Bounds on the derivatives of the Isgur-Wise function with a nonrelativistic light quark

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In a preceding study in the heavy quark limit of QCD, it has been demonstrated that the best lower bound on the curvature of the Isgur-Wise function $\xi(w)$ is $\xi''(1) > \frac{1}{5}[4\rho^2 + 3(\rho^2)^2] > \frac{15}{16}$. The quadratic term $(\rho^2)^2$ is dominant in a nonrelativistic expansion in the light quark, both $\xi''(1)$ and $(\rho^2)^2$ scaling like $(R^2 m_q^2)^2$, where m_q is the light quark mass and R the bound state radius. The nonrelativistic limit is thus a good guideline in the study of the shape of $\xi(w)$. In the present paper we obtain similar bounds on all the derivatives of $\xi_{\rm NR}(w)$, the IW function with the light quark nonrelativistic, and we demonstrate that these bounds are optimal. Our general method is based on the positivity of matrices of moments of the ground state wave function, that allows to bound the nth derivative $\xi_{\rm NR}^{(n)}(w)$ in terms of the mth ones $(m < n)$. We show that the method can be generalized to the true Isgur-Wise function of QCD $\xi(w)$.

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I. INTRODUCTION

Using the operator product expansion (OPE) in the heavy quark limit of QCD, new Bjorken-like sum rules (SR) have been obtained [1–3]. It has been shown that the Isgur-Wise (IW) function $\xi(w)$ is an alternate series in powers of $(w-1)$, and lower bounds have been found on the absolute magnitude of its derivatives. Important ingredients in the derivation of the SR are the consideration, following Uraltsev [4], of the nonforward amplitude, plus the systematic use of boundary conditions that ensure that only a finite number of j^P intermediate states (with their tower of radial excitations) contribute.

In particular, it has been found that the nth derivative is bounded by the $(n-1)$th one [2]

$$(-1)^n \xi^{(n)}(1) \geq \frac{2n+1}{4}(-1)^{n-1}\xi^{(n-1)}(1) \geq \frac{(2n+1)!!}{2^{2n}}, \tag{1}$$

where the second inequality follows from the recursive character of the first one, and generalizes the inequality for the slope $\rho^2 = -\xi'(1)$:

$$\rho^2 \geq \tfrac{3}{4} \tag{2}$$

that follows from Bjorken [5] and Uraltsev [4] SR. The first inequalities (1) read, for the curvature $\sigma^2 = \xi''(1)$

$$\sigma^2 \geq \tfrac{5}{4}\rho^2 \geq \tfrac{15}{16}. \tag{3}$$

In [3] one has obtained, from a wider class of SR, the following better bound on the curvature

$$\sigma^2 \geq \tfrac{1}{5}\,[4\rho^2 + 3(\rho^2)^2] \geq \tfrac{15}{16}, \tag{4}$$

where the absolute lower bound (independent of ρ^2) follows from (2). Radiative corrections to the bounds (3) and (4) have been computed by M. Dorsten [6].

It has been underlined in [3] that the quadratic term in (4) has a clear physical interpretation, as it is leading in a nonrelativistic (NR) expansion in the mass of the light quark:

$$\xi''_{\rm NR}(1) > \tfrac{3}{5}(\rho^2)^2 \tag{5}$$

where $\xi_{\rm NR}(w)$ denotes the Isgur-Wise function with the light quark in the NR limit.

It is clear that it is very important to have rigorous bounds on the derivatives of the IW function. The main general reason is that the shape of the latter is linked to the determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{cb}|$ through the exclusive processes $B \to D^{(*)}\ell\nu$. As pointed out in [3], a more quantitative reason is that, beyond the first derivative, higher derivatives will play a non-negligible role at the edge of the phase space, at high w, in the region where the data are presently rather precise, and will become more and more precise in the near future.

Therefore, it is not only of an academic interest to find bounds on higher derivatives. To this aim, one can begin by using systematically for higher derivatives the method exposed in [3]. It is then possible to find bounds of the form (4) for higher derivatives [7].

However, we have realized that, studying the NR situation, a more powerful method can be developed that leads to better bounds, exposed below. With this method, one finds for the second derivative, in the NR limit, the bound (5), but for higher derivatives $\xi_{\rm NR}^{(n)}(1)$ one finds more complicated bounds involving the mth ones $(m < n)$. It is important to notice that, unlike the method that we have used in QCD [1–3], where the IW functions to excited states played a crucial role, in the present paper we note that a simple general property (positivity) involving only

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the elastic IW function allows us to deduce all the bounds on its derivatives at zero recoil.

In QCD, in the heavy quark limit one can hope to obtain bounds such that, in the NR limit for the light quark, reduce to these better bounds. For the moment, the aim of the present paper is to obtain bounds for the derivatives of $\xi_{\rm NR}(w)$, opening the way to study the more complex case of the actual IW function $\xi(w)$ in QCD.

In Sec. II we set the relation between the derivatives $\xi_{\rm NR}^{(m)}(1)$ and the moments $\langle 0|z^m|0\rangle$, where $|0\rangle$ is the ground state radial wave function. In Sec. III we study the general constraints on the moments. In Sec. IV we shift to the corresponding constraints on the derivatives and the resulting bounds. In Section V we illustrate our bounds by particularizing to simple potentials and in Sec. VI we conclude. In Appendix A we demonstrate a mathematical identity used in the text, in Appendix B we show the frame dependence of subleading moments and in Appendix C we demonstrate the optimality of the bounds. In Appendix D we deduce a simple formula giving a weaker but completely explicit bound for all the even derivatives $\xi^{(2n)}(1)$, that we compare with our optimal bounds.

II. ISGUR-WISE FUNCTION IN THE NR LIMIT: RELATION BETWEEN ITS DERIVATIVES AND MOMENTS

A. Universal NR form factors in heavy-heavy transitions

As it is well known from nuclear physics, the first-quantized nonrelativistic operator corresponding to the electromagnetic current $e_Q\overline{Q}(x)\gamma^0 Q(x)$ of field theory is given by the expression $e_Q\delta(\mathbf{x}-\mathbf{r}_Q)$, where $\mathbf{r}_Q$ is the position of the active quark (see for example [8]). Here we are interested in the form factor corresponding to a general current $\overline{Q}'(x)\Gamma Q(x)$ for transitions between bound states of the type $(Q,q)\to(Q',q)$ with unequal masses. For simplicity, let us begin with the matrix element of the fourth component of the vector current $J^0(0)=\overline{Q}'(0)\gamma^0 Q(0)$.

Let us write the corresponding nonrelativistic (NR) form factor in a general frame. The form factor for the transition $(Q,q)\to(Q',q)$ will be given simply by the matrix element of the operator $\delta(\mathbf{r}_Q)$:

$$F(\mathbf{P}',\mathbf{P})=\langle\mathbf{P}'|\delta(\mathbf{r}_Q)|\mathbf{P}\rangle=\int\Psi^f_{P'}(\mathbf{r}_q,\mathbf{0})^*\Psi^i_P(\mathbf{r}_q,\mathbf{0})d\mathbf{r}_q, \tag{6}$$

where $\mathbf{r}_Q$ and $\mathbf{r}_q$ are, respectively, the positions of the active and the spectator quarks.

