

Time scale for loss of massive vector hair by a black hole and its consequences for proton decay

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It has long been known that matter charged under a broken $U(1)$ gauge symmetry collapsing to form a black hole will radiate away the associated external (massive) gauge field. We show that the time scale for the radiation of the monopole component of the field will be on the order of the inverse Compton wavelength of the gauge boson (assuming natural units). Since the Compton wavelength for a massive gauge boson is directly related to the scale of symmetry breaking, the time scale for a black hole to lose its gauge field “hair” is determined only by this scale. The time scale for Hawking radiation, however, is set by the mass of the black hole. These different dependencies mean that for any (sub-Planckian) scale of symmetry breaking we can define a mass below which black holes radiate quickly enough to discharge themselves via the Hawking process before the gauge field is radiated away. This has important implications for the extrapolation of classical black hole physics to Planck-scale virtual black holes. In particular, we comment on the implications for protecting protons from gravitationally mediated decay.

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I. INTRODUCTION

In the early 1970s Beckenstein and Teitelboim proved that a static bare black hole can be endowed with no exterior classical massive or massless scalar fields, nor with exterior classical massive vector fields [1]. At the same time, Price explored the mechanism by which matter collapsing to form a black hole could divest itself of a classical massless scalar field [2]. He concluded that the mass of the nascent black hole will set the time scale for a massless scalar monopole field to radiate away.

This leaves open the question of the time scale for the loss of a massive vector field. The question is nontrivial, since setting the vector mass to zero results in the case of electromagnetism (unbroken $U(1)$ gauge symmetry). It is well known that an external electric monopole field can persist even in the limit of a static black hole, where we recover the Reissner-Nordström solution. We are led, then, to consider whether a massive vector monopole will decay with a time scale set by the vector boson mass or whether (as in the case of a scalar field) the time scale is determined by the black hole mass. In the former case we recover electromagnetism in the continuous limit of small boson mass, while in the latter situation a massless photon represents a discontinuous jump from the physics of massive vector fields.

Coleman, Preskill and Wilczek, in their work on discrete gauge symmetries and black holes, state (without proof) that the correct answer is the continuous one and that the lifetime of a classical massive vector monopole field is set by the mass of the field itself [3]. In this work, we show that their assertion is correct. We demonstrate this by adapting the detailed calculational techniques

used by Price in his study of massless vector fields [2,4] to the case of a massive vector monopole field.

This answer has important consequences for black hole phenomenology. Black holes are postulated to radiate via the Hawking process [5]. When charged particles fall into a black hole, this process can theoretically recover the “lost” charge if and only if some external field generated by those charges remains [6–8]. Since black holes cannot sustain an external massive vector field forever, it has been assumed that particles charged only under a broken $U(1)$ gauge symmetry that fall into a black hole result in charge nonconservation—their charge will not be recovered through Hawking radiation. Now we see that this need not be the case for all black holes. The time scale for Hawking radiation is related only to the mass of the black hole. Therefore, because the external field’s decay time scale and the black hole’s radiation time scale depend on parameters that are totally independent of one another, we can define a regime in which the black hole will discharge quickly enough to respect a broken symmetry.

This result has particular relevance to the assumptions made about virtual (Planck-scale) black holes and the so-called “spacetime foam” (see, e.g. [9]). It has been assumed that since classical black holes do not respect a broken symmetry, quantum processes involving black holes will also violate such symmetries. In this work, however, we will show that the smallest classical black holes *will* in general respect broken symmetries. Thus, the extrapolation of conservation laws to quantum black holes must be reexamined.

II. DESPUN FIELD EQUATION FOR A MASSIVE VECTOR BOSON

We will consider the collapse of a star containing matter that is charged under a broken $U(1)$ symmetry by

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adapting the model defined in [2,4] for the cases of stars acting as a source of massless scalar and vector fields to our purpose. We will be concerned only with the field external to the star (and later the black hole) so we will work in the Schwarzschild geometry with line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \quad (1)$$

We will make the assumption that the charges in the collapsing star are small enough that the vector fields never have a significant impact on the external geometry.

To analyze the behavior of the external massive vector field, it will be very convenient to use the “despun” versions of the field equations, as emphasized by Price [4]. This formalism is briefly reviewed in Appendix A. In the case of a purely monopole field, the field equations reduce to

$$-\partial_{r^*}^2 \Phi_0 + \partial_t^2 \Phi_0 - \partial_{r^*} \left[\left(1 - \frac{2M}{r}\right) \frac{2}{r} \Phi_0 \right] = -\mu^2 \left(1 - \frac{2M}{r}\right) \Phi_0 \quad (2)$$

where r^* is defined by

$$r^* = r + 2M \ln \left(\frac{r}{2M} - 1 \right) + C \quad (3)$$

and C is an arbitrary constant. It is important to note that in these coordinates the horizon of the black hole will be found at $r^* = -\infty$. Note also that the quantity μ appearing in Eq. (2) is the inverse Compton wavelength of the vector boson field. We will assume throughout this paper that

$$\mu M \ll 1. \quad (4)$$

We will comment on this assumption in Sec. IV. For a monopole field, as explained in the Appendices, the despun quantity Φ_0 appearing in Eq. (2) is simply the radial component of the electric field (or the equivalent for the massive vector boson field). As a check on the form of our field equation, then, it is important to notice that in the limit $r \gg 2M$ Eq. (2) admits the flat space Proca solution

$$\Phi_0 = \frac{\mu e^{-\mu r}}{r} + \frac{e^{-\mu r}}{r^2}. \quad (5)$$

To make full use of Price’s results, we must make one further simplification. We make the change of variables $\Phi_0 = \Psi/r$ which gives the convenient form

$$-\partial_{r^*}^2 \Psi + \partial_t^2 \Psi = \left(1 - \frac{2M}{r}\right) \left[\frac{6M}{r^3} - \frac{2}{r^2} - \mu^2 \right] \Psi. \quad (6)$$

III. CALCULATION OF DECAY TIME SCALE

A. Reflection of the field from the curvature potential

We are now in a position to use the field equations of the previous section to determine how long a massive vector monopole field will persist outside of a collapsing star. In his analysis of scalar monopole fields, Price used both analytical and numerical methods to show that the time scale for decay is set by M , the mass of the collapsing star [2]. We will attempt to adapt both of his methods to the case of a massive vector field.

