattice theory, treated nonperturbatively and in the vicinity of the critical point at $G_c$ where the lattice continuum limit is formally defined, still describes a massless spin-two particle. A comparison will be made between the lattice results and those obtained recently in the continuum using a variety of perturbative ($2 + \epsilon$ expansion) and nonperturbative (renormalization group combined with a derivative expansion) methods. A second line of approach will be to compare the lattice results for the critical exponents of gravitation with what is known either exactly or approximately for other spin values (0, 1) in four dimensions, and look for a discernible trend.

The cornerstone for these kind of arguments is the basic idea of universality. It is known that the long-distance behavior of quantum field theories is to a great extent determined by the scaling behavior of the coupling constant under a change in momentum scale [3]. It is also well known that asymptotically free theories such as QCD lead to vanishing gauge couplings at short distances, while the opposite is true for QED and self-interacting scalar field theories in four dimensions. More generally, the fixed points of the renormalization group need not be at zero coupling, but can be located at some finite coupling, leading to nontrivial fixed points not necessarily accessible by perturbation theory, or possibly even more complex fixed lines and limit cycles [3,28].

The existence of nontrivial ultraviolet fixed points, at which the theory becomes scale invariant, corresponds in statistical mechanics language to the existence of one or more critical points. There the partition function develops nonanalyticities and singularities caused by infrared divergences associated with a divergent correlation length. Lattice gravity exhibits precisely such a transition [9,10] where, for example, the curvature fluctuation

$$\chi_{\mathcal{R}}(k) \sim \frac{\langle(\int\sqrt{g}R)^2\rangle - \langle\int\sqrt{g}R\rangle^2}{\langle\int\sqrt{g}\rangle} \qquad (4.1)$$

diverges at some $k_c$. Such a divergence signals a singularity in the partition function itself, since averages such as the average curvature $\mathcal{R}$ and the curvature fluctuation $\chi_{\mathcal{R}}$ are related to derivatives of $Z_L$ [of Eq. (1.1)], with respect to the gravitational coupling $k = 1/(16\pi G)$.

Simple scaling arguments allow one to determine the scaling behavior of correlation functions from the critical exponents which characterize the singular behavior of local averages in the vicinity of the critical point. The appearance of a singularity in the free energy $F(k)$ is caused by the divergence of the correlation length $\xi$, which close to the critical point at $k_c$ is assumed to behave as

$$\xi \equiv 1/m \sim_{k\to k_c} A_\xi(k_c - k)^{-\nu} \qquad (4.2)$$

and defines the exponent $\nu$. Since for the singular part of the free energy one expects $F_{\text{sing}}(k) \sim \xi^{-d}$ simply on dimensional grounds, one then obtains by differentiation with respect to $k$ for the curvature fluctuation

$$\chi_{\mathcal{R}}(k) \sim_{k\to k_c} A_{\chi_{\mathcal{R}}}(k_c - k)^{-(2-d\nu)}. \qquad (4.3)$$

The last expression allows, at least in principle, a direct determination of the critical exponent $\nu$. Large scale direct numerical studies of the lattice theory [10] give the values $\nu \approx 0.33$ and $G_c \approx = 0.626$, which suggests $\nu = 1/3$ for pure gravitation.

Apart from a detailed comparison between critical exponents (which will be done later in this paper), a number of direct and indirect arguments can be given

in support of the fact that the nonperturbative lattice theory still describes a massless spin-two particle in the vicinity of the critical point at $k_c$. Firstly the gravitational lattice action only propagates spin-two (transverse-traceless) degrees of freedom, as shown explicitly in the weak field expansion of the previous sections (the lattice gravitational functional measure is completely local, and does not contain any propagation terms, to any order of the weak field expansion). This result is further supported by rigorous work describing the convergence of the lattice action towards the continuum one for smooth enough manifolds [29,30]. Secondly, the static interaction of two heavy particles of mass $m$ described by two world lines kept at a fixed distance $d$ has been shown to scale consistently as the mass squared, as expected for gravitational type interactions [31]. In the following we will explore this delicate issue further, by pursuing the connection with available nonperturbative results in the continuum.

### A. Ultraviolet fixed point

One can contrast and compare the lattice results with what one obtains for quantum gravity in the continuum. Since gravity is not perturbatively renormalizable in four dimensions, one has to go to a lower dimension ($d = 2$) where the perturbative expansion becomes meaningful, and expand about that dimension. Similar expansions have been shown to be quantitatively very successful in scalar field theories [32,33], but the series are shorter and a significantly larger extrapolation is required in the gravitational case (for a general review of the diagrammatic field theory methods as applied to statistical mechanics models see [34–36]).

In the $2 + \epsilon$ perturbative expansion for gravity [37] (earlier references can be found in Refs. [38–40]) one analytically continues in the space-time dimension by using dimensional regularization, and applies perturbation theory about $d = 2$, where the theory is formally power counting renormalizable and Newton's constant is dimensionless. An expansion in the number of dimensions of course goes back to Wilson's original work [3], and since then similar methods have been shown to be quite successful in determining among others the critical properties of the $O(n)$-symmetric nonlinear sigma model above two dimensions [41]. This model is not perturbatively renormalizable either, yet describes a completely well-defined and physically relevant statistical spin system, namely, the universality class of the 3-d Heisenberg ferromagnet. The same dimensional expansion methods have been extended with some success to fermionic models as well [42].

In the gravitational case the dimensionful bare coupling is written as $G_0 = \Lambda^{2-d} G$, where $G$ is dimensionless and $\Lambda$ is an ultraviolet cutoff (not to be confused here with the scaled cosmological constant), corresponding on the lattice to a momentum cutoff of the order of the inverse average lattice spacing, $\Lambda \sim 1/l_0$. The method has of course its share of problems, as the Einstein action is a topological invariant in two dimensions, which leads to kinematic singularities in the propagator. In addition, to recover the physical case $d = 4$ requires a rather bold extrapolation from two dimensions. The series themselves are rather short and strong assumptions need to be made about the nature of possible singularities in the complex coupling constant plane (in particular the absence of singularities close to $d = 3$). Still, one can view this approach as providing some sort of gauge-invariant resummation of a specific set of subdiagrams which may or may not be ultimately relevant in $d = 4$.

