

Thermodynamics of $O(N)$ sigma models: $1/N$ corrections

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The thermodynamics of the $O(N)$ linear and nonlinear sigma models in $3 + 1$ dimensions is studied. We calculate the pressure to next-to-leading order in the $1/N$ expansion and show that at this order, temperature-independent renormalization is only possible at the minimum of the effective potential. The $1/N$ expansion is found to be a good expansion for N as low as 4, which is the case relevant for low-energy QCD phenomenology. We consider the cases with and without explicit symmetry breaking. We show that previous next-to-leading-order calculations of the pressure are either breaking down at the temperatures of interest, or based on unjustifiable high-energy approximations.

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I. INTRODUCTION

It is well known that although the QCD Lagrangian possesses a chiral symmetry in the limit of zero quark masses, the true QCD ground state does not respect this symmetry. The chiral symmetry is spontaneously broken by quantum effects. To be specific, the QCD Lagrangian with N_f massless quarks has a global $SU(N_f)_L \times SU(N_f)_R$ symmetry, which for the ground state at low temperatures is broken down to a $SU(N_f)_V$ symmetry. According to Goldstone's theorem, there is a massless, spinless particle for each generator of a broken global continuous symmetry. In this case this implies the occurrence of $N_f^2 - 1$ Goldstone bosons. In phenomenological applications N_f is either two or three, and one also has to take into account the explicit symmetry breaking due to the nonzero quark masses. Both the spontaneous and the explicit chiral symmetry breaking are apparent in the low-energy hadronic particle spectrum, where the expected number of relatively light mesons is observed (e.g., the three pions for $N_f = 2$). At sufficiently high temperatures one expects the chiral symmetry to be restored and lattice simulations of QCD suggest that this happens at a temperature of approximately 150 MeV depending on the number of quarks and their masses. Heavy-ion collisions at Relativistic Heavy Ion Collider and Large Hadron Collider are expected to reach such temperatures and will allow experimental studies of the deconfined, chirally symmetric phase of QCD [1].

Apart from using lattice simulations it has not yet been possible to calculate thermodynamic properties, such as the pressure, from QCD in the low-temperature hadronic

phase. However, this can be done using low-energy effective theories. Such effective theories for the low-energy particle spectrum, involving both mesons and baryons and displaying the above-mentioned pattern of chiral symmetry breaking, were constructed before QCD itself. In the case of two flavors, the situation is the simplest, since one can exploit the fact that the $SU(2)_L \times SU(2)_R$ symmetry is locally isomorphic to $SO(4)$. If baryons (nucleons in this case) are not included the simpler $O(4)$ linear sigma model can be used as a low-energy effective theory for describing the dynamics of three pion fields and one sigma field. These four fields form a four-dimensional vector ϕ in the fundamental representation of $O(4)$. At low temperature, the $O(4)$ symmetry is spontaneously broken down to $O(3)$, where the sigma field acquires a vacuum expectation value and the three pions are interpreted as the Goldstone bosons. For $N_f > 2$ there is no connection between the $SU(N_f)_L \times SU(N_f)_R$ model and the $O(N)$ linear sigma model, but the latter has been studied in great detail for general N due to its relevance to spin models.

In this paper the $O(N)$ linear sigma model (LSM) and $O(N)$ nonlinear sigma model (NLSM) in $3 + 1$ dimensions will be studied at finite temperature and to next-to-leading order in the $1/N$ expansion. At zero temperature, the $1/N$ expansion was applied to $O(N)$ sigma models a long time ago at leading order (LO) [2] and at next-to-leading order (NLO) [3]. At finite temperature the LO in $1/N$ has been studied in Ref. [4]. The effective potential in the large- N limit is that of an ideal gas and thus straightforward to compute. The NLO $1/N$ corrections to the free energy involve a momentum-dependent self-energy and cannot be evaluated analytically. In Ref. [5], the author therefore carried out a high-temperature expansion to obtain purely analytical results for the LSM. Similarly, in Ref. [6] the authors resorted to a "high-

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energy” approximation, which makes the calculations manageable. However, this approximation is uncontrolled and it is difficult to assess how reasonable it is, unless one calculates the full NLO $1/N$ corrections.

The $O(N)$ sigma models have also been studied in detail at finite temperature using various other approaches. A systematic study has been carried out by Chiku and Hatsuda using optimized perturbation theory [7]. The method was used to calculate spectral functions, properties of the effective potential, and dilepton emission rates. This method is convenient from the point of view of renormalization. The Cornwall-Jackiw-Tomboulis or 2PI formalism [8] has also been used to examine various properties of the $O(N)$ linear sigma models at finite temperature [9–16], see Ref. [17] for a recent review. For example, in several papers the temperature dependence of the pion and sigma masses, and of the vacuum expectation value of the sigma field, have been investigated. In the low-temperature phase, the $O(N)$ symmetry is spontaneously broken and it is expected that the symmetry is restored via a second-order phase transition. The calculations of the effective potential as a function of temperature have been carried out in the Hartree approximation and the large- N limit [10–15]. In these cases, the gap equations for the propagators are easy to solve since the self-energy reduces to a local mass term. In the Hartree approximation, the result has been shown to be problematic (and a first-order phase transition occurs), which has been remedied by including more diagrams in the truncation [18,19], resulting in a second-order phase transition. If one goes beyond the Hartree approximation or includes the NLO contributions in the $1/N$ expansion, the gap equations become nonlocal and very difficult to solve.

In this paper, we will study the one particle irreducible effective potential at NLO in the $1/N$ expansion without resorting to a high-temperature or high-energy approximation. Thus the analysis presented here is an extension of the papers by Jain [5] and by Bochkarev and Kapusta [6]. We will follow the approach to the NLSM in $1 + 1$ dimension of Ref. [20]. In the present case we do not need to deal with thermal infrared renormalons, which simplifies the renormalization procedure. Nevertheless, we will obtain similar conclusions about the renormalization of the effective potential in $3 + 1$ dimensions as in $1 + 1$ dimensions. It turns out that at NLO, temperature-independent renormalization is only possible at the minimum of the effective potential. This aspect of the $1/N$ expansion was missed in previous work [5,6], since the renormalization is considerably simplified or even ignored in the various approximations.

Since explicit chiral symmetry breaking plays a very important role in the actual hadron spectrum at low energy, we will consider also the case of explicit symmetry breaking. The results change considerably and more-

over, a critical temperature cannot be determined in that case, since the second-order phase transition turns into a smooth crossover.

The NLSM in $3 + 1$ dimensions is nonrenormalizable and should be viewed as an effective theory, which is valid up to a certain energy scale where new physics enters. Strictly speaking the LSM is renormalizable, but since it becomes a trivial theory in the limit where the cutoff goes to infinity, we treat it as a theory with a finite cutoff. Given a finite cutoff, we speak of divergences when terms are increasing in magnitude without bound as the cutoff is increased. The low-energy physics should be independent of such terms (decoupling) and one can subtract them in the renormalization procedure in order to avoid increasing sensitivity to the ultraviolet cutoff as it grows. On general grounds, one expects the temperature dependence to be insensitive to an increasing cutoff due to the exponential suppression provided by the Bose-Einstein distribution. Therefore one expects the renormalization to be possible in a temperature-independent way.