We begin with the analytical approach. Price was able to show that the field equation for the despun piece of any radiatable massless field will reduce to the same form exhibited by Eq. (6)—namely, the field equation will be a one-dimensional wave equation with an effective potential [2,4]. In the case of our massive vector monopole field, the effective potential is

$$V_{\text{eff}} = \left(1 - \frac{2M}{r}\right) \left[\frac{2}{r^2} + \mu^2 - \frac{6M}{r^3} \right]. \quad (7)$$

This effective potential plays an important role in the loss of the external field after collapse of the star [2,4,10,11]. As seen in Fig. 1, the barrier is highly localized at a specific value of r^* (corresponding to $r \sim 3.5M$). The precise value of r^* depends on the choice of the

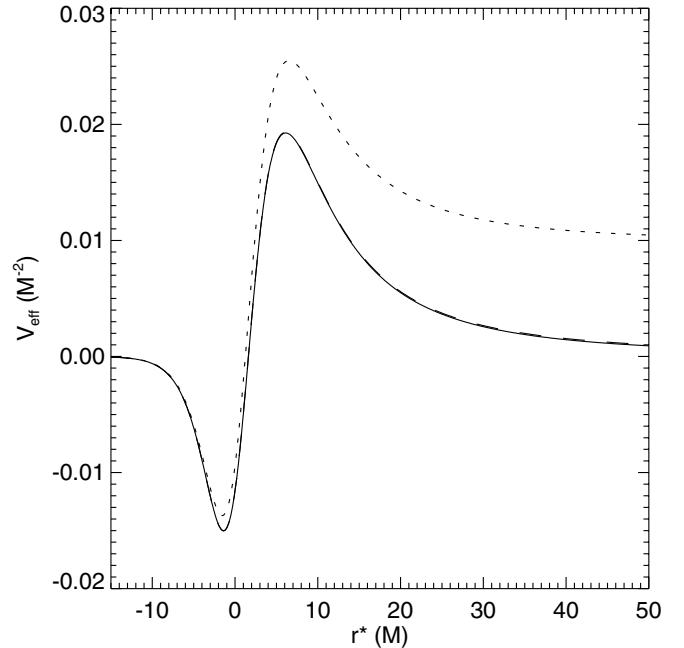


FIG. 1. Effective potential of Eq. (7) as a function of r^* , taking $C = 0$ in Eq. (3). The potential is plotted for three values of μ . The solid line represents $\mu = 0$ (unbroken symmetry), the dotted line $\mu = 0.1$, and the dashed line $\mu = 0.01$. The line for $\mu = 0.01$ is almost indistinguishable from the massless case. Note that the plot for $\mu = 0.1$ shows that the effects of μ on the potential are suppressed below $r^* = 3M$ ($r \sim 3.5M$).

arbitrary constant in Eq. (3). Waves propagating out from the surface of the star as it collapses beyond the location of the barrier will have to penetrate the potential barrier if the field far outside is to remain coupled to the sources within the star.

As emphasized in Price's approach [2,4,10,11], we will be most interested in the behavior of long wavelength waves as they approach the barrier. This is because time dilation requires that the field on the stellar surface settle down to a constant value as

$$\Psi = a + b \exp\left(-\frac{t - r^*}{4M}\right) \quad (8)$$

where a and b are constants. As the stellar surface approaches the horizon ($t \rightarrow \infty$ and $r^* \rightarrow -\infty$) the field becomes constant. This implies an effective redshift of the information propagating out from the star. If the external field is to remain in contact with the sources falling into the forming black hole, it can only do so via waves with wavelengths approaching infinity. We can find the transmission and reflection coefficients for such waves by examining the solution to the wave Eq. (6) in the three distinct regions (far outside the peak of the potential, the region where the potential is important, and well inside the peak of the potential) and then matching these solutions [10]. We will assume

$$\Psi = \psi(r) \exp(-i\omega t). \quad (9)$$

We begin with the solution well outside the peak of the potential. In mathematical terms, we want the limit $r^* \gg 2M$. Note that for very large values of r^* , Eq. (3) implies $r \sim r^*$. In this limit, the differential equation for $\psi(r)$ becomes

$$\frac{d^2}{dr^{*2}} \psi_{\text{far}} + \omega^2 \psi_{\text{far}} - \left[\frac{2}{(r^*)^2} + \mu^2 \right] \psi_{\text{far}} = 0 \quad (10)$$

which has a solution in terms of spherical Bessel functions of the first order [10,12]. To find the transmission coefficient for waves propagating out from the black hole, we must choose the linear combination which reduces to the form $T \exp(ikr^*)$ in the limit of very large r^* . This turns out to be

$$\psi_{\text{far}} = Th_1^{(1)}\left(\sqrt{\omega^2 - \mu^2} r^*\right). \quad (11)$$

Next we look at the solution in the region where the potential is important. Recall that we are assuming $\mu M \ll 1$ so that in this region the μ^2 contribution is negligible. Further, as we have stated, we are only interested in the reflection and transmission coefficients for long wavelength modes. Thus, upon assuming the time dependence of Eq. (9), we lose no information by taking the limit $\omega \ll M^{-1}$. With all these simplifications, the field equation for ψ near the peak of the potential becomes

$$\frac{d^2}{dr^{*2}} \psi_{\text{peak}} + \left(1 - \frac{2M}{r}\right) \left[\frac{6M}{r^3} - \frac{2}{r^2} \right] \psi_{\text{peak}} = 0. \quad (12)$$

This equation has exact solutions. One solution is easy to guess, as this equation is exactly the same field equation we would have found for a static, massless vector boson field. (The limits of small ω and small μ make this apparent.) This indicates that one valid solution is $\psi_{\text{peak}} \propto 1/r$. The full solution near the peak can be written

$$\psi_{\text{peak}} = \frac{A}{r} + B \left[4M^2 + Mr + \frac{r^2}{3} + \frac{8M^3}{r} \ln(r - 2M) \right] \quad (13)$$

where A and B are constants.

Finally, we consider the region inside the potential. Here, we are in the limit of large negative values of r^* . In this limit, the term $(1 - 2M/r)$ causes the potential to vanish exponentially rapidly in r^* . For this reason waves propagate freely near the horizon in r^* coordinates. We can therefore adopt the solution (for outgoing waves)

$$\psi_{\text{near}} = \exp(i\omega r^*) + R \exp(-i\omega r^*). \quad (14)$$

We now solve for the transmission and reflection coefficients by matching Eqs. (11), (13), and (14), to form a complete solution. We begin by matching the far zone to the region near the potential peak. Recall that we are interested in the limit of long wavelength, so we can approximate the spherical Hankel function of Eq. (11) as

$$\psi_{\text{far}} \sim \frac{(\omega^2 - \mu^2)(r^*)^2 T}{3} - i \frac{T}{\sqrt{\omega^2 - \mu^2} r^*} \quad (15)$$

where we have kept the leading order real and imaginary parts. Similarly, Eq. (13) is of the form

$$\psi_{\text{peak}} \sim \frac{A}{r^*} + \frac{B(r^*)^2}{3}. \quad (16)$$

Matching, we can see that

$$A = -i \frac{T}{\sqrt{\omega^2 - \mu^2}} \quad (17)$$

and

$$B = T(\omega^2 - \mu^2). \quad (18)$$

Next, we match the near zone to the region near the peak. Here, for long wavelength, Eq. (14) becomes

$$\psi_{\text{near}} \sim 1 + R + i\omega(1 - R)r^*. \quad (19)$$

Eq. (13) has the limit :

$$\psi_{\text{peak}} \sim \frac{A}{2M} + 2MBr^*. \quad (20)$$

Matching gives

$$A = 2M(1 + R) \quad (21)$$

and

$$B = \frac{i\omega}{2M}(1 - R). \quad (22)$$

By equating our two expressions for A and our two expressions for B we can arrive at two equations in the unknowns T and R which are solved to obtain

$$T = \frac{2i}{\frac{1}{2M\sqrt{\omega^2 - \mu^2}} + \frac{2M}{\omega}(\omega^2 - \mu^2)} \quad (23)$$

and

$$R = \frac{1 - \frac{4M^2}{\omega}(\omega^2 - \mu^2)^{3/2}}{1 + \frac{4M^2}{\omega}(\omega^2 - \mu^2)^{3/2}}. \quad (24)$$

