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Stationary and axisymmetric solutions of higher-dimensional general relativity

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We study stationary and axisymmetric solutions of General Relativity, i.e., pure gravity, in four or higher dimensions. D-dimensional stationary and axisymmetric solutions are defined as having $D-2$ commuting Killing vector fields. We derive a canonical form of the metric for such solutions that effectively reduces the Einstein equations to a differential equation on an axisymmetric $D-2$ by $D-2$ matrix field living in three-dimensional flat space (apart from a subclass of solutions that instead reduce to a set of equations on a $D-2$ by $D-2$ matrix field living in two-dimensional flat space). This generalizes the Papapetrou form of the metric for stationary and axisymmetric solutions in four dimensions, and furthermore generalizes the work on Weyl solutions in four and higher dimensions. We analyze then the sources for the solutions, which are in the form of thin rods along a line in the three-dimensional flat space that the matrix field can be seen to live in. As examples of stationary and axisymmetric solutions, we study the five-dimensional rotating black hole and the rotating black ring, write the metrics in the canonical form and analyze the structure of the rods for each solution.

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I. INTRODUCTION

Black holes in four-dimensional General Relativity have been the subject of intense research for several decades. One of the most important results on four-dimensional black holes in pure gravity, i.e., gravity without matter, is the uniqueness theorem stating that the rotating black hole solution of Kerr [1] is the unique solution for given mass and angular momentum [2–5]. This shows that the phase structure of black holes in four dimensions is very simple: Only one phase is available.

In recent years, attention has turned to the study of black holes in higher-dimensional General Relativity. It is by now clear that the phase structure of black holes is much more complicated when having more than four dimensions. For five-dimensional asymptotically flat black hole solutions, it was discovered by Emparan and Reall in [6] that in addition to the Myers-Perry rotating black hole solution [7], which has horizon topology S^3, there exists also a rotating black ring solution with horizon topology $S^2 \times S^1$. This means that for a given mass and angular momentum one can have as many as three different available phases, for five-dimensional asymptotically flat solutions of pure gravity. For pure gravity solutions asymptoting to Minkowski-space times a circle $\mathcal{M}^d \times S^1$, one has an even richer phase structure, involving phases with different horizon topologies and also phases with Kaluza-Klein bubbles.[1]

The complicated and rich phase structure of black holes in higher dimensions makes it desirable to develop new tools to find exact solutions. We focus in this paper on a particular class of solutions: Stationary and axisymmetric solutions of the vacuum Einstein equations in higher-dimensional General Relativity, i.e., in pure gravity. These solutions have $D-2$ commuting Killing vector fields where D is the dimension of the space-time. In four dimensions, this class of solutions includes the Kerr black hole [1], while in five dimensions both the rotating black hole with horizon topology S^3 [7] and the rotating black ring with horizon topology $S^2 \times S^1$ [6] are in this class.

We find in this paper a canonical form of the metric for stationary and axisymmetric solutions of the vacuum Einstein equations in higher-dimensional General Relativity. With the metric in the canonical form, the Einstein equations take a remarkably simple form: They reduce effectively to a differential equation on an axisymmetric $D-2$ by $D-2$ matrix field G living in a three-dimensional flat space, apart from a subclass of solutions that instead reduce to a set of equations on a $D-2$ by $D-2$ matrix field living in two-dimensional flat space.

We analyze the general structure of such solutions. In the three-dimensional space that G can be seen to live in the sources for G are in the form of thin rods along a line. We examine the general structure of the rods that constitute the sources of a given solution. We furthermore identify the asymptotic behavior of asymptotically flat solutions in four and five dimensions.

As examples of stationary and axisymmetric solutions, we consider the five-dimensional rotating black hole with horizon topology S^3 and the black ring with horizon topology $S^2 \times S^1$. We write down the metric in the canonical coordinates and analyze their rod-structure, i.e., the structure of their sources.

In four dimensions, the canonical form of the metric that we find for stationary and axisymmetric solutions is equivalent to the so-called *Papapetrou form* for the metric [10,11]. Papapetrou found that, under certain condi-

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[1] See [8,9] and references therein.

tions, the metric of four-dimensional stationary and axisymmetric pure gravity solutions can be written in the form[2]

$$ds^2 = -e^{2U}(dt + Ad\phi)^2 + e^{-2U}r^2 d\phi^2 + e^{2\nu}(dr^2 + dz^2). \tag{1.1}$$

The functions $U(r, z)$, $A(r, z)$ are solutions of

$$\begin{aligned} &\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)U = -\frac{e^{4U}}{2r^2}[(\partial_r A)^2 + (\partial_z A)^2], \\ &\partial_r\left(\frac{e^{4U}}{r}\partial_r A\right) + \partial_z\left(\frac{e^{4U}}{r}\partial_z A\right) = 0, \end{aligned} \tag{1.2}$$

and the function $\nu(r, z)$ is a solution of

$$\begin{aligned} \partial_r \nu &= -\partial_r U + r[(\partial_r U)^2 - (\partial_z U)^2] - \frac{e^{4U}}{4r}[(\partial_r A)^2 - (\partial_z A)^2], \\ \partial_z \nu &= -\partial_z U + 2r\partial_r U \partial_z U - \frac{e^{4U}}{2r}\partial_r A \partial_z A. \end{aligned} \tag{1.3}$$

Here $\partial/\partial t$ and $\partial/\partial\phi$ are the two Killing vector fields. Since the Eqs. (1.3) for ν are integrable, one can solve the Einstein equations by first finding U and A that solves (1.2), and then a ν can be found that solves (1.3).

The canonical form of the metric for stationary and axisymmetric solutions that we find in this paper is a generalization of the Papapetrou form (1.1) of the metric for four-dimensional solutions. Moreover, the simplified form of the Einstein equations that we find generalizes the Eqs. (1.2) and (1.3) for four dimensions.

For the special case when all the $D - 2$ Killing vector fields are orthogonal to each other, the canonical form of the metric that we find in this paper is equivalent to the form of the so-called generalized Weyl solutions of Emparan and Reall [16].[3] In Ref. [16] it is shown that, under certain conditions, the metric for D-dimensional pure gravity solutions with $D - 2$ commuting orthogonal Killing vector fields can be written in the form

$$\begin{aligned} &ds^2 = -e^{2U_1}dt^2 + \sum_{i=2}^{D-2} e^{2U_i}(dx^i)^2 + e^{2\nu}(dr^2 + dz^2), \\ &\sum_{i=1}^{D-2} U_i = \log r, \end{aligned} \tag{1.4}$$

with $t = x^1$. The functions $U_i(r, z)$ are solutions of the three-dimensional Laplace equations

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)U_i = 0, \tag{1.5}$$

for $i = 1, \ldots, D - 2$, while $\nu(r, z)$ is a solution of

$$\begin{aligned} \partial_r \nu &= -\frac{1}{2r} + \frac{r}{2}\sum_{i=1}^{D-2}[(\partial_r U_i) - (\partial_z U_i)^2], \\ \partial_z \nu &= r\sum_{i=1}^{D-2}\partial_r U_i \partial_z U_i. \end{aligned} \tag{1.6}$$

Here $\partial/\partial x^i$, $i = 1, ..., D - 2$, are the $D - 2$ orthogonal Killing vector fields. Solutions with metric (1.4) and with U_i and ν obeying (1.5) and (1.6) are called *generalized Weyl solutions.*

We see that using the form of the metric (1.4) for solutions with $D - 2$ commuting orthogonal Killing vector fields, solving the Einstein equations effectively reduces to the task of solving $D - 3$ free Laplace equations on a three-dimensional flat space. This is due to the fact that the Eqs. (1.6) for ν are integrable, so that one can find a ν solving (1.6) given any solution for U_i, $i = 1, \ldots, D - 2$.

It is important to remark that the method of generalized Weyl solutions generalizes Weyl's work on four-dimensional static and axisymmetric solutions [18]. Moreover, one also obtains Weyl's form of the metric for four-dimensional static and axisymmetric solutions by setting $A = 0$ in Papapetrou form (1.1). This is consistent with the fact that Eqs. (1.1), (1.2), and (1.3) become equivalent to Eqs. (1.4), (1.5), and (1.6) for $D = 4$ when $A = 0$, with $U_1 = U$ and $U_2 = \log r - U$.

Equations (1.5) can be seen as free Laplace equations for axisymmetric potentials living in a three-dimensional flat space. Solutions are then built up from thin rods located at the line $r = 0$ in the three-dimensional space, with a given rod being a source for one of the $D - 2$ potentials U_i [16]. In this paper we generalize the concept of rods to the more general class of stationary and axisymmetric solutions, i.e., solutions for which the Killing vector fields are not necessarily orthogonal. One of the new features is that for a given rod we can associate a direction in the $(D - 2)$-dimensional vector space spanned by the Killing vector fields. Solutions for which the directions of the rods are not orthogonal to each other are then also solutions where the Killing vector fields are not orthogonal to each other.

The outline of this paper is as follows: In Sec. II we derive a canonical form of the metric for stationary and axisymmetric pure gravity solutions. Using this, we find a simplified version of the Einstein equations, effectively reducing them to an equation on an axisymmetric $D - 2$ by $D - 2$ matrix field G living in flat three-dimensional space. Some of the details of the derivation are placed in the Appendices B, C, and D. In Appendix A we consider a special subclass of solutions that has the matrix field G

[2] See also [12–15].

[3] See [8] for a brief review of generalized Weyl solutions. See furthermore [17] for work on extending the generalized Weyl solutions of [16] to space-times with a cosmological constant.

living in two-dimensional flat space. In Appendix E we explore further the equation for the matrix field G.

In Sec. III we consider the behavior of the matrix field G near the $r = 0$ line in the flat three-dimensional space that G lives in. The sources for G lives on the $r = 0$ line in the form of rods. We analyze the general structure of these rods. See also Appendix F.

In Sec. IV we consider the asymptotic region, and we find out how to read off the asymptotic quantities for solutions that asymptotic to four-dimensional or five-dimensional Minkowski-space.

In Secs. V and VI we write down the metrics for the five-dimensional rotating black hole of Myers and Perry and the rotating black ring of Emperan and Reall in the canonical form. We furthermore analyze the rod-structure for these solutions. For the rotating black hole solutions, we make use of Appendix G on prolate spherical coordinates, while for the black ring solutions we make use of Appendix H which considers C-metric coordinates and how to transform these to the canonical coordinates of this paper.

In Sec. VII we have the conclusions.

II. STATIONARY AND AXISYMMETRIC SOLUTIONS

In this section we show that finding stationary and axisymmetric solutions of General Relativity in D dimensions without matter (i.e. pure gravity) can be reduced to solving a differential equation on an axisymmetric $D-2$ by $D-2$ matrix field in flat three-dimensional Euclidean space. As part of this, we find a particularly simple form of the metric for such solutions.

With respect to four-dimensional General Relativity, the results of this section generalizes the work of Papapetrou on stationary and axisymmetric metrics in four dimensions [10,11] (see Eqs. (1.1), (1.2), and (1.3) in the Introduction), which again is a generalization of the work of Weyl on static and axisymmetric metrics [18]. In higher-dimensional General Relativity, the results of this section generalizes the work of Emparan and Reall on metrics with $D-2$ orthogonal commuting Killing vector fields [16] (see Eqs. (1.4), (1.5), and (1.6) in the Introduction). We comment in more detail on the connection to previous work in the following. Finally, we note that the derivation of this section follows similar lines as that of Wald's derivation in [13] for four-dimensional stationary and axisymmetric metrics.

A. Deriving canonical form of metric and the Einstein equations

1. Formulation of problem

In this section we study D-dimensional manifolds which have $D-2$ commuting linearly independent Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$. With Lorentzian signature this corresponds to what we in this paper call *stationary and axisymmetric* space-times, where the term "stationary" means that one of our Killing vector fields are timelike, while the $D-3$ space-like Killing vector fields give what we call "axisymmetry" of the space-time.[4] That the Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$, commute means that

$$[V_{(i)}, V_{(j)}] = 0, \tag{2.1}$$

for $i, j = 1, ..., D-2$. We see that the Killing vector fields generate a $(D-2)$-dimensional Abelian group.

We restrict moreover ourselves to consider solutions of D-dimensional General Relativity without matter, i.e., we consider metrics that solve the vacuum Einstein equations

$$R_{\mu\nu} = 0. \tag{2.2}$$

In the following we find a canonical form of this class of metrics, and we find furthermore a reduced form of the Einstein Eqs. (2.2).

2. Finding two-dimensional orthogonal subspaces

Consider first a general D-dimensional space-time with $D-2$ commuting Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$. From the fact that the Killing vector fields are commuting, as expressed in Eq. (2.1), we get that we can find coordinates x^i, $i = 1, \ldots, D-2$, and u^a, $a = 1, 2$, so that

$$V_{(i)} = \frac{\partial}{\partial x^i}, \tag{2.3}$$

for $i = 1, \ldots, D-2$. Clearly, this means that the metric components in this coordinate system only depends on u^1 and u^2.

We need now the theorem [13,16]:

Theorem 2.1.—Let $V_{(i)}$, $i = 1, \ldots, D-2$, be $D-2$ commuting Killing vector fields such that:

(1) The tensor $V_{(1)}^{[\mu_1} V_{(2)}^{\mu_1} \cdots V_{(D-2)}^{\mu_{D-2}} D^\nu V_{(i)}^{\rho]}$ vanishes at at least one point of the space-time for a given $i = 1, \ldots, D-2$

(2) The tensor $V_{(i)}^\nu R_\nu^{[\rho} V_{(1)}^{\mu_1} V_{(2)}^{\mu_1} \cdots V_{(D-2)}^{\mu_{D-2}]} = 0$ for all $i = 1, \ldots, D-2$. Then the two-planes orthogonal to the Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$, are integrable. □

This theorem is stated and proven in four dimensions in [13] using Frobenius theorem on integrable submanifolds. Emparan and Reall generalized it to higher-dimensional manifolds in [16].

Assume now that the two conditions in Theorem 2.1 are obeyed. That the two-planes orthogonal to the Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$, are integrable means that for any given point of our D-dimensional manifold we have a two-dimensional submanifold that includes

[4] One can use our results for null Killing vector fields, but we will not elaborate on that case in this paper.

this point and moreover have the property that for any point of the submanifold the two-dimensional tangent-space is orthogonal to all of the Killing vector fields. By choosing coordinates on one of these two-dimensional submanifolds and dragging them along the integral curves of our Killing vector fields, we can find two coordinates y^1 and y^2 for our D-dimensional manifold so that $\partial/\partial x^i$ is orthogonal to $\partial/\partial y^a$ everywhere for all $i = 1, \ldots, D-2$ and $a = 1, 2$. This means the metric takes the form

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij} dx^i dx^j + \sum_{a,b=1}^{2} \hat{g}_{ab} dy^a dy^b, \tag{2.4}$$

where G_{ij} and $\hat{g}_{ab}$ only depends on y^1 and y^2.

From now on we restrict ourselves to solutions solving the vacuum Einstein Eqs. (2.2). This ensures immediately that Condition (2) in Theorem 2.1 is obeyed. We assume furthermore that Condition (1) in Theorem 2.1 is obeyed. Condition (1) can, for example, be argued to hold if one of the Killing vector fields is an angle, since then it is zero on the axis of rotation. This means for instance that solutions asymptoting to Minkowski-space $\mathcal{M}^D$ for $D = 4, 5$ obeys Condition (1) since they have angles in them. Clearly, the same is true for solutions asymptoting to $\mathcal{M}^{D-p} \times T^p$ for $D - p = 4, 5$.

3. The r and z coordinates

Define now the function $r(y^1, y^2)$ as

$$r = \sqrt{|\det(G_{ij})|}. \tag{2.5}$$

In Appendix A we treat the case in which $\det(G_{ij})$ is constant, giving rise to a special class of solutions. Instead, we assume here and in the following that $r(y^1, y^2)$ is not a constant function. From Appendix B we get then that $(\partial r/\partial y^1, \partial r/\partial y^2) \neq (0, 0)$ except in isolated points. We can then use the result of Appendix C that we can find a coordinate $z(y^1, y^2)$, along with two functions $\nu(y^1, y^2)$ and $\Lambda(y^1, y^2)$, so that

$$\sum_{a,b=1}^{2} \hat{g}_{ab} dy^a dy^b = e^{2\nu}(dr^2 + \Lambda dz^2). \tag{2.6}$$

Therefore, the full metric takes the form

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij} dx^i dx^j + e^{2\nu}(dr^2 + \Lambda dz^2), \tag{2.7}$$

where $\nu(r, z)$ and $\Lambda(r, z)$ are functions of r and z.