The paper is organized as follows: In Sec. II, we discuss effective actions of the LSM and NLSM in the $1/N$ expansion. In Sec. III, we calculate the effective potential and gap equations at NLO. In Sec. IV, we discuss our results for the pressure at NLO for general N and for the special case of $N = 4$. Also, the so-called high-energy approximation is discussed and compared with exact numerical results. In Sec. V, we elaborate on the choice of parameters for $N = 4$, in order to make contact with low-energy QCD phenomenology. In Sec. VI, we summarize and conclude.

II. EFFECTIVE ACTIONS

The Euclidean Lagrangian of the $O(N)$ -symmetric linear sigma model with a symmetry breaking term H is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{\lambda_b}{8N}(\phi_i \phi_i - N f_{\pi,b}^2)^2 - \sqrt{N} H \phi_N, \quad (1)$$

where $i = 1 \dots N$. Summation over repeated indices is implicitly understood. The subscript b denotes a bare quantity. The coupling constants are rescaled with factors of N in such a way that for large N the action naturally scales as N .

It is possible to eliminate the quartic interaction term from Eq. (1) by introducing an auxiliary field which is denoted by α , in order to allow for Gaussian integration. To this end, we add to the Lagrangian Eq. (1) the term

$$\mathcal{L}_\alpha = \frac{N}{2\lambda_b} \left[\alpha - \frac{i\lambda_b}{2N}(\phi_i \phi_i - N f_{\pi,b}^2) \right]^2, \quad (2)$$

such that one has

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{i}{2}\alpha(\phi_i \phi_i - N f_{\pi,b}^2) + \frac{N}{2\lambda_b} \alpha^2 - \sqrt{NH} \phi_N. \quad (3)$$

Using the equation of motion for α , one recovers the original Lagrangian in Eq. (1). In the limit $\lambda_b \rightarrow \infty$, one obtains the Lagrangian of the nonlinear sigma model.

If explicit symmetry breaking is absent ($H = 0$), the field ϕ acquires a vacuum expectation value by spontaneously breaking the symmetry. Because of the residual $O(N-1)$ symmetry, we can write $\phi = (\pi_1, \pi_2, \dots, \pi_{N-1}, \sigma)$, such that only $\phi_N = \sigma$ has a non-zero expectation value. For $H > 0$ the same argument applies, because the action is minimal when the σ field is the only one that acquires an expectation value.

Integrating over the π 's gives the following effective action

$$S_{\text{eff}} = \frac{1}{2}(N-1)\text{Tr} \log(-\partial^2 - i\alpha) + \int_0^\beta d\tau \times \int d^3x \left[\frac{1}{2}(\partial_\mu \sigma)^2 - \frac{i}{2}\alpha \sigma^2 + \frac{i}{2}N f_{\pi,b}^2 \alpha + \frac{N}{2\lambda_b} \alpha^2 - \sqrt{NH} \sigma \right]. \quad (4)$$

We next parametrize the quantum fields σ and α by writing them as a sum of space-time independent vacuum expectation values im^2 and $\bar{\sigma}$ and quantum fluctuating fields $\tilde{\alpha}$ and $\tilde{\sigma}$:

$$\alpha = im^2 + \frac{\tilde{\alpha}}{\sqrt{N}}, \quad (5)$$

$$\sigma = \sqrt{N}\bar{\sigma} + \tilde{\sigma}. \quad (6)$$

Using Eq. (2) one can show that the vacuum expectation value of α is purely imaginary. The vacuum expectation value of σ is proportional to \sqrt{N} , which follows from Eq. (1). Substituting Eqs. (5) and (6) into Eq. (4), the effective action S_{eff} can be written as

$$S_{\text{eff}} = \frac{1}{2}(N-1)\text{Tr} \log\left(-\partial^2 + m^2 - \frac{i\tilde{\alpha}}{\sqrt{N}}\right) - \beta V N H \bar{\sigma} + \int_0^\beta d\tau \int d^3x \left[\frac{1}{2}(\partial_\mu \tilde{\sigma})^2 + \frac{1}{2}\left(m^2 - \frac{i\tilde{\alpha}}{\sqrt{N}}\right) \times (\sqrt{N}\bar{\sigma} + \tilde{\sigma})^2 - \frac{N}{2} f_{\pi,b}^2 \left(m^2 - \frac{i\tilde{\alpha}}{\sqrt{N}}\right) - \frac{N}{2\lambda_b} \left(m^2 - \frac{i\tilde{\alpha}}{\sqrt{N}}\right)^2 - \sqrt{NH} \tilde{\sigma} \right]. \quad (7)$$

Expanding Eq. (7) in powers of $1/\sqrt{N}$ up to corrections of order $1/\sqrt{N}$, one finds¹

¹The terms that are linear in $\tilde{\alpha}$ and $\tilde{\sigma}$ vanish at the minimum of the effective potential.

$$\frac{S_{\text{eff}}}{\beta V} = \frac{1}{2}(N-1) \sum_P \log(P^2 + m^2) - \frac{Nm^2}{2}(f_{\pi,b}^2 - \bar{\sigma}^2) - \frac{Nm^4}{2\lambda_b} - NH\bar{\sigma} + \sqrt{N} \times \text{terms linear in } \tilde{\alpha} \text{ and } \tilde{\sigma} + \frac{1}{2} \sum_P \chi^T \begin{pmatrix} \frac{1}{2}\Pi(P, m) + \frac{1}{\lambda_b} & -i\tilde{\sigma} \\ -i\tilde{\sigma} & P^2 + m^2 \end{pmatrix} \chi^*, \quad (8)$$

where $\chi^T = (\tilde{\alpha}_P, \tilde{\sigma}_P)$ is a vector containing the Fourier transforms of $\tilde{\alpha}$ and $\tilde{\sigma}$, and the function $\Pi(P, m)$ is given by

$$\Pi(P, m) = \sum_Q \frac{1}{Q^2 + m^2} \frac{1}{(P+Q)^2 + m^2}. \quad (9)$$

We have introduced the sum-integral

$$\sum_Q \equiv T \sum_{q_0=2\pi nT} \int \frac{d^3q}{(2\pi)^3}, \quad (10)$$

which involves a summation over Matsubara frequencies q_0 and an integral over three-momenta q . For later convenience we also introduce a symbol for the difference between a sum-integral and an integral

$$\mathcal{F}_Q \equiv \sum_Q - \int_Q, \quad (11)$$

where $\int_Q = \int d^4Q/(2\pi)^4$.