We are left with a paradox. It seems that as ω approaches μ , the transmission coefficient drops to zero. Thus, we anticipate that, as discussed in [2,4], the external field will die away as the information from the source (the charge of the collapsing star) is reflected back and lost. It is important to note, however, that the transmission coefficient goes to zero in a way that is essentially independent of the precise value of μ (see Fig. 2). As long as μ is small compared to M , we see much the same behavior for T . This is a problem for the theory, since we find that even in the limit $\mu = 0$ taking $\omega \rightarrow 0$ gives $T = 0$. This would mean that even in the case of unbroken U(1) symmetry the external gauge field can be lost.

The loophole that saves the case of the photon (or other massless gauge boson) is contained in the setup of the problem rather than the solution. In the simultaneous

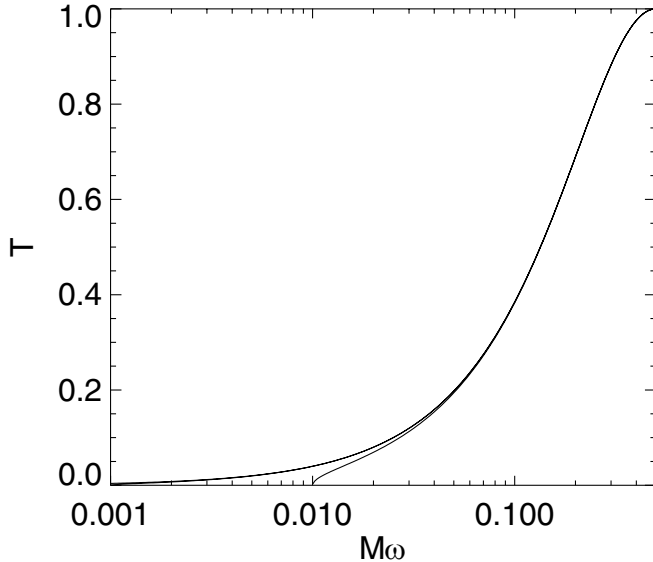


FIG. 2. Magnitude of the transmission coefficient ($|T|$) for outgoing massive vector waves to propagate through the effective potential barrier as a function of the wave frequency ω . Plots for the values $\mu M = 0.01, 0.001$, and 0.0001 are overlaid. Note that the curves are essentially indistinguishable.

limit $\mu = 0$ and $\omega = 0$, we do not actually have three regions worth of solutions to match to one another. In fact, when we take these limits, the exact solutions we derived for Eq. (13) will be valid at *any* r . This is a very important point, as we shall see.

The fact that the exception for massless bosons is not obvious leads us to be suspicious of the case of nonzero μ . The most dangerous case is $\omega \rightarrow \mu$, since here it seems that the potential in the far zone will strongly resemble that for the massless photon. There is an important difference, however. A nonzero mass requires that we retain the term $2M\mu^2/r$ in the potential. Thus, for $\omega \approx \mu$, the equation for ψ in the far zone becomes

$$\frac{d^2}{dr^{*2}}\psi_{\text{far}} + \left[(\omega^2 - \mu^2) + \mu^2 \frac{2M}{r^*} - \frac{2}{(r^*)^2} \right] \psi_{\text{far}} = 0. \quad (25)$$

This is in the form of a Coulomb wave equation, and can be solved with the Coulomb wave functions [12]

$$\psi_{\text{far}} = T \left[G_1 \left(-\eta, \sqrt{\omega^2 - \mu^2} r^* \right) + iF_1 \left(-\eta, \sqrt{\omega^2 - \mu^2} r^* \right) \right] \quad (26)$$

where

$$\eta = \frac{\mu^2 M}{\sqrt{\omega^2 - \mu^2}}. \quad (27)$$

When expanded near the interface with the region near the potential peak we find that this solution becomes (see Appendix C for a detailed derivation)

$$\begin{aligned} \psi_{\text{far}} \approx & \frac{iT\mu M^{1/2}}{3} \frac{\pi^{1/2} e^{\pi\eta/2}}{\sqrt{\sinh(\pi\eta)}} (1 + \eta^2)^{1/2} (\omega^2 - \mu^2)^{3/4} \\ & \times (r^*)^2 + \frac{T}{\mu M^{1/2}} \frac{\sqrt{\sinh(\pi\eta)}}{\pi^{1/2} e^{\pi\eta/2}} (1 + \eta^2)^{-1/2} \\ & \times (\omega^2 - \mu^2)^{-1/4} \frac{1}{r^*} \end{aligned} \quad (28)$$

This implies that we have replaced the A and B parameters of Eqs. (17) and (18) with the expressions:

$$A = \frac{T}{\mu M^{1/2}} \frac{\sqrt{\sinh(\pi\eta)}}{\pi^{1/2} e^{\pi\eta/2}} (1 + \eta^2)^{-1/2} (\omega^2 - \mu^2)^{-1/4} \quad (29)$$

and

$$B = iT\mu M^{1/2} \frac{\pi^{1/2} e^{\pi\eta/2}}{\sqrt{\sinh(\pi\eta)}} (1 + \eta^2)^{1/2} (\omega^2 - \mu^2)^{3/4} \quad (30)$$

Combining these expressions with Eqs. (21) and (22), which still hold, we can solve for T

$$\begin{aligned}
 T = 2 & \left[\frac{2\mu M^{3/2}}{\omega} \frac{\pi^{1/2} e^{\pi\eta/2}}{\sqrt{\sinh(\pi\eta)}} (1 + \eta^2)^{1/2} (\omega^2 - \mu^2)^{3/4} \right. \\
 & + \frac{1}{2\mu M^{3/2}} \frac{\sqrt{\sinh(\pi\eta)}}{\pi^{1/2} e^{\pi\eta/2}} (1 + \eta^2)^{-1/2} \\
 & \left. \times (\omega^2 - \mu^2)^{-1/4} \right]^{-1} \quad (31)
 \end{aligned}$$

and

$$\frac{1 - R}{1 + R} = \frac{4\mu^2 M^3}{\omega} \frac{\pi e^{\pi\eta}}{\sinh(\pi\eta)} (1 + \eta^2)(\omega^2 - \mu^2). \quad (32)$$

These expressions for T and R show the distinction between a massless photon and a massive vector boson in their explicit μ dependence. If $\mu \neq 0$, then the limit of vanishing $\omega^2 - \mu^2$ gives

$$\lim_{\omega \rightarrow \mu} \left(\frac{1 - R}{1 + R} \right) = 8M^2 \pi \mu^5 M^5. \quad (33)$$

This equation tells us that even for the case of vanishing $\omega^2 - \mu^2$ we can consistently define three regions of solution for ψ as long as $\mu \neq 0$. This fact allows us to construct transmission and reflection coefficients for the barrier. Therefore, we can be confident that the transmission of massive vector waves really does approach zero as ω becomes small. There are no technicalities to save an external massive vector field.