From Appendix D, where part of the Ricci tensor for the metric (2.7) is computed, we have from Eq. (D8)

$$\sum_{i,j=1}^{D-2} G^{ij} R_{ij} = -\frac{\partial_r \Lambda}{2e^{2\nu} \Lambda r}. \tag{2.8}$$

Since our solution should fulfill the vacuum Einstein equations $R_{\mu\nu} = 0$, this means that

$$\partial_r \Lambda = 0. \tag{2.9}$$

This gives that $\Lambda = \Lambda(z)$. Since we preserve the form of the metric (2.7) under a transformation $z' = f(z)$ we can therefore set $\Lambda(z) = 1$ by a coordinate transformation of z alone. Thus, we can define the z-coordinate by demanding $\Lambda = 1$. This fixes z up to transformations $z \to z + \text{constant}$.

4. Canonical form of metric

In conclusion, we have shown that for any Ricci-flat space-time with $D - 2$ commuting Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-2$, obeying Condition (1) of Theorem 2.1, we can find a coordinate system $(x^1, \ldots, x^{D-2}, r, z)$ such that $V_{(i)} = \partial/\partial x^i$ and such that the metric takes the *canonical form*

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij} dx^i dx^j + e^{2\nu}(dr^2 + dz^2), \tag{2.10}$$

with

$$r = \sqrt{|\det(G_{ij})|}, \tag{2.11}$$

where $G_{ij}(r, z)$ and $\nu(r, z)$ are functions only of r and z. In addition to the assumption that the Killing vector fields should obey Condition (1) of Theorem 2.1 we also assume here that $\det(G_{ij})$ is not constant on our space-time. The situation in which $\det(G_{ij})$ is constant is instead treated in Appendix A.

5. The Einstein equations

We now consider the vacuum Einstein equations $R_{\mu\nu} = 0$ for the metric (2.10) with the constraint (2.11) using the computed Ricci tensor (D9) in Appendix D.

Considering the $R_{ij} = 0$ equations we see from (D9) that the equations for G_{ij} are

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right) G_{ij} = \sum_{k,l=1}^{D-2} G^{kl} \partial_r G_{ki} \partial_r G_{lj} + \sum_{k,l=1}^{D-2} G^{kl} \partial_z G_{ki} \partial_z G_{lj}. \tag{2.12}$$

Considering the $R_{rr} - R_{zz} = 0$ and $R_{rz} = 0$ equations we see from (D9) that the equations for ν are

$$\partial_r \nu = -\frac{1}{2r} + \frac{r}{8} \sum_{i,j,k,l=1}^{D-2} G^{ij} G^{kl} \partial_r G_{ik} \partial_r G_{jl} - \frac{r}{8} \sum_{i,j,k,l=1}^{D-2} G^{ij} G^{kl} \partial_z G_{ik} \partial_z G_{jl}, \quad (2.13)$$

$$\partial_z \nu = \frac{r}{4} \sum_{i,j,k,l=1}^{D-2} G^{ij} G^{kl} \partial_r G_{ik} \partial_z G_{jl}.$$

Using now (2.13) together with (2.12) one can check that the integrability condition $\partial_z \partial_r \nu = \partial_r \partial_z \nu$ on $\nu(r, z)$ is obeyed. Thus, for a given solution $G_{ij}(r, z)$ of (2.12) the Eqs. (2.13) can be integrated to give $\nu(r, z)$.

Finally, there is the remaining nontrivial equation $R_{rr} + R_{zz} = 0$ coming from the Einstein equations. The explicit expression for this equation is easily found using (D9) and is seen to involve second derivatives of ν. Since $G_{ij}(r, z)$ and $\nu(r, z)$ already are determined by (2.12) and (2.13) it needs to be checked that $R_{rr} + R_{zz} = 0$ is consistent with (2.12) and (2.13). This can be checked by finding $\partial_r^2 \nu + \partial_z^2 \nu$ from (2.13). Inserting the result into $R_{rr} + R_{zz}$ from (D9) this is seen to be zero using (2.12).

Therefore, we have shown that one can find solutions of the vacuum Einstein equations for the canonical form for the metric (2.10) and (2.11) by finding a $G_{ij}(r, z)$ that satisfies (2.12). Then, subsequently one can always find a function $\nu(r, z)$ that satisfies (2.13), and thereby we have a complete solution satisfying all the Einstein equations.

6. Reduction to Papapetrou form and generalized Weyl solutions

We show here that the canonical form of the metric (2.10) and (2.11), along with the form of the Einstein Eqs. (2.12) and (2.13), reduces to the previously known cases.

We first consider the Papapetrou form (1.1) for four-dimensional stationary and axisymmetric solutions [10,11], with the Einstein equations in the form (1.2) and (1.3). Setting $D = 4$, we see that by setting $G_{11} = -e^{2U}$, $G_{12} = -e^{2U} A$ and $G_{22} = e^{-2U}(r^2 - A^2 e^{4U})$ with $x^1 = t$ and $x^2 = \phi$, we get the Papapetrou form (1.1) from (2.10). Furthermore, we see that (2.12) and (2.13) reduce to (1.2) and (1.3).

Consider now instead the generalized Weyl solutions of [16] which have $D - 2$ orthogonal commuting Killing vector fields. These have metric (1.4), and the Einstein equations are in the form (1.5) and (1.6). We see that setting $G_{11} = -e^{2U_1}$ and $G_{ii} = e^{2U_i}$ for $i = 2, .., D - 2$, we get the metric (1.4) from the canonical form (2.10). $\det G = -r^2$ gives then $\sum_{i=1}^{D-2} U_i = \log r$. For the Einstein equations, it is easily seen that (2.12) and (2.13) reduces to (1.5) and (1.6) (see also Appendix E). Thus, the canonical form (2.10) and (2.11) correctly reduce to the generalized Weyl solutions.

B. Compact notation for the equations for $G_{ij}(r, z)$

We have derived above that the metric of D-dimensional manifolds with $D - 2$ commuting Killing vector fields obeying the vacuum Einstein equations can be written in the canonical form (2.10) and (2.11). Moreover, the vacuum Einstein equations reduce to (2.12) and (2.13). We now show that we can write the equations for $G_{ij}(r, z)$ in a more compact form. This is highly useful for analysis of these equations.

For a given r and z we can view G_{ij} as a $D - 2$ times $D - 2$ real symmetric matrix, with G^{ij} as its inverse. In this way we can write (2.12) in matrix notation as

$$G^{-1}\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)G = (G^{-1}\partial_r G)^2 + (G^{-1}\partial_z G)^2, \quad (2.14)$$

with the constraint $|\det G| = r^2$ coming from (2.11).

We can make a further formal rewriting of (2.12) by recognizing that the derivatives respects the symmetries of a flat three-dimensional Euclidean space with metric

$$dr^2 + r^2 d\gamma^2 + dz^2. \quad (2.15)$$

Here γ is an angular coordinate of period 2π.[5] Therefore, if we define $\vec{\nabla}$ to be the gradiant in three-dimensional flat Euclidean space, we can write (2.12) as

$$G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla} G)^2. \quad (2.16)$$

Thus, by finding the axisymmetric solutions of the differential matrix Eq. (2.16) in three-dimensional flat Euclidean space, that obey the constraint $|\det G| = r^2$, we can find all stationary and axisymmetric solutions of the vacuum Einstein equations in D dimensions.[6]

We explore some of the mathematical properties of Eq. (2.16) in Appendix E. Here the compact form (2.16) of (2.12) prove highly useful.

III. BEHAVIOR OF SOLUTIONS NEAR $r = 0$

In Sec. II we derived the canonical form of the metric (2.10) and (2.11), along with the corresponding equations of motion, (EOMs) (2.12) and (2.13), for stationary and axisymmetric solutions of the vacuum Einstein equations. In this section we consider the behavior of such solutions close to $r = 0$.

[5] It is important to remark that γ is not an actual physical variable for the solution (2.10), but rather an auxiliary coordinate that is useful for understanding the structure of Eqs. (2.12).

[6] To be precise, we mean all solutions for which the $D - 2$ Killing vector fields obey Condition (1) of Theorem 2.1 and for which $\det(G_{ij})$ is a nonconstant function on our D-dimensional manifold.

A. Behavior of $G(r, z)$ near $r = 0$

We describe first how the z-axis at $r = 0$ is divided into intervals, called rods, according to the dimension of the kernel of G at $r = 0$. We find then coordinates in which a solution simplifies near a rod, making it possible to describe the solution in detail near the rod. We use this to define the rod-structure of a solution in Sec. III B.

1. Dividing the z-axis into rods

Consider a given solution $G(r, z)$. $G(r, z)$ is required to be continuous. Since $|\det G| = r^2$ we see that the product of the eigenvalues of $G(r, z)$ goes to zero for $r \to 0$. Therefore, we have that the eigenvalues of $G(0, z)$, which all are real since $G(0, z)$ is symmetric, include the eigenvalue zero for a given z. This means that the dimension of the kernel of $G(0, z)$ is greater than or equal to one for any z. We can write this more compactly as $\dim(\ker[G(0, z)]) \geq 1$.

A necessary condition for a regular solution is that precisely one eigenvalue of $G(0, z)$ is zero for a given z, except in isolated points. This statement is explained in Appendix F where we argue that if we have more than one eigenvalue going to zero as $r \to 0$, for a given z, we have a curvature singularity at that point. Therefore, in the following we consider only solutions which, for a given z, only have one eigenvalue going to zero for $r \to 0$, except at isolated values of z. Written compactly, this means $\dim(\ker[G(0, z)]) = 1$, except at isolated values of z. Denote now these isolated values of z as $a_1, a_2, \ldots, a_N$, with $a_1 < a_2 < \ldots < a_N$.

We see now that we divided the z-axis into the $N + 1$ intervals $[-\infty, a_1]$, $[a_1, a_2]$, $\ldots$, $[a_{N-1}, a_N]$ and $[a_N, \infty]$.[7] We call these $N + 1$ intervals the *rods* of the solution.

One can easily check that the above definition of rods reduces to the definition of [16] for the special case of generalized Weyl solutions, i.e., with $D - 2$ orthogonal Killing vector fields.

2. Behavior of $G(r, z)$ near a rod

In Sec. II we found that $G(r, z)$ should solve the equation $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ with the constraint that $|\det G| = r^2$. However, this breaks down as $r \to 0$, because for $r = 0$ we have that $\det G = 0$ so G is not invertible anymore. The reason for this is that we have sources added to the equation $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ at $r = 0$. The sources corresponds precisely to the rods defined above, i.e., the intervals with $\dim(\ker[G(r, z)]) = 1$. Moreover, if we view the solution $G(r, z)$ as a matrix-valued field in the unphysical three-dimensional flat Euclidean space with metric (2.15), a rod is really a source in the form of a rod of zero thickness in this unphysical space. In the following we examine in detail the equation $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ near a rod in order to describe more precisely the behavior of $G(r, z)$ near a rod.

Consider a solution $G(r, z)$ and a given rod $[z_1, z_2]$. Consider furthermore a given value of $z = z_*$ obeying $z_1 < z_* < z_2$. Since $G(0, z_*)$ is a symmetric real matrix we can diagonalize it using an orthogonal matrix Λ_* such that $\Lambda_*^T G(0, z_*)\Lambda_*$ is diagonal. Furthermore, since $G(0, z_*)$ has precisely one zero eigenvalue, we can choose Λ_* so that $[\Lambda_*^T G(0, z_*)\Lambda_*]_{11} = 0$.

Define now $\tilde{G}(r, z) = \Lambda_*^T G(r, z)\Lambda_*$. Clearly, $\tilde{G}(r, z)$ is a solution of (2.12) by Lemma E.7, and furthermore $\det\tilde{G} = \det G = \pm r^2$. Note that all entries of $\tilde{G}(r, z_*)$ are of order $\mathcal{O}(r^2)$ for $r \to 0$, expect the entries $\tilde{G}_{ii}(r, z_*)$, $i = 2, \ldots, D - 2$, which are finite and nonzero.

Since $\tilde{G}^{ij}(0, z_*)$ is not well-defined we need to consider the limit of $\tilde{G}^{ij}(r, z_*)$ for $r \to 0$ carefully. To this end, define the $D - 2$ by $D - 2$ matrix-valued function $M(r, z)$ by

$$M_{11} = \frac{\tilde{G}_{11}}{r^2}, \qquad M_{1i} = \frac{\tilde{G}_{1i}}{r}, \qquad M_{ij} = \tilde{G}_{ij}, \qquad i, j = 2, \ldots, D - 2 \tag{3.1}$$

for any (r, z). We see that this corresponds to a rescaling $x^1_{\rm new} = r x^1_{\rm old}$. Clearly, we have that $M(0, z_*)$ is diagonal, with nonzero eigenvalues. Moreover, we have that $\tilde{G}^{11} = M^{11}/r^2$, $\tilde{G}^{1i} = M^{1i}/r$ and $\tilde{G}^{ij} = M^{ij}$, $i, j = 2, \ldots, D - 2$. Since M_{1i}, $i = 2, \ldots, D - 2$, are of order $\mathcal{O}(r)$, we have that M^{1i}, $i = 2, \ldots, D - 2$, are of order $\mathcal{O}(r)$, and therefore that $\tilde{G}^{1i}(r, z_*)$, $i = 2, \ldots, D - 2$, stay finite (or goes to zero) in the limit $r \to 0$. Also, $\tilde{G}^{ij}(r, z_*) \to 0$ with $2 \leq i < j \leq D - 2$ and $\tilde{G}^{ii}(r, z_*) \to [\tilde{G}_{ii}(0, z_*)]^{-1}$ with $i = 2, \ldots, D - 2$, while $\tilde{G}^{11}(r, z_*)$ is of order $1/r^2$ for $r \to 0$.

Consider now the equation $\tilde{G}^{-1}\vec{\nabla}^2\tilde{G} = (\tilde{G}^{-1}\vec{\nabla}\tilde{G})^2$ for $z = z_*$ and $r \to 0$. We have

$$\vec{\nabla}^2\tilde{G}_{11} = \tilde{G}^{11}(\vec{\nabla}\tilde{G}_{11})^2 + 2\sum_{i=2}^{D-2}\tilde{G}^{1i}\vec{\nabla}\tilde{G}_{11}\cdot\vec{\nabla}\tilde{G}_{1i} + \sum_{i=2}^{D-2}\tilde{G}^{ii}(\vec{\nabla}\tilde{G}_{1i})^2, \tag{3.2}$$

up to terms that go to zero for $r \to 0$. Since we just found that $\tilde{G}^{1i}$ and $\tilde{G}^{ii}$ are finite for $z = z_*$ and $r \to 0$ (with $i = 2, \ldots, D - 2$) and since we require that $\tilde{G}_{ij}(r, z)$ and its derivatives are finite as a necessary condition for regular solutions,[8] we see that the left-hand side and the second and third term on the right-hand side of (3.2) are finite for $z = z_*$ and $r \to 0$. Since $\tilde{G}^{11} \to \infty$ we see therefore that we need $\vec{\nabla}\tilde{G}_{11} \to 0$ for $z = z_*$ and $r \to 0$. If we consider

[7] Note that it is possible to have an infinite number of intervals.

[8] Except in the endpoints of a rod where the derivatives are not necessarily well-defined. Hence the condition $z_1 < z_* < z_2$.

instead $\vec{\nabla}^2\tilde{G}_{ii}$ we see similarly that

$$\vec{\nabla}^2\tilde{G}_{ii} = \tilde{G}^{11}(\vec{\nabla}\tilde{G}_{1i})^2 + \text{ finite terms,} \tag{3.3}$$

so we get that $\vec{\nabla}\tilde{G}_{1i} \to 0$ for $z = z_*$ and $r \to 0$. Thus, we have derived that $\vec{\nabla}\tilde{G}_{1i}(0, z_*) = 0$ for $i = 1, \ldots, D-2$. In particular, this implies that $\partial_z\tilde{G}_{1i}(0, z_*) = 0$. Therefore, since this works for any $z_* \in]z_1, z_2[$ we get the following theorem:

Theorem 3.1.—Consider a rod $[z_1, z_2]$ for a solution $G(r, z)$. Then we can find an orthogonal matrix Λ_* such that the solution $\tilde{G}(r, z) = \Lambda_*^T G(r, z)\Lambda_*$ has the property that $\tilde{G}_{1i}(0, z) = 0$ for $i = 1, \ldots, D-2$ and $z \in [z_1, z_2]$.□

We take now a closer look at the EOMs (2.12) and (2.13) near $r = 0$. Consider a solution $G(r, z)$ and a particular rod $[z_1, z_2]$. Using Theorem 3.1 we always make a constant coordinate transformation of the x^i coordinates so that $G(r, z)$ has the property that $G_{1i}(0, z) = 0$ for $i = 1, \ldots, D-2$ and $z \in [z_1, z_2]$. To leading order, we can therefore write $G(r, z) = [\pm a(z)r^2] \oplus A(z)$ for $r \to 0$ with $z_1 < z < z_2$ where $a(z)$ is a function of z with $a(z) > 0$ for $z \in]z_1, z_2[$ and $A(z)$ is a $D-3$ by $D-3$ matrix-valued function of z. Thus, $G_{11} = \pm a(z)r^2$ for $r \to 0$. Note that $|\det[A(z)]| = 1/a(z)$.