A double expansion in $G$ and $\epsilon = d - 2$ then leads above two dimensions to a nonvanishing beta function

$$\beta(G) \equiv \frac{\partial G}{\partial \log \Lambda} = (d - 2)G - \beta_0 G^2 - \beta_1 G^3 + \cdots,$$
(4.4)

and consequently a nontrivial ultraviolet fixed point in $G$, since $\beta_0 > 0$ for pure gravity. Integrating Eq. (4.4) close to the fixed point, one obtains for $G > G_c$ a nonperturbative, dynamically generated mass scale

$$m = \Lambda \exp\left[-\int^G \frac{dG'}{\beta(G')}\right] \sim_{G \to G_c} \Lambda |G - G_c|^{-1/\beta'(G_c)}.$$
(4.5)

It should be noted at this point that Eq. (4.5) is essentially the same as Eq. (4.2), with slightly different notations. It also brings out the central importance of the exponent $\nu$, and how it relates to the scale dependence of the coupling $G$. The derivative of the beta function at the fixed point defines the critical exponent $\nu$ (which to lowest order is in fact independent of $\beta_0$),

$$\beta'(G_c) = -1/\nu.$$
(4.6)

In the previous expression $m$ is an arbitrary integration constant, with dimensions of a mass, and has to be associated with some physical scale to be determined (as in QCD) by physical considerations (we will argue that it is the analog of $\Lambda_{\overline{MS}}$ for gravitation). It would appear natural here to identify it with the inverse of a gravitational correlation length ($\xi = m^{-1}$), perhaps a length scale associated with some average long-distance curvature (more on this later). The above renormalization group result also illustrates in a direct way how the lattice continuum limit should be taken. It corresponds to taking the ultraviolet cutoff $\Lambda \to \infty$, and therefore $G \to G_c$, with $m$ held constant. For a fixed lattice cutoff, the continuum limit is approached by tuning $G$ to $G_c$.

The value of the universal critical exponent $\nu$ has important physical consequences, as it directly determines the running of the effective coupling $G(\mu)$, where $\mu$ is an arbitrary momentum scale. The renormalization

group tells us that in general the effective coupling will grow or decrease with length scale $r = 1/\mu$, depending on whether $G > G_c$ or $G < G_c$, respectively. The physical mass parameter $m$ is itself by definition scale independent, and therefore obeys a Callan-Symanzik renormalization group equation, which in the immediate vicinity of the fixed point takes on the simple form

$$\mu \frac{\partial}{\partial \mu} m(\mu) = \mu \frac{\partial}{\partial \mu} \{A_m \mu |G(\mu) - G_c|^\nu\} = 0 \quad (4.7)$$

with $A_m$ a numerical constant. As a consequence, for $G > G_c$, corresponding to the smooth phase, one expects for the running, effective gravitational coupling [10,15]

$$G(r) = G(0)\{1 + c(r/\xi)^{1/\nu} + O[(r/\xi)^{2/\nu}]\}, \quad (4.8)$$

with $c$ a calculable numerical constant of order one.[1] The physical renormalization group invariant mass $m = \xi^{-1}$ determines the magnitude of scaling corrections, and separates the short distance, ultraviolet regime from the large distance, infrared region. As already mentioned in the introduction, there are in fact indications that in the Euclidean lattice theory only the smooth phase with $G > G_c$ exists (since space-time becomes branched-polymerlike and therefore degenerates for $G < G_c$), which would then imply that the gravitational coupling can only *increase* with distance (this point will be discussed further in Sec. V).

Recently the continuum $2 + \epsilon$ expansion for gravitation has been pushed to two loops, giving close to two dimensions [37]

$$\beta(G) = (d - 2)G - \frac{2}{3}(25 - n_f)G^2$$
$$- \frac{20}{3}(25 - n_f)G^3 + \cdots, \quad (4.9)$$

for $n_f$ massless real scalar fields minimally coupled to gravity. After solving the equation $\beta(G_c) = 0$ to establish the location of the fixed point, one obtains for pure gravity ($n_f = 0$)

$$G_c = (3/50)(d - 2) - (9/250)(d - 2)^2 + \cdots \quad (4.10)$$

and therefore close to two dimensions

$$\nu^{-1} = -\beta'(G_c) = (d - 2) + \frac{9}{5}(d - 2)^2 + O(d - 2)^3, \quad (4.11)$$

which gives to lowest order $\nu^{-1} = 2$ independently of $d$, and $\nu^{-1} = 4.4$ at the next order in $d = 4$. The uncertainty in these results can perhaps best be judged by comparing to similar calculations in the scalar case, for which much

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[1]At very short distances $r \sim l_P$ one finds finite perturbative corrections to the potential as well, which can be computed analytically using weak coupling diagrammatic techniques [43].

longer series exist, and for which rather sophisticated resummation methods based on Pade-Borel transforms, conformal mappings, and incorporating asymptotic large order estimates, are available [32,33] (the methods of statistical field theory are discussed in detail in Refs. [34,35]). Unfortunately in general the convergence properties of the $2 + \epsilon$ expansion for the nonlinear sigma model are not encouraging, even when comparing to well-established results in $d = 3$ ($\epsilon = 1$) [41].

The $2 + \epsilon$ expansion is not the only method that has been applied in the continuum to extract quantitative information about nonperturbative properties of gravitation. In this context we should mention another set of related results for the critical exponents of quantum gravitation. Recently in a separate, approximate renormalization group calculation based on the Einstein-Hilbert action truncation [44] one finds in the limit of vanishing bare cosmological constant $\nu^{-1} = 2d(d - 2)/(d + 2) = 2.667$ in $d = 4$, and $\nu^{-1} \approx 1.667$ in a more elaborate truncation. In this paper the sensitivity of the results to the choice of gauge-fixing term and to the specific shape of the momentum cutoff is investigated as well. These more recent results extend earlier calculations for the exponent $\nu$ done by similar operator truncation methods, and described in detail in Refs. [45,46]. A quantitative comparison of these various continuum results with the lattice answer for $\nu^{-1}$ will be postponed until later in this paper.

### B. Geometric argument for $\nu = 1/3$

A simple geometric argument can be given in support of the exact value $\nu = 1/3$ for pure quantum gravitation. The vacuum-polarization induced scale dependence of the gravitational coupling $G(r)$ as given in Eq. (4.8) implies the following quantum corrected static gravitational potential, for a point source of mass $M$ located at the origin:

$$V(r) = -G(r)\frac{mM}{r}$$
$$= -G(0)\frac{mM}{r}\{1 + c(r/\xi)^{1/\nu} + \mathcal{O}[(r/\xi)^{2/\nu}]\} \quad (4.12)$$

and for intermediate distances $l_p \ll r \ll \xi$. As a result, the vacuum-polarization effects due to virtual graviton loops cause an effective antiscreening of the primary gravitational source $M$. Thus the effect of the running gravitational coupling $G(r)$ is to give rise to a new nonperturbative quantum contribution to the potential, proportional to $r^{1/\nu-1}$. Remarkably for $\nu = 1/3$ the additional contribution, now proportional to $r^2$, can be interpreted as being due to what ultimately appears as a uniform mass distribution surrounding the original source. Its origin lies with a nonperturbative graviton vacuum-polarization contribution, localized around the

point source, and of strength

$$\rho_0 = \frac{3cM}{4\pi\xi^3}. \tag{4.13}$$

Of course such a simple geometric interpretation fails unless the critical exponent $\nu$ for gravitation is exactly one-third. In fact in any dimensions $d \geq 4$ one would expect based on the geometric argument that $-\beta'(G_c) = \nu^{-1} = d - 1$, if the leading correction to the gravitational potential is due to a uniformly distributed, antiscreening cloud of virtual gravitons. These arguments rely of course on the lowest order result $V(r) \sim \int d^{d-1} p e^{ipx}/p^2 \sim r^{3-d}$ for single graviton exchange in $d > 3$ dimensions.