Now that we have decided that massive vector waves can be said to reflect from the curvature barrier, we must decide what that statement means. In particular, we need to understand the time scales for reflection and transmission. It is important to note that our construction of the transmission and reflection coefficients does not allow us to pinpoint the location where reflection takes place. Rather, we must consider the reflection to be a result of the accumulated effect of the potential barrier over its entire width. We can say with certainty, however, that it is the edge of the barrier that is farthest from the black hole which plays the decisive role. If we can consistently define such an edge as the place where the terms generated by the mass of the Proca field begin to dominate the curvature potential, we expect reflection and loss of the field. If we cannot define such an edge, then a static $1/r^2$ law holds and we expect to find a Reissner-Nordström black hole.

We can estimate the approximate location of this edge by comparing the magnitude of the terms in the potential of the far zone. It is apparent from Eq. (6) that the changeover from a curvature dominated potential to a mass-dominated potential will happen when

$$\frac{2}{r^2} \sim \mu^2 \quad (34)$$

which gives a characteristic location

$$r \approx \mu^{-1}. \quad (35)$$

To approximate the time taken for waves to begin reflecting, we can construct two separate arguments leading to the same basic result. The clearest argument is that unless ω is smaller than μ , the μ term is negligible in the field equation. Thus, our first inclination that the field is not massless must come from waves with a characteristic time scale μ^{-1} . Therefore, we expect the external field to persist (as in the massless case) until a time of order μ^{-1} has elapsed. Another way to arrive at this conclusion is to consider what we have already said about the mechanics of reflection. Since it is the far edge of the potential at $r \sim \mu^{-1}$ which will make the final difference between reflection and transmission, an outgoing wave cannot in effect be reflected until it has time to feel the effects of the *entire* potential barrier, all the way out to $r \sim \mu^{-1}$. Assuming then that the wave travels near the speed of light (this will at least be good for a lower bound) the time that must elapse between the onset of collapse and the first reflection of outgoing waves originating near the collapsing surface is of order μ^{-1} .

B. Numerical solution

The previous section has motivated the idea that an external massive gauge field arising from sources that collapse into a black hole will decay with a characteristic time scale set by the mass of the gauge field. Now, we wish to provide a more concrete demonstration. To do so, we can numerically integrate the field equation for Φ_0 .

The first step in this process is to establish the initial conditions for the integration. We will follow Price [2] in considering the case of a static star with initial radius $R_i \sim 4M$ which suddenly begins to collapse into a black hole of mass M . This situation implies that at the moment when collapse begins, the field far away from the star should resemble the flat space static Proca solution. Thus, we expect that the “electric field” far from the star will have the form

$$E = Q e^{-\mu r} \left[\frac{\mu}{r} + \frac{1}{r^2} \right] \quad (36)$$

where Q is the total net Proca charge (in some normalized units) contained within the star. Since we know that Φ_0 is equal to the “electric field” for the Schwarzschild space Proca solution (see the discussion in Appendix B and around Eq. (5)), we can use the flat space form for E to set the field value and first derivative of Φ_0 at some point far from the initial surface of the star. Then, by numerically integrating the field equation for Φ_0 outside the star (Eq. (2)), we can find the initial conditions for Φ_0 everywhere outside the star before collapse.

To make this integration simpler, we will assume that our end result will resemble the flat space Proca solution

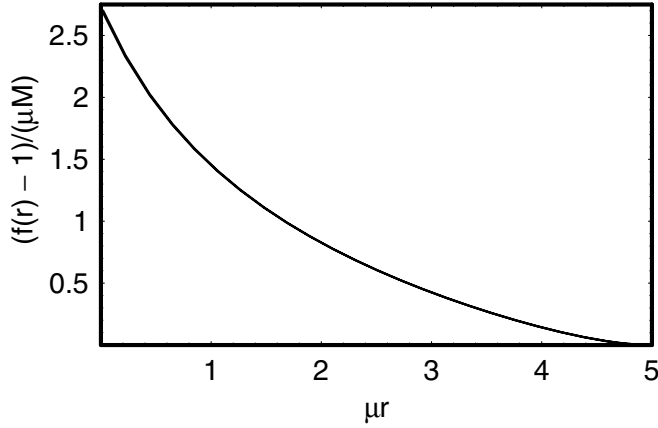


FIG. 3. Results of integration for $f(r)$ of Eq. (38). This figure shows the results for $\mu = 0.01$ and $\mu = 0.001$ overlaid (indistinguishable).

everywhere (this assumption will be justified in our results). With this assumption, it becomes useful to make the substitution

$$\Phi_0(r, t_i) \equiv f(r)e^{-\mu r} \left(\frac{1}{r^2} + \frac{\mu}{r} \right) \quad (37)$$

which results in the field equation for $f(r)$

$$\frac{2M - r}{r^3(1 + \mu r)} \left\{ - \left(1 - \frac{2M}{r} \right) (1 + \mu r) r^2 f''(r) + 2[r(1 + \mu r + \mu^2 r^2) - M(3 + 3\mu r + 2\mu^2 r^2)] f'(r) + 2M\mu^2 r(2 + \mu r) f(r) \right\} = 0. \quad (38)$$

In terms of $f(r)$, the boundary conditions for our numerical integration are particularly simple. Far from the location of the star ($r \gg M$) we expect $f = 1$ and $f' = 0$. We choose to take $f(5/\mu) = 1$; $f'(5/\mu) = 0$ as our far boundary condition and integrate in toward the initial radius of the star. The results of this integration are summarized in Figs. 3–5.

From these results, one can see that the initial conditions for Φ_0 just outside the star will deviate from the Proca solution at the order μM . Now we must consider how the field evolves in the area vacated by the collapsing star once collapse to a black hole has begun (the region between R_i and the eventual horizon $2M$). Here we must appeal to an analogy with electromagnetism.