There is a simple argument to find this nonrelativistic expression of the current $J^0=\overline{Q}'\gamma^0 Q$. Its matrix elements between one-particle states of given momenta $\mathbf{p}$ and $\mathbf{p}'$ in the nonrelativistic limit, where $\mathbf{p}'/m_{Q'}$ and $\mathbf{p}/m_Q$ are small, are $\langle\mathbf{p}',s'|J^0|\mathbf{p},s\rangle=\delta_{s',s}$. As readily verified, these matrix elements are precisely those of the multiplication operator (in configuration space) by the function $\delta(\mathbf{r})$. Indeed, one has:

$$\langle\mathbf{p}',s'|\delta(\mathbf{r})|\mathbf{p},s\rangle=\delta_{s',s}\int e^{-i\mathbf{p}'\cdot\mathbf{r}}\delta(\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{r}=\delta_{s's}. \tag{7}$$

The nonrelativistic limit of the current is therefore $J^0=\delta(\mathbf{r})$.

The wave functions, that factorize in center-of-mass and internal wave functions, are given by

$$\begin{aligned}\Psi^i_P(\mathbf{r}_q,\mathbf{r}_Q)&=\exp[i(m_q\mathbf{r}_q+m_Q\mathbf{r}_Q)\cdot\mathbf{v}]\psi^i(\mathbf{r}_q-\mathbf{r}_Q),\\ \Psi^f_{P'}(\mathbf{r}_q,\mathbf{r}_{Q'})&=\exp[i(m_q\mathbf{r}_q+m_{Q'}\mathbf{r}_{Q'})\cdot\mathbf{v}']\psi^f(\mathbf{r}_q-\mathbf{r}_{Q'}),\end{aligned} \tag{8}$$

where

$$\mathbf{v}=\frac{\mathbf{P}}{m_q+m_Q},\qquad \mathbf{v}'=\frac{\mathbf{P}'}{m_q+m_{Q'}}, \tag{9}$$

are the nonrelativistic velocities and one gets for the form factor (6) the following expression, exhibiting Galilean invariance, since the NR form factor $F(\mathbf{P},\mathbf{P}')$ is a function of the variable $(\mathbf{v}-\mathbf{v}')^2$:

$$F(\mathbf{P}',\mathbf{P})=\langle\psi^f|\exp[im_q(\mathbf{v}-\mathbf{v}')\cdot\mathbf{r}_q]|\psi^i\rangle\equiv f[(\mathbf{v}-\mathbf{v}')^2]. \tag{10}$$

Up to now, Eqs. (6)–(10) are valid for the current $J^0(0)=\overline{Q}'(0)\gamma^0Q(0)$ and for any value of the masses m_Q, $m_{Q'}$, and m_q. If we now assume the hierarchy

$$1/R\ll m_q\ll m_Q,m_{Q'}, \tag{11}$$

we will be in the situation of a heavy-heavy $Q\to Q'$ quark transition with a light spectator, nonrelativistic quark q. The first condition $1/R\ll m_q$, where R is the radius of the bound state, ensures that the quark q is nonrelativistic, while from the second condition $m_q\ll m_Q,m_{Q'}$, the quark q is light relative to the active quarks Q and Q'. This latter condition implies a heavy quark symmetry SU($2N_h$) where N_h is the number quarks that are heavy relatively to the quark q. Therefore, these conditions imply Isgur-Wise scaling, with all form factors being given by the universal form factor $f[(\mathbf{v}-\mathbf{v}')^2]$ (10). Notably, the flavor independence is due to the fact that the internal wave function, ψ_f and ψ_i become independent of the heavy quark mass.

Expanding the form factor $f[(\mathbf{v}-\mathbf{v}')^2]$ in powers of $(\mathbf{v}-\mathbf{v}')^2$ we can write then

$$\begin{aligned}f[(\mathbf{v}-\mathbf{v}')^2]&=\sum_n\frac{1}{n!}f^{(n)}(0)(\mathbf{v}-\mathbf{v}')^{2n}\\ &=\sum_n(-1)^n\frac{1}{(2n)!}(m_q)^{2n}\langle 0|z^{2n}|0\rangle(\mathbf{v}-\mathbf{v}')^{2n},\end{aligned} \tag{12}$$

and we obtain the relations between the derivatives of the form factor and the moments:

$$f^{(n)}(0) = (-1)^n \frac{n!}{(2n)!}(m_q)^{2n}\langle 0|z^{2n}|0\rangle, \qquad (13)$$

or, by spherical symmetry,

$$f^{(n)}(0) = (-1)^n \frac{n!}{(2n+1)!}(m_q)^{2n}\mu_{2n}, \qquad (14)$$

with

$$\mu_{2n} = \langle 0|r^{2n}|0\rangle. \qquad (15)$$

B. Relation between the universal NR form factor and the NR limit of the Isgur-Wise function

Thus, the NR form factor $F(\mathbf{P}', \mathbf{P})$ is a function of the variable $(\mathbf{v} - \mathbf{v}')^2$, while the relativistic Isgur-Wise function $\xi(w)$ depends on w, where

$$w = v \cdot v' \qquad \text{with} \qquad v = \frac{P}{M}, \quad v' = \frac{P'}{M'}, \qquad (16)$$

and (M, P), (M', P') are the masses and four-momenta of the initial and final mesons.

There are several facts that ask for care in the identification between the NR form factor $f[(\mathbf{v} - \mathbf{v}')^2]$ and the Isgur-Wise function $\xi(w)$ in its nonrelativistic limit $\xi_{\rm NR}(w)$.

Let us leave aside for the moment the fact that the velocities (16) differ from their nonrelativistic limits (9) by the binding energy that is neglected in the latter.

Of course, $f[(\mathbf{v} - \mathbf{v}')^2]$ and $\xi_{\rm NR}(w)$ cannot be generally identified, since w is not a function of $(\mathbf{v} - \mathbf{v}')^2$:

$$w = v \cdot v' = \sqrt{1 + \mathbf{v}'^2}\sqrt{1 + \mathbf{v}^2} - \mathbf{v} \cdot \mathbf{v}'$$
$$= 1 + \frac{1}{2}\left[(\mathbf{v} - \mathbf{v}')^2 - \left(\sqrt{1 + \mathbf{v}^2} - \sqrt{1 + \mathbf{v}'^2}\right)^2\right]. \qquad (17)$$

To relate w and $(\mathbf{v} - \mathbf{v}')^2$ one needs to choose a frame. The natural frame is the rest frame of the initial particle, i. e. $\mathbf{v} = 0$. One has, in this frame:

$$w = [1 + (\mathbf{v} - \mathbf{v}')^2]^{1/2}. \qquad (18)$$

The relation between w and $(\mathbf{v} - \mathbf{v}')^2$ being nonlinear, the relations between the derivatives relatively to w and to $(\mathbf{v} - \mathbf{v}')^2$ are complicated. As we show in Appendix B, the derivative of order n relatively to w depends on the derivatives of order $m \leq n$ relatively to $(\mathbf{v} - \mathbf{v}')^2$ and conversely.

Another frame, the equal-velocity frame (EVF) where the velocities are equal and opposite $\mathbf{v}' = -\mathbf{v}$, gives from (17) :

$$w = 1 + \frac{1}{2}(\mathbf{v} - \mathbf{v}')^2. \qquad (19)$$

Thus, in this frame, the relation between w and $(\mathbf{v} - \mathbf{v}')^2$ is *linear*, and the nth derivative relatively to w is proportional to the nth derivative relatively to $(\mathbf{v} - \mathbf{v}')^2$.