Equivalently, the running of $G$ can be characterized as being due to a tiny nonvanishing (and positive) nonperturbative gravitational vacuum contribution to the cosmological constant, with

$$\lambda_\xi = \frac{3cM}{\xi^3} \tag{4.14}$$

and therefore an associated effective curvature of magnitude $\mathcal{R} \sim G\lambda_\xi \sim GM/\xi^3$. It is amusing that for a very large mass distribution, the above expression for the curvature can only be reconciled with the naive dimensional estimate $\mathcal{R} \sim 1/\xi^2$, provided for the gravitational coupling $G$ itself one has $G \sim \xi/M$ [47].

### C. Random gravitational paths

Within the Feynman path integral formulation of quantum field theory, a well-known relationship exists between the properties of random paths and those of field correlations (see for example [48]). In this section the analogy will be exploited in trying to gain more insight on the specific values for the gravitational critical exponents.

In the simpler case of self-interacting scalar field theories a rigorous argument can be given [49] based on an exact geometric characterization of criticality in the $\lambda\phi^4$ theory and the Ising model, in and above four dimensions. The key element of the argument lies in the recognition of the fact that random walks representing the propagation of free particles in Euclidean space-time have fractal [50] dimensions $d_H = \nu^{-1} = 2$, with vanishing probability of self-intersection above $d = 4$. As a consequence these models are governed by mean field theory above $d = 4$, with mild logarithmic corrections to free field behavior at $d = 4$. They provide rigorous support for the original claim that in the infinite cutoff limit all scalar field theories are trivial in four dimensions [3].

Let us first illustrate these results for the simplest case of a free scalar field in $d$ dimensions with action

$$S = \frac{1}{2} \int_{x,y} \phi(x) M(x, y) \phi(y). \tag{4.15}$$

On a lattice one has for the matrix $M_{ij} = D_{ij} - S_{ij}$, where

$S$ is the (nearest-neighbor) hopping part, and the rest is the diagonal part $D_{ij} = (2d + m_0^2)\delta_{ij}$, with $d$ the dimensions and $m_0$ the bare mass. The propagator connecting point 1 to point 2 is then given in terms of the kernel $S$ by

$$G_{12} = \frac{1}{m_0^2 + 2d} \sum_{n=0}^{\infty} \frac{S}{m_0^2 + 2d} \tag{4.16}$$

or, equivalently, in terms of a sum over paths

$$G_{12} = \sum_{\text{paths } 1 \to 2} e^{-ml_{12}(\text{path})} \sim r_{12}^{-(d-1)/2} e^{-r_{12}/\xi} \tag{4.17}$$

where $l_{12}(\text{path})$ is the length of the random path connecting points 1 and 2. In the second part of the expression we have indicated the asymptotic behavior of the free propagator for large distances, which brings in the correlation length $\xi = m_0^{-1}$.

In its simplest form, the lattice partition function needed to generate the above random curves is given by

$$Z(\beta) = \mathcal{N} \int \prod_{i=1}^{N} d^D X_i \exp\left\{-\beta \sum_{i=1}^{N} |\mathbf{X}_i - \mathbf{X}_{i+1}|^\alpha\right\} \tag{4.18}$$

which for $\alpha = 1$ generates closed ($\mathbf{X}_{N+1} = \mathbf{X}_1$) piecewise linear curves embedded in $D$ Euclidean dimensions. For $\alpha = 2$ it is equivalent to the generating function for a one-dimensional $D$-component massless field theory with unit lattice spacing, with infrared divergences appearing as the size $N$ goes to infinity. In the limit of a large number of steps $N$ one obtains for the size of the random walk

$$\langle X^2 \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \mathbf{X}_i^2 \rangle \sim_{N \to \infty} N^{2/d_H} \tag{4.19}$$

with $d_H = 2$ for free Brownian motion, independently of $\alpha$. The Hausdorff dimension $d_H$ characterizes the deviation of an ensemble of random paths from what one would expect based on their topological dimension of one. Furthermore $d_H$ is known to be a universal number, i.e., independent of the specific choice for the measure over random paths, and describing general geometric properties of random curves in the limit of very long paths. The relation $\nu^{-1} = d_H$ for free fields follows as a direct consequence of the representation of the field correlation function in terms of a sum over random paths with fixed end points, as given by Eq. (4.17).

Below four dimensions nontrivial continuum behavior is expected for scalar fields, including non-Gaussian exponents and nontrivial fractal dimensions. Thus, for example, for an interacting scalar field in two dimensions $\nu = 1$ (Onsager solution of the Ising model), while for a self-avoiding random walk one has instead $d_H = \nu^{-1} = 4/3$ exactly in $d = 2$ [51]. Other nontrivial constraints, such as the requirement that the random walks do not

backtrack, are also expected to change the fractal dimension $d_H$.

In the gravitational case one is dealing with random paths associated with a massless particle of spin two. As a result new constraints on the nature of the random paths come into play, which are not present in the simpler case of a spinless scalar field. As discussed in Feynman and Hibbs [52], these constraints are in fact already rather complicated for the case of a particle of spin one-half, and give rise even in this simplest case to a set of nontrivial complex weights needed to correctly reproduce the continuum expression for the Dirac propagator. In four dimensions such paths involve Dirac projection operators $1 \pm \gamma_\mu$ [53].

On general grounds one would then be inclined to identify the value $\nu = 1/3$ found for four-dimensional gravitation with a fractal dimension of random gravitational paths $d_H = 3$. Unfortunately (or fortunately) the value $\nu = 1/3$ itself does not correspond to any known field theory or statistical mechanics model in four dimensions. For dilute branched polymers it is known that $\nu = 1/2$ in three dimensions [54], and $\nu = 1/4$ at the upper critical dimension $d = 8$ [55], so one would expect a value close to $1/3$ somewhere in between. A value for the fractal dimension close to 1 would indicate the paths have almost linear Euclidean geometry, while at the opposite end a very large fractal dimension would indicate the paths are largely collapsed to a very small region about the origin. The paths in this latter case are highly folded and to some extent self-intersecting. Therefore the value $d_H = 3$ found for quantum gravity would suggest a far greater degree of folding compared to the spinless case, for which $d_H = 2$.