If we consider Eqs. (2.12) we see that $\vec{\nabla}^2 G_{11} = \pm 4a(z) + \mathcal{O}(r)$ and that $G^{11}(\partial_r G_{11})^2 = \pm 4a(z) + \mathcal{O}(r)$, so this is consistent. Considering $\partial_r\nu$ in (2.13) we see that since $\partial_r G_{11} = \pm 2a(z)r$ we have that $\partial_r\nu = 0$ to leading order. Considering instead $\partial_z\nu$ in (2.13), we get

$$\partial_z\nu = \frac{1}{2}\frac{a'}{a} + \mathcal{O}(r). \tag{3.4}$$

Thus, to leading order for $r \to 0$ we have $e^{2\nu} = c^2 a(z)$ where c is a positive number. Therefore, for $r \to 0$ with $z_1 < z < z_2$ the metric (2.10) has the form

$$ds^2 = \sum_{i,j=2,\ldots,D-2} A_{ij}(z)dx^i dx^j + a(z)[\pm r^2(dx^1)^2 + c^2(dr^2 + dz^2)]. \tag{3.5}$$

This is the behavior of the canonical metric (2.10) near a rod.

Notice now that if G_{11}/r^2 is positive for $r \to 0$ the coordinate x^1 is spacelike and the metric (3.5) has a conical singularity for $r \to 0$, unless x^1 is periodic with period $2\pi c$. For a regular solution, this means that if we have a rod in a spacelike direction we have necessarily that this direction is periodic with the period constrained from avoiding the conical singularity.

If G_{11}/r^2 is negative for $r \to 0$ the coordinate x^1 is timelike and we see that there is a horizon at $r = 0$ since $G_{11} = 0$. Moreover, using the above argument for the spacelike direction, we see that the Wick rotated coordinate ix^1 must be periodic with period $2\pi c$. This means that the horizon has a temperature $T = 1/(2\pi c)$ associated to it.

B. The rod-structure of a solution

In this section we define what we mean by the *rod-structure* of a solution, and we discuss the general structure of rods, in view of the considerations of Sec. III A.

1. Specifying the rod-structure of a solution

Let a solution $G_{ij}(r, z)$ of Eqs. (2.11) and (2.12) be given with $N + 1$ rods which meet in the z-values $a_1 < a_2 < \ldots < a_N$. Introduce here the notation $a_0 = -\infty$ and $a_{N+1} = \infty$ in order to write the equations below more compactly. The solution $G(r, z)$ thus have the $N + 1$ rods $[a_{k-1}, a_k]$ with $k = 1, \ldots, N+1$.

Define now for the solution $G_{ij}(r, z)$ the $N + 1$ vectors $v_{(k)}$ in $\mathbb{R}^{D-2}$, $k = 1, \ldots, N+1$, by

$$G(0, z)v_{(k)} = 0 \quad \text{for} \quad z \in [a_{k-1}, a_k], \quad k = 1, \ldots, N+1, \tag{3.6}$$

with $v_{(k)} \neq 0$ for all $k = 1, \ldots, N+1$. In other words, $v_{(k)} \in \ker[G(0, z)]$. We call $v_{(k)}$ the *direction* of the corresponding rod $[a_{k-1}, a_k]$.

We define then the *rod-structure* of the solution $G_{ij}(r, z)$ as the specification of the rod intervals $[a_{k-1}, a_k]$ plus the corresponding directions $v_{(k)}$, $k = 1, \ldots, N+1$.

Obviously, since $v_{(k)}$ is defined as an eigenvector, it is only defined up to a multiplicative factor (different from zero). In other words, one should really regard $v_{(k)}$ as an element of the real projective space $\mathbb{R}P^{D-3}$.

We now demonstrate that it follows from the considerations of Sec. III A that the above definition of the rod-structure is meaningful. This involves showing that $v_{(k)}$for a given k, as defined in (3.6) exists and is unique, as element in $\mathbb{R}P^{D-3}$

Observe first that by Theorem 3.1 we get for each of the $N + 1$ rods an orthogonal matrix $\Lambda_{(k)}$, $k = 1, \ldots, N+1$, so that $[\Lambda_{(k)}^T G(0, z)\Lambda_{(k)}]_{1i} = 0$ for $z \in [a_{k-1}, a_k]$. Define the unit vector $e = (1, 0, \ldots, 0)$ in $\mathbb{R}^{D-2}$. Note that from the above we have then that $\Lambda_{(k)}^T G(0, z)\Lambda_{(k)}e = 0$ for $z \in [a_{k-1}, a_k]$ with $k = 1, \ldots, N+1$. We can now define the $N + 1$ vectors $v_{(k)} = \Lambda_{(k)}e$, $k = 1, \ldots, N+1$. Clearly, then these $N + 1$ vectors $v_{(k)}$ obey (3.6). Thus, we have shown that we can always find $N + 1$ vectors $v_{(k)}$ obeying (3.6).

To see that each of the $N + 1$ vectors $v_{(k)}$ are unique, seen as elements of $\mathbb{R}P^{D-3}$, it is enough to notice that we know from Sec. III A that $\dim(\ker[G(0, z)]) = 1$ for $z \in]a_{k-1}, a_k[$.

2. Discussion of existence and uniqueness of solutions

We discuss here whether a solution is uniquely given by its rod-structure, and whether there exists a solution for any given rod-structure. We consider here only solutions of Euclidean signature, but one can easily extend the considerations to solutions of Lorentzian signature.

If we consider the special case of the generalized Weyl solutions of [16], corresponding to $G(r, z)$ being a diagonal matrix, we clearly have the directions of the rods can be chosen to have the form $v_{(k)} = (0, \dots, 0, \pm 1, 0, \dots, 0)$. It is then known from the analysis of [16] that we can specify a solution completely by the N parameters $a_1 < \dots < a_N$ and $N + 1$ vectors $v_{(k)}$, i.e., a solution is completely specified by its rod-structure.

We now speculate that this statement can be generalized, i.e., that also in the more general class of solution considered here, a solution is specified uniquely by its rod-structure. Thus, we claim in detail that: A solution with $N + 1$ rods is completely determined by specifying the parameters $a_1 < \dots < a_N$ and directions $v_{(k)}$, $k = 1, \dots, N + 1$, i.e., it is not possible to find two physically different solutions with $N + 1$ rods that have the same N parameters a_k and $N + 1$ directions $v_{(k)}$. Intuitively, this statement seems valid since one would expect that the system of Eqs. (2.12) determine $G(r, z)$ once we have determined the sources for $G(r, z)$ at $r = 0$. And, the values a_k and directions $v_{(k)}$ seems precisely to specify that.

Note that if this statement is true, it is moreover true that it is not possible to find two physically different solutions with $N + 1$ rods that have the same N parameters a_k, up to a global translation of all N parameters, and the same $N + 1$ directions $v_{(k)}$, up to a global rotation of all $N + 1$ directions.

One can also turn things around and ask whether there exists a solution with $N + 1$ rods given the N parameters $a_1 < \dots < a_N$ and $N + 1$ directions $v_{(k)}$, $k = 1, \dots, N + 1$ (not imposing the solution to be regular). This would be interesting to examine further. However, there is an obvious restriction on the directions of the first and last rod $[-\infty, a_1]$ and $[a_N, \infty]$. For a given asymptotic space, which the solution is required to asymptote to for $\sqrt{r^2 + z^2} \to \infty$ with $z/\sqrt{r^2 + z^2}$ fixed (see Sec. IV), the directions of these two rods should be correlated, and can therefore not be chosen independently.

C. Analysis of the rod-structure

In this section we summarize how to analyze the rod-structure, and add some useful nomenclature and general comments. We consider here solutions $G_{ij}(r, z)$ of (2.12) with $\det G = r^2$.

In Sec. III A we learned that in order to avoid curvature singularities it is a necessary condition on a solution that the kernel of the matrix $G(0, z)$ for a given z should be one-dimensional, except for isolated values of z. We therefore restrict ourselves to solutions where this applies. Naming the isolated z-values as $a_1, \dots, a_N$, we see that the z-axis splits up into the $N + 1$ intervals $[-\infty, a_1]$, $[a_1, a_2]$,…,$[a_{N-1}, a_N]$, $[a_N, \infty]$. The first task in understanding the rod-structure of a solution is thus to find these intervals, called rods.

Consider now a specific rod $[z_1, z_2]$. From Theorem 3.1 (see also Sec. III B) we know that we can find a vector

$$v = v^i \frac{\partial}{\partial x^i}, \tag{3.7}$$

so that

$$\sum_{j=1}^{D-2} G_{ij}(0, z) v^j = 0, \tag{3.8}$$

for $i = 1, \dots, D - 2$ and $z \in [z_1, z_2]$. This vector v is called the direction of the rod $[z_1, z_2]$. Then, if $G_{ij} v^i v^j / r^2$ is negative (positive) for $r \to 0$ we say the rod $[z_1, z_2]$ is *timelike* (*spacelike*).

Consider now a spacelike rod $[z_1, z_2]$. For $r \to 0$ with $z \in]z_1, z_2[$ we have a potential conical singularity. Let η be a coordinate, made as a linear combination of x^i, $i = 1, \dots, D - 2$, with

$$\frac{\partial}{\partial \eta} = v = v^i \frac{\partial}{\partial x^i}. \tag{3.9}$$

Then in order to cure the conical singularity at the rod, the coordinate η should have period

$$\Delta \eta = 2\pi \lim_{r \to 0} \sqrt{\frac{r^2 e^{2\nu}}{G_{ij} v^i v^j}}, \tag{3.10}$$

with $z \in [z_1, z_2]$. This is seen from the analysis of Sec. III A. We see from this that a spacelike rod corresponds to a compact direction. For a timelike rod, one can similarly find an associated temperature, by doing a Wick rotation. Therefore, a timelike rod corresponds to a horizon (see Sec. III A).

We introduce here some additional nomenclature for rods. Consider a rod $[z_1, z_2]$. If this is a finite interval we call $[z_1, z_2]$ a finite rod. If either $z_1 = -\infty$ or $z_2 = \infty$ but not both of them, we call $[z_1, z_2]$ a semi-infinite rod. $[-\infty, \infty]$ is instead called the infinite rod.

As discussed in [16], a finite timelike rod corresponds to an event horizon, at least if there are no semi-infinite timelike rods for the solution. Similarly, a finite spacelike rod corresponds to a Kaluza-Klein direction if there are no semi-infinite spacelike rods in that direction. Moreover, a (semi-)infinite spacelike rod corresponds to an axis of rotation, with the associated coordinate being the rotation angle, while a semi-infinite timelike rod corresponds to an acceleration horizon.

IV. ASYMPTOTICALLY FLAT SPACE-TIMES

In this section we consider asymptotically flat space-times. More specifically, we consider the four- and five-dimensional Minkowski-spaces $\mathcal{M}^4$ and $\mathcal{M}^5$, and we consider the asymptotic behavior of solutions that asymptote to $\mathcal{M}^4$ and $\mathcal{M}^5$.

The Minkowski-spaces $\mathcal{M}^4$ and $\mathcal{M}^5$ are special in that they are the only Minkowski-spaces that one can describe using the ansatz (2.10) and (2.11). This is easily seen by counting the number of Killing vector fields. An obvious generalization of the considerations of this section would be to consider the Kaluza-Klein space-times $\mathcal{M}^4 \times S^1$ and $\mathcal{M}^5 \times S^1$, or other space-times with even more compact directions, i.e. $\mathcal{M}^4 \times T^p$ or $\mathcal{M}^5 \times T^p$. We leave this for the future.[9]

In the following we put Newtons constant $G_N = 1$. To reinstate G_N one should substitute $M \to G_N M$ and $J \to G_N J$.

A. Perturbation of diagonal metric

Before describing asymptotically flat spaces, we first develop a tool that will prove useful. We consider in this section a perturbation $\delta G(r, z)$ of a solution $G(r, z)$ of Eqs. (2.12), with $G(r, z)$ being diagonal, such that $G(r, z) + \delta G(r, z)$ also is a solution of Eqs. (2.12). This will be useful below since asymptotic behavior of a solution typically involves the solution asymptoting towards a diagonal metric like, for example, the metric of Minkowski-space. The results here can also be used in a broader context to find corrections to solutions.

Now, Eqs. (2.12) for the perturbation $\delta G(r, z)$ becomes

$$\vec{\nabla}^2 \delta G_{ij} = \left(\frac{\vec{\nabla} G_{ii}}{G_{ii}} + \frac{\vec{\nabla} G_{jj}}{G_{jj}} \right) \cdot \vec{\nabla} \delta G_{ij} - \frac{\vec{\nabla} G_{ii}}{G_{ii}} \cdot \frac{\vec{\nabla} G_{jj}}{G_{jj}} \delta G_{ij}. \tag{4.1}$$

We see here that the equations for δG_{ij} are completely decoupled, i.e., we can solve for each component of $\delta G(r, z)$ separately. The only constraint is that $|\det(G + \delta G)| = r^2$. Using that $|\det G| = r^2$ this constraint can be written as $\mathrm{tr}(G^{-1}\delta G) = 0$ which we again can write as

$$\sum_{i=1}^{D-2} \frac{\delta G_{ii}}{G_{ii}} = 0. \tag{4.2}$$

We see thus that only the diagonal components of δG_{ii} are subject to a constraint. We note that for the diagonal components of $\delta G(r, z)$ the Eqs. (4.1) can be written

$$\vec{\nabla}^2 \left(\frac{\delta G_{ii}}{G_{ii}} \right) = 0, \tag{4.3}$$

for $i = 1, \ldots, D-2$.

B. Four-dimensional asymptotic Minkowski-space

We consider in this section the four-dimensional Minkowski-space $\mathcal{M}^4$ and the asymptotic structure of solutions asymptoting to $\mathcal{M}^4$.

We first describe $D = 4$ Minkowski-space $\mathcal{M}^4$. In terms of $G(r, z)$ we have that $\mathcal{M}^4$ is given by

$$G_{11} = -1, \qquad G_{22} = r^2, \tag{4.4}$$

Thus, we have an infinite spacelike rod $[-\infty, \infty]$. In accordance with (2.13) we choose $e^{2\nu} = 1$. Demanding regularity of the solution near $r = 0$ we get using (3.10) that $x^2 = \phi$ should have period 2π. Making the coordinate transformation

$$r = \rho \sin\theta, \qquad z = \rho \cos\theta, \tag{4.5}$$

we get the metric in spherical coordinates

$$ds^2 = -dt^2 + \rho^2 \sin^2\theta d\phi^2 + d\rho^2 + \rho^2 d\theta^2, \tag{4.6}$$

where we put $x^1 = t$ and $x^2 = \phi$.

If we consider a $D = 4$ asymptotically Minkowski-space solution we have for $\rho \to \infty$ the corrections to the metric

$$\begin{aligned} g_{tt} &= -1 + \frac{2M}{\rho} + \mathcal{O}(\rho^{-2}), \\ g_{t\phi} &= -2J \frac{\sin^2\theta}{\rho} [1 + \mathcal{O}(\rho^{-1})], \end{aligned} \tag{4.7}$$

with $g_{\phi\phi} = \rho^2 \sin^2\theta [1 + \mathcal{O}(\rho^{-1})]$.