In the EVF, we obtain

$$\sum_n \frac{1}{n!}\xi_{NR}^{(n)}(1)(w-1)^n$$
$$= \xi_{\rm NR}(w) = \langle\psi|\exp[im_q(\mathbf{v} - \mathbf{v}')\cdot\mathbf{r}_q]|\psi\rangle$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(m_q)^{2n}\langle 0|z^{2n}|0\rangle(\mathbf{v} - \mathbf{v}')^{2n}$$
$$= \sum_{n=0}^{\infty}(-1)^n 2^n\frac{1}{(2n)!}(m_q)^{2n}\langle 0|z^{2n}|0\rangle(w-1)^n. \qquad (20)$$

Therefore, in this frame one gets the relation between the derivatives of $\xi_{\rm NR}(w)$ and the moments:

$$\xi_{NR}^{(n)}(1) = (-1)^n 2^n\frac{n!}{(2n+1)!}(m_q)^{2n}\langle 0|r^{2n}|0\rangle. \qquad (21)$$

This relation is exact in this frame, and coincides with the leading term in the NR limit in all collinear frames, as the rest frame (see Appendix B).

Therefore, in a NR expansion for the light quark, one can only claim to obtain frame-independent results for the derivatives of $\xi_{\rm NR}(w)$ in the leading NR order for the moments. From now on we will then rely only on the relation (21).

However, $\mathbf{v}$, $\mathbf{v}'$ in relation (20) are not identical to their nonrelativistic limits (9). This fact does not invalidate the relations given above, since w in the EVF (19) differs from its NR expression in terms of NR velocities by terms of order ε/m_Q where ε is the binding energy. This can be summarized by expanding w in terms of the NR velocities (9) and binding energies:

$$w = 1 + \frac{1}{2}\left(\frac{\mathbf{P}}{m_q + m_Q} - \frac{\mathbf{P}'}{m_q + m_{Q'}}\right)^2$$
$$+ \text{subleading terms in velocities} + O(\varepsilon/m_Q). \qquad (22)$$

Therefore, since these differences are subleading we can make the identification

$$\xi_{\rm NR}(w_{\rm NR}) = f[(\mathbf{v} - \mathbf{v}')^2], \qquad (23)$$

where $w_{\rm NR}$ is the leading term of w given by the preceding expansion, and relation (21) holds indeed for the leading terms.

From Eq. (21) one may be surprised that the NR expansion leads to a result for $\xi^{(n)}(1)$ increasing with $m_q R \sim (v/c)^{-1}$, seemingly at odds with the notion of a NR expansion. In fact, the two last formulas in expression (20) show that the form factor is expanded in a series of powers $(m_q)^{2n}\langle 0|z^{2n}|0\rangle[(\mathbf{v} - \mathbf{v}')/c]^{2n} \sim (m_q R)^{2n}[(\mathbf{v} - \mathbf{v}')/c]^{2n}$ where R is the bound state radius $R \sim (p_{\rm int})^{-1}$, $p_{\rm int}$ being the *internal* quark momentum. On the other hand $\mathbf{v} - \mathbf{v}'$ is the *external* center-of-mass velocity transfer. Therefore, since $m_q R \sim m_q/p_{\rm int} = [(v/c)_{\rm int}]^{-1}$, the form factor is expanded in powers of the type $[(v/c)_{\rm int}]^{-2n}[(v/c)_{\rm ext}]^{2n}$. The negative order in

$[(v/c)_{\rm int}]^{-2n}$ is therefore compensated by the corresponding positive powers of $[(v/c)_{\rm ext}]^{2n}$. The derivatives $\xi_{\rm NR}^{(n)}(1)$ in Eq. (21) are given by the inverse powers $[(v/c)_{\rm int}]^{-2n}$, that multiply in (20) $[(\mathbf{v}-\mathbf{v}')/c]^{2n} = 2^n(w-1)^n$. In a general frame, the coefficient of $(\mathbf{v}-\mathbf{v}')^{2n}$ or $(w-1)^n$ will contain an expansion in powers of $[(v/c)_{\rm int}]^{-2n+m}$ $(m \geq 0)$. However, the subleading terms $(m > 0)$ are frame dependent, as shown in Appendix B.

III. CONSTRAINTS ON THE MOMENTS

Let us define the moments

$$\mu_n = \langle 0|r^n|0\rangle, \tag{24}$$

and consider the even moments μ_{2n}, related to $\langle 0|z^{2n}|0\rangle$ from rotational invariance

$$\mu_{2n} = (2n+1)\langle 0|z^{2n}|0\rangle. \tag{25}$$

We will now formulate *necessary* constraints on the μ_{2n} resulting from the fact that they are indeed moments, i.e., that there exists a function $\varphi(r)$ such that

$$\mu_{2n} = \int_0^\infty r^{2n}|\varphi(r)|^2 dr. \tag{26}$$

It turns out that these conditions are *sufficient*, but this is only proved in Appendix C, implying that *the constraints are optimal.*

(i) A necessary condition is that for any nonzero polynomial P

$$P(r^2) = \sum_{i=0}^n a_i r^{2i} \geq 0 \quad (r \geq 0) \quad \Rightarrow$$
$$\int_0^\infty P(r^2)|\varphi(r)|^2 dr = \sum_{i=0}^n a_i \mu_{2i} > 0. \tag{27}$$

From this condition, that is not very explicit, one deduces the following conditions (ii), (iii), and (iv), that are equivalent. Condition (iv) is explicit.

(ii) For any $n \geq 0$ and nonvanishing $a_0, \cdots a_n$ one has

$$\sum_{i,j=0}^n (a_i)^* a_j \mu_{2i+2j} > 0 \quad \text{and}$$
$$\sum_{i,j=0}^n (a_i)^* a_j \mu_{2i+2j+2} > 0. \tag{28}$$

One demonstrates (ii) from (i) by considering the polynomials $P(r^2) = |\sum_{i=0}^n a_i r^{2i}|^2$ and $P(r^2) = r^2|\sum_{i=0}^n a_i r^{2i}|^2$. Conversely, (i) results from (ii).

(iii) For any $n \geq 0$, the matrices $(\mu_{2i+2j})_{0\leq i,j\leq n}$ and $(\mu_{2i+2j+2})_{0\leq i,j\leq n}$ are positive definite.
This condition is just a rephrasing of condition (ii).

(iv) For any $n \geq 0$, one has

$$\det[(\mu_{2i+2j})_{0\leq i,j\leq n}] > 0, \tag{29}$$

$$\det[(\mu_{2i+2j+2})_{0\leq i,j\leq n}] > 0. \tag{30}$$

To obtain (iv) from (iii) it is enough to note that a positive definite matrix has strictly positive eigenvalues, and that the determinant is the product of its eigenvalues.