One could further develop these arguments and, in analogy with the scalar case, conclude that below six space-time dimensions two random gravitational paths will have a nonvanishing probability of self-intersection. These arguments would imply that the "upper critical dimension" for gravity is six, above which the theory becomes in some sense noninteracting and therefore trivial.[2] Unfortunately this argument is probably flawed, as $d_H = 3$ holds only in $d = 4$, and presumably not at the upper critical dimension, if one indeed exists. On the other hand, the large-$d$ geometric estimate discussed previously, $\nu^{-1} = d - 1$, equates to two (the fractal dimen-
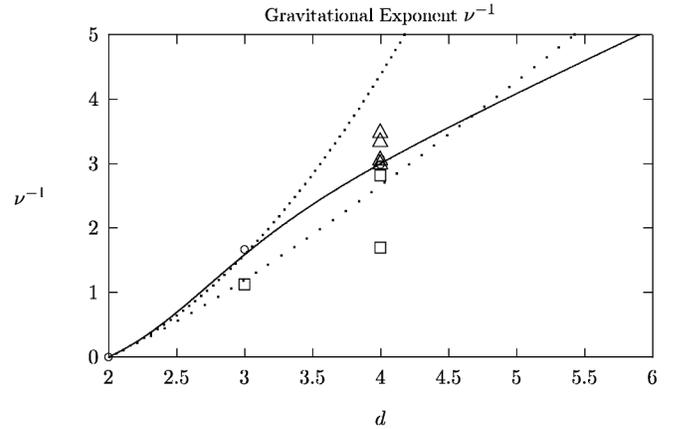


FIG. 1. Gravitational critical exponent $\nu^{-1} = -\beta'(G_c)$ as a function of dimension. Direct determinations from the Regge lattice (small circles at two, three and four dimensions), in the continuum using renormalization group truncation methods (squares), and by extrapolating lattice results from lower spin (triangles) are compared (see Tables I and II). The solid line is an interpolation through the Regge lattice results, incorporating the asymptotic behavior $d - 1$ for large $d$. The thin-dotted line is the analytic $2 + \epsilon$ result of Eq. (4.11). The dotted line is the continuum renormalization group result of Ref. [44]. The origin, methodology and comparison of the these various results are discussed further in the text.

sion for an unconstrained, spinless random walk) in three space-time dimensions, where it is in fact known that there can be no propagating, genuine spin-two degrees of freedom.

## D. Approaching quantized gravitation from spin zero and spin one

While only limited results exist for the nonperturbative scaling dimensions of quantum gravity in four dimensions, the same is not quite true for spin one (compact pure Abelian gauge theory) and spin zero (self-interacting scalar field theory). It has been known since the work of Wilson [3] that all local one-component scalar field theories in four dimensions are described by the Gaussian fixed point. The fact that these theories become noninteracting (up to logarithmic corrections) at large distances in four dimensions implies that the critical exponents and scaling dimensions coincide with those of a free field. In particular one finds for the universal critical exponent

$$\nu^{-1} = 2 \qquad (s = 0) \qquad (4.20)$$

a result which in fact can be proven rigorously [49] (for some unresolved issues and an unconventional point of view regarding the self-interacting scalar field theory in four dimensions see [56]). In three dimensions the interacting $\lambda\phi^4$ scalar field theory shares the same universal long-distance properties with the Ising ferromagnet. For

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[2]One can obtain a direct estimate for the upper critical dimension by equating $2\nu^{-1}(d) = d$, which after interpolating in $d$ the known Regge lattice results gives as a solution the surprisingly low value $d \approx 2.929$ (see also Fig. 1). This in turn would lead to the somewhat paradoxical conclusion that quantum gravity, in spite of being perturbatively nonrenormalizable, is weakly interacting in the infrared above three dimensions, in the same sense that self-interacting scalar field theories, in spite of not being perturbatively renormalizable, become weakly interacting at low energies above four dimensions.

both incarnations a wealth of numerical and analytical data exists on the critical exponents, and for the purposes of the present discussion the relevant result here is $\nu^{-1} \approx 27/17 = 1.5882$ [33].

In the massless spin-one case the results for the critical exponent $\nu$ are somewhat less unambiguous, and furthermore no exact results are available yet. If one considers just ordinary (noncompact) electrodynamics without matter, then the universality class is that of a free (Gaussian) field. Wilson's compact lattice electrodynamics (without matter),[3] on the other hand retains, at least in the naive continuum limit, the same degrees of freedom as ordinary electrodynamics (a field $U_{n\mu} = \exp i A_{n\mu}$ on each lattice link, with $A_{n\mu}$ periodic with period $2\pi$), but adds gauge-invariant nonlinear interactions [which in the naive continuum limit appear as terms of the type $(D_\mu F^{\mu\nu})^2$, $(F^{\mu\nu})^4$, etc.]. These extra terms make the theory self-interacting and nontrivial, and in Wilson's lattice formulation give rise to a phase transition between a Coulomb phase and a strong coupling confining phase [57]. An open question has remained for some time as to the nature of the transition and the value of the critical exponents, although specific scenarios have been put forward in the quoted references. An analytical variational real space renormalization group analysis of the Abelian U(1) lattice gauge theory in $d = 4$ [58] gave for the nontrivial mass gap (or leading thermal) exponent

$$\nu^{-1} = 2.5(2) \qquad (s = 1). \qquad (4.22)$$

The errors there can be estimated from an analysis of the results for $\nu$ using the same renormalization group methods in the 3-d U(1) spin system, which gave $\nu = 0.6702(6)$ [58], compared to the present best theoretical value [33] $\nu = 0.6698(15)$ based on the $\epsilon$ expansion about four dimensions as well as the 3-d $\phi^4$ field theory, and also in good agreement with the latest experimental value $\nu = 0.6706(5)$, as quoted again in the recent comprehensive review [33].

More recently [59,60] detailed numerical simulation studies of the Abelian U(1) compact lattice gauge theory have been performed with various additional action terms besides the standard Wilson action plaquette term. Away from the "Wilson line" (where the adjoint coupling $\gamma \neq 0$, and where the transition, being first order, is more difficult to analyze regarding the true singularity of the free energy located at the end of the metastable phase) the

value $\nu^{-1} = 2.57(6)$ is found for an adjoint gauge coupling $\gamma = -1/2$. Closer to the first-order Wilson line $\gamma = 0$ they find the larger value $\nu^{-1} = 2.81(6)$, but which perhaps should be discarded in view of the first-order nature of the intervening transition for $\gamma = 0$ [61], unless as mentioned above a more refined analytic continuation towards the true critical point is performed, in order to extract the required critical exponent characteristic of the end-point singularity.[4]

There is one item though that clouds the picture presented above regarding the spin content of the pure Abelian Wilson lattice gauge theory, and that is the presence of monopole degrees of freedom, which can be clearly exhibited in the Villain or periodic Gaussian formulation of the action [57]. When the lattice partition function is suitably transformed using the Poisson resummation formula, these scalar degrees of freedom can be exhibited directly and their properties investigated. Thus in the scaling region of the lattice theory one finds both photon and (integer valued) magnetic monopole degrees of freedom. We refer, for example, to Ref. [63] for a discussion of the rich features of the theory, which include monopole condensation and a whole spectrum of "gauge balls" in the strong coupling phase. In the following we shall assume that it still remains legitimate to compare the critical exponents for the compact Abelian pure gauge theory to the other two cases in question ($s = 0$ and $s = 2$), but caution the reader that such a comparison might not be entirely warranted. At the same time we hope that the present discussion will perhaps renew interest in a high-accuracy determination of the critical exponent(s) (including possibly multicritical behavior) for the compact U(1) gauge theory, which is, after all, the simplest four-dimensional lattice gauge theory still endowed with a continuous local gauge symmetry.