III. EFFECTIVE POTENTIAL AND GAP EQUATIONS

One can obtain the effective potential through next-to-leading order in the $1/N$ expansion from Eq. (8) by performing the Gaussian integral over the fluctuating fields $\tilde{\alpha}$ and $\tilde{\sigma}$. Up to corrections of order $1/N$, the effective potential can be written as

$$\mathcal{V}(m^2, \bar{\sigma}) = N \mathcal{V}_{\text{LO}}(m^2, \bar{\sigma}) + \mathcal{V}_{\text{NLO}}(m^2, \bar{\sigma}), \quad (12)$$

where

$$\mathcal{V}_{\text{LO}}(m^2, \bar{\sigma}) = \frac{m^2}{2}(f_{\pi,b}^2 - \bar{\sigma}^2) + \frac{m^4}{2\lambda_b} + H\bar{\sigma} - \frac{1}{2} \sum_P \log(P^2 + m^2), \quad (13)$$

$$\mathcal{V}_{\text{NLO}}(m^2, \bar{\sigma}) = -\frac{1}{2} \sum_P \log I(P, m). \quad (14)$$

Here,

$$I(P, m) = 16\pi^2 \Pi(P, m) + \frac{32\pi^2}{\lambda_b} + \frac{32\pi^2 \bar{\sigma}^2}{P^2 + m^2}. \quad (15)$$

To derive the effective potential, we subtracted divergent constants which are independent of $\bar{\sigma}$, m and the temperature. Equivalently, these terms can be removed by

adding a vacuum counterterm $\Delta\mathcal{E}$ to the effective potential. In the following, we simply drop these terms.

In thermodynamic equilibrium, the system will be in the state that minimizes the effective potential with respect to m^2 and maximizes it with respect to $\bar{\sigma}$. This difference is due to the fact that the vacuum expectation value of α is imaginary and that of σ is real. Differentiating the NLO effective potential with respect to m^2 and $\bar{\sigma}$, one obtains

$$\int_P \frac{1}{P^2 + m^2} - \frac{2m^2}{\lambda_b} + \frac{1}{N} \int_P \frac{\frac{d\Pi(P, m)}{dm^2} - \frac{2\bar{\sigma}^2}{(P^2 + m^2)^2}}{\Pi(P, m) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}^2}{P^2 + m^2}} = (f_{\pi, b}^2 - \bar{\sigma}^2), \quad (16)$$

$$\left(m^2 + \frac{2}{N} \int_P \frac{1}{P^2 + m^2} \frac{1}{\Pi(P, m) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}^2}{P^2 + m^2}} \right) \bar{\sigma} = H. \quad (17)$$

These equations are often referred to as gap equations. Solving the gap equations gives m and $\bar{\sigma}$ as a function of the parameters f_π , H , and λ , and of the temperature.

The inverse $\tilde{\alpha}$ and $\tilde{\sigma}$ propagators can be obtained from Eq. (8). One finds

$$D_{\tilde{\alpha}}^{-1}(P, m) = \frac{1}{2} \Pi(P, m) + \frac{1}{\lambda_b} + \frac{\bar{\sigma}^2}{P^2 + m^2}, \quad (18)$$

$$D_{\tilde{\sigma}}^{-1}(P, m) = P^2 + m^2 + \frac{2\bar{\sigma}^2}{\Pi(P, m) + 2/\lambda_b}. \quad (19)$$

The values for m^2 and $\bar{\sigma}$ are determined by solving the gap equations. Using the $\tilde{\alpha}$ propagator, one finds that the inverse π propagator is given by

$$D_\pi^{-1}(P, m) = P^2 + m^2 + \frac{2}{N} \int_Q \frac{1}{(P+Q)^2 + m^2} \times \frac{1}{\Pi(Q, m) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}^2}{Q^2 + m^2}}. \quad (20)$$

From this equation and the gap equation (17) one sees that also at NLO in the broken phase where $\bar{\sigma} \neq 0$ (for $H = 0$), the pions are massless, in accordance with Goldstone's theorem.

From Eq. (19) one can see that in the unbroken phase, the σ mass becomes equal to the LO mass of the π field, which is m^2 . It may appear therefore that the σ and π masses are not equal at next-to-leading order, but this is not a correct conclusion. We note that the σ field only starts to propagate at NLO, so its $1/N$ mass corrections require a next-to-next-to-leading-order calculation. We emphasize that in the calculation of the NLO pressure one only needs the LO masses, as we will see explicitly below in Eq. (53).

In Secs. III A and III B we will explicitly calculate the leading order and next-to-leading order contributions to

the effective potential in 3 + 1 dimensions. We will evaluate integrals using an ultraviolet momentum cut-off Λ and assume that $\Lambda \gg m, 2\pi T$.

A. Leading-order contribution

The leading-order contribution to the effective potential is

$$\mathcal{V}_{\text{LO}} = \frac{m^2}{2} \left(f_{\pi, b}^2 - \frac{\Lambda^2}{16\pi^2} - \bar{\sigma}^2 \right) + \frac{T^4}{64\pi^2} J_0(\beta m) + \frac{m^4}{64\pi^2} \left[\frac{32\pi^2}{\lambda_b} + \log\left(\frac{\Lambda^2}{m^2}\right) + \frac{1}{2} \right] + H\bar{\sigma}. \quad (21)$$

Equation (21) contains ultraviolet divergences in the sense explained in the introduction. These divergences can be dealt with by defining the renormalized parameters f_π^2 and λ as

$$f_\pi^2 = f_{\pi, b}^2 - \Lambda^2/16\pi^2, \quad (22)$$

$$\frac{32\pi^2}{\lambda} = \log\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{32\pi^2}{\lambda_b}, \quad (23)$$

where $\lambda = \lambda(\mu)$. After this renormalization, the leading-order effective potential becomes

$$\mathcal{V}_{\text{LO}} = \frac{m^2}{2} (f_\pi^2 - \bar{\sigma}^2) + \frac{m^4}{64\pi^2} \left[\frac{32\pi^2}{\lambda} + \log\left(\frac{\mu^2}{m^2}\right) + \frac{1}{2} \right] + \frac{T^4}{64\pi^2} J_0(\beta m) + H\bar{\sigma}, \quad (24)$$

where the function $J_0(\beta m)$ is

$$J_0(\beta m) = \frac{32}{3T^4} \int_0^\infty dp \frac{p^4}{\omega_p} n(\omega_p). \quad (25)$$

Here, $n(\omega_p) = [\exp(\beta\omega_p) - 1]^{-1}$ is the Bose-Einstein distribution function. Note that one makes an error in the evaluation of J_0 by integrating up to infinite momenta instead of up to Λ . However, this error is negligible as long as $\Lambda \gg m, 2\pi T$. This remark also applies to the functions J_1 , K_0^\pm , and K_1^\pm defined below. For an investigation of how to deal with a finite cutoff in the calculation of sum-integrals cf. Ref. [21].

The renormalization group equation for the running coupling λ that follows from Eq. (23) is

$$\beta(\lambda) = \mu \frac{d\lambda}{d\mu} = \frac{\lambda^2}{16\pi^2}. \quad (26)$$

Note that the β -function is exact to all orders in λ^2 in the large- N limit but differs from the perturbative one obtained at one loop. However, at NLO they agree.

Since the potential term in the Lagrangian should always have a minimum in order to have a stable theory, λ_b must be positive (cf. [11] for a detailed discussion). From Eq. (23) it immediately follows that there is a maximal value for the cutoff given by

$$\Lambda_{\max} = \mu \exp\left(\frac{16\pi^2}{\lambda}\right). \quad (27)$$

Therefore this theory should be viewed as an effective theory, which is valid up to the cutoff given by Eq. (27). Taking the cutoff to infinity is equivalent to taking λ to zero, which implies that the theory is trivial. One should keep in mind that the renormalized leading-order effective potential does not depend explicitly on Λ but is only valid for m and T much smaller than Λ_{\max} . When $\Lambda = \Lambda_{\max}$ the linear sigma model reduces to the nonlinear sigma model, since in this case $\lambda_b = \infty$.