Let us first write the determinants (29) and (30) for the lower values of n, namely

$$\mu_2 > 0 \tag{31}$$

$$\det\begin{pmatrix} 1 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} > 0 \tag{32}$$

$$\det\begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix} > 0 \tag{33}$$

$$\det\begin{pmatrix} 1 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{pmatrix} > 0 \tag{34}$$

$$\det\begin{pmatrix} \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \\ \mu_6 & \mu_8 & \mu_{10} \end{pmatrix} > 0 \tag{35}$$

$$\det\begin{pmatrix} 1 & \mu_2 & \mu_4 & \mu_6 \\ \mu_2 & \mu_4 & \mu_6 & \mu_8 \\ \mu_4 & \mu_6 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{10} & \mu_{12} \end{pmatrix} > 0 \tag{36}$$

$$\cdots$$

where (32), (34), and (36) belong to the class of positivity conditions (29), and (31), (33), and (35) to the class (30).

From (32) and (33) we find, respectively

$$\mu_4 > \mu_2^2, \tag{37}$$

$$\mu_6 > \frac{\mu_4^2}{\mu_2}. \tag{38}$$

To get the constraint on μ_8 from (34) in terms of positive definite quantities, we make use of the following identities:

$$\mu_4 \det\begin{pmatrix} 1 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{pmatrix} = \det\begin{pmatrix} 1 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} \det\begin{pmatrix} \mu_4 & \mu_6 \\ \mu_6 & \mu_8 \end{pmatrix} - \left[\det\begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix}\right]^2, \tag{39}$$

$$\det\begin{pmatrix} 1 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{pmatrix} = \det\begin{pmatrix} \mu_4 & \mu_2 & \mu_6 \\ \mu_2 & 1 & \mu_4 \\ \mu_6 & \mu_4 & \mu_8 \end{pmatrix} = \det\begin{pmatrix} \mu_4 & \mu_2 \\ \mu_2 & 1 \end{pmatrix}\det\begin{pmatrix} 1 & \mu_4 \\ \mu_4 & \mu_8 \end{pmatrix} - \left[\det\begin{pmatrix} \mu_2 & 1 \\ \mu_6 & \mu_4 \end{pmatrix}\right]^2, \quad (40)$$

that follows from the general identity among determinants of the Appendix A. We find:

$$\mu_8 > \frac{\mu_6^2 - 2\mu_2\mu_4\mu_6 + \mu_4^3}{\mu_4 - \mu_2^2} = \frac{(\mu_2\mu_6 - \mu_4^2)^2}{\mu_4(\mu_4 - \mu_2^2)} + \frac{\mu_6^2}{\mu_4} = \frac{(\mu_6 - \mu_2\mu_4)^2}{\mu_4 - \mu_2^2} + \mu_4^2, \quad (41)$$

where the first equality follows from (34) and (39) and the second from (34) and (40).

To proceed in the same way with the 10th moment, we make use of the inequality (35) and the relations among determinants similar to (39) and (40), and this yields

$$\mu_{10} > \frac{\mu_2\mu_8^2 - 2\mu_4\mu_6\mu_8 + \mu_6^3}{\mu_2\mu_6 - \mu_4^2} = \frac{(\mu_4\mu_8 - \mu_6^2)^2}{\mu_6(\mu_2\mu_6 - \mu_4^2)} + \frac{\mu_8^2}{\mu_6} = \frac{(\mu_2\mu_8 - \mu_4\mu_6)^2}{\mu_2(\mu_2\mu_6 - \mu_4^2)} + \frac{\mu_6^2}{\mu_2}. \quad (42)$$

Things become more complicated for higher moments, but the method proceeds in the same way. The 12th moment is dealt with the inequality (36) and the results among determinants (A8)–(A10).

IV. BOUNDS ON THE DERIVATIVES

Let us summarize the inequalities among the moments deduced in the previous section. We adopt in (41) and (42) the expressions given by the last equalities in the right-hand side. This will be instructive, as it will become clear below. We have obtained

$$\mu_2 > 0, \quad (43)$$

$$\mu_4 > \mu_2^2, \quad (44)$$

$$\mu_6 > \frac{\mu_4^2}{\mu_2}, \quad (45)$$

$$\mu_8 > \frac{(\mu_6 - \mu_2\mu_4)^2}{\mu_4 - \mu_2^2} + \mu_4^2, \quad (46)$$

$$\mu_{10} > \frac{(\mu_2\mu_8 - \mu_4\mu_6)^2}{\mu_2(\mu_2\mu_6 - \mu_4^2)} + \frac{\mu_6^2}{\mu_2}. \quad (47)$$

From the *frame-independent* relation between moments and derivatives obtained from (21)–(25) (cf. Appendices B and D):

$$\xi_{NR}^{(n)}(1) = (-1)^n 2^n \frac{n!}{(2n+1)!}(m_q)^{2n}\mu_{2n}, \quad (48)$$

and from (43)–(47), we obtain, respectively, the following inequalities among the derivatives:

$$-\xi_{\rm NR}^{(1)}(1) > 0, \quad (49)$$

$$\xi_{\rm NR}^{(2)}(1) > \tfrac{3}{5}\,[\xi_{\rm NR}^{(1)}(1)]^2, \quad (50)$$

$$-\xi_{\rm NR}^{(3)}(1) > -\frac{5}{7}\frac{[\xi_{\rm NR}^{(2)}(1)]^2}{\xi_{\rm NR}^{(1)}(1)}, \quad (51)$$

$$\xi_{\rm NR}^{(4)}(1) > \frac{7}{9}\frac{[-\xi_{\rm NR}^{(3)}(1) + \frac{3}{7}\xi_{\rm NR}^{(1)}(1)\xi_{\rm NR}^{(2)}(1)]^2}{\xi_{\rm NR}^{(2)}(1) - \frac{3}{5}[\xi_{\rm NR}^{(1)}(1)]^2} + \frac{5}{21}[\xi_{\rm NR}^{(2)}(1)]^2, \quad (52)$$

$$-\xi_{\rm NR}^{(5)}(1) > \frac{9}{11}\frac{\left[\xi_{\rm NR}^{(4)}(1) - \frac{5}{9}\frac{\xi_{\rm NR}^{(2)}(1)\xi_{\rm NR}^{(3)}(1)}{\xi_{\rm NR}^{(1)}(1)}\right]^2}{-\xi_{\rm NR}^{(3)}(1) + \frac{5}{7}\frac{[\xi_{\rm NR}^{(2)}(1)]^2}{-\xi_{\rm NR}^{(1)}(1)}} - \frac{35}{99}\frac{[\xi_{\rm NR}^{(3)}(1)]^2}{\xi_{\rm NR}^{(1)}(1)}. \quad (53)$$

Importantly, we observe that the left-hand side and the right-hand side of all the inequalities (49)–(53) scale in the same way in the parameter $R^2m_q^2$, where m_q is the light quark mass and R the bound state radius, since the derivatives $\xi_{NR}^{(n)}(1)$, from (21), scale like $(R^2m_q^2)^n$. The inequalities (49) and (50) are the nonrelativistic limit of the bounds of the true IW function (2) and (4).

Comparing with the results of the method used in Appendix D, we have found stronger results for all the derivatives.

For the even derivatives, we find for $2n > 2$ a new term that strengthens the lower bound. This can be seen from the bounds (52) and (D9) or $\xi_{\rm NR}^{(4)}(1)$. We have found a new term besides the term proportional to $[\xi_{\rm NR}^{(2)}(1)]^2$. However, for the curvature $\xi_{\rm NR}^{(2)}(1)$, relevant for the nonrelativistic limit of the curvature of the true IW function $\xi(w)$, we find the same bound (5) as with the former simpler method of Appendix D.