Given the above values for the critical exponent $\nu^{-1}$ for the massless spin-zero and spin-one fields, but bearing in mind the above-mentioned cautionary statements, it is tempting to use them to estimate independently the scaling dimensions for gravitation, using

$$\nu^{-1}(s) = s\nu^{-1}_{s=1} + (1 - s)\nu^{-1}_{s=0} \qquad (4.23)$$

(see Table I). Assuming $\nu^{-1} = 2$ for $s = 0$ and $\nu^{-1} = 5/2$ for $s = 1$ one then obtains $\nu^{-1} = 3$ for spin two, in good agreement with the previous discussion. In addition the simple formula

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[3]The partition function for Wilson's compact Abelian U(1) lattice gauge theory, without matter, is given by

$$Z = \prod_{n,\mu} \int_0^{2\pi} dA_{n\mu} \exp\left\{\beta \sum_{n,\mu,\nu} (\cos F_{n\mu\nu} - 1)\right\} \qquad (4.21)$$

with $A_{n\nu}$ the angle on the directed link from $n$ to $n + \hat{\mu}$, and $F_{n\mu\nu} = \Delta_\mu A_{n\nu} - \Delta_\nu A_{n\mu}$ with $\Delta_\mu$ a finite difference operator in the $\mu$ direction.

[4]One might wonder to what extent the quoted numerical results for the critical exponents, often obtained on relatively small lattices, are reliable. To further estimate the uncertainties in the numerical determination for $\nu$ in $d = 4$ one can, for example, compare to a recent high-accuracy determination of $\nu$ in the spin-zero case (Ising model) in $d = 4$, which yields $\nu^{-1} = 1.992(6)$ [62] on similarly sized lattices, and which is quite close to the expected exact value $\nu^{-1} = 2$.

TABLE I. Critical exponent $\nu^{-1}$ for a massless spin-two particle in four dimensions, as obtained indirectly either by extrapolation from other dimensions ($d = 3$ in row 1) or from information on other spin values (rows 2–4). Included in the table is also one direct (lattice) determination in $d = 3$. The unweighted average of all extrapolated values listed in the second column is $\nu^{-1} = 2.98$.

| Reference | $\nu^{-1}$ in $d = 3$ | $\nu^{-1}$ in $d = 4$ |
|---|---|---|
| [25] | 1.67(6) | 3.34 |
| [58] | ⋯ | 2.98(25) |
| [59] | ⋯ | 2.74(6) |
| [60] | ⋯ | 2.86(8) |
| Exact | 1.5882 | 3 |

$$\nu^{-1} = 2 + \frac{s}{2} \qquad (4.24)$$

gives for the exponent $\alpha/\nu = (2 - d\nu)/\nu = s$ in four dimensions, and therefore a divergence of the second derivative of the free energy of the remarkably simple form $C \sim \xi^s$.

Yet another, independent way of estimating the critical exponent for four-dimensional quantum gravity involves looking at the two lower-dimensional cases of pure gravity in $d = 2$ (where $\nu^{-1} = 0$) and pure gravity in $d = 3$ (where $\nu^{-1} \approx 1.67$) [25]. A linear extrapolation to four dimensions would then give $\nu^{-1} = 3.3$ which is quite consistent with what has been said in the previous discussion. It is worth noting here that the value for the exponent for three-dimensional gravity is tantalizingly close to the scalar field case. In the $2 + \epsilon$ expansion one finds $\nu^{-1} = 1.6$ while some relatively old direct numerical simulations in $d = 3$ give $\nu^{-1} = 1.67$. Both values are quite close to the $3 - d$ scalar field exponent $\nu^{-1} = 27/17 = 1.5882$ [33], which would be in line with the conjecture that in the infrared limit three-dimensional gravity belongs to the same universality class as a self-interacting single-component scalar field, with the scalar curvature playing the role of the scalar field $R \sim \Box \phi$, as in fact suggested some time ago by the authors of Ref. [64].

As one last exercise one can look at the case of fractional spin, which presumably corresponds to massless self-interacting fermions. In the spin-one-half case, which should apply to self-interacting fermions in four dimensions (such as those represented by the nonrenormalizable 4-d Gross-Neveu [42] and similar four-fermion models), one obtains $\nu^{-1} = 9/4 = 2.25$ in $d = 4$, which should be compared to the known values $\nu^{-1} \approx 27/17 = 1.5882$ in $d = 3$ (interacting 3-d fermions as described by Ising model exponents), and $\nu = 1$ in $d = 2$ (based on the rigorous equivalence between the two-dimensional critical Ising model and a free Majorana fermion). Had one extrapolated linearly these known results to four dimensions, one would have estimated 2.18, a value quite close

TABLE II. Direct determinations of the critical exponent $\nu^{-1}$ for quantum gravity, using a variety of analytical and numerical methods in three and four space-time dimensions. The unweighted average of all direct determinations for quantum gravity in four dimensions listed above gives $\nu^{-1} = 2.93$.

| Reference | $\nu^{-1}$ in $d = 3$ | $\nu^{-1}$ in $d = 4$ |
|---|---|---|
| [25] | 1.67(6) | - |
| [10] | ⋯ | 3.08(62) |
| [10] | ⋯ | 2.98(7) |
| [37] | 1.6 | 4.4 |
| [45] | 1.11(5) | 1.68(26) |
| [46] | ⋯ | 2.8(6) |
| [44] | 1.2 | 2.666 |
| Exact | 1.5882 | 3 |

to 9/4 (to within 3%). It is of course not obvious at this point how to interpret the above result in terms of a fermion random walk, which would have fractal dimension $d_h = \nu^{-1} = 9/4$. But the trend in the exponents is at least consistent with the expectation that the fractal dimension increases with embedding dimension $d$, as there are more dimensions to expand into.