The leading-order renormalized gap equations follow from differentiating Eq. (24) with respect to m^2 and $\bar{\sigma}$ and are given by

$$G = 16\pi^2 f_\pi^2, \quad (28)$$

$$H = m^2 \bar{\sigma}, \quad (29)$$

where

$$G = T^2 J_1(\beta m) + 16\pi^2 \bar{\sigma}^2 - m^2 \log\left(\frac{\mu^2}{m^2}\right) - \frac{32\pi^2 m^2}{\lambda}. \quad (30)$$

Here, we have defined the function $J_1(\beta m)$ as

$$J_1(\beta m) = \frac{8}{T^2} \int_0^\infty dp \frac{p^2}{\omega_p} n(\omega_p). \quad (31)$$

If $H = 0$, one can show by using the gap equation (29) that either $m = 0$ or $\bar{\sigma} = 0$. From the gap equation (28) it follows that for $m = 0$ the expectation value of σ has the temperature dependence

$$\bar{\sigma} = \sqrt{f_\pi^2 - \frac{T^2}{12}}. \quad (32)$$

At $T = T_c \equiv \sqrt{12}f_\pi$ there is a second-order phase transition [6]. Below T_c the $O(N)$ symmetry is broken spontaneously to $O(N-1)$ since $\bar{\sigma} \neq 0$. Above T_c the $O(N)$ symmetry is restored and one has $\bar{\sigma} = 0$ and $m \neq 0$.

B. Next-to-leading-order contribution

In this section we will show that it is not possible to renormalize the next-to-leading-order effective potential in a temperature-independent way. It turns out that we can only renormalize the effective potential at the minimum, since the temperature-dependent divergences become temperature-independent by using the leading-order gap equations. To show this, we will extract the divergent parts of the effective potential, which can be done analytically.

In order to isolate all divergences, in principle we need to evaluate $\Pi(P, m)$ including corrections of order m^2/Λ^2 , since such terms can also give rise to divergences in the effective potential. However, since the linear sigma

model is an effective theory, Eq. (1) should be viewed as the part containing only the relevant operators. For instance, we have not included irrelevant operators of dimension six, which also contribute to $\Pi(P, m)$ at order $1/\Lambda^2$. Therefore, for consistency with Eq. (1) we do not consider order $1/\Lambda^2$ terms in $\Pi(P, m)$ [22]. We only retain the unsuppressed terms of $\Pi(P, m)$ in an expansion in $1/\Lambda^2$, since this expression is much less complicated than the exact one. We find

$$\begin{aligned} \Pi(P, m) = & \frac{1}{16\pi^2} \left[\log\left(\frac{\Lambda^2}{m^2}\right) + 1 + \sqrt{\frac{P^2 + 4m^2}{P^2}} \right. \\ & \left. \times \log\left(\frac{\sqrt{P^2 + 4m^2} - \sqrt{P^2}}{\sqrt{P^2 + 4m^2} + \sqrt{P^2}}\right) \right] + \Pi_T(P, m), \end{aligned} \quad (33)$$

where the temperature-dependent part of $\Pi(P, m)$ equals

$$\Pi_T(P, m) = \frac{1}{8\pi^2 p} \int_0^\infty dq \frac{q}{\omega_q} \log\left(\frac{q^2 + pq + A^2}{q^2 - pq + A^2}\right) n(\omega_q). \quad (34)$$

Here

$$A^2 = \frac{P^4 + 4m^2 p_0^2}{4P^2}. \quad (35)$$

In the limit $P \gg m, T$, we can approximate $\Pi_T(P, m)$ by

$$\begin{aligned} \Pi_T(P, m) \approx & \frac{1}{8\pi^2} \left[\frac{T^2}{P^2} J_1(\beta m) - \frac{4m^2 T^2 p_0^2}{P^6} J_1(\beta m) \right. \\ & \left. - \frac{(3P^2 - 4p^2)T^4}{P^6} J_0(\beta m) \right]. \end{aligned} \quad (36)$$

The next-to-leading-order effective potential has only ultraviolet divergences. Using the leading-order renormalization of λ_b , it is easily seen that $I(P, m)$ becomes finite. Also, the difference

$$\int_P \log I(P, m), \quad (37)$$

is finite (cf. Sec. IV B). Therefore, all possible divergences of \mathcal{V}_{NLO} can be isolated by calculating

$$-\frac{1}{2} \int_P \log I_{\text{HE}}(P, m), \quad (38)$$

where $I_{\text{HE}}(P, m)$ is the high-energy (HE) approximation to $I(P, m)$. It gives the large- P behavior of $I(P, m)$. After averaging over angles, we find

$$\log I_{\text{HE}} = \log C_1 + \frac{1}{P^2} \frac{C_2}{C_1} - \frac{1}{2P^4} \left(\frac{C_2}{C_1}\right)^2 + \frac{1}{P^4} \frac{C_3}{C_1}, \quad (39)$$

where

$$C_1 = \log\left(\frac{\mu^2}{P^2}\right) + 1 + \frac{32\pi^2}{\lambda}, \quad (40)$$

$$C_2 = -2m^2 \left[1 + \log\left(\frac{P^2}{m^2}\right) \right] + 32\pi^2 \bar{\sigma}^2 + 2T^2 J_1(\beta m), \quad (41)$$

$$C_3 = +m^4 \left[2 \log\left(\frac{P^2}{m^2}\right) - 1 \right] - m^2 [32\pi^2 \bar{\sigma}^2 + 2T^2 J_1(\beta m)]. \quad (42)$$

By integrating the function $\log I_{\text{HE}}$ over P , we obtain all the divergences of the NLO effective potential. The logarithmic and power divergences are given by the quantity D , which is

$$D = \frac{1}{16\pi^2} \left\{ \Lambda^2 e^{1+32\pi^2/\lambda_b} \text{li}\left(\frac{1}{e^{1+32\pi^2/\lambda_b}}\right) G - m^2 \Lambda^2 \left[1 + 2e^{1+32\pi^2/\lambda_b} \text{li}\left(\frac{1}{e^{1+32\pi^2/\lambda_b}}\right) \right] + 2m^4 \log\left(\frac{\Lambda^2}{m^2}\right) \right\}, \quad (43)$$

while the terms that have a small cut-off dependence through their dependence on λ_b , are given by the quantity E , which is

$$E = \frac{1}{16\pi^2} \left[3m^2 \left(-G + \frac{3}{2} m^2 \right) \log\left(1 + \frac{32\pi^2}{\lambda_b} \right) + (G - 2m^2)^2 \frac{1}{1 + \frac{32\pi^2}{\lambda_b}} \right]. \quad (44)$$

Since G depends explicitly on the temperature, it is impossible to renormalize the next-to-leading-order effective potential in a temperature-independent way. However, at the minimum, one can use the leading-order gap equation (28), to show that $G = 16\pi^2 f_\pi^2$. Hence, the divergences become independent of the temperature at the minimum and we can renormalize in a temperature-independent manner. We discuss this next.