As for the odd derivatives, the trivial bound (D10) has been changed in a very substantial way, since we find the lower bounds (51) and (53).

Finally, let us emphasize that the lower bounds (49)–(53) are optimal. The optimality of these bounds is demonstrated in the Appendix C.

V. SOME ILLUSTRATIONS

For the sake of a simple illustration of the bounds, let us consider the harmonic oscillator in the equal-velocity frame:

$$\xi_{\rm NR}^{\rm h.o.}(w) = \exp[-(w-1)m_q^2R^2], \tag{54}$$

where the bound state radius R is normalized in a convenient way to have this simple expression. The nth derivative reads

$$\xi_{\rm NR}^{(n)}(1) = (-1)^n(m_q^2R^2)^n. \tag{55}$$

Then, the bound (49) reads simply $m_q^2R^2 > 0$ and (50)–(53) will become for the nth derivative

$$\begin{aligned} &n=2 \qquad 1 > \frac{3}{5}, \qquad n=3 \qquad 1 > \frac{5}{7}, \\ &n=4 \qquad 1 > \frac{55}{63}, \qquad n=5 \qquad 1 > \frac{91}{99}. \end{aligned} \tag{56}$$

Interestingly, we find that the bounds become better and better as we consider higher derivatives. For the 5th derivative the bound is already very strict.

However, it is not granted that these features will remain for more realistic potentials. Therefore, it can be useful to examine another simple potential, although not confining, namely, the Coulomb potential. In this case we have a dipole form factor

$$\xi_{\rm NR}^{\rm Coulomb}(w) = \frac{1}{[1+(w-1)m_q^2R^2]^2}. \tag{57}$$

The derivatives read

$$\xi_{NR}^{(n)}(1) = (n+1)!(-1)^n(m_q^2R^2)^n, \tag{58}$$

and the inequalities (50)–(53) give, respectively

$$\begin{aligned} &n=2 \qquad 1 > \frac{2}{5}, \qquad n=3 \qquad 1 > \frac{15}{28}, \\ &n=4 \qquad 1 > \frac{263}{378}, \qquad n=5 \qquad 1 > \frac{3626}{4719}. \end{aligned} \tag{59}$$

These inequalities are somewhat less strict than in the harmonic oscillator case but here also they improve for higher derivatives.

We can expect that in the case of a realistic phenomenological $Q\bar{q}$ potential, with a confining and a short distance parts, the situation will be in between the harmonic oscillator and the Coulomb potentials.

VI. GENERALIZATION OF THE METHOD TO QCD.

We have obtained lower bounds on the derivatives at zero recoil of the nonrelativistic Isgur-Wise function $\xi_{\rm NR}(w)$, i.e., the IW function with a NR light quark. Our main motivation has been to find the leading term in a NR expansion of the derivatives at zero recoil of the true IW function $\xi(w)$ that should be obtained in the heavy quark limit of QCD. The parameter in this expansion is $(v^2/c^2)_{\rm int}$ or, equivalently $1/R^2m_q^2$, where m_q is the light quark mass and R is the bound state radius. In previous work [3] we did obtain in the heavy quark limit of QCD such an expansion for the slope and the curvature, inequalities (2) and (4),

$$-\xi^{(1)}(1) > \tfrac{3}{4}, \qquad \xi^{(2)}(1) > \tfrac{1}{5}\ \{-4\xi^{(1)}(1) + 3[\xi^{(1)}(1)]^2\}. \tag{60}$$

Since $-\xi^{(1)}(1)$ and $\xi^{(2)}(1)$, scale, respectively, like $R^2m_q^2$ and $(R^2m_q^2)^2$, in the NR limit these inequalities become respectively (49) and (50). The inequalities (60) contain terms, specific to QCD in the heavy quark limit, that are subleading in a NR expansion.

Our aim would be, in the long run, to obtain bounds for the nth derivative of the IW function in the heavy quark limit of QCD that must contain the subleading terms in a NR expansion. We know that in the strict NR limit we must recover the bounds (49)–(53) obtained in the present paper.

To obtain these bounds in QCD we could try the method of [3] in a systematic way, that uses sum rules for the nonforward amplitude, relating a sum over intermediate states and the OPE, that depends on three variables $w_i = v_i \cdot v'$, $w_f = v_f \cdot v'$, $w_{if} = v_i \cdot v_f$, that lie in a certain domain [1]. Differentiating the SR relatively to (w_i, w_f, w_{if}) and going to the frontier of the domain one gets relations that allow to obtain (60) [3]. This method can be pursued further and obtain bounds for the higher derivatives [7]. However, the obtained bounds, in their NR limit, are weaker than (51)–(53).

We have developed here a more powerful method, based on the positivity of matrices of moments of the ground state wave function, that allows to go further for the derivatives $n > 2$ in the NR limit. To generalize the present method to QCD in the heavy quark limit one should investigate whether the derivatives $\xi^{(n)}(1)$ can be expressed in terms of positive definite quantities that are true moments as in the nonrelativistic expression (26). Then, one could draw the consequences that follow from the positivity of the relevant matrices. A step in this direction is the conjecture that, at least in the meson case [1], all SR in the heavy quark limit of QCD are satisfied in the Bakamjian-Thomas (BT) class of relativistic quark models [9]. We have realized this in practice for the lower derivatives $\xi^{(n)}(1)$, $n = 1, 2, 3$. These models are relativistic for the states and also for the current matrix elements in the heavy quark limit, exhibiting Isgur-Wise scaling. One can hope to start from the NR quark model and go to BT models, and from those to the heavy quark limit QCD.

Another, more direct way to proceed to the heavy quark limit of QCD is to start from the sum rules obtained in [1–3], realizing that one can obtain the NR bounds of the present paper from the equivalent sum rules of the nonrelativistic limit. Indeed, in the NR limit we have a SR of the form

$$\sum_{n''} f_{n,n''}(\mathbf{k}) f_{n'',n'}(\mathbf{k}') = f_{n,n'}(\mathbf{k} + \mathbf{k}'), \tag{61}$$

that follows, very simply, from

$$f_{n,n'}(\mathbf{k}) = \langle n | e^{i\mathbf{k}\cdot\mathbf{r}} | n' \rangle. \tag{62}$$

In QCD in the heavy quark limit we have sum rules of the same form (61), but *without* the explicit expression (62). However, to derive the inequalities of this paper, (61) is sufficient. Indeed, from (61) one gets

$$\sum_{i,j} c_i c_j^* f_{0,0}(\mathbf{k}_i - \mathbf{k}_j) = \sum_n \sum_{i,j} c_i c_j^* f_{0,n}(\mathbf{k}_i) f_{n,0}(-\mathbf{k}_j) = \sum_n \Big|\sum_i c_i f_{0,n}(\mathbf{k}_i)\Big|^2 \geq 0. \tag{63}$$