In the case of a zero mass field, Φ_0^{EM} would be Q/r^2 everywhere outside the initial surface of the star. In this case, the Reissner-Nordström solution tells us that upon collapse of the star the Φ_0^{EM} field will simply evolve to equal Q/r^2 everywhere external to the black hole. This is important, because numerical integrations for the Proca field show that if $\mu^2 R_i^2 \sim \mu^2 M^2 \ll 1$ we obtain initial conditions that closely resemble those for a massless EM field. In fact, our initial conditions on Φ_0 near the surface

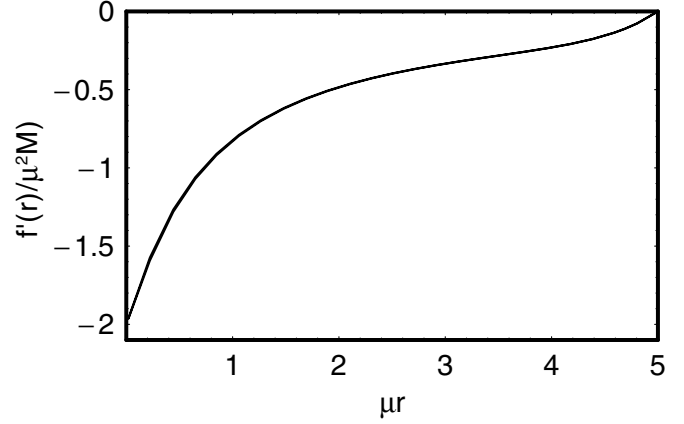


FIG. 4. Results of integration for $f'(r)$ of Eq. (38). This figure shows the results for $\mu = 0.01$ and $\mu = 0.001$ overlaid (indistinguishable).

of the star can be summarized

$$\Phi_0(Q, R_i, t_i) \equiv \Phi_0^{\text{EM}}(\tilde{Q}, R_i, t_i) \quad (39)$$

$$\partial_r \Phi_0(Q, R_i, t_i) = \partial_r \Phi_0^{\text{EM}}(\tilde{Q}, R_i, t_i) [1 + \mathcal{O}(\mu^2 R_i^2)] \quad (40)$$

$$\partial_r^2 \Phi_0(Q, R_i, t_i) = \partial_r^2 \Phi_0^{\text{EM}}(\tilde{Q}, R_i, t_i) [1 + \mathcal{O}(\mu^2 R_i^2)] \quad (41)$$

where $\tilde{Q} = Q[1 + \mathcal{O}(\mu R_i)]$ renormalizes the EM profile (this is allowed because the overall normalization is irrelevant to the evolution of the EM field—we would obtain a Q/r^2 profile regardless of the value of Q). If we now take R_i of order $4M$ then we see from Fig. 1 that at radial distances $r < R_i$ the effective potential governing the evolution of Φ_0 rapidly approaches that for electromagnetism (differing by less than order $\mu^2 M^2$). Putting these initial conditions and the effective potential into the differential equation governing the evolution of the field (Eq. (2)) shows us that the evolution of the Φ_0 field in the region vacated by the collapsing star will be governed by

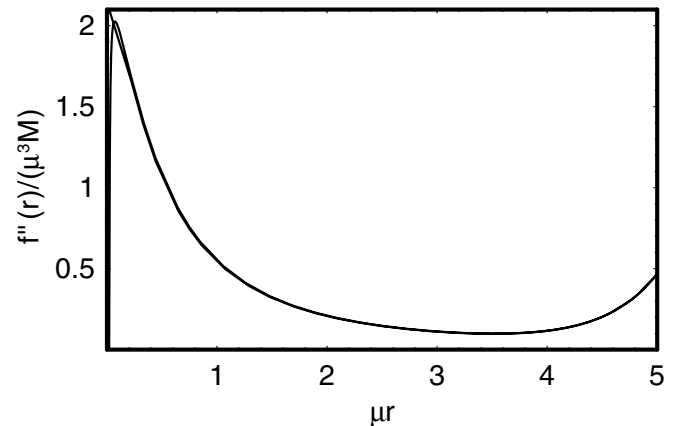


FIG. 5. Results of integration for $f''(r)$ of Eq. (38). This figure shows the results for $\mu = 0.01$ and $\mu = 0.001$ overlaid (indistinguishable).

a differential equation that differs from that for a massless EM field only at the order $\mu^2 M^2$. Further, these differences will be suppressed by the factor $(1 - 2M/r)$ which rapidly becomes small in this region. Similarly, the integration of the field equation will use boundary conditions that differ from the EM case only at the order $\mu^2 M^2$ and the collapse will characteristically require a time $M \ll \mu^{-1}$. Taken together, all of this implies that we expect the profile of a massive vector monopole field to differ by $\mathcal{O}(\mu^2 M^2)$ from the EM case in the near neighborhood of the event horizon ($r \sim M$) immediately after the star has completed its collapse to a black hole.

Our analogy with the EM case has shown us that the Proca field in the region vacated by the collapsing star will have a profile that is within order $\mu^2 M^2$ of the EM field result \tilde{Q}/r^2 immediately following the collapse. This, in turn, differs only at the order μM from the flat

space Proca profile. We have already shown through explicit numerical integration that the field outside the original surface of the star will also be within order μM of the flat space Proca profile. We can therefore assume that up to errors of order $\mu M \ll 1$

$$\Phi_0(t = 0) = Q \exp(-\mu r) \left[\frac{\mu}{r} + \frac{1}{r^2} \right] \quad (42)$$

describes the Proca field profile *everywhere* outside the horizon of the black hole immediately after collapse. Since we are trying to show that the field will persist well after the black hole has collapsed, we will begin our simulations after the black hole has already formed.

Using the static Proca field solution as our initial conditions, we numerically integrate the full field Eq. (2) for several values of μM . The results of these integrations are summarized in Figs. 6 and 7. These figures clearly show