The divergence in the first line of Eq. (43) is independent of m in the minimum. This divergence can be removed by vacuum renormalization. The divergent terms which are proportional to m^2 can be removed by defining the renormalized parameter f_π as

$$f_\pi^2 = f_{\pi,b}^2 - \left(1 + \frac{2}{N} \right) \frac{\Lambda^2}{16\pi^2} - \frac{1}{N} \frac{\Lambda^2}{4\pi^2} \times \left[e^{1+32\pi^2/\lambda_b} \text{li}\left(\frac{1}{e^{1+32\pi^2/\lambda_b}}\right) \right]. \quad (45)$$

The remaining divergence is proportional to m^4 and is removed by renormalizing λ_b as follows

$$\frac{32\pi^2}{\lambda} = \frac{32\pi^2}{\lambda_b} + \left(1 + \frac{8}{N} \right) \log\left(\frac{\Lambda^2}{\mu^2}\right). \quad (46)$$

From Eq. (46), we obtain the β -function governing the running of λ :

$$\beta(\lambda) = \frac{\lambda^2}{16\pi^2} \left(1 + \frac{8}{N} \right), \quad (47)$$

which coincides with the standard one-loop β -function in perturbation theory. One could argue from the renormalization that one can only trust the $1/N$ expansion for $N \gg 8$. Although the $1/N$ correction to the β function indeed has a large coefficient, this is not the case for the effective potential itself as we will see below.

As mentioned, the terms in E have a small cut-off dependence through their dependence on λ_b . We do not renormalize them, since they do not grow without bound with increasing cutoff and are not strictly speaking divergences. The effective potential does not become increasingly sensitive to them with increasing cutoff.² The term in the first line of Eq. (44) becomes smaller if we increase Λ and the absolute value of the other term from E increases as a function of Λ but is bounded by a finite number which is independent of λ_b and Λ . Moreover, renormalizing these terms would invalidate the $1/N$ expansion, because of their magnitude. This is similar to ordinary perturbation theory, where one is only allowed to do finite renormalizations that do not invalidate the perturbative expansion. A final reason for not renormalizing these terms is the connection with the nonlinear sigma model ($\lambda_b = \infty$). In that case, the terms from E are not divergent and we would choose to renormalize f_π just as in Eq. (45) with $\lambda_b = \infty$.

The NLO correction changes the critical temperature T_c . Since the NLO gap equations are complicated, we are not able to get an analytical expression for the NLO critical temperature. In the limit of small λ_b and $H = 0$ the gap equations however simplify to

$$\oint_P \frac{1}{P^2 + m^2} - \frac{2m^2}{\lambda_b} = (f_{\pi,b}^2 - \bar{\sigma}^2), \quad (48)$$

$$\left(m^2 + \frac{\lambda_b}{N} \oint_P \frac{1}{P^2 + m^2} \right) \bar{\sigma} = 0. \quad (49)$$

From the gap equations it follows that the critical temperature at NLO is

$$T_c = \sqrt{\frac{12}{1 + 2/N}} f_\pi. \quad (50)$$

This result is the same as those obtained in Refs. [5,6]. This result is probably only correct in the weak-coupling

²Since $\Lambda \leq \Lambda_{\text{max}}$, one also has to increase Λ_{max} accordingly, by considering decreasing values of λ .

limit and T_c may depend on λ at NLO in $1/N$. As we will see in the next section the transition is of second (or higher) order.

IV. PRESSURE

The pressure $\mathcal{P}(T)$ is equal to the value of the effective potential at the minimum at temperature T minus its value at the minimum at zero temperature. As we showed in the previous section, we can renormalize the effective potential at the minimum. The pressure is therefore a well-defined quantity. In order to determine the NLO effective potential in the minimum, we need the gap equation only to leading order [3]. Writing the solutions to the gap equations as

$$m^2 = m_{\text{LO}}^2 + m_{\text{NLO}}^2/N, \quad (51)$$

$$\bar{\sigma} = \bar{\sigma}_{\text{LO}} + \bar{\sigma}_{\text{NLO}}/N, \quad (52)$$

and Taylor expanding the effective potential (12), we obtain [up to $\mathcal{O}(1/N)$ corrections]

$$\begin{aligned} \mathcal{V}(m^2, \bar{\sigma}) &= N \mathcal{V}_{\text{LO}}(m_{\text{LO}}^2, \bar{\sigma}_{\text{LO}}) + \mathcal{V}_{\text{NLO}}(m_{\text{LO}}^2, \bar{\sigma}_{\text{LO}}) \\ &+ m_{\text{NLO}}^2 \left. \frac{\partial \mathcal{V}_{\text{LO}}(m^2)}{\partial m^2} \right|_{m^2=m_{\text{LO}}^2} \\ &+ \bar{\sigma}_{\text{NLO}} \left. \frac{\partial \mathcal{V}_{\text{LO}}(\bar{\sigma})}{\partial \bar{\sigma}} \right|_{\bar{\sigma}=\bar{\sigma}_{\text{LO}}}. \end{aligned} \quad (53)$$

The last two lines of Eq. (53) vanish by using the leading-order gap equations. In the following, we will write the pressure \mathcal{P} as

$$\mathcal{P} \equiv N \mathcal{P}_{\text{LO}} + \mathcal{P}_{\text{NLO}}. \quad (54)$$

From the discussion above, it follows that

$$\mathcal{P}_{\text{LO}} = \mathcal{V}_{\text{LO}}^T(m_T^2, \bar{\sigma}_T) - \mathcal{V}_{\text{LO}}^{T=0}(m_0^2, \bar{\sigma}_0), \quad (55)$$

$$\mathcal{P}_{\text{NLO}} = \mathcal{V}_{\text{NLO}}^T(m_T^2, \bar{\sigma}_T) - \mathcal{V}_{\text{NLO}}^{T=0}(m_0^2, \bar{\sigma}_0), \quad (56)$$

where m_T^2 and $\bar{\sigma}_T$ are the solutions of the leading-order gap equations (28) and (29) at temperature T .

In the following, we will present the results for the numerical evaluation of the leading- and the next-to-leading-order contributions to the pressure for general N . In Secs. IVC and IVD, we specialize to $N = 4$. As we will motivate in Sec. V, we will use the following values for the parameters: $\lambda(\mu = 100 \text{ MeV}) = 30$, $f_\pi = 47 \text{ MeV}$ (note that our f_π is $1/2$ times the more conventional definition), and if there is explicit symmetry breaking $H = (104 \text{ MeV})^3$. Otherwise H vanishes. A realistic choice of parameters would allow us to compare to lattice-QCD simulations of the $N_f = 2$ case for $T \lesssim T_c$, although this would require extrapolation of the lattice results down to the actual pion masses.

A. Leading-order contribution to the pressure

Using Eqs. (24) and (55), we can easily calculate the leading-order pressure. In Fig. 1, we show the leading-order pressure normalized by T^4 . If $H = 0$, the pions are massless below T_c and the leading-order effective potential (24) reduces to the free energy of an ideal gas of massless particles: $\mathcal{V}_{\text{LO}} = \pi^2 T^4/90$.