From this relation one can infer, for any function $\varphi(\mathbf{k})$,

$$\int d\mathbf{k} d\mathbf{k}' \varphi(\mathbf{k}')^* f_{0,0}(\mathbf{k} - \mathbf{k}') \varphi(\mathbf{k}) \geq 0, \tag{64}$$

and therefore, for the Fourier transform

$$\int d\mathbf{r} |\tilde{\varphi}(\mathbf{r})|^2 \tilde{f}_{0,0}(\mathbf{r}) \geq 0, \tag{65}$$

and hence

$$\tilde{f}_{0,0}(\mathbf{r}) \geq 0. \tag{66}$$

These are the conditions that we need to obtain constraints on the moments and hence bounds on the derivatives of the form factor $f_{0,0}(\mathbf{k} - \mathbf{k}')$, because writing the form factor in terms of its Fourier transform

$$f_{0,0}(\mathbf{k} - \mathbf{k}') = \int d\mathbf{r} \tilde{f}_{0,0}(\mathbf{r}) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}, \tag{67}$$

and taking into account that $f_{0,0}(\mathbf{k} - \mathbf{k}')$ must be an even function, we obtain (Oz is defined along the momentum transfer $\mathbf{k} - \mathbf{k}'$):

$$f_{0,0}(\mathbf{k} - \mathbf{k}') = \sum_n (-1)^n \frac{1}{(2n)!} (\mathbf{k} - \mathbf{k}')^{2n} \frac{\mu_{2n}}{2n+1}, \tag{68}$$

i.e., an expansion of the form factor in terms of moments of the form (20), with the identification

$$\mu_{2n} = \langle 0 | r^{2n} | 0 \rangle = \int d\mathbf{r} \tilde{f}_{0,0}(\mathbf{r}) r^{2n}, \tag{69}$$

since the positivity condition $\tilde{f}_{0,0}(\mathbf{r}) \geq 0$ (66) holds and, from (67):

$$\int \tilde{f}_{0,0}(\mathbf{r}) d\mathbf{r} = 1, \tag{70}$$

We do recover essentially the previous results (43)–(47) *using only the sum rules* (61). If $\tilde{f}_{0,0}(\mathbf{r})$ is a function, it can be seen as the square of a wave function, and all the *strict* inequalities of the NR type (43)–(47) would follow. However, this is not implied by the SR, and weaker results could follow, namely, the inequalities may not be strict. For example, one could have a distribution like

$$\tilde{f}_{0,0}(\mathbf{r}) = \frac{1}{4\pi r_0^2} \delta(|\mathbf{r}| - r_0), \tag{71}$$

which is not the square of a wave function, that would imply

$$\mu_{2n} = r_0^{2n}, \tag{72}$$

and, for example, the strict inequality (44) would become an equality. For example, in the true QCD case, the lower bound (2) for ρ^2 could become an equality. By the way, this would correspond to the approximation considered in Ref. [10].

Our strategy will then be to start from the SR in the heavy quark limit of QCD that are equivalent to the NR ones (61), and proceed along the same lines. We can presume that the method will give the optimal bounds for the derivatives of the true Isgur-Wise function $\xi(w)$.

VII. CONCLUSION

To conclude, we have obtained the best possible general bounds on the derivatives of the Isgur-Wise function $\xi_{\rm NR}(w)$, i.e., considering the light quark as nonrelativistic, in terms of lower derivatives. These bounds must be the nonrelativistic limit of the bounds on the derivatives of the true Isgur-Wise function $\xi(w)$, and constitute a guideline in the derivation of the latter. Moreover, we argue that the method developed here, that exploits the positivity of matrices of moments, can be generalized, starting from SR in the heavy quark limit of QCD, to obtain the best bounds on all the derivatives of $\xi(w)$.

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APPENDIX A: AN IDENTITY BETWEEN DETERMINANTS

In Sec. III we have made use of the identity among determinants

$$\det[(a_{ij})_{1\leq i,j\leq n}]\det[(a_{ij})_{2\leq i,j\leq n-1}] = \det[(a_{ij})_{1\leq i,j\leq n-1}]\det[(a_{ij})_{2\leq i,j\leq n}] - \det[(a_{ij})_{1\leq i\leq n-1,2\leq j\leq n}]\det[(a_{ij})_{2\leq i\leq n,1\leq j\leq n-1}], \tag{A1}$$

or, in a more readable way:

$$
\det\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} \det\begin{pmatrix} a_{2,2} & \cdots & a_{2,n-1} \\ \cdots & \cdots & \cdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix}
$$
$$
= \det\begin{pmatrix} a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} \det\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-2} & a_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-2,1} & \cdots & a_{n-2,2} & a_{n-2,n-1} \\ a_{n-1,1} & \cdots & a_{n-1,2} & a_{n-1,n-1} \end{pmatrix}
$$
$$
- \det\begin{pmatrix} a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-2,2} & \cdots & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \end{pmatrix} \det\begin{pmatrix} a_{2,1} & \cdots & a_{2,n-2} & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} \\ a_{n,1} & \cdots & a_{n,n-2} & a_{n,n-1} \end{pmatrix}. \tag{A2}
$$

To demonstrate this relation, let us introduce the column vectors

$$
x_i = \sum_{j=1}^{n} a_{i,j} e_j, \tag{A3}
$$

where the $a_{i,j}$ are the elements of the matrices (A1) or (A2). Multiplying (A2) by $(e_1 \wedge \cdots \wedge e_n) \otimes (e_1 \wedge \cdots \wedge e_n)$, this formula writes

$$
\begin{aligned}
(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \otimes (e_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_n) &= (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_n) \otimes (e_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \\
&\quad -(e_n \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_1).
\end{aligned} \tag{A4}
$$

Assuming that the vectors $x_1, \cdots x_n$ are independent, one can expand e_1 and e_n:

$$
e_1 = \alpha_1 x_1 + \cdots + \alpha_n x_n, \qquad e_n = \beta_1 x_1 + \cdots + \beta_n x_n. \tag{A5}
$$

The left-hand side of (A4) becomes

$$
\begin{aligned}
(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \otimes (e_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_n) &= (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \\
&\quad \otimes[(\alpha_1 x_1 + \alpha_n x_n) \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge (\beta_1 x_1 + \beta_n x_n)] \\
&= (\alpha_1\beta_n - \alpha_n\beta_1)(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \\
&\quad \otimes(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n),
\end{aligned} \tag{A6}
$$

while the terms in the right-hand side become

$$
\begin{aligned}
(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_n) \otimes (e_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) &= \alpha_1\beta_n(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \\
&\quad \otimes(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n), \\
(e_n \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge e_1) &= \alpha_n\beta_1(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n) \\
&\quad \otimes(x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1} \wedge x_n).
\end{aligned} \tag{A7}
$$

The identity is therefore demonstrated if the vectors $x_1, \cdots x_n$ are independent.

If the vectors are dependent one can show that both the left-hand side and the right-hand side vanish identically.