The various estimates for the critical exponent are compared in Tables I (indirect determinations) and II (direct determinations). Table II provides a list of critical exponents for gravitation as obtained by direct perturbative and nonperturbative methods in three and four dimensions. As mentioned before, direct numerical simulations for the lattice model of Eq. (1.1) in four dimensions give for the critical point $G_c \approx 0.626$ in units of the ultraviolet cutoff, and one finds [10]

$$\nu^{-1} = 2.99(8) \qquad (s = 2) \qquad (4.25)$$

which is used for comparison in Table II and III.[5]

To conclude this section, one could reverse the line of the above arguments relating to the critical exponents for gravitation, and instead estimate the spin of the massless lattice graviton by considering the dependence of the measured exponent $\nu$ on the spin. Again optimistically assuming a roughly linear dependence of the exponent $\nu^{-1}$ on the spin, and using the more accurate values at $s = 0$ and $s = 1$ one obtains, from $\nu^{-1} = 2.98(7)$ [10], about $s = 1.96 \pm 0.38$, which is not far from the expected value of spin two.

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[5]Furthermore, the critical point obtained from the analytic continuation of the strong coupling (small $k$) branch of the free energy lies at the end of the metastable phase of the Euclidean theory, which is not necessarily a concern here as one is ultimately interested in the pseudo-Riemannian theory. Indeed one would not have expected otherwise, in view of the well-known and seemingly unavoidable conformal instability of the Euclidean theory.

TABLE III.   Summary table for the critical exponent $\nu^{-1}$ as a function of spin and dimension.

| $\nu^{-1}$ | $d = 2$ | $d = 3$ | $d = 4$ |
|---|---|---|---|
| Spin $s = 0$ | 1 | 1.588 | 2 |
| Spin $s = 1$ | 0 | 0 | 2.5 |
| Spin $s = 2$ | 0 | 1.588 | 3 |

## V. EXPONENTS AND LONG-DISTANCE QUANTIZED GRAVITATION

The result of Eq. (4.8) implies that the gravitational constant is no longer constant as in the classical theory, but instead slowly changes with scale due to the presence of weak vacuum-polarization effects,

$$G(r) = G(0)\{1 + c(r/\xi)^{1/\nu} + O[(r/\xi)^{2/\nu}]\}. \quad (5.1)$$

The exponent $\nu$, related to the derivative of the beta function evaluated at the nontrivial ultraviolet fixed point via the relation $\beta'(G_c) = -1/\nu \approx -3$ (see the previous discussion in Sec. IV and Ref. [10]) for pure quantum gravity, is supposed to characterize the universal long-distance properties of quantum gravitation, and is therefore expected to be independent of the specifics related to the nature of the ultraviolet regulator, introduced to make the quantum theory well defined.

The mass scale $m = \xi^{-1}$ in Eq. (5.1) determines the magnitude of quantum deviations from the classical theory, and separates the short distance, ultraviolet regime with characteristic momentum scale $\mu \ll m$ where nonperturbative quantum corrections are negligible, from the long-distance regime where quantum corrections are significant.[6] It should be emphasized here that most of these considerations are in fact quite general, to the extent that they rely mainly on rather general principles of the renormalization group and are in fact not tied to any particular value for the exponent $\nu$, although the value $\nu = 1/3$ clearly has some aesthetic appeal. Furthermore the dimensionless constant $c$ is, at least in principle, a calculable number. In Ref. [65] $c$ was estimated from the curvature correlation function at $c = 0.014(4)$, while more recently in Ref. [31] it was estimated to be $c = 0.056(27)$ from the correlation of Wilson lines. It is important to note that while the exponent $\nu$ is universal, $c$ in general depends on the specific choice of regularization scheme (i.e., lattice regularization versus dimensional regularization or momentum subtraction scheme).

It is worthwhile to note that the result of Eq. (5.1), which as discussed in Sec. IV is a direct consequence of Eq. (4.3) and the value for $\nu$ found in the lattice theory [defined by the partition function of Eq. (1.1) with higher derivative coupling ($a \to 0$) and functional measure parameter $\sigma = 0$], only applies to the simplest case of pure Einstein gravity with a bare cosmological term.[7] But one does not expect this to be the correct theory at sufficiently short distances $r \sim l_P$, where higher derivative curvature terms will come into play, either through direct inclusion and gravitational radiative corrections, or via matter field and the conformal anomaly. In this limit the gravitational potential will be further modified by exponential and logarithmic terms, as discussed in Ref. [66].

Let us recall here that in SU($N$) gauge theories and, in particular, in QCD, the theory of the strong interactions, a similar set of results is known to hold [18]. The crucial difference lies in the fact that there the scale evolution of the coupling constant can be systematically computed in perturbation theory due to asymptotic freedom, a statement which reflects the fact that such theories become free at short distances (up to logarithmic corrections). In non-Abelian gauge theories one has for weak coupling

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda_{\overline{MS}})} + 2\beta_0 \log\left(\frac{\mu}{\Lambda_{\overline{MS}}}\right) + \cdots \quad (5.2)$$

with $\beta_0 = \frac{1}{16\pi^2}(\frac{11N}{3} - \frac{2}{3}n_f)$ where $N$ is the number of colors, $n_f$ is the number of flavors of massless fermions, $\mu = 1/r$ is an arbitrary momentum scale, and $\Lambda_{\overline{MS}} = 217 \pm 25$ MeV is a scale parameter which determines the overall size of scaling violations. The dots indicate higher order loop corrections. Of course QCD does not determine $\Lambda_{\overline{MS}}$ (it appears as an integration constant of the Callan-Symanzik renormalization group equations), and therefore it has to be fixed by experiment from a measurement of the size of scaling violations, i.e., via the observed deviations from free field behavior at sufficiently high energies. It is a remarkable fact that a good fraction of QCD and electroweak standard model phenomenology simply follows from the result in Eq. (5.2) and its electroweak analog.[8]

If one pursues in a straightforward way the analogy with non-Abelian gauge theories one is led to conclude that in quantum gravitation the quantity $\xi$ plays the same role as $\Lambda_{\overline{MS}}$ in QCD, $\xi \leftrightarrow 1/\Lambda_{\overline{MS}}$. One major difference between the two theories lies of course in the fact that in one case the ultraviolet fixed point is at the origin $g^2 = 0$, while in the other it is not. As a result one has logarithmic

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[6]The very existence of a nonperturbative correlation length $\xi$ implies that the growth of the coupling $G(r)$ with distance cannot persist indefinitely. At distances much larger than $\xi$ qualitatively new behavior sets in, since a finite correlation length is expected to rapidly cut off any correlation effects.

[7]Light matter fields will modify the exponent $\nu$, and therefore the result of Eq. (5.1), provided their mass is small enough to contribute significantly to vacuum-polarization loops, $m \sim \xi^{-1}$.

[8]In QED the scale dependence of the vacuum-polarization effects is of course quite small, with the fine structure constant only changing from $\alpha(0) \sim 1/137.036$ at atomic distances to about $\alpha(m_{Z_0}) \sim 1/128.978$ at energies comparable to the $Z^0$ mass.

quantum corrections to free field behavior in QCD, but power law corrections in gravitation.