B. Next-to-leading-order contribution to the pressure

To calculate the next-to-leading-order contribution to the pressure, we decompose \mathcal{P}_{NLO} as follows

$$\mathcal{P}_{\text{NLO}} = D(m_T) - D(m_0) + F_1 + F_2, \quad (57)$$

where $D(m)$ is the term containing logarithmic and power ultraviolet divergences given in Eq. (43), and F_1 and F_2 are finite terms defined below.

The term F_1 has a weak cut-off dependence and is defined by

$$\begin{aligned} F_1 &= -\frac{1}{2} \int_P \left\{ \log \left[\Pi(P, m_T) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}_T^2}{P^2 + m_T^2} \right] \right. \\ &- \log \left[\Pi(P, m_0) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}_0^2}{P^2 + m_0^2} \right] \\ &- D(m_T) + D(m_0). \end{aligned} \quad (58)$$

We calculated F_1 numerically by rewriting the terms involving D as an integral. We can then subtract the integrands, instead of the large values of the integral. In this way, it is easier to avoid large numerical errors.

The function F_2 is defined by

$$F_2 = -\frac{1}{2} \int_P \log \left[\Pi(P, m_T) + \frac{2}{\lambda_b} + \frac{2\bar{\sigma}_T^2}{P^2 + m_T^2} \right]. \quad (59)$$

In order to calculate the function F_2 , we have modified the Abel-Plana formula [23] to obtain the relation

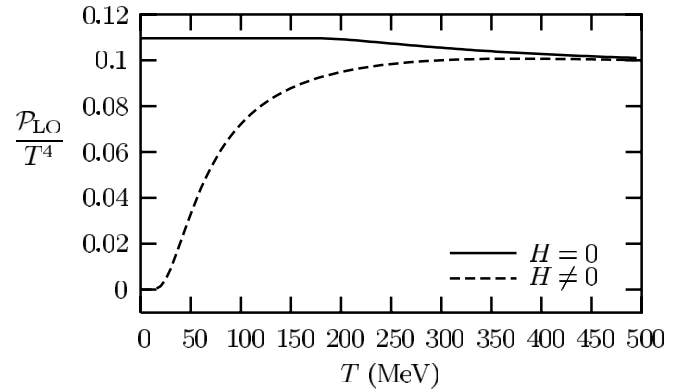


FIG. 1. Leading-order pressure \mathcal{P}_{LO} normalized to T^4 as a function of temperature without and with explicit symmetry breaking.

$$\begin{aligned}\Delta_N &= \sum_{n=N}^{\infty} f(p_0 = 2\pi n) - \frac{1}{2\pi} \int_a^{\infty} dp_0 f(p_0) \\ &= \frac{1}{\pi} \int_0^{\infty} d\rho \operatorname{Im} f(a + i\rho) \frac{1}{e^\rho + 1},\end{aligned}\quad (60)$$

where $a = 2\pi(N - 1/2)$. This formula is valid as long as $f(p_0)$ has no poles or cuts for $\operatorname{Re}(p_0) \geq a$, $f(p_0) \in \mathbb{R}$ for $p_0 \in \mathbb{R}$ and $f(p_0)$ grows slower than an exponential for $p_0 \rightarrow \infty$. This relation is useful for numerical calculations, since it prevents us from subtracting two large quantities. In our case $f(p_0) = \log I(P, m)$ which is even in p_0 and has a cut for $\operatorname{Re}(p_0) = 0$. Then we can use that

$$\begin{aligned}\Delta_{-\infty} &= 2\Delta_N + \sum_{n=-N+1}^{N-1} f(p_0 = 2\pi n) \\ &\quad - \frac{1}{2\pi} \int_{-a}^a dp_0 f(p_0),\end{aligned}\quad (61)$$

where $N \geq 1$ because of the cut. We checked that changing N has no effects on the results. After calculating $\Delta_{-\infty}$, we integrate over three-momentum p up to Λ_{\max} , which gives a finite result for F_2 . We observe that the difference of a sum-integral and an integral is dominated by the low-momentum modes. This shows that the high-energy approximation in Ref. [6] applied to their ‘‘interaction pressure’’ (implicitly defined as such a difference and directly related to F_2) is invalid, since the high-momentum modes are assumed to give the main contribution.

After renormalization, we find that the next-to-leading-order contribution to the pressure is

$$\mathcal{P}_{\text{NLO}} = \frac{m_T^4}{8\pi^2} \log\left(\frac{\mu^2}{m_T^2}\right) - \frac{m_0^4}{8\pi^2} \log\left(\frac{\mu^2}{m_0^2}\right) + F_1 + F_2,\quad (62)$$

which is shown in Fig. 2. At $T = 0$, we can calculate $\mathcal{P}_{\text{NLO}}/T^4$ exactly and use it as a check of the numerical calculations. At $T = 0$, clearly $F_1 = 0$, and hence

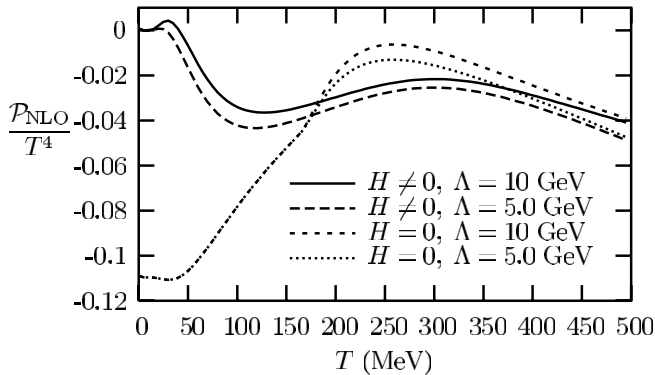


FIG. 2. Next-to-leading-order contribution to the pressure normalized to T^4 , as function of temperature for $H = 0$ and $H = (104 \text{ MeV})^3$, for different values of the cut-off Λ .

$\mathcal{P}_{\text{NLO}}/T^4 = F_2/T^4$. At $T = 0$, it is easy to see that for low P , $I(P, m)$ is dominated by $32\pi^2 \bar{\sigma}^2/(P^2 + m^2)$. This gives

$$F_2 \approx \frac{1}{2} \int_P^{\Lambda} \log(P^2 + m^2). \quad (63)$$

For $H = 0$, the mass m vanishes and so $\mathcal{P}_{\text{NLO}}/T^4 = -\pi^2/90$. For $H = (104 \text{ MeV})^3$ we have that $\mathcal{P}_{\text{NLO}}/T^4 = 0$, since T is in that case much smaller than m , such that the pressure is exponentially suppressed.

The pressure for $H = 0$ is approaching the $H \neq 0$ pressure at high temperatures, indicating that the effects of the explicit symmetry breaking terms become smaller at higher temperatures. This is because H is a temperature-independent constant.