In the (r, z) canonical coordinates the asymptotic region corresponds to $\sqrt{r^2 + z^2} \to \infty$ with $z/\sqrt{r^2 + z^2}$ finite. In the canonical coordinates we have therefore from (4.7) the asymptotic behavior

$$\begin{aligned} G_{11} &= -1 + \frac{2M}{\sqrt{r^2 + z^2}} + \mathcal{O}[(r^2 + z^2)^{-1}], \\ G_{12} &= -\frac{2Jr^2}{(r^2 + z^2)^{3/2}} + \mathcal{O}[(r^2 + z^2)^{-1}], \\ G_{22} &= r^2 \left\{ 1 + \frac{2M}{\sqrt{r^2 + z^2}} + \mathcal{O}[(r^2 + z^2)^{-1}] \right\}, \end{aligned} \tag{4.8}$$

for $\sqrt{r^2 + z^2} \to \infty$ with $z/\sqrt{r^2 + z^2}$ finite. For G_{22} we used (4.2) and (4.3) of Sec. IVA. We see from (4.8) that the leading asymptotic behavior of $G(r, z)$ is determined completely from M and J. For $e^{2\nu}$, the asymptotic behavior is simply that $e^{2\nu} \simeq 1$ for $\sqrt{r^2 + z^2} \to \infty$ with

[9] Strictly speaking, one can consider higher-dimensional Minkowski-spaces $\mathcal{M}^D$ with $D \geq 6$, for example, by making the split up $\mathcal{M}^D = \mathcal{M}^4 \times \mathbb{R}^{D-4}$, with the $\mathbb{R}^{D-4}$ part spanned by the Killing vector fields. However, one can not use that to write any nontrivial solutions which asymptotes to $\mathcal{M}^D$, since any solution would be independent of the $\mathbb{R}^{D-4}$ part. Instead, one should consider $\mathcal{M}^4 \times T^{D-4}$ or $\mathcal{M}^5 \times T^{D-5}$.

$z/\sqrt{r^2+z^2}$ finite.

1. The period of ϕ

In the above we considered the period of $x^2 = \phi$ to be 2π. We can also consider the more general case where $x^2 = \phi$ has period $\Delta\phi = 2\pi\varepsilon$. Then the asymptotic behavior of a solution is

$$G_{11} \simeq -1 + \frac{2M}{\varepsilon}\frac{1}{\sqrt{r^2+z^2}}, \qquad G_{12} \simeq -\frac{2J}{\varepsilon^2}\frac{r^2}{(r^2+z^2)^{3/2}}, \qquad e^{2\nu} \simeq \varepsilon^2, \tag{4.9}$$

for $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite, where we used here a less precise notation than above for the sake of brevity.

C. Five-dimensional asymptotic Minkowski-space

We consider in this section the five-dimensional Minkowski-space $\mathcal{M}^5$ and the asymptotic structure of solutions asymptoting to $\mathcal{M}^5$.

We first describe $D = 5$ Minkowski-space $\mathcal{M}^5$. In terms of $G(r, z)$ we have that $\mathcal{M}^5$ is described by

$$G_{11} = -1, \quad G_{22} = \sqrt{r^2+z^2} - z, \quad G_{33} = \sqrt{r^2+z^2} + z. \tag{4.10}$$

This corresponds to two semi-infinite rods $[-\infty, 0]$ and $[0, \infty]$. In accordance with Eqs. (2.13) we choose

$$e^{2\nu} = \frac{1}{2\sqrt{r^2+z^2}}. \tag{4.11}$$

Demanding regularity of the solution near $r = 0$ we get using (3.10) that both x^2 and x^3 are periodic with period 2π. Making the coordinate transformation

$$r = \frac{1}{2}\rho^2 \sin 2\theta, \qquad z = \frac{1}{2}\rho^2\cos 2\theta, \tag{4.12}$$

we get the metric in spheroidal coordinates

$$ds^2 = -dt^2 + \rho^2\sin^2\theta d\phi^2 + \rho^2\cos^2\theta d\psi^2 + d\rho^2 + \rho^2 d\theta^2, \tag{4.13}$$

where we put $x^1 = t$, $x^2 = \phi$ and $x^3 = \psi$. We remind the reader that regularity of the solution requires both $x^2 = \phi$ and $x^3 = \psi$ to be periodic with period 2π.

If we consider a $D = 5$ asymptotically Minkowski-space solution we have for $\rho \to \infty$ the corrections to the metric

$$\begin{aligned} g_{tt} &= -1 + \frac{8M}{3\pi}\frac{1}{\rho^2} + \mathcal{O}(\rho^{-4}), \\ g_{t\phi} &= -\frac{4J_1}{\pi}\frac{\sin^2\theta}{\rho^2}[1 + \mathcal{O}(\rho^{-2})], \\ g_{t\psi} &= -\frac{4J_2}{\pi}\frac{\cos^2\theta}{\rho^2}[1 + \mathcal{O}(\rho^{-2})], \end{aligned} \tag{4.14}$$

with $g_{\phi\phi} = \rho^2\sin^2\theta[1 + \mathcal{O}(\rho^{-2})]$ and $g_{\psi\psi} = \rho^2\cos^2\theta[1 + \mathcal{O}(\rho^{-2})]$. Using this together with (4.1) and (4.2), we get the asymptotics in the (r, z) canonical coordinates

$$\begin{aligned} G_{11} &= -1 + \frac{4M}{3\pi}\frac{1}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}], \qquad G_{23} = \zeta\frac{r^2}{(r^2+z^2)^{\frac{3}{2}}} + \mathcal{O}[(r^2+z^2)^{-1}], \\ G_{12} &= -\frac{J_1}{\pi}\frac{\sqrt{r^2+z^2} - z}{r^2+z^2} + \mathcal{O}[(r^2+z^2)^{-1}], \qquad G_{13} = -\frac{J_2}{\pi}\frac{\sqrt{r^2+z^2} + z}{r^2+z^2} + \mathcal{O}[(r^2+z^2)^{-1}], \\ G_{22} &= (\sqrt{r^2+z^2} - z)\left\{1 + \frac{2}{3\pi}\frac{M+\eta}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}]\right\}, \\ G_{33} &= (\sqrt{r^2+z^2} + z)\left\{1 + \frac{2}{3\pi}\frac{M-\eta}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}]\right\}, \end{aligned} \tag{4.15}$$

for $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite, where ζ and η are constants. Note that η changes under the transformation $z \to z +$ constant and is thus not a gauge-invariant parameter, unlike ζ. Finally, we remark that the asymptotics of $e^{2\nu}$ is

$$e^{2\nu} \simeq \frac{1}{2\sqrt{r^2+z^2}}, \tag{4.16}$$

for $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite.

1. The periods of ϕ and ψ

The periods of $x^2 = \phi$ and $x^3 = \psi$ were chosen to be 2π in the above. We consider here the more general case where the period $\Delta\phi$ of $x^2 = \phi$ and the period $\Delta\psi$ of $x^3 = \psi$ are given by $\Delta\phi = \Delta\psi = 2\pi\varepsilon$. Then the asymptotics of $G_{ij}(r, z)$ and $e^{2\nu}$ takes the form

$$G_{11} \simeq -1 + \frac{4M}{3\pi\varepsilon^2}\frac{1}{\sqrt{r^2+z^2}}, \qquad G_{23} \simeq \frac{\zeta}{\varepsilon^4}\frac{r^2}{(r^2+z^2)^{3/2}}, \qquad e^{2\nu} \simeq \frac{\varepsilon^2}{2\sqrt{r^2+z^2}},$$
$$G_{12} \simeq -\frac{J_1}{\pi\varepsilon^3}\frac{\sqrt{r^2+z^2}-z}{r^2+z^2}, \qquad G_{22} \simeq (\sqrt{r^2+z^2}-z)\left[1+\frac{2}{3\pi\varepsilon^2}\frac{M+\eta}{\sqrt{r^2+z^2}}\right],$$
$$G_{13} \simeq -\frac{J_2}{\pi\varepsilon^3}\frac{\sqrt{r^2+z^2}+z}{r^2+z^2}, \qquad G_{33} \simeq (\sqrt{r^2+z^2}+z)\left[1+\frac{2}{3\pi\varepsilon^2}\frac{M-\eta}{\sqrt{r^2+z^2}}\right], \tag{4.17}$$

for $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite, where we used here a less precise notation than above for the sake of brevity.

V. ROTATING BLACK HOLE SOLUTIONS

In this section we consider rotating black hole solutions in four and five dimensions and describe them using the canonical form of the metric (2.10) and (2.11).

A. Kerr solution

We first consider the four-dimensional Kerr solution [1] which corresponds to a rotating black hole. The topology of the event horizon is that of a two-sphere S^2. It is already known how to write the Kerr solution in the canonical form (2.10) and (2.11) (see for example [12,14,15]), but we review this here for completeness, and since it illustrates the methods developed in Sec. III.

The Kerr-metric in Boyer-Linquist coordinates is

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\frac{\rho^2+a^2-\Delta}{\Sigma}dtd\phi + \frac{(\rho^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}d\rho^2 + \Sigma d\theta^2, \tag{5.1}$$

with

$$\Delta = \rho^2 - 2M\rho + a^2, \qquad \Sigma = \rho^2 + a^2\cos^2\theta. \tag{5.2}$$

The coordinates for the two Killing directions are $x^1 = t$ and $x^2 = \phi$. From $\det(G) = -\Delta\sin^2\theta$ we get the r-coordinate, and it is a straightforward exercise to find a z-coordinate so that the metric fits into the ansatz (2.10). We find

$$r = \sqrt{\Delta}\sin\theta, \qquad z = (\rho - M)\cos\theta. \tag{5.3}$$

Using this, we can in principle write the Kerr-metric in the canonical form (2.10) and (2.11). However, it is useful to instead first write the Kerr-metric in the prolate spherical coordinates (see Appendix G). From the definition (G1) of the prolate spherical coordinates (x, y) we see that

$$\alpha^2(x^2-1)(1-y^2) = \Delta\sin^2\theta, \qquad \alpha xy = (\rho - M)\cos\theta, \tag{5.4}$$

Using the ansatz $x = x(\rho)$ and $y = y(\theta)$ we get

$$x = \frac{\rho - M}{\sqrt{M^2-a^2}}, \quad y = \cos\theta, \quad \alpha = \sqrt{M^2-a^2}. \tag{5.5}$$

We compute

$$G_{11} = -\frac{x^2\cos^2\lambda + y^2\sin^2\lambda - 1}{(1+x\cos\lambda)^2 + y^2\sin^2\lambda},$$
$$G_{12} = -2a\frac{(1-y^2)(1+x\cos\lambda)}{(1+x\cos\lambda)^2+y^2\sin^2\lambda}, \tag{5.6}$$
$$e^{2\nu} = \frac{(1+x\cos\lambda)^2+y^2\sin^2\lambda}{(x^2-y^2)\cos^2\lambda},$$

where we defined

$$\sin\lambda = \frac{a}{M}. \tag{5.7}$$

The G_{22} component can be found from

$$G_{22} = \frac{G_{12}^2 - \alpha^2(x^2-1)(1-y^2)}{G_{11}}. \tag{5.8}$$

We obtained now G_{ij} and $e^{2\nu}$ as functions of x and y. From this it is straightforward to use Eq. (G7) to get G_{ij} and $e^{2\nu}$ as functions of r and z.

1. Asymptotic region

Using (G9) we find that in the asymptotic region $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite, we have

$$G_{11} = -1 + \frac{2M}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}],$$
$$G_{12} = -2Ma\frac{r^2}{(r^2+z^2)^{3/2}} + \mathcal{O}[(r^2+z^2)^{-1}]. \tag{5.9}$$

From (4.8) we see that this means that M is the mass, which justifies our use of this symbol in the solution, and that the angular momentum is $J = Ma$.

Note that $e^{2\nu} \simeq 1$ in the asymptotic region. From Sec. IV B we know that this means that $\Delta\phi = 2\pi$, i.e., that ϕ is required to have period 2π. This can also be found directly from the solution near $r = 0$ using the analysis of Sec. III A.

2. Rod-structure

We now analyze the rod-structure of the Kerr solution according to the methods of Sec. III. We have:

(i) The two semi-infinite spacelike rods $[-\infty, -\alpha]$ and $[\alpha, \infty]$. For $z \in [-\infty, -\alpha]$ and $r = 0$ we see from (G7) that $x = -z/\alpha$ and $y = -1$. Similarly, for $z \in [\alpha, \infty]$ and $r = 0$ we have that $x = z/\alpha$ and

$y = 1$. Considering Eqs. (5.6) and (5.8) we see that for both intervals $G_{12} = G_{22} = 0$ while $G_{11} \neq 0$. By Eq. (3.8) this means that the two rods both are in the direction $v = (0, 1)$, i.e., that the rods are in the $\partial/\partial x^2$ direction and therefore spacelike.

(ii) The finite timelike rod $[-\alpha, \alpha]$. For $z \in [-\alpha, \alpha]$ we see from (G7) that $x = 1$ and $y = z/\alpha$. Considering Eqs. (5.6) and (5.8) we see that $\sum_{j=1}^{2} G_{ij} v^j = 0$ for $z \in [-\alpha, \alpha]$ with

$$v = (1, \Omega), \qquad \Omega = \frac{\sin\lambda}{2M(1 + \cos\lambda)}. \tag{5.10}$$

This means that we have a rod $[-\alpha, \alpha]$ along the direction (5.10). Since $G_{ij} v^i v^j / r^2$ is negative for $r \to 0$ the rod is timelike. Note that Ω in (5.10) is the *angular velocity* of the event horizon.[10] One finds easily that this rod corresponds to an event horizon of topology S^2. This is a consequence of the fact that the rods on each side of the $[-\alpha, \alpha]$ rod are in the same spacelike direction, i.e., the $\partial/\partial x^2$ direction.

For the timelike rod $[-\alpha, \alpha]$ we see from (5.10) that if we change coordinates as $\tilde{x}^1 = x^1$ and $\tilde{x}^2 = x^2 - \Omega x^1$, then in these coordinates the $[-\alpha, \alpha]$ rod is along the $\partial/\partial \tilde{x}^1$ direction. This means that $\tilde{x}^1$ and $\tilde{x}^2$ are two of the coordinates of the comoving coordinates for the Kerr solution since the comoving coordinates precisely gives a diagonal metric at the horizon. In other words, finding the direction of the $[-\alpha, \alpha]$ rod precisely corresponds to finding the comoving coordinates near the horizon.

Finally, in the accordance with the ideas of Sec. III B, we note that we can make an alternative parametrization of the Kerr solution by stating that we have three rods, the two rods $[-\infty, -\alpha]$ and $[\alpha, \infty]$ in the x^2 direction and the rod $[-\alpha, \alpha]$ in the $(1, \Omega)$ direction. Then the whole Kerr solution can be parametrized uniquely by the two parameters α and Ω.

B. Five-dimensional Myers-Perry solution

The five-dimensional Myers-Perry solution [7] corresponds to a five-dimensional spinning black hole.[11] This is an asymptotically flat stationary solution of the vacuum Einstein equations with an event horizon that has the topology of a three-sphere S^3.

The metric of the five-dimensional Myers-Perry black hole is

$$\begin{aligned} ds^2 = &-dt^2 + \frac{\rho_0^2}{\Sigma}[dt - a_1 \sin^2\theta d\phi - a_2 \cos^2\theta d\psi]^2 \\ &+ (\rho^2 + a_1^2)\sin^2\theta d\phi^2 + (\rho^2 + a_2^2)\cos^2\theta d\psi^2 \\ &+ \frac{\Sigma}{\Delta} d\rho^2 + \Sigma d\theta^2, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} \Delta &= \rho^2\left(1 + \frac{a_1^2}{\rho^2}\right)\left(1 + \frac{a_2^2}{\rho^2}\right) - \rho_0^2, \\ \Sigma &= \rho^2 + a_1^2 \cos^2\theta + a_2^2 \sin^2\theta. \end{aligned} \tag{5.12}$$

The coordinates for the three Killing directions are $x^1 = t$, $x^2 = \phi$ and $x^3 = \psi$. We now transform this metric to the canonical form (2.10) and (2.11). We compute that $\det G = -\frac{1}{4}\rho^2 \Delta \sin^2 2\theta$. From this we can determine r, and z can be found by demanding the metric to be of the form (2.10). We get

$$\begin{aligned} r &= \frac{1}{2}\rho\sqrt{\Delta}\sin 2\theta, \\ z &= \frac{1}{2}\rho^2\left(1 - \frac{\rho_0^2 - a_1^2 - a_2^2}{2\rho^2}\right)\cos 2\theta. \end{aligned} \tag{5.13}$$

This determines in principle how the Myers-Perry metric (5.11) should transform to the form (2.10). However, as for the Kerr-metric, it is convenient to express the Myers-Perry metric in prolate spherical coordinates, defined in Appendix G, instead. From the definition of the prolate spherical coordinates (G1) we see that

$$\begin{aligned} \alpha^2(x^2 - 1)(1 - y^2) &= \frac{1}{4}\rho^2 \Delta \sin^2 2\theta, \\ \alpha x y &= \frac{1}{2}\rho^2\left(1 - \frac{\rho_0^2 - a_1^2 - a_2^2}{2\rho^2}\right)\cos 2\theta. \end{aligned} \tag{5.14}$$

If we try the ansatz $x = x(\rho)$ and $y = y(\theta)$ we get

$$\begin{aligned} x &= \frac{2\rho^2 + a_1^2 + a_2^2 - \rho_0^2}{\sqrt{(\rho_0^2 - a_1^2 - a_2^2)^2 - 4a_1^2 a_2^2}}, \qquad y = \cos 2\theta, \\ \alpha &= \frac{1}{4}\sqrt{(\rho_0^2 - a_1^2 - a_2^2)^2 - 4a_1^2 a_2^2}. \end{aligned} \tag{5.15}$$

Using this, we can write G_{ij} and $e^{2\nu}$ in terms of the prolate spherical coordinates. We get

[10] Note that v in (5.10) precisely is the null Killing vector for the Killing horizon (the event horizon), since $v = \sum_{i=1}^{2} v^i V_{(i)}$, and since $v^2 = G_{ij} v^i v^j = 0$ for $r = 0$ and $z \in [-\alpha, \alpha]$. In other words, for a Killing horizon the null Killing vector is the same as the direction of the timelike rod.