In particular, in Sec. III we refer to the following identities:

$$\det\begin{pmatrix}\mu_4 & \mu_6\\ \mu_6 & \mu_8\end{pmatrix}\det\begin{pmatrix}1 & \mu_2 & \mu_4 & \mu_6\\ \mu_2 & \mu_4 & \mu_6 & \mu_8\\ \mu_4 & \mu_6 & \mu_8 & \mu_{10}\\ \mu_6 & \mu_8 & \mu_{10} & \mu_{12}\end{pmatrix} = \det\begin{pmatrix}\mu_4 & \mu_6 & \mu_8\\ \mu_6 & \mu_8 & \mu_{10}\\ \mu_8 & \mu_{10} & \mu_{12}\end{pmatrix}\det\begin{pmatrix}1 & \mu_2 & \mu_4\\ \mu_2 & \mu_4 & \mu_6\\ \mu_4 & \mu_6 & \mu_8\end{pmatrix} - \left[\det\begin{pmatrix}\mu_2 & \mu_4 & \mu_6\\ \mu_4 & \mu_6 & \mu_8\\ \mu_6 & \mu_8 & \mu_{10}\end{pmatrix}\right]^2, \tag{A8}$$

$$\det\begin{pmatrix}1 & \mu_4\\ \mu_4 & \mu_8\end{pmatrix}\det\begin{pmatrix}\mu_4 & \mu_2 & \mu_6 & \mu_8\\ \mu_2 & 1 & \mu_4 & \mu_6\\ \mu_6 & \mu_4 & \mu_8 & \mu_{10}\\ \mu_8 & \mu_6 & \mu_{10} & \mu_{12}\end{pmatrix} = \det\begin{pmatrix}\mu_4 & \mu_2 & \mu_6\\ \mu_2 & 1 & \mu_4\\ \mu_6 & \mu_4 & \mu_8\end{pmatrix}\det\begin{pmatrix}1 & \mu_4 & \mu_6\\ \mu_4 & \mu_8 & \mu_{10}\\ \mu_6 & \mu_{10} & \mu_{12}\end{pmatrix} - \left[\det\begin{pmatrix}\mu_2 & 1 & \mu_4\\ \mu_6 & \mu_4 & \mu_8\\ \mu_8 & \mu_6 & \mu_{10}\end{pmatrix}\right]^2, \tag{A9}$$

$$\det\begin{pmatrix}\mu_4 & \mu_2\\ \mu_2 & 1\end{pmatrix}\det\begin{pmatrix}\mu_8 & \mu_6 & \mu_4 & \mu_{10}\\ \mu_6 & \mu_4 & \mu_2 & \mu_8\\ \mu_4 & \mu_2 & 1 & \mu_6\\ \mu_{10} & \mu_8 & \mu_6 & \mu_{12}\end{pmatrix} = \det\begin{pmatrix}\mu_8 & \mu_6 & \mu_4\\ \mu_6 & \mu_4 & \mu_2\\ \mu_4 & \mu_2 & 1\end{pmatrix}\det\begin{pmatrix}\mu_4 & \mu_2 & \mu_8\\ \mu_2 & 1 & \mu_6\\ \mu_8 & \mu_6 & \mu_{12}\end{pmatrix} - \left[\det\begin{pmatrix}\mu_6 & \mu_4 & \mu_2\\ \mu_4 & \mu_{12} & 1\\ \mu_{10} & \mu_8 & \mu_6\end{pmatrix}\right]^2. \tag{A10}$$

APPENDIX B: FRAME DEPENDENCE OF THE SUBLEADING MOMENTS

Let us begin from the general expressions (18) and (20) in the initial hadron rest frame, $\mathbf{v} = 0$, without neglecting powers $\langle 0|z^{2p}|0\rangle(w-1)^p$ $(p < m)$. From the relation

$$(\mathbf{v}-\mathbf{v}')^{2m} = (w^2-1)^m = \sum_{i=0}^{m} 2^{m-i}\binom{m}{i}(w-1)^{m+i}, \tag{B1}$$

one obtains the following relation between derivatives and moments:

$$\xi_{\rm NR}^{(n)}(1) = \sum_{m=n/2}^{n} (-1)^m 2^{2m-n}\frac{n!m!}{(2m)!(n-m)!(2m-n)!} \times (m_q)^{2m}\langle 0|z^{2m}|0\rangle. \tag{B2}$$

conversely, to obtain the moments in terms of the derivatives, it is enough to replace the variable w by

$$y = w^2 - 1. \tag{B3}$$

Then

$$\xi_{\rm NR}(w) = \xi_{\rm NR}[(1+y)^{1/2}] \tag{B4}$$

can be expanded in powers of $y = w^2 - 1$:

$$\xi_{\rm NR}[(1+y)^{1/2}] = \sum_{n=0}^{\infty}\frac{1}{n!}\xi_{\rm NR}^{(n)}(1)\sum_{m=n}^{\infty} c_{n,m}y^m = \sum_{m=0}^{\infty}(-1)^m\frac{1}{(2m)!}(m_q)^{2m}\langle 0|z^{2m}|0\rangle y^m. \tag{B5}$$

From this expression we can read the expression of the moments in terms of the derivatives

$$\langle 0|z^{2m}|0\rangle = (-1)^m(2m)!(m_q)^{-2m}\sum_{n=0}^{m}\frac{1}{n!}c_{n,m}\xi_{\rm NR}^{(n)}(1). \tag{B6}$$

The coefficients $c_{n,m}$ are defined by

$$[(1+y)^{1/2}-1]^n = \sum_{m=n}^{\infty} c_{n,m}y^m \tag{B7}$$

and therefore given by

$$c_{n,m} = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{i/2}{m}. \tag{B8}$$

This sum can be calculated and gives:

$$c_{n,m} = (-1)^{m-n}2^{-2m+n}\left[\binom{2m-n-1}{m-n} - \binom{2m-n-1}{m-n-1}\right]. \tag{B9}$$

Explicitly, one has

$$c_{0,m} = \delta_{m,0} \qquad c_{n,m} = (-1)^{m-n} 2^{-2m+n} n \frac{(2m-n-1)!}{m!(m-n)!}, \tag{B10}$$

where the second relation holds except for $n = m = 0$, since $c_{0,0} = 1$.

From relations (B6) and (B10) we obtain the final relation giving the moments in terms of the derivatives

$$\langle 0|z^{2m}|0\rangle = (m_q)^{-2m}\left[\delta_{m,0}\xi_{\rm NR}(1) + \sum_{n=1}^{m}(-1)^n 2^{-2m+n}\frac{(2m)!(2m-n-1)!}{m!(m-n)!(n-1)!}\xi_{\rm NR}^{(n)}(1)\right]. \tag{B11}$$

The relations (B2) and (B11) are the main results of this section.

Explicitly, one obtains, for the lower derivatives and moments:

$$\begin{aligned}
\xi_{\rm NR}(1) &= \langle 0|1|0\rangle,\\
\xi'_{\rm NR}(1) &= -m_q^2\langle 0|z^2|0\rangle,\\
\xi''_{\rm NR}(1) &= \tfrac{1}{3}m_q^4\langle 0|z^4|0\rangle - m_q^2\langle 0|z^2|0\rangle,\\
\xi^{(3)}_{\rm NR}(1) &= -\tfrac{1}{15}m_q^6\langle 0|z^6|0\rangle + m_q^4\langle 0|z^4|0\rangle,\\
\xi^{(4)}_{\rm NR}(1) &= \tfrac{1}{105}m_q^8\langle 0|z^8|0\rangle - \tfrac{2}{5}m_q^6\langle 0|z^6|0\rangle + m_q^4\langle 0|z^4|0\rangle,\\
\xi^{(5)}_{\rm NR}(1) &= -\tfrac{1}{945}m_q^{10}\langle 0|z^{10}|0\rangle + \tfrac{2}{21}m_q^8\langle 0|z^8|0\rangle - m_q^6\langle 0|z^6|0\rangle.
\end{aligned} \tag{B12}$$