C. Pressure of the O(4) linear sigma model

In order to make contact with two-flavor low-energy QCD, we specialize to $N = 4$. In Fig. 3, we show the pressure for $N = 4$ and $H = 0$ to next-to-leading order as function of T normalized by T^4 . The LO pressure below T_c equals the pressure of a gas of four massless non-interacting scalars. This follows immediately from Eqs. (24) and (25). At NLO the sigma field becomes massive. For temperatures much lower than m_σ , the contribution to the pressure from the sigma is Boltzmann suppressed and we have (to good approximation) $\mathcal{P} = \pi^2 T^4/30$, which is the pressure of a gas of three massless noninteracting scalars. From the calculations we conclude that the transition to NLO is of second (or higher) order since the derivative of the pressure is not diverging.

In Fig. 4, we show the pressure for $N = 4$ and $H = (104 \text{ MeV})^3$ to next-to-leading order as function of T normalized by T^4 .

In Figs. 3 and 4 we have chosen the cut-off $\Lambda = 5.0 \text{ GeV}$. Because we wish to make contact with low-energy QCD, a few comments on this choice are in order. For the low-energy chiral Lagrangian, the cutoff is usually taken to be $8\pi f_\pi$ (using our definition of f_π), which

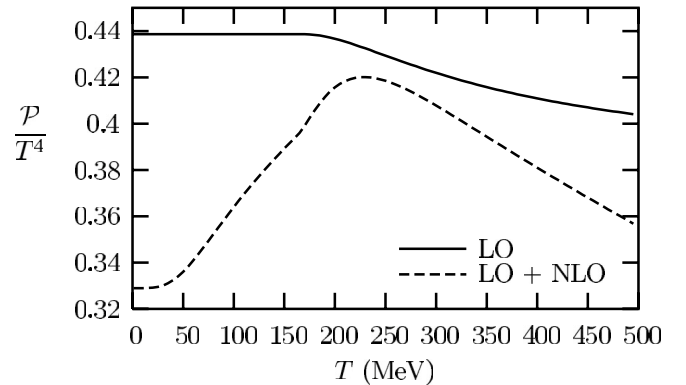


FIG. 3. LO and NLO pressure normalized to T^4 , for $N = 4$ as a function of temperature, for $H = 0$ and $\Lambda = 5.0 \text{ GeV}$.

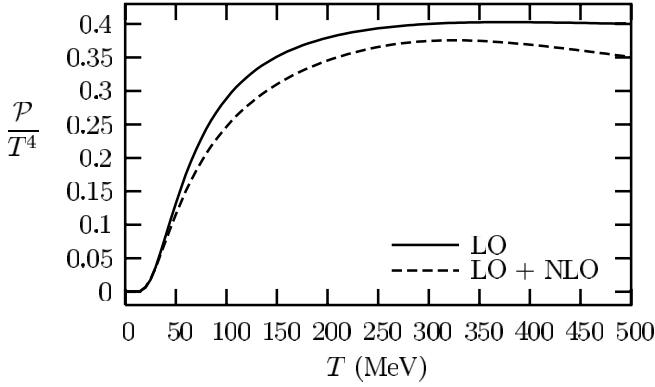


FIG. 4. LO and NLO pressure for $N = 4$ normalized to T^4 , as a function of temperature for $H = (104 \text{ MeV})^3$ and $\Lambda = 5.0 \text{ GeV}$.

is around 1.2 GeV. However, for the present purpose this value would be at the limit of applicability, since the critical temperature at which chiral symmetry is (approximately) restored is only about a factor of 8 smaller and we have to satisfy the requirement that $2\pi T \ll \Lambda$. In this way one ensures that one sums over sufficient Matsubara modes.³ Therefore, we have taken the cutoff considerably larger to reduce the sensitivity to the cutoff but emphasize that only for the region below T_c can one expect the result to be of relevance for the QCD pressure.

D. Pressure of the O(4) nonlinear sigma model

In the limit $\lambda_b = \infty$, we obtain the Lagrangian for the nonlinear sigma model. So there are no counterterms for logarithmic divergences. We will only renormalize $f_{\pi,b}$ as in Eq. (45) with $\lambda_b = \infty$. This implies that F_2 has a small cut-off dependence.

In Fig. 5 we show the pressure of the O(4) nonlinear sigma model without explicit symmetry breaking ($H = 0$), through next-to-leading order in $1/N$. We have calculated the pressure for different values of the cutoff. The LO result for $\Lambda = 20 \text{ GeV}$ is included. For comparison, we also show the pressure resulting from the approximations employed by Bochkarev and Kapusta, Ref. [6]. A considerable difference between our results and those of Bochkarev and Kapusta is observed.

We next discuss the approximations made in Ref. [6]. The $T = 0$ part of sum-integrals are omitted such that every \int is replaced by \int . Hence $\Pi(P, m)$ is replaced by $\Pi_T(P, m)$. In the high-energy approximation the latter is approximated by

$$\Pi_T(P, m) \approx \frac{T^2 J_1(\beta m)}{32\pi^2 A^2}, \quad (64)$$

where J_1 and A^2 are given by Eqs. (31) and (35), respec-

³The amount of Matsubara modes to be summed over depends on how one implements the cutoff on sum-integrals [21].

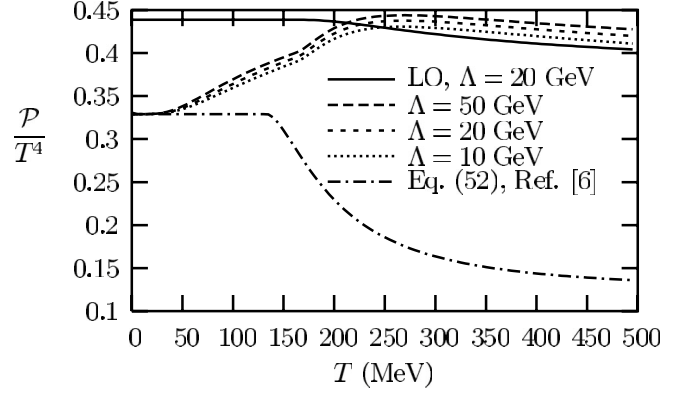


FIG. 5. NLO pressure of the nonlinear sigma model for $N = 4$ normalized to T^4 , as a function of temperature for different values of the cut-off Λ . For comparison we have included the LO pressure and a curve corresponding to the NLO pressure expression from Ref. [6].

tively. The term involving $\bar{\sigma}^2$ in Eq. (15) is omitted and the pressure reduces to

$$\mathcal{P} = \frac{Nm^2}{2} (f_\pi^2 - \bar{\sigma}^2) - \frac{N}{2} \int \frac{d^4 p}{(2\pi)^4} \log(P^2 + m^2) - \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \log \frac{P^2}{P^4 + 4m^2 p_0^2}. \quad (65)$$

Defining the functions

$$K_0^\pm(\beta m) = \frac{32}{3T^4} \int_0^\infty dp \frac{p^4}{\omega_p} n(\omega_\pm), \quad (66)$$

where $\omega_\pm = \sqrt{p^2 + m^2} \pm m$, the pressure becomes

$$\mathcal{P} = \frac{Nm^2}{2} (f_\pi^2 - \bar{\sigma}^2) + \frac{NT^4 J_0(\beta m)}{64\pi^2} + \frac{\pi^2}{90} T^4 - \frac{T^4}{64\pi^2} [K_0^+(\beta m) + K_0^-(\beta m)]. \quad (67)$$

Expanding Eq. (67) in powers of m/T and rescaling with factors of N , one obtains Eq. (52) of Ref. [6].