[11] The five-dimensional Myers-Perry black hole generalizes the static Schwarzschild-Tangherlini black hole [19].

$$
\begin{aligned}
&G_{11}=-\frac{4\alpha x+(a_1^2-a_2^2)y-\rho_0^2}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}, \qquad G_{12}=-\frac{a_1\rho_0^2(1-y)}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}, \qquad G_{13}=-\frac{a_2\rho_0^2(1+y)}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2},\\
&G_{23}=\frac{1}{2}\frac{a_1a_2\rho_0^2(1-y^2)}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}, \qquad G_{22}=\frac{1-y}{4}\left[4\alpha x+\rho_0^2+a_1^2-a_2^2+\frac{2a_1^2\rho_0^2(1-y)}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}\right],\\
&G_{33}=\frac{1+y}{4}\left[4\alpha x+\rho_0^2-a_1^2+a_2^2+\frac{2a_2^2\rho_0^2(1+y)}{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}\right], \qquad e^{2\nu}=\frac{4\alpha x+(a_1^2-a_2^2)y+\rho_0^2}{8\alpha^2(x^2-y^2)}.
\end{aligned}
\tag{5.16}
$$

Using Eq. (G7) it is now a straightforward exercise to write the components G_{ij} and $e^{2\nu}$ as functions of the canonical (r,z) coordinates.

1. Asymptotic region

Regarding (5.16) as functions of the canonical coordinates (r,z), we find that $G_{ij}(r,z)$ in the asymptotic region, $\sqrt{r^2+z^2}\to\infty$ with $z/\sqrt{r^2+z^2}$ finite, behaves as

$$
\begin{aligned}
&G_{11}=-1+\frac{\rho_0^2}{2\sqrt{r^2+z^2}}+\mathcal{O}[(r^2+z^2)^{-1}], \qquad G_{12}=-\frac{a_1\rho_0^2}{4}\frac{\sqrt{r^2+z^2}-z}{r^2+z^2}+\mathcal{O}[(r^2+z^2)^{-1}],\\
&G_{13}=-\frac{a_2\rho_0^2}{4}\frac{\sqrt{r^2+z^2}+z}{r^2+z^2}+\mathcal{O}[(r^2+z^2)^{-1}], \qquad G_{23}=\frac{a_1a_2\rho_0^2r^2}{8(r^2+z^2)^{3/2}}+\mathcal{O}[(r^2+z^2)^{-1}],\\
&G_{22}=(\sqrt{r^2+z^2}-z)\left\{1+\frac{\rho_0^2+a_1^2-a_2^2}{4\sqrt{r^2+z^2}}+\mathcal{O}[(r^2+z^2)^{-1}]\right\},\\
&G_{33}=(\sqrt{r^2+z^2}+z)\left\{1+\frac{\rho_0^2-a_1^2+a_2^2}{4\sqrt{r^2+z^2}}+\mathcal{O}[(r^2+z^2)^{-1}]\right\}.
\end{aligned}
\tag{5.17}
$$

We can now use (4.15) to read off the asymptotic quantities. We get

$$
M=\frac{3\pi}{8}\rho_0^2, \qquad J_1=\frac{\pi}{4}a_1\rho_0^2, \qquad J_2=\frac{\pi}{4}a_2\rho_0^2,
$$
$$
\zeta=\frac{1}{8}a_1a_2\rho_0^2, \qquad \eta=\frac{3\pi}{8}(a_1^2-a_2^2). \tag{5.18}
$$

Note that one can see from the above results that $e^{2\nu}\simeq 1/(2\sqrt{r^2+z^2})$. From Sec. IV C we have that this means $x^2=\phi$ and $x^3=\psi$ are periodic with period 2π.

2. Rod-structure

We now analyze the rod-structure of the five-dimensional Myers-Perry solution according to the methods of Sec. III. We have

(i) The semi-infinite spacelike rod $[-\infty,-\alpha]$. For $z\in[-\infty,-\alpha]$ and $r=0$ we see from (G7) that $x=-z/\alpha$ and $y=-1$. From (5.16) we see then that $G_{13}=G_{23}=G_{33}=0$. By Eq. (3.8) we see that this rod has the direction $v=(0,0,1)$, i.e., it is in the $\partial/\partial x^3$ direction.

(ii) The finite timelike rod $[-\alpha,\alpha]$. For $z\in[-\alpha,\alpha]$ we see from (G7) that $x=1$ and $y=z/\alpha$. Using (5.16), we see that $\sum_{j=1}^3 G_{ij}v^j=0$ for $z\in[-\alpha,\alpha]$ with v being the vector

$$
v=(1,\Omega_1,\Omega_2), \tag{5.19}
$$

with

$$
\begin{aligned}
\Omega_1&=\frac{\rho_0^2+a_1^2-a_2^2-4\alpha}{2a_1\rho_0^2},\\
\Omega_2&=\frac{\rho_0^2-a_1^2+a_2^2-4\alpha}{2a_2\rho_0^2}.
\end{aligned}
\tag{5.20}
$$

Therefore, the rod $[-\alpha,\alpha]$ is in the direction v given by (5.19). Note that Ω_1 and Ω_2 are the angular velocities of the Myers-Perry black hole. That the rod $[-\alpha,\alpha]$ is timelike can be seen by noting that $G_{ij}v^iv^j/r^2$ is negative for $r\to 0$. One can check that this rod corresponds to an event horizon with topology S^3. This follows from the fact that the $[-\alpha,\alpha]$ rod has spacelike rods on each side in two different directions.

(iii) The semi-infinite spacelike rod $[\alpha,\infty]$. For $z\in[\alpha,\infty]$ and $r=0$ we see that $x=z/\alpha$ and $y=1$. From (5.16) we see then that $G_{12}=G_{22}=G_{23}=0$. This means that the rod $[\alpha,\infty]$ is in the direction $v=(0,1,0)$, i.e., in the $\partial/\partial x^2$ direction.

We see that one can use the direction (5.19) to transform to coordinates $(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)=(x^1,x^2-\Omega_1x^1,x^3-\Omega_2x^1)$ so that the $[-\alpha,\alpha]$ rod is along the $\partial/\partial\tilde{x}^1$ direction. This means that $(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)$ are comoving coordinates for the event horizon. Thus, finding the direction of the rod $[-\alpha, \alpha]$ corresponds to finding the comoving coordinates on the horizon of the Myers-Perry black hole.

We note furthermore that we can make an alternative parametrization of the five-dimensional Myers-Perry solution. Clearly, the direction $v = (1, \Omega_1, \Omega_2)$ in (5.19) is given uniquely by the two parameters Ω_1 and Ω_2. Letting then α be the third parameter, we have that the five-dimensional Myers-Perry solution is characterized uniquely by the three parameters α, Ω_1 and Ω_2, in accordance with the ideas of Sec. III B.

C. Myers-Perry solution with one angular momentum

In the following we give details on the five-dimensional Myers-Perry solution with one angular momentum. In our conventions, we obtain the Myers-Perry solution with one angular momentum below from the Myers-Perry solution with two angular momenta above by setting $a_1 = a$ and $a_2 = 0$. The metric is

$$ds^2 = -dt^2 + \frac{\rho_0^2}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + (\rho^2 + a^2) \times \sin^2\theta d\phi^2 + \rho^2\cos^2\theta d\psi^2 + \frac{\Sigma}{\Delta}d\rho^2 + \Sigma d\theta^2, \quad (5.21)$$

where

$$\Delta = \rho^2 - \rho_0^2 + a^2, \qquad \Sigma = \rho^2 + a^2\cos^2\theta. \quad (5.22)$$

From (5.13) we see that the (r, z) coordinates are given by

$$r = \frac{1}{2}\rho\sqrt{\Delta}\sin 2\theta, \qquad z = \left(\frac{1}{2}\rho^2 - \frac{\rho_0^2 - a^2}{4}\right)\cos 2\theta. \quad (5.23)$$

The prolate spherical coordinates are given by

$$x = \frac{2\rho^2}{\rho_0^2 - a^2} - 1, \qquad y = \cos 2\theta, \qquad \alpha = \frac{\rho_0^2 - a^2}{4}, \quad (5.24)$$

as one can see from (5.15). From Eqs. (5.16) we obtain that the metric in prolate spherical coordinates is given by

$$\begin{aligned}
G_{11} &= -\frac{x\cos^2\lambda + y\sin^2\lambda - 1}{x\cos^2\lambda + y\sin^2\lambda + 1}, \qquad G_{12} = -\frac{2\sqrt{\alpha}\tan\lambda(1-y)}{x\cos^2\lambda + y\sin^2\lambda + 1},\\
G_{22} &= \frac{\alpha}{\cos^2\lambda}(1-y)\left[x\cos^2\lambda + 1 + \sin^2\lambda + \frac{2\sin^2\lambda(1-y)}{x\cos^2\lambda + y\sin^2\lambda + 1}\right],\\
G_{33} &= \alpha(x+1)(1+y), \qquad e^{2\nu} = \frac{x\cos^2\lambda + y\sin^2\lambda + 1}{2\alpha\cos^2\lambda(x^2 - y^2)},
\end{aligned} \quad (5.25)$$

where we have defined

$$\sin\lambda = \frac{a}{\rho_0}. \quad (5.26)$$

Using Eq. (G7) we get furthermore the metric written in the canonical form (2.10) and (2.11) as functions of the canonical (r, z) coordinates

$$\begin{aligned}
G_{11} &= -\frac{R_+ + R_-\cos 2\lambda - 2\alpha}{R_+ + R_-\cos 2\lambda + 2\alpha}, \qquad G_{12} = -\frac{2\sqrt{\alpha}\tan\lambda(2\alpha - R_+ + R_-)}{R_+ + R_-\cos 2\lambda + 2\alpha},\\
G_{22} &= \frac{2\alpha - R_+ + R_-}{4\alpha}\left[R_+ + R_- + 2\alpha\frac{1+\sin^2\lambda}{\cos^2\lambda} + \frac{4\alpha\tan^2\lambda(2\alpha - R_+ + R_-)}{R_+ + R_-\cos 2\lambda + 2\alpha}\right],\\
G_{33} &= R_+ + z + \alpha, \qquad e^{2\nu} = \frac{R_+ + R_-\cos 2\lambda + 2\alpha}{4R_+R_-\cos^2\lambda},
\end{aligned} \quad (5.27)$$

with

$$R_+ = \sqrt{r^2 + (z+\alpha)^2}, \qquad R_- = \sqrt{r^2 + (z-\alpha)^2}. \quad (5.28)$$

We see that the whole solution (5.27) is determined by the two parameters α and λ.

From the analysis of the asymptotic region of the Myers-Perry solution with two angular momenta, we see that the asymptotic quantities are now

$$M = \frac{3\pi}{8}\rho_0^2, \qquad J_1 = \frac{\pi}{4}a\rho_0^2, \qquad J_2 = 0, \qquad \zeta = 0, \qquad \eta = \frac{3\pi}{8}a^2. \quad (5.29)$$

We list for completeness here the rod-structure of the solution (5.27). This can easily be obtained using the results of the analysis for the case of two angular momenta.

(i) The semi-infinite spacelike rod $[-\infty, -\alpha]$. This rod is in the direction $v = (0, 0, 1)$, i.e., in the $\partial/\partial x^3$ direction.

(ii) The finite timelike rod $[-\alpha, \alpha]$. This rod is in the direction v given by

$$v = (1, \Omega, 0), \qquad \Omega = \frac{a}{\rho_0^2} = \frac{\sin\lambda\cos\lambda}{2\sqrt{\alpha}}. \quad (5.30)$$

(iii) The semi-infinite spacelike rod $[\alpha, \infty]$. This rod is in the direction $v = (0, 1, 0)$, i.e., in the $\partial/\partial x^2$ direction.

VI. BLACK RING SOLUTIONS

In this section we consider the rotating black ring [6]. The rotating black ring is the first known example of a stationary and regular asymptotically flat five-dimensional solution with an event horizon that is not topologically a three-sphere S^3. Instead the horizon is topologically a ring $S^2 \times S^1$.

We first describe in Sec. VI A the general black ring solution which generically has a conical singularity. We write its metric in the canonical form (2.10) and (2.11) and discuss the rod-structure. We consider then briefly the special case of the static black ring solution and furthermore how to obtain the Myers-Perry rotating black hole with one angular momentum.

In Sec. VI A we present the regular rotating black ring, write its metric in the canonical form (2.10) and (2.11) and discuss its properties.

A. General black ring solution

We begin by reviewing briefly the general black ring metric. So far, the general black ring metric has been written only in the so-called C-metric coordinates. In the C-metric coordinates of [20], the general black ring metric is[12]

$$ds^2 = -\frac{F(v)}{F(u)}\left(dt - C\kappa\frac{1+v}{F(v)}d\phi\right)^2 + \frac{2\kappa^2 F(u)}{(u-v)^2} \times\left[-\frac{G(v)}{F(v)}d\phi^2 + \frac{G(u)}{F(u)}d\psi^2 + \frac{du^2}{G(u)} - \frac{dv^2}{G(v)}\right]. \quad (6.1)$$

[12]We use here the C-metric coordinates of [20], since they are particular convenient for our purposes. There have been given three different, but equivalent, C-metric coordinates for the black ring in the literature: i) The original coordinates of [6]. ii) The coordinates described in [21] where one takes the solution of [6] and rewrite it so that structure functions are factorable. iii) The coordinates of [20] where the structure functions also are factorizable, but where det$G$ is simpler.

Here $F(\xi)$ and $G(\xi)$ are the structure functions, which takes the form

$$F(\xi) = 1 + b\xi, \qquad G(\xi) = (1-\xi^2)(1+c\xi), \quad (6.2)$$

where the parameters b and c lie in the ranges

$$0 < c \le b < 1. \quad (6.3)$$

Furthermore, in (6.1), the constant C is given in terms of b and c by

$$C = \sqrt{2b(b-c)\frac{1+b}{1-b}}. \quad (6.4)$$

The u and v coordinates in (6.1) have the ranges

$$-1 \le u \le 1, \qquad v \le -1. \quad (6.5)$$

Note that the solution (6.1) generically has conical singularities at $u = 1$, $u = -1$ and $v = -1$ [20]. These will be analyzed below using the methods of Sec. III. We note here that while the potential singularities at $u = -1$ and $v = -1$ will be cured by choosing the periods of $x^2 = \phi$ and $x^3 = \psi$ appropriately, we do not fix the singularity at $u = 1$ before in Sec. VI B where we consider the regular rotating black ring. Thus, in the following the black ring solution is generically singular at $u = 1$.

1. Metric in canonical coordinates

We now find the metric in the canonical coordinates (2.10) and (2.11). In the following we use extensively the results of Appendix H. In Appendix H the general relation between C-metric coordinates (u, v) and the canonical coordinates (r, z) is discussed in detail. Furthermore, for the specific case relevant here several useful relations between the C-metric coordinates (u, v) and the canonical coordinates (r, z) are given.

We take the coordinates for the Killing directions to be $x^1 = t$, $x^2 = \phi$ and $x^3 = \psi$. Using the results of Appendix H we see that the r and z coordinates takes the form

$$r = \frac{2\kappa^2\sqrt{-G(u)G(v)}}{(u-v)^2}, \qquad z = \frac{\kappa^2(1-uv)(2+cu+cv)}{(u-v)^2}. \quad (6.6)$$

This is obtained by first computing det$G$, which gives r. In Appendix H it is then found for this particular r, given by the structure function $G(\xi)$ in (6.2), that z can be chosen as in (H12).