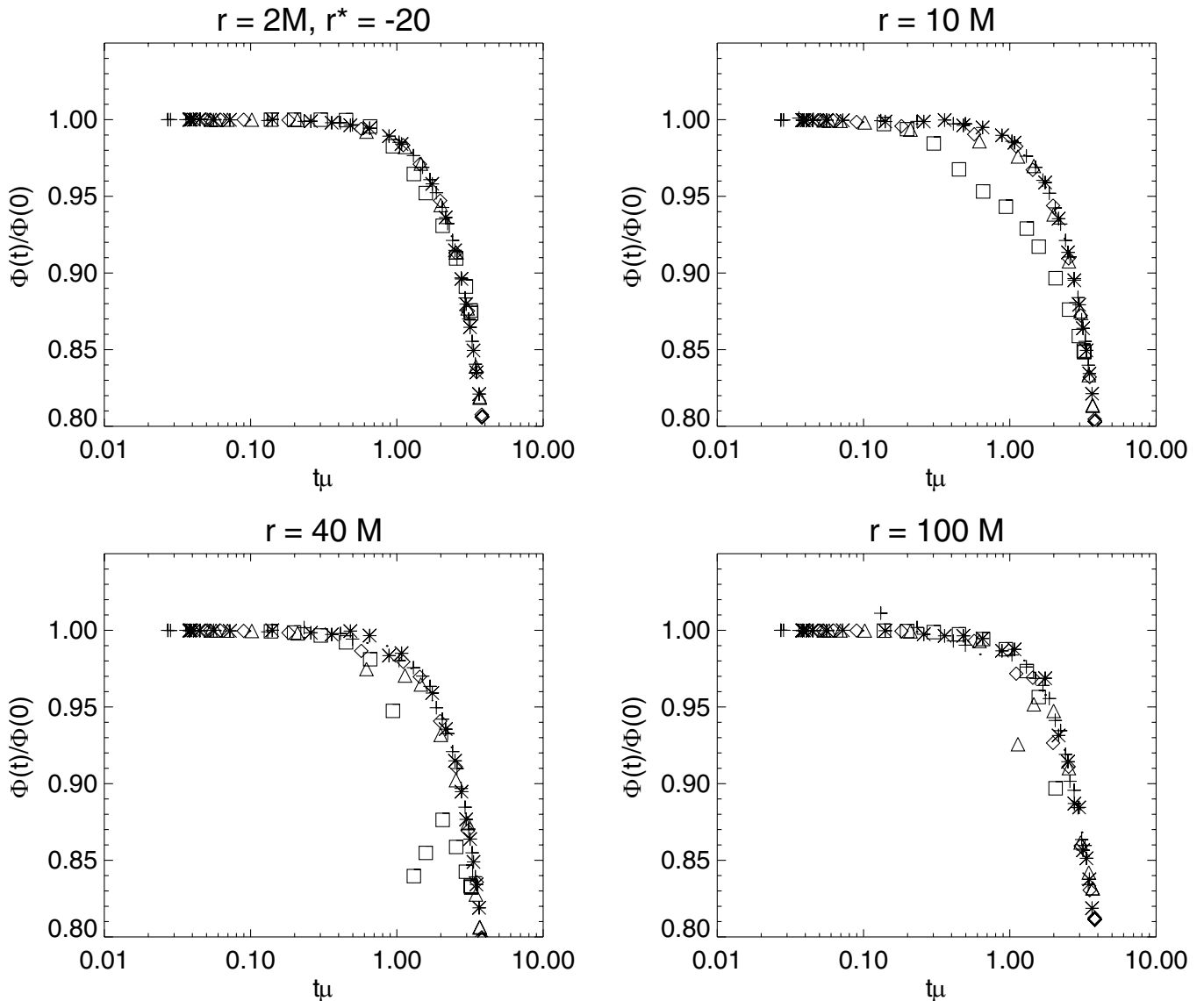


FIG. 6. Evolution of the Φ field for various values of μM : 0.001 (+); 0.0025 (*); 0.005 (·); 0.0075 (◇); 0.01 (Δ); 0.025 (□).

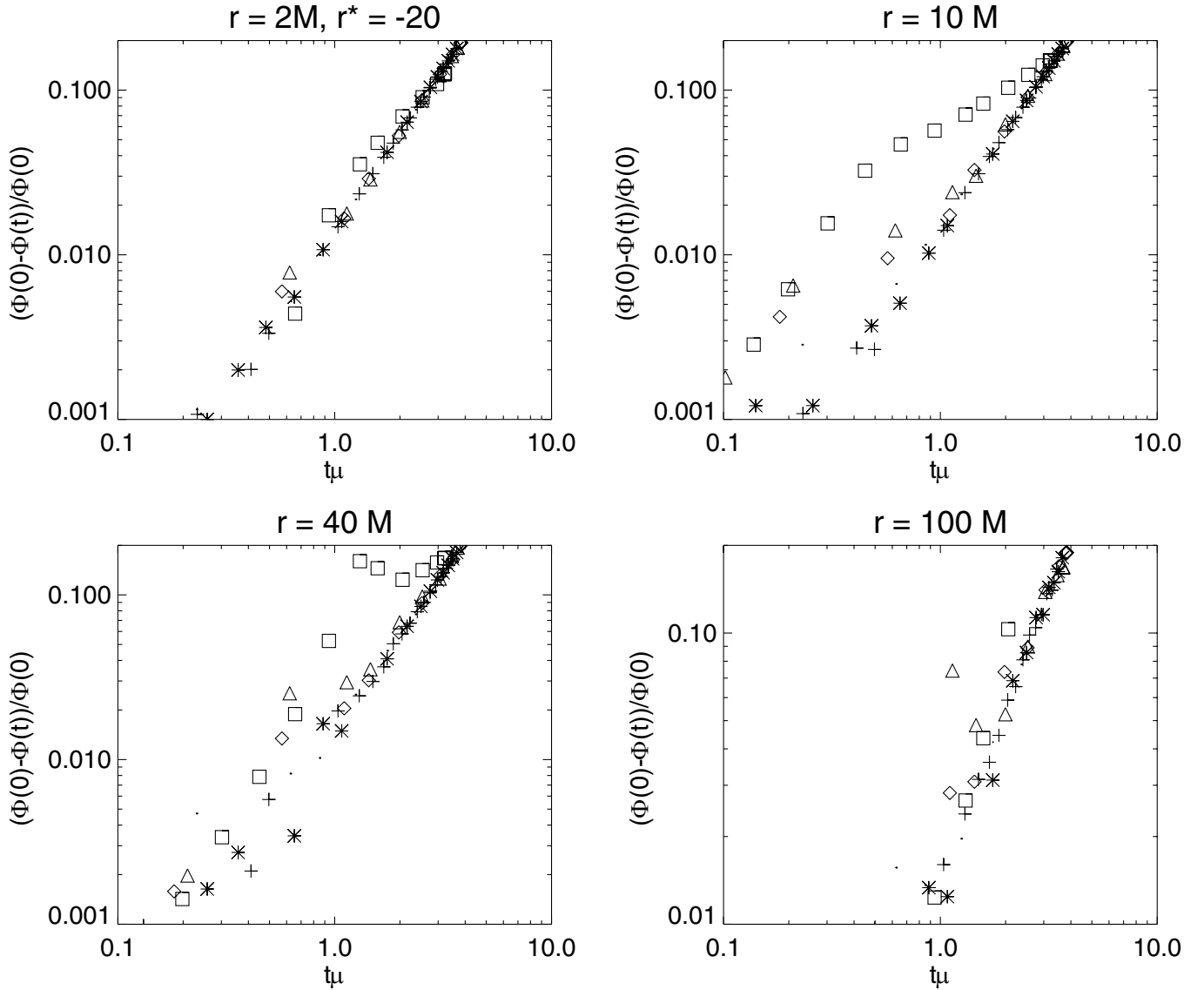


FIG. 7. Log plot of the evolution of $\Phi_0 - \Phi(t)$ for various values of μM : 0.001 (+); 0.0025 (*); 0.005 (·); 0.0075 (◇); 0.01 (△); 0.025 (□).

that the time scale for loss of the field is characteristically of order μ^{-1} , and that the field near the black hole remains almost completely undisturbed for a time approximately equal to μ^{-1} as long as $\mu M < 1/40$. For $\mu M > 1/40$ we can make no definite statement about the field behavior, since in this regime the field evolution is sufficiently violent that a more sophisticated numerical integration technique is needed (also, one should begin to consider evolution of the field during the collapse of the star).