For completeness, we also give the gap equations in this approximation:

$$16\pi^2 f_\pi^2 = T^2 J_1(\beta m) + 16\pi^2 \bar{\sigma}^2 - \frac{T^2}{N} [K_1^+(\beta m) + K_1^-(\beta m)], \quad (68)$$

$$m^2 \bar{\sigma} = 0, \quad (69)$$

where the functions K_1^\pm are

$$K_1^\pm(\beta m) = \pm \frac{8}{T^2} \int_0^\infty dp \frac{p^2}{\omega_p} \frac{\omega_\pm}{m} n(\omega_\pm). \quad (70)$$

There are several problems with the approach in Ref. [6]. First, it is incorrect to ignore zero-temperature contributions to the pressure and it also obscures renor-

malization issues. Then, as we saw in Sec. IV B, the arguments for applying the high-energy approximation are not valid. Furthermore the term proportional to $\bar{\sigma}$ is not suppressed by $1/N$ compared to the other terms appearing in $I(P, m)$. Fourthly, as the solutions to the gap Eq. (67) indicate for $T > T_c$ m/T becomes significantly larger than 1, hence the m/T expansion breaks down. If one were to use Eq. (68) instead, one finds that the pressure even becomes negative above $T \gtrsim 300$ MeV. Another problem is that for $T < T_c$ their pressure is equal to that of a massless gas. However, this is incorrect since the sigma meson is massive and included at NLO. Hence one expects a deviation from the ideal-gas pressure at $T < T_c$. Finally, at high temperatures we expect that the NLO pressure will become approximately equal to the LO pressure because chiral symmetry will be restored. This is not the case for the pressure of Ref. [6].

We briefly comment on the paper by Jain [5]. In that paper the author is calculating the thermodynamic potential to NLO in the $O(N)$ linear sigma model using a high-temperature expansion. Since this approximation breaks down at low temperatures, we will refrain from comparing with our results.

V. CHOICE OF PARAMETERS

In the preceding sections we have shown plots for particular choices of the parameters, namely, $\lambda(\mu = 100 \text{ MeV}) = 30$, $f_\pi = 47 \text{ MeV}$, and if there is explicit symmetry breaking, we take $H = (104 \text{ MeV})^3$. In this section we will motivate these choices. For simplicity we partly use leading-order calculations for fixing the parameters.

We start with choosing the values for f_π and m_π to be roughly equal to their measured values: $f_\pi = 47 \text{ MeV}$ (note that our f_π differs from the more conventional definition by a factor of $1/2$) and $m_\pi = 138 \text{ MeV}$ (the average of the measured masses of the π^0 , π^+ and π^-). We will use this for choosing our parameter H as follows. Given a choice of λ at a given scale μ we solve the LO renormalized gap equations (28) and (29), for $\bar{\sigma}$ and m^2 , such that $m^2 = m_\pi^2$ (which is the correct identification at LO). For our choice of $\lambda(\mu = 100 \text{ MeV}) = 30$, this results in $H = (104 \text{ MeV})^3$.

The choice of λ is motivated by considerations on the maximal value of the cutoff and the sigma mass. To obtain this mass, one has to find the poles of the propagators in Minkowski space. The physical mass m_{ph} is often defined by the solution to the equation

$$-m_{\text{ph}}^2 + m^2 + \text{Re}\Sigma(p_0 = im_{\text{ph}} + \epsilon, p = 0, m) = 0, \quad (71)$$

where Σ is the self-energy. Using Eq. (19) and choosing $\mu = m_\sigma$, we find that at $T = 0$ and for $H = 0$

$$m_\sigma^2 = \frac{32\pi^2 f_\pi^2}{1 + \frac{32\pi^2}{\lambda(m_\sigma)} + \frac{\pi^2}{1+32\pi^2/\lambda(m_\sigma)}}. \quad (72)$$

Equation (72) implies that $m_\sigma \leq \sqrt{16\pi} f_\pi \approx 333 \text{ MeV}$, which is lower than 600 MeV . For $H \neq 0$ a similar bound applies. In that case we find that the maximal value of the sigma mass can be found by solving the following equation for m_σ

$$m_\sigma^2 = \left[2 + \sqrt{1 + A^2(m_\sigma^2/m_\pi^2)} \right] m_\pi^2, \quad (73)$$

where

$$A(x) = \left(\frac{16\pi f_\pi^2}{m_\pi^2} - \frac{1}{\pi} \right) \frac{1}{\sqrt{1 - 4/x}} - \frac{1}{\pi} \log \left(\frac{1 - \sqrt{1 - 4/x}}{1 + \sqrt{1 - 4/x}} \right). \quad (74)$$

We find by solving this equation that the maximal value of m_σ is equal to 433 MeV . This is also smaller than the average measured value of 600 MeV . The reason that we find an unphysical bound for the sigma mass could be that we consider $N_f = 2$ and may miss out on essential three-flavor physics.

The sigma mass turns out to be maximal if $\lambda(\mu = 100 \text{ MeV}) = 80$. The problem with this choice of λ is that in that case the maximal value of the cutoff is 720 MeV . This is very low and allows us only to do calculations up to around $T = 50 \text{ MeV}$. Therefore we choose a lower value: $\lambda(\mu = 100 \text{ MeV}) = 30$. Using that parameter choice $\Lambda_{\text{max}} = 19 \text{ GeV}$ and the sigma mass is equal to 256 MeV and 350 MeV in the case of $H = 0$ and $H = (104 \text{ MeV})^3$, respectively.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have considered the thermodynamics of the $O(N)$ linear and nonlinear sigma models to NLO in the $1/N$ expansion.

At NLO we have shown that one can renormalize the effective potential in a temperature-independent manner only at the minimum of the effective potential. This is another example of the ambiguity in the definition of off-shell Green's functions. A perhaps more familiar example comes from the calculation of the zero-temperature effective potential in gauge theories. In this case, the effective potential depends on the gauge-fixing condition except at the minimum [24–26]. By renormalizing the NLO effective potential in the minimum we found the beta function for λ to NLO. This beta function is consistent with the perturbative calculation.

We calculated numerically the pressure for the linear and nonlinear sigma model to NLO as a function of temperature. Our calculations show that for the calculation of the pressure $1/N$ is a good expansion, even if $N = 4$. With a relatively realistic choice of the parameters we

made a prediction for the pressure of QCD at low temperatures. Our results for the pressure disagree significantly with the calculations of those in Ref. [6]. This is due to the fact that we are not neglecting zero-temperature contributions and that we treat the NLO contribution without resorting to any high-energy approximation.

We also found that in the linear sigma model the sigma mass has an upper bound. This bound depends only on the parameters f_π and m_π . For a realistic choice of these parameters, this implies that the sigma mass is smaller than 433 MeV. This does not necessarily have consequences for the real sigma meson, since we did not take into account the full three-flavor physics.

Having solved the $O(N)$ model for the thermodynamics, it is natural to apply it to other quantities such as spectral functions. The methods developed here should also be useful for more complicated models incorporating additional features of low-energy QCD, e.g., $U(3)_A \times U(3)_V$.

ACKNOWLEDGMENTS

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