To determine the actual physical value for the non-perturbative scale $\xi$ further physical input is needed. It seems natural to identify $1/\xi^2$ with either some average spatial curvature, or perhaps more appropriately with the Hubble constant determining the macroscopic expansion rate of the present Universe [10,15], via the correspondence

$$\xi = 1/H_0, \tag{5.3}$$

in a system of units for which the speed of light is equal to one. This correspondence can be elaborated upon further. In the standard homogeneous isotropic Friedmann-Robertson-Walker model of classical relativistic cosmology one uses the line element

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right\} \tag{5.4}$$

with $k = 0, \pm 1$ and $H(t_0) = (\dot{R}/R)_{t_0}$ denoting today's expansion rate as determined from the field equations. It is well known that the presence of a small cosmological constant induces an exponential expansion of the scale factor at large times. In this very distant future, dominated by a nonvanishing cosmological constant, an equivalent description can be given in terms of the static isotropic de Sitter metric

$$ds^2 = (1 - H_\infty^2 r^2)dt^2 - (1 - H_\infty^2 r^2)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.5}$$

with a horizon radius $H_\infty = \lim_{t\to\infty}H(t)$. From Einstein's classical field equations one has

$$H_\infty^2 = \frac{8\pi G}{3}\lambda \equiv \frac{\Lambda}{3} \tag{5.6}$$

so the existence of an $H_\infty$ is equivalent to assuming the presence of a nonvanishing cosmological constant $\lambda$ (here we follow common convention in defining the scaled cosmological constant $\Lambda$, which should not be confused with the ultraviolet cutoff). It is presumably this quantity which should be identified with $\xi$. Given the rather crude nature of our arguments, in the following we shall not distinguish between $H_0$ and $H_\infty$, and simply take $H_0^{-1} \sim 10^{28}$ cm as today's estimate for the size of the visible Universe.[9]

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[9]While the observational evidence for a nonvanishing cosmological constant is quite recent, simplicial lattice theories of Euclidean quantum gravity can, as far as one knows, only be formulated with a *nonvanishing* positive bare cosmological constant ($\lambda > 0$). In the absence of such a constant the path integral does not converge for large edge lengths, and *no* stable ground state exists in the Euclidean theory [8,9,13,16].

The appearance of the renormalization group invariant quantity $\xi$ in the quantum evolution of the coupling $G$, a very large quantity by the identification of Eq. (5.3), suggests that the leading scale-dependent correction, which gradually increases the strength of the effective gravitational interaction as one goes to larger and larger length scales, should be extremely small. One would therefore expect the deviations from classical general relativistic behavior for most physical quantities to be in the end practically negligible, at least until one reaches very large distances $r \sim \xi$.

At this stage we should comment on an apparent paradox associated with the identification of the correlation length $\xi$ with $1/H_0$. Naively one would expect, simply on the basis of dimensional arguments, that the curvature scale close to the fixed point be determined by the correlation length[10]

$$\mathcal{R} \sim_{\mathcal{R}\to 0} 1/\xi^2, \tag{5.7}$$

but one cannot in general exclude the appearance of some nontrivial exponent. Indeed one finds for the vacuum expectation value of the Ricci scalar [see Eqs. (1.5) and (4.2)]

$$\mathcal{R}(\xi) \sim_{k\to k_c} \frac{1}{l_P^{2-d+1/\nu}\xi^{d-1/\nu}} \tag{5.8}$$

with $\nu = 1/3$ in four dimensions, and therefore $\mathcal{R} \sim 1/l_P\xi$. Only close to two dimensions one recovers the classical result, for which $\nu \sim 1/(d-2)$.

At first one might be tempted to identify the expectation value of the local scalar curvature with the quantity $H_0^2$, but further thought reveals that this correspondence is inconsistent with the identification $\xi = 1/H_0$ proposed before, and would only be legitimate if the local scalar curvature average $\mathcal{R}$ were to indeed correctly describe the rotation of vectors, parallel transported around very large loops. But the analogy with the local action density $\langle F_{\mu\nu}^2\rangle$ in non-Abelian gauge theories seems to suggest that such an identification might in fact be incorrect [67], and that the long-distance contribution to the curvature is not given by the local average in Eq. (5.8), but should instead be measured directly by computing the parallel transport of vectors around very large loops (with characteristic size much larger that the Planck length, $A \sim \xi^2 \gg l_p^2$). This is rather laborious and has not been done yet. Indeed at the other end one expects, for very short distances comparable to the size of the ultraviolet cutoff, significant fluctuations in the curvature with fluctuations of the order of $\mathcal{R} \sim 1/l_P^2$. In other words, the above arguments would suggest that the observable average curvature should be scale dependent.

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[10]There is at least in principle an even more naive expectation, namely $\mathcal{R} \sim 1/l_P^2$, which is excluded though by all numerical studies of simplicial lattice gravity.

On the other hand for the curvature correlation at fixed geodesic distance $d$ one obtains from simple scaling, and for "short distances" ($r \ll \xi$),

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\delta(|x-y|-d)\rangle_c \sim_{d \ll \xi} \frac{1}{d^{2(d-1/\nu)}} \sim \frac{A}{d^2}, \tag{5.9}$$

for $\nu = 1/3$ in four dimensions, and with $A$ a calculable constant of order one [10] (here $|x-y|$ denotes the geodesic distance between the points $x$ and $y$, which is a function of the background metric, and an integration over $x$ and $y$ is understood [65]). The above result is an immediate consequence of the relationship between the curvature-curvature correlation function and the second derivative of the partition function with respect to $G$, which determines the curvature fluctuation and thus the curvature correlation function at zero momentum. If one then considers the (scalar) curvature $R$ averaged over a very small spherical volume $V_r = 4\pi r^3/3$,

$$\overline{\sqrt{g}R} = \frac{1}{V_r} \int_{V_r} d^3\mathbf{x}\sqrt{g(\mathbf{x},t)}R(\mathbf{x},t) \tag{5.10}$$

one can compute the corresponding variance as

$$[\delta(\sqrt{g}R)]^2 = \frac{1}{V_r^2} \int_{V_r} d^3\mathbf{x} \int_{V_r} d^3\mathbf{y}\langle \sqrt{g}R(\mathbf{x})\sqrt{g}R(\mathbf{y})\rangle_c$$
$$= \frac{9A}{4r^2}. \tag{5.11}$$

As a result the rms fluctuation of $\sqrt{g}R$ averaged over a small spherical region of size $r$ is given by

$$\delta(\sqrt{g}R) = \frac{3\sqrt{A}}{2}\frac{1}{r}, \tag{5.12}$$

with a Fourier transform power spectrum of the form

$$P_\mathbf{k} = |\sqrt{g}R_\mathbf{k}|^2 = \frac{4\pi^2 A}{2V}\frac{1}{k}. \tag{5.13}$$

These results only hold for relatively short distances, and presumably get modified at distances $r \sim \xi$. The semiclassical answer would look quite different; in this limit $\nu \sim 1/(d-2)$ and therefore for the curvature correlation the distance dependence would tend to $1/d^4$ close to $d = 2$, where it makes sense to make a comparison.