From (6.1) and (6.2) we get using (H18) of Appendix H, giving u and v as functions of r and z, that $G_{ij}(r, z)$ is

$$G_{11} = -\frac{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 - 2b(1-c^2)\kappa^2}{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\kappa^2},$$
$$G_{12} = -\frac{2C\kappa(1-c)[R_3 - R_1 + (1+c)\kappa^2]}{(1+b)(1-c)R_1 + (1-b)(1+c)R_2 - 2(b-c)R_3 + 2b(1-c^2)\kappa^2}, \quad (6.7)$$
$$G_{33} = \frac{(R_1 + R_2 + 2c\kappa^2)[R_1 - R_3 + (1+c)\kappa^2][R_2 + R_3 - (1-c)\kappa^2]}{2\kappa^2[(1-c)R_1 - (1+c)R_2 - 2cR_3]},$$

where we have defined R_1, R_2 and R_3 by

$$R_1 = \sqrt{r^2 + (z + c\kappa^2)^2}, \qquad R_2 = \sqrt{r^2 + (z - c\kappa^2)^2}, \qquad R_3 = \sqrt{r^2 + (z - \kappa^2)^2}, \quad (6.8)$$

as also defined in (H15). For simplicity, we do not write G_{22} explicitly here, but note that it is given implicitly as a function of (r, z) by

$$G_{22} = -\frac{r^2}{G_{11}G_{33}} + \frac{G_{12}^2}{G_{11}}. \quad (6.9)$$

Using now furthermore (H10), we get

$$e^{2\nu} = [(1+b)(1-c)R_1 + (1-b)(1+c)R_2 + 2(c-b)R_3 + 2b(1-c^2)\kappa^2] \times \frac{(1-c)R_1 + (1+c)R_2 + 2cR_3}{8(1-c^2)^2 R_1 R_2 R_3}. \quad (6.10)$$

This completes the general black ring solution as written in canonical coordinates (2.10) and (2.11). Note that using (H17) in Appendix H it is easy to see that G_{33} can be written in the alternative form

$$G_{33} = \frac{(R_3 + z - \kappa^2)(R_2 - z + c\kappa^2)}{R_1 - z - c\kappa^2}. \quad (6.11)$$

2. Rod-structure

We now analyze the rod-structure of the general black ring metric. This includes an analysis of the possible conical singularities of the solution. The rod-structure is as follows:

(i) The semi-infinite spacelike rod $[-\infty, -c\kappa^2]$. For $r = 0$ and $z \in [-\infty, -c\kappa^2]$ we have that $R_1 - R_3 + (1+c)\kappa^2 = 0$ which using (6.7) is seen to give that $G_{33} = 0$. This means we have a rod $[-\infty, -c\kappa^2]$ in the direction $v = (0, 0, 1)$, i.e., in the $\partial/\partial x^3$ direction. Using (3.10) we see furthermore that $x^3 = \psi$ needs to have period

$$\Delta\psi = 2\pi\frac{\sqrt{1-b}}{1-c}, \quad (6.12)$$

to avoid a conical singularity for $r = 0$ and $z \in [-\infty, -c\kappa^2]$. Since $u = -1$ is equivalent to $R_1 - R_3 + (1+c)\kappa^2 = 0$ we see that this conical singularity corresponds to the one at $u = -1$ mentioned above.

(ii) The finite timelike rod $[-c\kappa^2, c\kappa^2]$. For $r = 0$ and $z \in [-c\kappa^2, c\kappa^2]$ we see that $R_1 + R_2 - 2c\kappa^2 = 0$. One can then check that $\sum_{j=1}^3 G_{ij}v^j = 0$ for $r = 0$ and $z \in [-c\kappa^2, c\kappa^2]$ with v being the vector

$$v = (1, \Omega, 0), \qquad \Omega = \frac{b-c}{(1-c)C\kappa}. \quad (6.13)$$

From this we see that we have a rod $[-c\kappa^2, c\kappa^2]$ along the direction v given in (6.13). The rod $[-c\kappa^2, c\kappa^2]$ is timelike since $G_{ij}v^iv^j/r^2$ is negative for $r \to 0$. Note that Ω in (6.13) is the angular velocity of the general black ring solution. One can check that this rod corresponds to an event horizon with topology $S^2 \times S^1$. This follows from the fact that the $[-c\kappa^2, c\kappa^2]$ rod has rods in the $\partial/\partial x^3$ direction on each side, so that the z and x^3 coordinates parametrize the S^2 while the x^2 coordinate parametrize the S^1.

(iii) The finite spacelike rod $[c\kappa^2, \kappa^2]$. For $r = 0$ and $z \in [c\kappa^2, \kappa^2]$ we have that $R_2 + R_3 - (1-c)\kappa^2 = 0$. Using (6.7) we see that this gives that $G_{33} = 0$. This means we have a rod $[c\kappa^2, \kappa^2]$ in the direction $v = (0, 0, 1)$, i.e., in the $\partial/\partial x^3$ direction. Using (3.10) we see furthermore that $x^3 = \psi$ needs to have period

$$\Delta\psi = 2\pi\frac{\sqrt{1+b}}{1+c}, \quad (6.14)$$

to avoid a conical singularity for $r = 0$ and $z \in [c\kappa^2, \kappa^2]$. However, since we have already fixed the period of $x^3 = \psi$ by (6.12), curing the conical singularity associated with the rod $[c\kappa^2, \kappa^2]$ requires putting $b = 2c/(1+c^2)$. We do not fix b in terms of c here, thus we consider here solutions that can have conical singularities for $r = 0$ and $z \in [c\kappa^2, \kappa^2]$. In Sec. VI B we consider the subset of solutions for which we do not have any conical singularities. Finally, note that since $u = 1$ is equivalent to $R_2 + R_3 - (1-c)\kappa^2 = 0$ we see that this conical singularity corresponds to the one at $u = 1$ mentioned above.

(iv) The semi-infinite spacelike rod $[\kappa^2, \infty]$. For $r = 0$ and $z \in [\kappa^2, \infty]$ we have that $R_1 - R_3 - (1+c)\kappa^2 = 0$. Using (6.7) we see that this gives that $G_{12} = G_{22} = 0$. This means we have a rod $[\kappa^2, \infty]$

in the direction $v = (0, 1, 0)$, i.e., in the $\partial/\partial x^2$ direction. Using (3.10) we see furthermore that $x^2 = \phi$ needs to have period

$$\Delta\phi = 2\pi \frac{\sqrt{1-b}}{1-c}, \tag{6.15}$$

to avoid a conical singularity for $r = 0$ and $z \in [\kappa^2, \infty]$. Since $v = -1$ is equivalent to $R_1 - R_3 - (1+c)\kappa^2 = 0$ we see that this conical singularity corresponds to the one at $v = -1$ mentioned above.

3. Static black ring

We consider here briefly the case of the static black ring, obtained by setting $b = c$. The static black ring was first discussed in [16]. The static black ring is in the class of generalized Weyl solutions of [16] since its metric is diagonal.

Putting $b = c$ in (6.1) one easily gets the neutral black ring in C-metric coordinates. Note that $C = 0$, $G(\xi) = (1-\xi^2)F(\xi)$, $F(\xi) = 1 + c\xi$ and $0 < c < 1$. Using (6.7), (6.8), (6.9), and (6.10), we see that the static black ring metric in canonical coordinates (2.10) and (2.11) takes the form [16]

$$\begin{aligned} G_{11} &= -\frac{R_1 + R_2 - 2c\kappa^2}{R_1 + R_2 + 2c\kappa^2} = -\frac{R_1 - z - c\kappa^2}{R_2 - z + c\kappa^2}, \\ G_{22} &= R_3 - z + \kappa^2, \\ G_{33} &= \frac{(R_3 + z - \kappa^2)(R_2 - z + c\kappa^2)}{R_1 - z - c\kappa^2}, \\ e^{2\nu} &= \frac{(R_1 + R_2 + 2c\kappa^2)[(1-c)R_1 + (1+c)R_2 + 2cR_3]}{8(1-c^2)R_1R_2R_3}. \end{aligned} \tag{6.16}$$

The static black ring metric have previously been written in canonical coordinates in [16] since it falls in the class of generalized Weyl solutions considered there. The rod-structure of the static black ring is:

(i) The semi-infinite spacelike rod $[-\infty, -c\kappa^2]$ in the $\partial/\partial x^3$ direction.
(ii) The finite timelike rod $[-c\kappa^2, c\kappa^2]$ in the $\partial/\partial x^1$ direction.
(iii) The finite spacelike rod $[c\kappa^2, \kappa^2]$ in the $\partial/\partial x^3$ direction.
(iv) The semi-infinite spacelike rod $[\kappa^2, \infty]$ in the $\partial/\partial x^2$ direction.

We see that all the rods are rectangular relative to each other. The rod-structure of the static black ring was previously described in [16].

4. Getting the Myers-Perry black hole from the general black ring solution

We show here how one obtains the five-dimensional Myers-Perry rotating black hole solution with one angular momentum, that we considered in Sec. V C, from the general black ring solution. This has previously been described in Ref. [20] in terms of the C-metric coordinates used in the metric (6.1). Here we do it instead in terms of the canonical form of the metric (6.7), (6.8), (6.9), and (6.10).

We first note that we need to take the limit $c \to 1$, since the $[c\kappa^2, \kappa^2]$ rod should be absent for the black hole solution. By considering explicit expressions for the mass M and angular momentum J_1, plus the fact that $c \le b < 1$, one can see that $(1-b)/(1-c)$ and $\kappa^2/(1-c)$ should be fixed as $c \to 1$. One can furthermore see that we can find λ and α, so that

$$c = 1 - \epsilon, \qquad b = 1 - \epsilon\cos^2\lambda, \qquad \kappa = \frac{\sqrt{\alpha}}{\cos\lambda}\sqrt{\epsilon}, \tag{6.17}$$

with the limit being defined as $\epsilon \to 0$. Since x^2 and x^3 have periods (6.15) and (6.12) we need to make the rescaling

$$x^2 = \frac{\cos\lambda}{\sqrt{\epsilon}}\tilde{x}^2, \qquad x^3 = \frac{\cos\lambda}{\sqrt{\epsilon}}\tilde{x}^3, \tag{6.18}$$

so that now $\tilde{x}^2$ and $\tilde{x}^3$ have period 2π for $\epsilon \to 0$. From the definition of the canonical (r, z) coordinates, we see that this means we should make the rescaling

$$r = \frac{\epsilon}{\cos^2\lambda}\tilde{r}, z = \frac{\epsilon}{\cos^2\lambda}\tilde{z}. \tag{6.19}$$

This gives that $\sqrt{r^2 + (z \pm \kappa^2)^2} = \epsilon\sqrt{\tilde{r}^2 + (\tilde{z} \pm \alpha)^2}/\cos^2\lambda$. Using this with the metric (6.7), (6.8), (6.9), and (6.10) it is easy to see that one gets the metric (5.27) of a Myers-Perry rotating black hole with one angular momentum.

B. Regular rotating black ring

We now consider the regular black ring solution. In Sec. VI A we cured the conical singularities at the $[-\infty, -c\kappa^2]$ and $[\kappa^2, \infty]$ rods by imposing $x^2 = \phi$ to have period (6.12) and $x^3 = \psi$ to have period (6.15). However, we did not fix the potential conical singularity at the $[c\kappa^2, \kappa^2]$ rod. To ensure regularity at the $[c\kappa^2, \kappa^2]$ rod, $x^3 = \psi$ should have period (6.14), which means we need to impose

$$b = \frac{2c}{1 + c^2}. \tag{6.20}$$

Therefore, with (6.20) imposed, and with $x^2 = \phi$ and $x^3 = \psi$ having their periods given by

$$\Delta\phi = \Delta\psi = \frac{2\pi}{\sqrt{1 + c^2}}, \tag{6.21}$$

the rotating black ring solution (6.1) is regular [6,20]. Note that the constant C in (6.4) now takes the form

$$C = \frac{2c(1+c)}{1+c^2}\sqrt{\frac{1+c}{1-c}}. \tag{6.22}$$

From (6.7) we get that the regular rotating black ring metric in canonical coordinates (2.10) and (2.11) is given by

$$\begin{aligned}
G_{11} &= -\frac{(1+c)R_1 + (1-c)R_2 - 2cR_3 - 4c\kappa^2}{(1+c)R_1 + (1-c)R_2 - 2cR_3 + 4c\kappa^2},\\
G_{12} &= -\frac{4c\kappa\sqrt{1+c}}{\sqrt{1-c}}\\
&\quad\times\frac{R_3 - R_1 + (1+c)\kappa^2}{(1+c)R_1 + (1-c)R_2 - 2cR_3 + 4c\kappa^2},\\
G_{33} &= \frac{(R_3 + z - \kappa^2)(R_2 - z + c\kappa^2)}{R_1 - z - c\kappa^2},\\
e^{2\nu} &= [(1+c)R_1 + (1-c)R_2 - 2cR_3 + 4c\kappa^2]\\
&\quad\times\frac{(1-c)R_1 + (1+c)R_2 + 2cR_3}{8(1-c^4)R_1R_2R_3}.
\end{aligned} \tag{6.23}$$

One can furthermore find G_{22} using (6.9).

1. Rod-structure

The rod-structure of the regular rotating black ring solution is easily obtained from the rod-structure of the general black ring solution analyzed in Sec. VI A by imposing (6.20). We list here therefore only a short summary of the rod-structure of the regular rotating black ring:

(i) The semi-infinite spacelike rod $[-\infty, -c\kappa^2]$. This rod is in the direction $v = (0, 0, 1)$, i.e., in the $\partial/\partial x^3$ direction.

(ii) The finite timelike rod $[-c\kappa^2, c\kappa^2]$. This rod is in the direction

$$v = (1, \Omega, 0), \qquad \Omega = \frac{1}{2\kappa}\sqrt{\frac{1-c}{1+c}}. \tag{6.24}$$

(iii) The finite spacelike rod $[c\kappa^2, \kappa^2]$. This rod is in the direction $v = (0, 0, 1)$, i.e., in the $\partial/\partial x^3$ direction.

(iv) The semi-infinite spacelike rod $[\kappa^2, \infty]$. This rod is in the direction $v = (0, 1, 0)$, i.e., in the $\partial/\partial x^2$ direction.

2. Asymptotic region

For the regular rotating black ring solution (6.23) in the asymptotic region $\sqrt{r^2+z^2} \to \infty$ with $z/\sqrt{r^2+z^2}$ finite, we find

$$\begin{aligned}
G_{11} &= -1 + \frac{4c\kappa^2}{1-c}\frac{1}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}],\\
G_{12} &= -2c\kappa^3\left(\frac{1+c}{1-c}\right)^{3/2}\frac{\sqrt{r^2+z^2}-z}{r^2+z^2} + \mathcal{O}[(r^2+z^2)^{-1}],\\
G_{22} &= (\sqrt{r^2+z^2}-z)\left\{1 + \frac{(1+c+2c^2)\kappa^2}{1-c}\frac{1}{\sqrt{r^2+z^2}}\right.\\
&\quad\left. + \mathcal{O}[(r^2+z^2)^{-1}]\right\},\\
G_{33} &= (\sqrt{r^2+z^2}+z)\left\{1 + \frac{(2c-1)\kappa^2}{\sqrt{r^2+z^2}} + \mathcal{O}[(r^2+z^2)^{-1}]\right\}.
\end{aligned} \tag{6.25}$$

Using (4.17) we then get

$$\begin{aligned}
M &= \frac{3\pi c\kappa^2}{(1-c)(1+c^2)},\\
J_1 &= 2\pi c\kappa^3\left(\frac{1+c}{(1-c)(1+c^2)}\right)^{3/2},\\
\eta &= \frac{3\pi\kappa^2(1-c+2c^2)}{2(1-c)(1+c^2)},
\end{aligned} \tag{6.26}$$

along with $J_2 = 0$ and $\zeta = 0$, where we used that $\varepsilon = 1/\sqrt{1+c^2}$ from (6.21). Note that

$$\frac{J_1^2}{M^3} = \frac{4(1+c)^3}{27\pi c}. \tag{6.27}$$

We see that this has the minimum at $c = 1/2$ with value $1/\pi$. For $c \to 0$ it goes to infinity, while for $c = 1$ it has the value $32/(27\pi)$. This is in accordance with [6,20,22].

