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Geodesic behavior of sudden future singularities

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In this paper we analyze the effect of recently proposed classes of sudden future singularities on causal geodesics of FLRW spacetimes. Geodesics are shown to be extendible and just the equations for geodesic deviation are singular, although tidal forces are not strong enough to produce a Big Rip. For the sake of completeness, we compare with the typical sudden future singularities of phantom cosmologies.

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Recently it has been suggested [1] that in an expanding FLRW universe a curvature singularity may appear at a finite time before Big Crunch for matter contents that satisfy both weak and strong energy conditions. This family of models has been further enlarged [2], and the same sort of behavior has also been found in inhomogeneous models [3]. It has been remarked, however, that the dominant energy condition must be violated in order to produce such sudden singularities [4], and that the inclusion of quantum corrections may appease their strength [5].

In these models, the energy density of the formal perfect fluid is finite at the singularity, but the pressure is infinite. More specifically, in the models proposed in [1,2] the scale factor and its first derivative are also finite, whereas second and higher order derivatives become infinite (in the models presented in [5] the singularity does not appear in the scale factor and its first three derivatives are finite).

These sorts of sudden future singularities are quite different from those in phantom cosmologies [6], because for the latter not only does the second derivative of the scale factor blow up at the singularity, but also do the energy density, the scale factor and its derivatives from the first order up.

In this paper we want to analyze the behavior of the sudden future singularities in [1–5] from a different point of view. Instead of regarding the curvature scalar polynomials we shall take a look at causal geodesics, since they describe the trajectories and the fate of nonaccelerated observers on these universes. This is not a difficult task since FLRW cosmologies,

$$ds^2 = -dt^2 + a(t)\{f^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\},$$
$$f^2(r) = \frac{1}{1-kr^2}, \quad k = 0, \pm 1, \tag{1}$$

are homogeneous and isotropic and therefore have a six-dimensional group of isometries generated, for instance, by the Killing fields

$$\xi_1 = \frac{\sin\theta\cos\phi}{f(r)}\partial_r + \frac{\cos\theta\cos\phi}{rf(r)}\partial_\theta - \frac{\sin\phi}{rf(r)\sin\theta}\partial_\phi, \tag{2a}$$
$$\xi_2 = \frac{\sin\theta\sin\phi}{f(r)}\partial_r + \frac{\cos\theta\sin\phi}{rf(r)}\partial_\theta + \frac{\cos\phi}{rf(r)\sin\theta}\partial_\phi, \tag{2b}$$
$$\xi_3 = \frac{\cos\theta}{f(r)}\partial_r - \frac{\sin\theta}{rf(r)}\partial_\theta, \tag{2c}$$
$$\zeta_1 = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi, \tag{2d}$$
$$\zeta_2 = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \tag{2e}$$
$$\zeta_3 = \partial_\phi, \tag{2f}$$

which yield six different constants of geodesic motion, i.e., three linear momenta and three angular momenta:

$$P_1 = a(t)\left\{\frac{r}{f(r)}(\cos\theta\cos\phi\dot\theta - \sin\theta\sin\phi\dot\phi) + f(r)\sin\theta\cos\phi\dot r\right\}, \tag{3a}$$
$$P_2 = a(t)\left\{\frac{r}{f(r)}(\cos\theta\sin\phi\dot\theta + \sin\theta\cos\phi\dot\phi) + f(r)\sin\theta\sin\phi\dot r\right\}, \tag{3b}$$
$$P_3 = a(t)\left\{f(r)\cos\theta\dot r - \frac{r}{f(r)}\sin\theta\dot\theta\right\}, \tag{3c}$$
$$L_1 = a(t)r^2(\cos\phi\dot\theta - \sin\theta\cos\theta\sin\phi\dot\phi), \tag{3d}$$
$$L_2 = a(t)r^2(\sin\phi\dot\theta + \sin\theta\cos\theta\cos\phi\dot\phi), \tag{3e}$$
$$L_3 = a(t)r^2\sin^2\theta\dot\phi, \tag{3f}$$

for a geodesic parametrized by its proper time τ, so that $d\tau^2 = -ds^2$. The dots stand for derivation with respect to this proper time. We define now

$$\delta \equiv \dot t^2 - a(t)\{f^2(r)\dot r^2 + r^2(\dot\theta^2 + \sin^2\theta\dot\phi^2)\}, \tag{4}$$

where δ is zero for null geodesics and one for timelike geodesics. With such an amount of conserved quantities, geodesic equations reduce to first order differential equations:

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$$\dot{t}^2 = \delta + \frac{P^2 + kL^2}{a(t)}, \tag{5a}$$

$$\dot{r} = \frac{P_1 \sin\theta \cos\phi + P_2 \sin\theta \sin\phi + P_3 \cos\theta}{a(t)f(r)}, \tag{5b}$$

$$\dot{\theta} = \frac{L_1 \cos\phi + L_2 \sin\phi}{a(t)r^2}, \tag{5c}$$

$$\dot{\phi} = \frac{L_3}{a(t)r^2 \sin^2\theta}, \tag{5d}$$

in terms of total linear momentum and angular momentum

$$P^2 = P_1^2 + P_2^2 + P_3^2, \qquad L^2 = L_1^2 + L_2^2 + L_3^2. \tag{6}$$

The system may be further simplified, since due to spherical symmetry every geodesic may be fit in the hypersurface $\theta = \pi/2$, with $L_1 = L_2 = 0 = P_3$, by a suitable choice of the coordinates, then

$$\dot{t}^2 = \delta + \frac{P^2 + kL^2}{a(t)}, \tag{7a}$$

$$\dot{r} = \frac{P_1 \cos\phi + P_2 \sin\phi}{a(t)f(r)}, \tag{7b}$$

$$\dot{\phi} = \frac{L_3}{a(t)r^2}, \tag{7c}$$

It can be easily noticed that these equations are singular if and only if $a(t)$ has a zero, which corresponds to either a Big Bang or a Big Crunch singularity. Therefore, if we consider models with sudden future singularities like those in [1],

$$a(t) = 1 + \left(\frac{t}{t_s}\right)^q (a_s - 1) - \left(1 - \frac{t}{t_s}\right)^n, \tag{8}$$

with constants a_s, t_s, $0 < q \leq 1$, $1 < n < 2$, we realize that the geodesics just see the Big Bang singularity at $t = 0$, but not the sudden singularity at $t = t_s$, where the scale factor does not vanish. This is obvious, since these universes are C^1-differentiable manifolds but for the Big Bang.

Generalizations to (8) have been also considered. For instance, in [2], the following evolution was put forward (among others):

$$a(t) = a_s - 1 + \exp[\lambda(t - t_s)] - \left(1 - \frac{t}{t_s}\right)^n, \tag{9}$$

with $\lambda > 0$ and n in the same range as above. Similarly, in [5] a quantum inspired model was proposed for which $a(t)$ has functionally the form of (8), but with $3 < n < 4$ instead, so that these universes are C^3-differentiable manifolds but for the Big Bang.

Furthermore, since in these settings a, a' are finite at t_s and the singularity appears just in higher order derivatives of a, the acceleration vector of the geodesic, $(\ddot{t}, \ddot{r}, \ddot{\theta}, \ddot{\phi})$, which comprises the effect of inertial forces, is also regular. Only the third derivative of the parametrization of the geodesic is singular at t_s, but we just require first and second derivatives to define geodesic equations. Causal geodesics in such universes do not see the singularities but through geodesic deviation effects, since they are due to the Riemann tensor. Point particles travelling along causal geodesics do not experience any singularity, but extended objects might suffer infinite tidal forces at $t = t_s$.

According to Tipler's definition [7] a strong curvature singularity is encountered at a point p if every volume element defined by three linearly independent, vorticity-free, geodesic deviation vectors along every causal geodesic through p vanishes at this point. This definition comes to say that an extended finite object is crushed to zero volume by tidal forces at a strong singularity. Generalizations of this widely accepted definition may be found in [8,9].

In [10], necessary and sufficient conditions for the appearance of strong curvature singularities are shown. For instance, if a causal geodesic meets a strong singularity at a value τ_s of its affine parametrization, expressions of the form

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R^i_{0j0}(\tau'')|, \tag{10}$$

will diverge along the geodesic on approaching τ_s. The components of the Riemann tensor are understood to be written in a frame parallelly transported along the geodesic. Similar results involving double integrals of the component R_{00} of the Ricci tensor or triple integrals of components C^i_{0j0} of the Weyl tensor are written for lightlike geodesics.

For Krolak's definition, necessary conditions are milder, since they involve a simple integral of components of the curvature tensor:

$$\int_0^\tau d\tau' |R^i_{0j0}(\tau')|. \tag{11}$$

For null geodesics conditions are relaxed in a similar way.

In the case of the sudden singularities in [1,2] the components of the Riemann tensor diverge as a'', since a' and a are finite; and in the worst case they diverge as a power $n - 2$, for $1 < n < 2$. Therefore after one integration of the components of the Riemann tensor, the power will be positive and the integral will not diverge. Of course, the situation is even more favorable if singularities do not arise in a'' but in higher derivatives like in those in [5].

Hence we have shown that sudden singularities are not strong according to Tipler and Krolak's definitions and therefore tidal forces do not crush all finite bodies. This is quite important, since it means that the spacetime may be extended across sudden singularities [7] and cannot be considered the final fate of these universes.

Let us come to conclusions now. In this paper we have shown that causal geodesics are not affected by the sudden future singularities in some recently put forward models, since these singularities are not seen by geodesic equations. Recall that geodesic incompleteness is the standard definition for singularities in General Relativity [11].

Furthermore, considering just curvature singularities, it has been shown that they are weak according to Tipler's and Krolak's definitions, and therefore finite objects are not necessarily torn on crossing the singularities.

In contrast, since in the typical sudden future singularities of phantom cosmologies there is a blow up of the scale factor and all its derivatives, such singularities are indeed seen by geodesic equations, thus altering causal geodesics, and leading to destruction of structure (or Big Rip) [6,12].

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