One can go one step further and use Einstein's field equations to relate the local curvature to the local mass density. From the field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{5.14}$$

for a perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu \tag{5.15}$$

one obtains for the Ricci scalar, in the limit of negligible pressure,

$$R(x) \approx 8\pi G\rho(x). \tag{5.16}$$

As a result one obtains from Eq. (5.9) for the density fluctuations a power law decay of the form

$$\langle \rho(x)\rho(y)\rangle_c \sim_{|x-y| \ll \xi} \frac{1}{|x-y|^2}. \tag{5.17}$$

One can list a few other classical general relativistic results which are presumably affected by a running gravitational constant. It should be clear from the above discussion that in order for the quantum corrections to become quantitatively significant, one needs to look at rather large distance scales, comparable to $\xi$ or equivalently $r \sim 1/H_0$. For example, in standard classical cosmology one writes

$$H^2(t) = \frac{8\pi G(t)}{3}[\rho_\Lambda(t) + \rho_{DM}(t) + \rho_B(t)] \tag{5.18}$$

with $G$ usually assumed to be constant, and with the $\rho$'s representing various density contributions. On the left-hand side one usually neglects terms of order $k/R_0^2$ arising from the curvature of the hypersurface of homogeneity. In view of the what has been said before though it seems natural that $G(t)$ in the above expression should be taken at the largest length scale $H_0^{-1}$. Then one obtains for the overall coefficient a quantity slightly larger than the laboratory value $\sim G(0)$, namely $G(H_0^{-1}) \approx G(0) \times (1+c) > G(0)$. On the lattice one finds a rather small value for $c \approx 0.06$. One should recall however, as stated earlier, that while the exponent $\nu$ is universal, the quantity $c$ is not, and in general depends on the specific regularization scheme. More specifically, in ordinary lattice gauge theories one finds large but calculable finite renormalization factors, relating the lattice gauge coupling to the continuum coupling [68]. A more reasonable expectation would therefore be that $G(H_0^{-1})$ is related to $G(0)$ by a constant of proportionality which is roughly of order one. Additional cosmological and astrophysical arguments and proposed tests can also be found in Ref. [69].

In addition, one would have to find a consistent way to formulate the running of $G$ in invariant terms, which should give a more precise meaning to the quantity $r$ and makes its coordinate independence manifest. The problem of finding a consistent, invariant effective nonlocal action which incorporates the quantum running of $G$, using an extension of the methods of Ref. [70], is presently under study and the results will be presented in a separate forthcoming publication.

## VI. CONCLUDING REMARKS

In this paper we have examined some aspects of the connection between lattice and continuum models for quantum gravity. In particular the aim of the paper was to elucidate the relationship between the more recent simplicial lattice results, which do not rely on the weak

field expansion and are therefore inherently nonperturbative, and the semiclassical Euclidean functional integral expansion in the continuum.

The first issue addressed was the very definition of the notion of spin content in the lattice theory. Proceeding from the Euclidean Feynman path integral approach, we have constructed the lattice analog of the semiclassical expansion for the ground-state functional of linearized gravity. Two procedures were followed. The first one relied on constructing directly a lattice expression for the exponent of the ground-state functional, obtained by transcribing the continuum expression in terms of lattice variables. There one proceeds from the lattice expression for the gravitational action, restricted to its contribution on a fixed time slice, and supplemented by the appropriate vacuum gauge conditions. A crucial ingredient in this method is the correct identification of the correspondence between continuum degrees of freedom (the metric) and the lattice variables (the squared edge lengths). The resulting lattice expression is then equivalent to the continuum one by construction.

The second procedure relies instead only on the expression for the lattice gravitational action, as expanded in the weak field limit, and determines the explicit lattice form for the ground-state functional for linearized gravity by performing explicitly the necessary lattice Gaussian functional integrals. The resulting discrete expression can then be compared to the continuum one by systematically reexpressing the edge lengths in terms of the metric. It is encouraging that the resulting lattice expression completely agrees with what is found by using the previous method.

It is advantageous in performing the above calculation to introduce spin projection operators, which separate out the spin-zero, spin-one and spin-two components of the gravitational action. As a by-product one can show that the lattice gravitational action only propagates massless spin-two (or transverse-traceless) degrees of freedom in the weak field limit, as is the case in the continuum. Furthermore, as expected the lattice ground-state functional for linearized gravity only contains these physical modes.

We further discussed the fact that an explicit construction of the ground-state wave functional of linearized lattice gravity in terms of the lattice transverse-traceless modes makes it possible, at least in principle, to compare the lattice and continuum results in the limit of small curvatures. After imposing appropriate boundary conditions at infinity, by suitably restricting the values for the edge lengths on the lattice boundary such that the deficit angle is zero there, one would then have to enforce the lattice vacuum gauge conditions of Eq. (2.48) as well, so as to make contact with the semiclassical lattice functional of Eq. (2.42). Since no gauge fixing is required for determining invariant averages obtained via the partition function of Eq. (1.1), the gauge conditions would have to be imposed on each edge length configuration, by progressively applying local gauge transformations, such as the ones discussed in Ref. [14]. It is then expected that after such a transformation the edge distributions on a fixed time slice should follow closely the distribution of Eq. (2.42), if indeed as expected the only surviving physical modes are transverse traceless. We hope to return to this interesting problem in a future publication.

In subsequent sections of the paper we have systematically examined the relationship between recent nonperturbative results obtained in the lattice theory and the corresponding calculations as performed in the continuum theory. The latter suggest the presence of a nontrivial ultraviolet fixed point in $G$, and in some cases have led to definite predictions for the universal critical exponent of quantum gravitation, which can therefore be compared—even quantitatively—to the lattice results.

Besides relying on the recent lattice and continuum results for quantum gravitation, one can also independently try to estimate the gravitational scaling dimensions based on what is known based on exact and approximate renormalization group methods for spin zero (self-interacting scalar field in four dimensions) and spin one (Abelian noncompact gauge theories), for which a wealth of information is available on the critical indices. We have argued that these results too are remarkably consistent with what is known about the gravitational exponents in four dimensions, both from the discrete as well as from the continuum side. We have also presented a simple geometric argument which interprets the value for the gravitational exponent $\nu^{-1} = 3$.

In the last section of the paper we have discussed some (almost immediate) physical implications of recent lattice and continuum results, with an emphasis on the small expected deviations from classical general relativity expected at sufficiently large scales due to the running of $G$. We have argued that it is an almost inevitable consequence of the existence of an ultraviolet fixed point that the gravitational coupling becomes scale dependent, with power law corrections involving the correlation length. In analogy with non-Abelian gauge theories, and in the absence of any other likely physical candidate, it seem natural to identify the nonperturbative scale determining the size of deviations from classical gravitation with $1/H_0$, as suggested in Ref. [15].

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