VII. DISCUSSION AND CONCLUSIONS

The main results of this paper are as follows. We found in Sec. II that the metric of stationary and axisymmetric pure gravity solutions in D dimensions can be written in the form (see Eqs. (2.10) and (2.11))

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij}dx^i dx^j + e^{2\nu}(dr^2 + dz^2), \quad r^2 = |\det G|,$$

apart from a subclass of solutions with constant $\det G$ that is considered in Appendix A. The equation on the $D-2$ by $D-2$ dimensional symmetric matrix G was found to take the simple form (see Eqs. (2.16))

$$G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2,$$

where $\vec{\nabla}$ is the gradient on a three-dimensional flat Euclidean space, with metric (2.15). The function ν can then be found from G using the integrable Eqs. (2.13).

In Sec. III we considered then the behavior of G for $r \to 0$. We generalized the concept of rods of [16] so that it can be used for stationary and axisymmetric solutions. One of the key points is that for each rod $[z_1, z_2]$ one has a direction in the $(D-2)$-dimensional vector space spanned by the Killing vector fields.

In Sec. IV we analyzed the asymptotic region of four- and five-dimensional asymptotically flat solutions. In particular we identified how to read off the asymptotic quantities.

Finally, in Secs. V and VI we wrote down the metrics of the five-dimensional rotating black hole of Myers and Perry and the rotating black ring of Emparan and Reall in the canonical form (2.10) and (2.11). Furthermore, we analyzed the structure of the rods according to Sec. III and moreover the asymptotic region according to Sec. IV.

The results of this paper have at least three interesting applications:

(i) Finding new stationary and axisymmetric solutions using the canonical form of the metric and the Einstein equations. For example one can look for new five-dimensional black ring solutions with two angular momenta, or for new solutions with a rotating black hole attached to a Kaluza-Klein bubble, as advocated in [8].

(ii) Understanding the rod-structure of known stationary and axisymmetric solutions.

(iii) Understanding better the uniqueness properties for higher-dimensional black holes. In four dimensions, the Carter-Robinson uniqueness theorem [3,5] on the Kerr rotating black hole rests on using the Papapetrou form (1.1) of the metric. We expect therefore similar arguments to be applicable in higher dimensions, although they of course cannot prove any kind of strict uniqueness for five-dimensional rotating black holes due to the existence of rotating black rings.

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APPENDIX A: SPECIAL CLASS OF SOLUTIONS

In this appendix we consider stationary and axisymmetric solutions to the vacuum Einstein equations for which $\det(G_{ij})$ is constant, with G_{ij} defined by Eq. (2.4).

We can always find coordinates (r, z) so that the metric (2.4) can be written

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij} dx^i dx^j + e^{2\nu}(dr^2 + dz^2). \tag{A1}$$

This is possible since any two-dimensional manifold is conformally flat. The metric (A1) is obviously the same as (2.10). However, the difference is that the constraint (2.11) is replaced by restricting $\det(G_{ij})$ to be constant.

Note first that demanding $\det(G_{ij})$ to be constant leads to the identities

$$\begin{gathered}\sum_{i,j=1}^{D-2} G^{ij}\partial_a G_{ij} = 0,\\ \sum_{i,j=1}^{D-2} G^{ij}\partial_a\partial_b G_{ij} = \sum_{i,j,k,l=1}^{D-2} G^{ij}\partial_a G_{jk}G^{kl}\partial_b G_{li},\end{gathered} \tag{A2}$$

with $a, b = r, z$. Computing the Ricci tensor for the metric (A1) and using the constraint that $\det(G_{ij})$ is constant, we get that the vacuum Einstein equations can be written

$$\begin{aligned}(\partial_r^2 + \partial_z^2)G_{ij} &= \sum_{k,l=1}^{D-2} \partial_r G_{ik}G^{kl}\partial_r G_{lj} + \sum_{k,l=1}^{D-2} \partial_z G_{ik}G^{kl}\partial_z G_{lj},\\ \sum_{i,j,k,l=1}^{D-2} G^{ij}\partial_r G_{jk}G^{kl}\partial_r G_{li} &= \sum_{i,j,k,l=1}^{D-2} G^{ij}\partial_z G_{jk}G^{kl}\partial_z G_{li},\\ \sum_{i,j,k,l=1}^{D-2} G^{ij}\partial_r G_{jk}G^{kl}\partial_z G_{li} &= 0\end{aligned} \tag{A3}$$

$$(\partial_r^2 + \partial_z^2)\nu = -\frac{1}{8}\sum_{i,j,k,l=1}^{D-2}(G^{ij}\partial_r G_{jk}G^{kl}\partial_r G_{li} + G^{ij}\partial_z G_{jk}G^{kl}\partial_z G_{li}). \tag{A4}$$

In conclusion, we can find solutions in the form of (A1) with $\det(G_{ij})$ being constant by first finding a $G_{ij}(r, z)$ solving (A3) and then finding a solution for $\nu(r, z)$ of (A4).

1. Four-dimensional examples

In four dimensions, we have a well-known class of solutions to Eqs. (A3) and (A4) in the form of a particular kind of pp-wave solutions. These pp-wave solutions have

$$G_{11} = -1 - H(r, z), \qquad G_{22} = 1 - H(r, z), \qquad G_{12} = -H(r, z). \tag{A5}$$

We see immediately that $\det(G_{ij}) = -1$. Furthermore, one can check that the Eqs. (A3) with $D = 4$ are solved, provided $H(r, z)$ obeys

$$(\partial_r^2 + \partial_z^2)H(r, z) = 0. \tag{A6}$$

Finally, $\nu(r, z) = 0$ solves Eq. (A4). Therefore, the pp-wave metrics

$$ds^2 = -dt^2 + dx^2 - H(dt + dx)^2 + dr^2 + dz^2, \tag{A7}$$

with $H(r, z)$ obeying Eq. (A6), are in the class of solutions described by the metric (A1) with $\det(G_{ij})$ being constant [12]. Note moreover that any $\nu(r, z)$ solving $(\partial_r^2 + \partial_z^2)\nu = 0$ also gives a solution.

APPENDIX B: ANALYSIS OF $\det(G_{ij})$

In this appendix we study the behavior of $\det(G_{ij})$ as a function. If we start with the metric (2.4) we can always put it in the form

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij}dx^i dx^j + C(du^2 + dv^2), \qquad \text{(B1)}$$

where $C(u, v)$ and $G_{ij}(u, v)$ are functions only of u and v. That we can bring the metric (2.4) to this form is easily seen from the fact that any two-dimensional manifold is conformally flat. Define now

$$f = \sqrt{|\det(G_{ij})|}. \qquad \text{(B2)}$$

In the following we study the function $f(u, v)$. By computing the Ricci tensor for the metric (B1) we get

$$\sum_{i,j=1}^{D-2} G^{ij}R_{ij} = -\frac{1}{Cf}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)f. \qquad \text{(B3)}$$

Now, since we consider solutions that are Ricci-flat, we get that

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)f = 0. \qquad \text{(B4)}$$

If we define the complex variable $\omega = u + iv$, along with the derivatives $\partial = \frac{\partial}{\partial u} + i\frac{\partial}{\partial v}$ and $\bar{\partial} = \frac{\partial}{\partial u} - i\frac{\partial}{\partial v}$, we see that $\bar{\partial}\partial f = 0$. Therefore, ∂f is a holomorphic function. We know from elementary complex analysis (see for example [23]) that either the zeroes of a holomorphic function are isolated or the function is identically zero (assuming the set that the function is defined on is simply connected). Since $\partial f = \frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v}$ we can draw the conclusion:

(i) Either $f(u, v)$ is a constant function or $(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}) \neq (0, 0)$ except in isolated points.

APPENDIX C: DIAGONALIZING A TWO-DIMENSIONAL METRIC

In this appendix we prove, for the sake of clarity and completeness, the rather basic result that given a well-behaved function on a two-dimensional Riemannian manifold one can diagonalize the metric with the given function being one of the coordinates.

Consider a two-dimensional Riemannian manifold M with a coordinate system (y^1, y^2). Write the metric as

$$ds^2 = \hat{g}_{ab}dy^a dy^b. \qquad \text{(C1)}$$

Let $z^1(y^1, y^2)$ be a given function with $(\partial z^1/\partial y^1, \partial z^1/\partial y^2) \neq (0, 0)$. We want to show that we can find a function $z^2(y^1, y^2)$ so that (z^1, z^2) is a new coordinate system and so that the metric in z^a coordinates is diagonal, i.e., so that $g_{12} = 0$, where we write the metric as $ds^2 = g_{ab}dz^a dz^b$. Equivalently, we can demand that $g^{12} = 0$. This is the same as

$$\hat{g}^{ab}\frac{\partial z^1}{\partial y^a}\frac{\partial z^2}{\partial y^b} = 0. \qquad \text{(C2)}$$

Now, define the vector field $V = V^1\frac{\partial}{\partial y^1} + V^2\frac{\partial}{\partial y^2}$ by

$$V^a = \hat{g}^{ab}\frac{\partial z^1}{\partial y^b}. \qquad \text{(C3)}$$

Consider now the integral curves of V. Define an equivalence relation $\sim$ on M where two points $p, q \in M$ are equivalent, i.e. $p \sim q$, if they are connected by an integral curve. Then we can define the quotient space $M/\sim$. Clearly, $M/\sim$ is a one-dimensional space. Let now z^2 be a coordinate on $M/\sim$. We then extend the scalar field z^2 on $M/\sim$ to a scalar field z^2 on M. Clearly, this scalar field z^2 on M has the property that z^2 is constant on the integral curves of V. Since z^2 is constant on the integral curves of V we get that

$$V^a\frac{\partial z^2}{\partial y^a} = 0, \qquad \text{(C4)}$$

which is the same as (C2). We have therefore proven that for any given function $z^1(y^1, y^2)$ with $(\partial z^1/\partial y^1, \partial z^1/\partial y^2) \neq (0, 0)$ we can find a function $z^2(y^1, y^2)$ such that

$$ds^2 = \hat{g}_{ab}dy^a dy^b = A(dz^1)^2 + B(dz^2)^2, \qquad \text{(C5)}$$

and so that (z^1, z^2) is a coordinate system on the two-dimensional manifold.

APPENDIX D: COMPUTATION OF RICCI TENSOR

1. Computation of the Ricci tensor with general Λ

We consider first the D-dimensional metric

$$ds^2 = \sum_{i,j=1}^{D-2} G_{ij}dx^i dx^j + e^{2\nu}(dr^2 + \Lambda dz^2), \qquad \text{(D1)}$$

with

$$r = \sqrt{|\det(G_{ij})|}, \qquad \text{(D2)}$$

where G_{ij}, ν and Λ are functions of r and z only. The nonzero components of the Christoffel symbols for the metric (D1) are

$$\Gamma^r_{ij} = -\frac{1}{2} e^{-2\nu}\partial_r G_{ij}, \Gamma^z_{ij} = -\frac{1}{2} e^{-2\nu}\Lambda^{-1}\partial_z G_{ij},$$
$$\Gamma^i_{rj} = \frac{1}{2}\sum_{k=1}^{D-2} G^{ik}\partial_r G_{jk}, \qquad \Gamma^i_{zj} = \frac{1}{2}\sum_{k=1}^{D-2} G^{ik}\partial_z G_{jk},$$
$$\Gamma^r_{rr} = \partial_r\nu, \qquad \Gamma^z_{zz} = \partial_z\nu + \frac{1}{2\Lambda}\partial_z\Lambda, \tag{D3}$$
$$\Gamma^r_{rz} = \partial_z\nu, \qquad \Gamma^z_{zr} = \partial_r\nu + \frac{1}{2\Lambda}\partial_r\Lambda,$$
$$\Gamma^r_{zz} = -\Lambda\partial_r\nu - \frac{1}{2C}\partial_r\Lambda, \qquad \Gamma^z_{rr} = -\frac{1}{\Lambda}\partial_z\nu.$$

Note now that since $r = \sqrt{|\det G_{ij}|}$ we have

$$\sum_{i,j=1}^{D-2} G^{ij}\partial_r G_{ij} = \frac{2}{r}, \qquad \sum_{i,j=1}^{D-2} G^{ij}\partial_z G_{ij} = 0. \tag{D4}$$

Using this we get

$$\sum_{i,j=1}^{D-2} G^{ij}\Gamma^r_{ij} = -\frac{1}{r}e^{-2\nu}, \qquad \sum_{i,j=1}^{D-2} G^{ij}\Gamma^z_{ij} = 0,$$
$$\sum_{i=1}^{D-2}\Gamma^i_{ri} = \frac{1}{r}, \qquad \sum_{i=1}^{D-2}\Gamma^i_{zi} = 0. \tag{D5}$$

We compute then

$$2e^{2\nu}R_{ij} = -\partial_r^2 G_{ij} - \frac{1}{r}\partial_r G_{ij} - \frac{\partial_r\Lambda}{2\Lambda}\partial_r G_{ij} - \frac{1}{\Lambda}\partial_z^2 G_{ij} + \frac{\partial_z\Lambda}{2\Lambda^2}\partial_z G_{ij} + \sum_{k,l=1}^{D-2} G^{kl}\partial_r G_{ki}\partial_r G_{lj} + \frac{1}{\Lambda}\sum_{k,l=1}^{D-2} G^{kl}\partial_z G_{ki}\partial_z G_{lj}. \tag{D6}$$

Notice now that from the fact that $r = \sqrt{|\det G_{ij}|}$ we have

$$-\sum_{i,j=1}^{D-2} G^{ij}\partial_r^2 G_{ij} + \sum_{i,j,k,l=1}^{D-2} G^{ij}G^{kl}\partial_r G_{ki}\partial_r G_{lj} = \frac{2}{r^2},$$
$$-\sum_{i,j=1}^{D-2} G^{ij}\partial_z^2 G_{ij} + \sum_{i,j,k,l=1}^{D-2} G^{ij}G^{kl}\partial_z G_{ki}\partial_z G_{lj} = 0. \tag{D7}$$

Using (D7) together with (D6), we get

$$\sum_{i,j=1}^{D-2} G^{ij}R_{ij} = -\frac{\partial_r\Lambda}{2e^{2\nu}\Lambda r}. \tag{D8}$$

2. Computation of the Ricci tensor with $\Lambda = 1$

We now set $\Lambda = 1$ in the metric (D1). The nonzero components of the Ricci tensor can then be computed to be

$$2e^{2\nu}R_{ij} = -\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)G_{ij} + \sum_{k,l=1}^{D-2} G^{kl}\partial_r G_{ki}\partial_r G_{lj} + \sum_{k,l=1}^{D-2} G^{kl}\partial_z G_{ki}\partial_z G_{lj},$$
$$R_{rr} = -\partial_r^2\nu - \partial_z^2\nu + \frac{1}{r^2} + \frac{1}{r}\partial_r\nu - \frac{1}{4}\sum_{i,j,k,l=1}^{D-2} G^{ij}G^{kl}\partial_r G_{ik}\partial_r G_{jl},$$
$$R_{zz} = -\partial_r^2\nu - \partial_z^2\nu - \frac{1}{r}\partial_r\nu - \frac{1}{4}\sum_{i,j,k,l=1}^{D-2} G^{ij}G^{kl}\partial_z G_{ik}\partial_z G_{jl}, \tag{D9}$$
$$R_{rz} = \frac{1}{r}\partial_z\nu - \frac{1}{4}\sum_{i,j,k,l=1}^{D-2} G^{ij}G^{kl}\partial_r G_{ik}\partial_z G_{jl}.$$

APPENDIX E: PROPERTIES OF THE EQUATIONS FOR $G_{ij}(r, z)$

In this appendix we derive several useful properties of the Eqs. (2.12) for $G_{ij}(r, z)$. We use in the following the formal rewriting of these equations in the form of Eq. (2.16).