IV. CONSEQUENCES

In this section we wish to discuss possible values of μM . We therefore switch from the natural units of general relativity ($G = \hbar = c = 1$) to the natural units of field theory ($\hbar = c = 1$, $G = 1/M_{\text{pl}}^2$). The major difference is

that M no longer has the units of radius. Instead, the Schwarzschild radius is $2M/M_{\text{pl}}^2$. Thus, rather than the condition $\mu M \ll 1$, the correct expression in field theory units is: $\mu M/M_{\text{pl}}^2 \ll 1$.

Once we have shown that a massive vector field around a star collapsing to a black hole will persist for a time of order μ^{-1} , we can draw an important conclusion. The lifetime of a black hole goes as M^3 [5]. Since a Planck-mass black hole is expected to live one Planck time, we have the approximate relation

$$t_{\text{evap}} \sim \frac{M^3}{M_{\text{pl}}^4}. \quad (43)$$

This approximation is essentially borne out by the calculations of Page [13], up to a numerical factor which depends on the particle species that are emitted by the

black hole. For the smallest black holes, one must include many known species not accounted for in Page's work and perhaps some wholly unknown species. These extra particles will tend to lessen the lifetime of small black holes, bringing the results of [13] closer to our own approximation.

As long as t_{evap} is less than μ^{-1} , we anticipate that the external field will persist for the *entire life* of the black hole (unless, of course, the charge is recovered before the complete evaporation of the black hole). If the black hole can evaporate while the external field persists, we anticipate the recovery of the charge [6–8]. We can therefore estimate the black hole mass below which a symmetry broken at the scale μ will obey charge conservation. The result is

$$M < \frac{M_{\text{pl}}^{4/3}}{\mu^{1/3}}. \quad (44)$$

In a traditional universe with the Planck-scale at about 10^{19} GeV, this implies that a symmetry broken at a scale of 10^4 GeV (just beyond the reach of current colliders) would be respected by black holes up to 10^5 Planck masses (approximately 1 g).

V. EXTRAPOLATIONS

It has historically been assumed that virtual or quantum black holes would not respect a broken symmetry because classical black holes do not. Here we have shown, however, that there is a strong case to be made for the claim that the smallest classical black holes will respect a broken symmetry. What constitutes a “small” black hole depends on the scale at which the symmetry is broken. Symmetries broken at small scales (high energies) require less massive (shorter-lived) black holes than symmetries broken at low energy. Since Planck-scale black holes are smaller and shorter-lived than any classical black hole, this result calls into question the conclusion that broken symmetries are not respected by quantum gravity.

This result could be of particular importance if there is a low scale for quantum gravity. Here, virtual black holes have been predicted to result in fast proton decay [14]. Now, however, we see that protons might be protected by a broken gauge symmetry. In fact, it is conceivable that the scale for such symmetry breaking can be pushed nearly to the quantum gravity scale, leaving the possibility that it decouples from physics at currently explored energies even in a low Planck scale universe.

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APPENDIX A: DERIVATION OF THE FIELD EQUATIONS

We begin our derivation of the field equations for a massive vector field in Schwarzschild spacetime by following Price in making use of the spinor formalism to write the field equation for a massive gauge boson in terms of scalar quantities. To obtain the correct expression for the field tensor $F_{\mu\nu}$ it is necessary to supplement the expressions given in Price's work [4] with the formalism found in the article by Pirani [15]. In this way we arrive at the expression

$$F^{\mu\nu} = \sigma_{AX}^{\mu} \sigma_{BY}^{\nu} F^{A\dot{X}B\dot{Y}} \quad (A1)$$

where Latin capitals denote spinor indices (see [15] for more detail).

We can now choose the connection $\sigma_{A\dot{X}}^{\mu}$ to be determined by the null tetrad chosen to parametrize our spacetime [16]. We follow Price in using the null tetrad

$$l^{\mu} = \left[\left(1 - \frac{2M}{r}\right)^{-1}, 1, 0, 0 \right] \quad (A2)$$

$$n^{\mu} = \frac{1}{2} \left[1, -\left(1 - \frac{2M}{r}\right), 0, 0 \right] \quad (A3)$$

$$m^{\mu} = \frac{1}{\sqrt{2}} \left(0, 0, \frac{1}{r}, \frac{i}{r \sin\theta} \right) \quad (A4)$$

We then choose

$$\sigma_{00}^{\mu} = l^{\mu}; \quad \sigma_{11}^{\mu} = n^{\mu}; \quad \sigma_{01}^{\mu} = m^{\mu}; \quad \sigma_{10}^{\mu} = \bar{m}^{\mu} \quad (A5)$$

where \bar{m}^{μ} denotes the complex conjugate of m^{μ} . The conversion to scalar fields is now made via the definition [4]

$$F_{A\dot{X}B\dot{Y}} = \frac{1}{2} (\epsilon_{AB} \Phi_{\dot{X}\dot{Y}}^* + \epsilon_{\dot{X}\dot{Y}} \Phi_{AB}). \quad (A6)$$

Using the rules for contracting spinor indices (see [15]), we can express the field tensor in the form

$$\begin{aligned} F^{\mu\nu} = & \frac{1}{2} [-l^{\mu} n^{\nu} (\Phi_{10} + \Phi_{10}^*) + l^{\nu} n^{\mu} (\Phi_{01} + \Phi_{01}^*) \\ & + m^{\mu} \bar{m}^{\nu} (\Phi_{10} - \Phi_{01}^*) + \text{c.c.} + (l^{\mu} m^{\nu} - l^{\nu} m^{\mu}) \Phi_{11} \\ & + \text{c.c.} + (m^{\mu} n^{\nu} - m^{\nu} n^{\mu}) \Phi_{00}^* + \text{c.c.}] \end{aligned} \quad (A7)$$

where c.c. denotes the complex conjugate of the term immediately preceding. Note that this expression for $F^{\mu\nu}$ is manifestly antisymmetric and real as long as $\Phi_{10} = \Phi_{01}$, which is guaranteed by the total symmetry of spin indices [4,15]. We desire an antisymmetric, real $F^{\mu\nu}$ so that we can make the standard definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (\text{A8})$$

We henceforth simplify our notation by introducing: $\Phi_0 \equiv \Phi_{10}$, $\Phi_{+1} \equiv \Phi_{00}$, and $\Phi_{-1} \equiv \Phi_{11}$.