Let $A(r, z)$ and $B(r, z)$ be $D-2$ times $D-2$ matrices obeying

$$[A(r, z), B(r', z')] = 0, \tag{E1}$$

for any (r, z) and (r', z'). Note that this means that $A(r, z)$ or any derivative of $A(r, z)$ commutes with $B(r, z)$ or any derivative of $B(r, z)$. Write now $G = AB$. Then we have

$$G^{-1}\vec{\nabla}^2 G - (G^{-1}\vec{\nabla}G)^2 = A^{-1}\vec{\nabla}^2 A - (A^{-1}\vec{\nabla}A)^2 + B^{-1}\vec{\nabla}^2 B - (B^{-1}\vec{\nabla}B)^2. \tag{E2}$$

From this equation we get the following lemma:

Lemma E.1.—Let $A(r, z)$ and $B(r, z)$ be $D-2$ times $D-2$ matrices that commutes as in (E1). If A and B obey the differential equations

$$A^{-1}\vec{\nabla}^2 A = (A^{-1}\vec{\nabla}A)^2, \qquad B^{-1}\vec{\nabla}^2 B = (B^{-1}\vec{\nabla}B)^2, \quad \text{(E3)}$$

then the matrix $G = AB$ obeys $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$.□

The consequence of this lemma is that we can combine solutions into new solutions, as long as (E1) is obeyed. An important use of lemma E.1 is the following corollary:

Corollary E.2.—Let $A(r, z)$ be a $D-2$ times $D-2$ matrix and let $f(r, z)$ be a function. If G and f obey the differential equations

$$A^{-1}\vec{\nabla}^2 A = (A^{-1}\vec{\nabla}A)^2, \vec{\nabla}^2 f = 0, \qquad \text{(E4)}$$

then the matrix $G = e^f A$ obeys $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$.□

Another important situation where the lemma E.1 can be applied and where the implication of lemma E.1 in fact can be reversed is expressed in the following lemma:

Lemma E.3.—Let the $D-2$ times $D-2$ matrix $G(r, z)$ be such that $G = A \oplus B$, where $A(r, z)$ is a k times k matrix and $B(r, z)$ is a $D-2-k$ times $D-2-k$ matrix. I.e. G is the geometric direct sum of A and B. In this case, it is clear that A and B obey the differential equations

$$A^{-1}\vec{\nabla}^2 A = (A^{-1}\vec{\nabla}A)^2, \qquad B^{-1}\vec{\nabla}^2 B = (B^{-1}\vec{\nabla}B)^2, \quad \text{(E5)}$$

if and only if G obeys $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ □.

This lemma can of course be used successively for the cases where $G(r, z)$ can be split up to the direct sum of several matrices acting on linearly independent subspaces, i.e. $G = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. An important special case of this is when G is diagonal. We have the following corollary of lemma E.3:

Corollary E.4.—Let the $D-2$ times $D-2$ matrix $G(r, z)$ be a diagonal matrix

$$G = \mathrm{diag}[\pm\exp(2U_1), \exp(2U_2), \ldots, \exp(2U_{D-2})], \qquad \text{(E6)}$$

where $U_i(r, z)$, $i = 1, \ldots, D-2$, are functions. Then

$$\vec{\nabla}^2 U_i = 0, \qquad i = 1, \ldots, D-2, \qquad \text{(E7)}$$

if and only if $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$. □

When $G(r, z)$ is diagonal it corresponds to a generalized Weyl solution (1.4) (see [16]), since the $D-2$ Killing vector fields are orthogonal. We see that (2.12) correctly reduce to (1.5). Moreover, it is clear that $\det(G_{ij}) = \pm r^2$ is equivalent to $\sum_{i=1}^{D-2} U_i = \log r$.

We have also a general result for the inverse of a matrix:

Lemma E.5.—An n by n invertible matrix $G(r, z)$ obeys the equation $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ if and only if the inverse matrix G^{-1} obeys the corresponding equation

$$G\vec{\nabla}^2 G^{-1} = (G\vec{\nabla}G^{-1})^2. \qquad \text{(E8)}$$

□

This lemma can be used to find new solutions from already known solutions. Of course, one has to remember that the complete G matrix moreover should have $|\det G| = r^2$. The following corollary is one way to take this into account:

Corollary E.6.—Let $G(r, z)$ be a $D-2$ by $D-2$ matrix with $|\det G| = r^2$ and $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$. Then the matrix $M = r^{4/D-2}G^{-1}$ obeys $\det M = \det G$ and $M^{-1}\vec{\nabla}^2 M = (M^{-1}\vec{\nabla}M)^2$. □

We note that we can multiply constant matrices on solutions:

Lemma E.7.—Let the $D-2$ by $D-2$ matrix $G(r, z)$ solve the equation $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ and let A and B be constant invertible matrices. Then the matrix $M = AGB$ obeys $M^{-1}\vec{\nabla}^2 M = (M^{-1}\vec{\nabla}M)^2$ □.

Finally, an important and useful theorem that concerns systems with an orthogonal Killing vector is the following:

Theorem E.8.—Consider the class of metrics with $G_{1i} = 0$, $i = 2, \ldots, D-2$, i.e., with the Killing vector $V_{(1)} = \frac{\partial}{\partial x^1}$ being orthogonal to the $D-3$ other Killing vector fields. Then we can always write G as

$$G = se^{2U} \oplus e^{(-2/D-3)U}M = \begin{pmatrix} se^{2U} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & e^{-(2/D-3)U}M & \\ 0 & & & \end{pmatrix}, \qquad \text{(E9)}$$

where $s = \pm 1$, $U(r, z)$ is a function and $M(r, z)$ is a $D-3$ by $D-3$ symmetric real matrix with $\det M = s\det G$ so that $|\det M| = r^2$. Moreover, G obeys $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ if and only if

$$\vec{\nabla}^2 U = 0, \qquad M^{-1}\vec{\nabla}^2 M = (M^{-1}\vec{\nabla}M)^2. \qquad \text{(E10)}$$

Proof.— It is trivial to see that we can always write G on the form (E9). That $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla}G)^2$ if and only (E10) is true follows from using lemma E.3 together with lemma E.2. □

Theorem E.8 is useful since it allows one to take a $D = n$ dimensional solution and creating new nontrivial $D = n+1$ dimensional solutions. Moreover, for a $D = n$ dimensional solution with a Killing vector orthogonal to all other $D-3$ Killing vector fields we can reduce the system of equations to be a 3-dimensional Laplace equation together with, but decoupled from, the equations for a $D = n-1$ dimensional solution.

APPENDIX F: SINGULARITIES AT $r = 0$

We consider in this appendix what happens for solutions that have more than one eigenvalue of $G(r, z)$ going to zero for $r \to 0$. We restrict for simplicity here to the case with two eigenvalues going to zero for $r \to 0$. One can easily extend the argument to consider more than two eigenvalues.

We begin by considering the solution

$$G_{11} = r^{2a}, \qquad G_{22} = r^{2-2a}, \qquad e^{2\nu} = r^{-2a(1-a)}, \quad \text{(F1)}$$

where $0 \leq a \leq 1$. This solves the Eqs. (2.12) and (2.13). We compute the curvature invariant

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 16a^2(1-a)^2(1-a+a^2)r^{-4(1-a+a^2)}. \quad \text{(F2)}$$

Since $1 - a + a^2$ is strictly positive we see that there is a curvature singularity at $r = 0$, unless $a = 0$ or $a = 1$ which in both cases corresponds to only having one eigenvalue going to zero. Therefore, for this solution we see that having two eigenvalues going to zero invariably leads to a curvature singularity.

Consider now a general solution for which we have an interval $[z_1, z_2]$ so that for any $z \in [z_1, z_2]$ we have that two eigenvalues of $G(r, z)$ going to zero for $r \to 0$. Then using the same type of arguments as in Sec. III A we can make a constant orthogonal transformation of $G(r, z)$ so that $G_{1i}(0, z) = G_{2i}(0, z) = 0$ for $i = 1, 2, \ldots, D-2$, for a given $z_1 < z < z_2$. Therefore, given the structure of the Eqs. (2.12) and (2.13), we see that we effectively can reduce this system to the example given by Eq. (F1). In conclusion, we have shown that for any solution having two eigenvalues of $G(r, z)$ that go to zero as $r \to 0$ for a given z, we get curvature singularities, except perhaps in isolated points on the z-axis corresponding to the endpoints of the interval given above. As mentioned above, one can easily extend these arguments to consider more than two eigenvalues going to zero.

APPENDIX G: PROLATE SPHERICAL COORDINATES

We define here the *prolate spherical coordinates* (x, y) which for certain stationary and axisymmetric solutions are convenient to use. The prolate spherical coordinates were introduced for four-dimensional stationary and axisymmetric solutions in [24] (see also [12,14,15]). The prolate spherical coordinates are used to describe rotating black hole solutions in Sec. V.

The prolate spherical coordinates (x, y) are defined in terms of the canonical (r, z) coordinates by

$$r = \alpha\sqrt{(x^2-1)(1-y^2)}, \qquad z = \alpha xy, \quad \text{(G1)}$$

where $\alpha > 0$ is a constant. We take x and y to have the ranges

$$x \geq 1, \qquad -1 \leq y \leq 1. \quad \text{(G2)}$$

We have

$$dr^2 + dz^2 = \alpha^2(x^2 - y^2)\left[\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right]. \quad \text{(G3)}$$

Note that Eqs. (2.12) can be written in prolate spherical coordinates as

$$\begin{aligned}&\partial_x[(x^2-1)\partial_x G] + \partial_y[(1-y^2)\partial_y G] \\ &\quad = (x^2-1)(\partial_x G)G^{-1}\partial_x G + (1-y^2)(\partial_y G)G^{-1}\partial_y G.\end{aligned} \quad \text{(G4)}$$

We now give the transformation from (x, y) coordinates to (r, z) coordinates. Defining

$$R_+ = \sqrt{r^2 + (z+\alpha)^2}, \qquad R_- = \sqrt{r^2 + (z-\alpha)^2}, \quad \text{(G5)}$$

one can easily check using (G1) that

$$R_+ = \alpha(x + y), \qquad R_- = \alpha(x - y). \quad \text{(G6)}$$

Therefore, we see that

$$x = \frac{R_+ + R_-}{2\alpha}, \qquad y = \frac{R_+ - R_-}{2\alpha}. \quad \text{(G7)}$$

Furthermore, we note that

$$\begin{aligned} R_\pm + z \pm \alpha &= \alpha(x \pm 1)(1 + y), \\ R_\pm - (z \pm \alpha) &= \alpha(x \mp 1)(1 - y). \end{aligned} \quad \text{(G8)}$$

If we consider the asymptotic region $\sqrt{r^2 + z^2} \to \infty$ with $z/\sqrt{r^2 + z^2}$ finite, we see that

$$x \simeq \frac{1}{\alpha}\sqrt{r^2 + z^2}, \qquad y \simeq \frac{z}{\sqrt{r^2 + z^2}}. \quad \text{(G9)}$$

Thus, the asymptotic region in terms of the prolate spherical coordinates is $x \to \infty$ and y being finite.

APPENDIX H: C-METRIC COORDINATES

We consider in this section the coordinate transformation from general C-metric coordinates to the canonical (r, z) coordinates.[13] The general C-metric coordinates (u, v) are here defined in relation to (r, z) by

$$\begin{aligned} r &= \frac{2\kappa^2\sqrt{-G(u)G(v)}}{(u-v)^2}, \\ e^{2\nu}(dr^2 + dz^2) &= \zeta(u, v)\left(\frac{du^2}{G(u)} - \frac{dv^2}{G(v)}\right), \end{aligned} \quad \text{(H1)}$$

where κ is a constant, and $\zeta(u, v)$ and $G(\xi)$ are functions that depend on the particular solution that we consider. The goal is now to find the z coordinate. From $g^{rz} = 0$ we find

$$\frac{\partial z}{\partial v} = -\frac{g_{vv}}{g_{uu}}\frac{\partial r}{\partial u}\left(\frac{\partial r}{\partial v}\right)^{-1}\frac{\partial z}{\partial u}. \quad \text{(H2)}$$

Using this together with $g_{uv} = 0$ one can easily derive that

[13] See [25] for a review of the C-metric and the transformation from C-metric coordinates to canonical (r, z) coordinates in the context of four-dimensional Weyl solutions.

$$\frac{\partial z}{\partial u} = s\sqrt{\frac{g_{uu}}{g_{vv}}}\frac{\partial r}{\partial v}, \qquad \frac{\partial z}{\partial v} = -s\sqrt{\frac{g_{vv}}{g_{uu}}}\frac{\partial r}{\partial u}, \qquad s = \pm 1. \tag{H3}$$

This gives

$$s\frac{\partial z}{\partial u} = -\frac{\kappa^2 G'(v)}{(u-v)^2} - \frac{4\kappa^2 G(v)}{(u-v)^3}, \quad s\frac{\partial z}{\partial v} = -\frac{\kappa^2 G'(u)}{(u-v)^2} + \frac{4\kappa^2 G(u)}{(u-v)^3}. \tag{H4}$$

The integrability condition

$$\frac{\partial}{\partial v}\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\frac{\partial z}{\partial v}, \tag{H5}$$

is satisfied if and only if

$$G''(v) + \frac{6G'(v)}{u-v} + \frac{12G(v)}{(u-v)^2} = G''(u) - \frac{6G'(u)}{u-v} + \frac{12G(u)}{(u-v)^2}. \tag{H6}$$

Integrating (H4), we get

$$sz = b(v) + \frac{\kappa^2 G'(v)}{(u-v)} + \frac{2\kappa^2 G(v)}{(u-v)^2} = c(u) - \frac{\kappa^2 G'(u)}{(u-v)} + \frac{2\kappa^2 G(u)}{(u-v)^2}, \tag{H7}$$

where $b(\xi)$ and $c(\xi)$ are two functions. Using the integrability condition (H6) we see that

$$s(z - z_0) = \frac{\kappa^2 G''(v)}{6} + \frac{\kappa^2 G'(v)}{(u-v)} + \frac{2\kappa^2 G(v)}{(u-v)^2} = \frac{\kappa^2 G''(u)}{6} - \frac{\kappa^2 G'(u)}{(u-v)} + \frac{2\kappa^2 G(u)}{(u-v)^2}, \tag{H8}$$

where z_0 is a constant. Now, if we take $G(\xi)$ to be of the form

$$G(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4, \tag{H9}$$

one can check that (H6) is obeyed.

Note that $e^{2\nu}$ and $\zeta(u, v)$ in (H1) are connected through the formula

$$\frac{\zeta(u,v)}{e^{2\nu}} = \frac{\kappa^4}{(u-v)^4}\left\{G(u)\left[G'(v) + \frac{4G(v)}{u-v}\right]^2 - G(v)\left[G'(u) - \frac{4G(u)}{u-v}\right]^2\right\}. \tag{H10}$$

This formula is useful for finding $e^{2\nu}$.

Consider now the particular choice of $G(\xi)$

$$G(\xi) = (1 - \xi^2)(1 + c\xi), \tag{H11}$$

which correspond to the case of the five-dimensional black ring metric described in Sec. VI. If we take $s = 1$ and $z_0 = 1/6$ we get

$$z = \frac{\kappa^2(1 - uv)(2 + cu + cv)}{(u-v)^2}. \tag{H12}$$

We now look for constants q and β that solves the equation

$$r^2 + (z - q\kappa^2)^2 = \frac{\kappa^4[\beta - cuv - 2q(u+v)]^2}{(u-v)^2}. \tag{H13}$$

There are precisely three solutions to this equation:

$$q = -c, \quad \beta = 2 + c; \qquad q = c, \quad \beta = -2 + c; \qquad q = 1, \quad \beta = -c. \tag{H14}$$

Write now

$$R_i = \sqrt{r^2 + (z - z_i)^2}, \qquad z_1 = -c\kappa^2, \qquad z_2 = c\kappa^2, \qquad z_3 = \kappa^2. \tag{H15}$$

We get

$$R_1 = \frac{\kappa^2[2 + c(1 + u + v - uv)]}{(u-v)}, \quad R_2 = \frac{\kappa^2[2 + c(-1 + u + v + uv)]}{(u-v)}, \quad R_3 = \frac{\kappa^2[-c(1 + uv) - (u+v)]}{(u-v)}. \tag{H16}$$

We have furthermore that

$$\begin{aligned} R_1 + z - z_1 &= \frac{2\kappa^2(1+u)(1-v)(1+cu)}{(u-v)^2}, \\ R_1 - z + z_1 &= \frac{2\kappa^2(1-u)(-1-v)(1+cv)}{(u-v)^2}, \\ R_2 + z - z_2 &= \frac{2\kappa^2(1+u)(1-v)(1+cv)}{(u-v)^2}, \\ R_2 - z + z_2 &= \frac{2\kappa^2(1-u)(-1-v)(1+cu)}{(u-v)^2}, \\ R_3 + z - z_3 &= \frac{2\kappa^2(1-u^2)(1+cv)}{(u-v)^2}, \\ R_3 - z + z_3 &= \frac{2\kappa^2(v^2-1)(1+cu)}{(u-v)^2}. \end{aligned} \tag{H17}$$

Finally, we can solve for u and v to obtain

$$u = \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 + 2(1-c^2)\kappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3}, \quad v = \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 - 2(1-c^2)\kappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3}. \tag{H18}$$

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