We now have the tools necessary to derive the field equations for the Proca field. To obtain a convenient form, there is one more trick to use. We combine the homogeneous and inhomogeneous Proca equations (Maxwell equations for a massive vector field) into the form [15]

$$F_{;\nu}^{\mu\nu} + \frac{i}{2\sqrt{-g}}(\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta})_{;\nu} = \mu^2 A^\mu. \quad (\text{A9})$$

This yields the equations (assuming Φ_0 has the form of a spherical harmonic)

$$D(r^2\hat{\Phi}_0) = r\hat{\Phi}_{+1} + \mu^2 r^2[A^0 - S^{-1}(r)A^1] \quad (\text{A10})$$

$$2\Delta(r^2\hat{\Phi}_0) = 2r\hat{\Phi}_{-1} - r^2\mu^2[S(r)A^0 + A^1] \quad (\text{A11})$$

$$2D(r\hat{\Phi}_{-1}) = -l(l+1)\hat{\Phi}_0 + \mu^2 r^2\Theta[i\sin\theta A^3 - A^2] \quad (\text{A12})$$

$$2\Delta[rS(r)\hat{\Phi}_{+1}] = -l(l+1)S(r)\hat{\Phi}_0 + \mu^2 r^2 S(r)\Theta[A^2 + i\sin\theta A^3]. \quad (\text{A13})$$

Here we have defined a number of quantities to simplify the form of the final equations. They are

$$D \equiv \left(1 - \frac{2M}{r}\right)^{-1} [\partial_t + \partial_{r^*}] \quad (\text{A14})$$

$$\Delta \equiv \frac{1}{2}[\partial_t - \partial_{r^*}] \quad (\text{A15})$$

$$\Theta \equiv \left(\frac{1}{\sin\theta}\partial_\theta \sin\theta + \frac{i}{\sin\theta}\partial_\phi\right) \quad (\text{A16})$$

$$S(r) \equiv \left(1 - \frac{2M}{r}\right). \quad (\text{A17})$$

Further, we have employed the despun versions of the fields as defined in [4]

$$\hat{\Phi}_0 \equiv \Phi_0 \quad (\text{A18})$$

$$\hat{\Phi}_{+1} \equiv \frac{1}{\sqrt{2}}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + \cot\theta\right)\Phi_{+1} \quad (\text{A19})$$

$$\hat{\Phi}_{-1} \equiv \frac{1}{\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \cot\theta\right)\Phi_{-1}. \quad (\text{A20})$$

One benefit of using these definitions is that it is immediately obvious that the equations for a massive vector boson field reduce to Price's equations for a massless vector boson field (given as Equation (42) in [4]) when we take the limit $\mu \rightarrow 0$.

APPENDIX B: MONOPOLE FIELDS

So far we have not assumed a monopole field. In fact, the results of Appendix A are general in the sense that any field can be decomposed into spherical harmonics. In this Appendix we will specialize to the case of spherical symmetry. The monopole produces the obvious simplification that $l = 0$. There are also restrictions on the form of $F^{\mu\nu}$ imposed by spherical symmetry (assuming $F^{\mu\nu}$ is given in terms of potentials as in Eq. (A8)). Because the Φ fields are defined in terms of $F^{\mu\nu}$ (see Eq. (A7)), spherical symmetry guarantees

$$\Phi_0 = \Phi_0^* \quad (\text{B1})$$

$$\Phi_{+1} = \Phi_{-1} = 0. \quad (\text{B2})$$

Further, by comparing Eqs. (A7) and (A8), we can see that in the spherically symmetric case

$$F_{10} = \Phi_0. \quad (\text{B3})$$

By analogy with the case of electromagnetism, we see that Φ_0 parametrizes the effective "electric field" of the broken symmetry.

Using these relationships, we can combine Eqs. (A10) and (A11) into a single, second order differential equation. The resulting field equation is exactly Eq. (2).

APPENDIX C: COULOMB WAVE FUNCTIONS

Equation (25) is solved by the Coulomb wave functions $F_1(-\eta, \rho)$ and $G_1(-\eta, \rho)$, where in our case [12]

$$\eta = \frac{\mu^2 M}{\sqrt{\omega^2 - \mu^2}} \quad (\text{C1})$$

and:

$$\rho = \sqrt{\omega^2 - \mu^2} r^*. \quad (\text{C2})$$

To consistently define a transmitted wave, we wish to find the linear combination of F_1 and G_1 which asymptotes to e^{ikr^*} as r^* (and hence ρ) approaches infinity. For $\rho \rightarrow \infty$, we find [12]

$$\lim_{\rho \rightarrow \infty} F_1(-\eta, \rho) = \sin(\rho + \phi) \quad (\text{C3})$$

where ϕ is a phase angle. Similarly, in the same limit

$$\lim_{\rho \rightarrow \infty} G_1(-\eta, \rho) = \cos(\rho + \phi). \quad (\text{C4})$$

Thus, our transmitted wave will have the form

$$\psi_{\text{far}} = T[G_1(-\eta, \rho) + iF_1(-\eta, \rho)] \quad (\text{C5})$$

just as asserted in Eq. (26).

Now, we wish to look at the expansion of ψ_{far} in the limit of small ρ , which will be matched to the solution for ψ near the peak of the curvature potential ($r^* \sim \text{few} \times M$). For $\rho \ll 1$, we have [12]

$$F_1(-\eta, \rho) \approx C_1(-\eta)\rho^2 \quad (\text{C6})$$

where

$$C_1(-\eta) = \frac{\exp(\pi\eta/2)|\Gamma(2-i\eta)|}{3}. \quad (\text{C7})$$

We can express $|\Gamma(2-i\eta)|$ in the form [12]

$$|\Gamma(2-i\eta)|^2 = (1+\eta^2) \frac{\pi\eta}{\sinh(\pi\eta)} \quad (\text{C8})$$

which gives

$$F_1(-\eta, \rho) \approx \frac{\sqrt{\pi}}{3} \frac{e^{\pi\eta/2}}{\sqrt{\sinh(\pi\eta)}} \eta^{1/2} (1+\eta^2)^{1/2} \rho^2. \quad (\text{C9})$$

Similarly, for $\rho \ll 1$, we can expand G_1 to find [12]

$$G_1(-\eta, \rho) \approx \frac{1}{3C_1(-\eta)\rho} \quad (\text{C10})$$

where C_1 is the same function as above, giving

$$G_1(-\eta, \rho) \approx \frac{1}{\sqrt{\pi}} \frac{\sqrt{\sinh(\pi\eta)}}{e^{\pi\eta/2}} \eta^{-1/2} (1+\eta^2)^{-1/2} \frac{1}{\rho}. \quad (\text{C11})$$

These limiting forms give us exactly Eq. (28).

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