$$\begin{aligned}
\langle 0|1|0\rangle &= \xi_{\rm NR}(1),\\
\langle 0|z^2|0\rangle &= -m_q^{-2}\xi'_{\rm NR}(1),\\
\langle 0|z^4|0\rangle &= 3m_q^{-4}[\xi''_{\rm NR}(1) - \xi'_{\rm NR}(1)],\\
\langle 0|z^6|0\rangle &= 15m_q^{-6}[-\xi^{(3)}_{\rm NR}(1) + 3\xi''_{\rm NR}(1) - 3\xi'_{\rm NR}(1)],\\
\langle 0|z^8|0\rangle &= 105m_q^{-8}[\xi^{(4)}_{\rm NR}(1) - 6\xi^{(3)}_{\rm NR}(1) + 15\xi''_{\rm NR}(1) - 15\xi'_{\rm NR}(1)],\\
\langle 0|z^{10}|0\rangle &= 945m_q^{-10}[-\xi^{(5)}_{\rm NR}(1) + 106\xi^{(4)}_{\rm NR}(1) - 45\xi^{(3)}_{\rm NR}(1) + 105\xi''_{\rm NR}(1) - 105\xi'_{\rm NR}(1)].
\end{aligned} \tag{B13}$$

We observe that in expressions (B2) and (B11) there is a leading term in the nonrelativistic expansion and in (B12) and (B13), the first term in the expansion is the leading term.

The purpose of the present detailed calculation is to point out that actually the subleading terms are *frame dependent* and that therefore, one can only get information on the *leading term* in the nonrelativistic expansion, the unique term that appears in the equal-velocity frame used in Sec. II.

Moreover, there is continuity between the rest frame and the equal-velocity frame, since w can be expressed in terms of $(\mathbf{v} - \mathbf{v}')^2$ in all collinear frames

$$\alpha\mathbf{v} + \beta\mathbf{v}' = 0, \qquad \alpha + \beta = 1, \tag{B14}$$

one gets indeed

$$w = \sqrt{1 + (1-\alpha)^2(\mathbf{v} - \mathbf{v}')^2}\sqrt{1 + \alpha^2(\mathbf{v} - \mathbf{v}')^2} + \alpha(1-\alpha)(\mathbf{v} - \mathbf{v}')^2, \tag{B15}$$

with $\alpha = 1$ in the rest frame and $\alpha = \frac{1}{2}$ in the equal-velocity frame. The first order in $(\mathbf{v} - \mathbf{v}')^2$ yields, independently of α,

$$w \cong 1 + \tfrac{1}{2}(\mathbf{v} - \mathbf{v}')^2, \tag{B16}$$

giving the leading order relation between derivatives of $\xi_{\rm NR}(w)$ and moments (21).

APPENDIX C: OPTIMALITY OF THE CONSTRAINTS

We have seen that the nonrelativistic Isgur-Wise function

$$\xi_{\rm NR}(w) = \langle\psi_0|e^{-im_q(\mathbf{v}'-\mathbf{v})\cdot\mathbf{r}}|\psi_0\rangle \qquad [w = \tfrac{1}{2}(\mathbf{v}' - \mathbf{v})^2 + 1], \tag{C1}$$

has its derivatives at $w = 1$ related to the moments

$$\mu_{2n} = \langle\psi_0|r^{2n}|\psi_0\rangle, \tag{C2}$$

by

$$\xi^{(n)}_{\rm NR}(1) = (-1)^n\frac{2^n n!}{(2n+1)!}m_q^{2n}\mu_{2n}, \tag{C3}$$

and that these moments satisfy the following tower of constraints:

$$\det(\mu_{2i+2j})_{0\le i,j\le n} > 0, \tag{C4}$$

$$\det(\mu_{2i+2j+2})_{0\leq i,j\leq n} > 0, \tag{C5}$$

which consists, for each $n \geq 0$, of a lower bound on μ_{2n} depending on the moments μ_{2k} for $0 \leq k \leq n-1$.

In this appendix, we show that this cannot in general (for arbitrary wave function) be improved. Namely that, given μ_{2k} for $0 \leq k \leq n-1$ satisfying (C4) and (C5) , the moment μ_{2n} can have *any* value larger than this lower bound.

To that goal, we forget (C2) by now, and consider *arbitrary* numbers μ_{2n} $(n = 0, 1, 2, \cdots)$ satisfying the constraints (C4) and (C5). In this appendix we prove that, for *any* $N \geq 0$, there exists a wave function $\psi_0^{(N)}$ such that one has

$$\mu_{2n} = \langle \psi_0^{(N)} | r^{2n} | \psi_0^{(N)} \rangle \qquad \text{for all } 0 \leq k \leq N. \tag{C6}$$

We shall not be able here to know if there is a wave function ψ_0 satisfying (C2) for all n, but our more limited result (C6) is enough to prove the point.

To simplify notations, we introduce

$$x = r^2 \tag{C7}$$

as a variable taking positive values. Let us proceed with our demonstration underlying the following points.

(i) Introduce, in the vector space of polynomials in x, the linear form defined by the values μ_{2k} on the monomials x^k (which constitutes an algebraic basis of this vector space). It is given by

$$\langle P \rangle = \sum_{k=0}^{n} a_k \mu_{2k} \qquad \text{for} \quad P(x) = \sum_{k=0}^{n} a_k x^k. \tag{C8}$$

As a preliminary and crucial step, we have:

(ii) This linear form $\langle P \rangle$ is strictly positive. Namely, one has

$$P(x) \geq 0 \quad \text{for all x} \geq 0, \quad \text{and} \quad \text{P} \neq 0 \Rightarrow \langle \text{P} \rangle > 0. \tag{C9}$$

To prove this, observe first that (C4) and (C5) imply that $(\mu_{2i+2j})_{0\leq i,j\leq n}$ and $(\mu_{2i+2j+2})_{0\leq i,j\leq n}$ are positive definite matrices, or explicitly that one has

$$\sum_{i,j=0}^{n} a_i a_j^* \mu_{2i+2j} > 0, \qquad \sum_{i,j=0}^{n} a_i a_j^* \mu_{2i+2j+2} > 0, \tag{C10}$$

for any coefficients $a_0, \cdots, a_n$ not all vanishing, and using the definition (C8) of $\langle P \rangle$, these properties (C10) translate into:

$$\langle |Q|^2 \rangle > 0, \qquad \langle x|Q|^2 \rangle > 0, \tag{C11}$$

for any nonvanishing polynomial Q.

Then, any P satisfying $P(x) \geq 0$ for all $x \geq 0$ is a linear combination with positive coefficients of polynomials of the form $|Q|^2$ or $x|Q|^2$. Indeed, considering the roots of P, we have

$$P = \prod_i (x + c_i) \prod_j (x - c'_j)^2 \prod_k |x - z_k|^2, \tag{C12}$$

with $c_i \geq 0$, $c'_j > 0$, Im $z_k \neq 0$, since complex roots z_k occur in conjugate pairs, strictly positive roots c'_j occur in even multiplicity (else a change of sign at $x = c'_j$), and negative roots $-c_i$ are arbitrary.

(iii) Next introduce a scalar product in the vector space of polynomials by

$$\langle P | P' \rangle = \langle P^* P' \rangle. \tag{C13}$$

The scalar product properties are easily verified. Notably, the important fact that $\langle P|P \rangle = 0$ implies $P = 0$ results from (C4).

We may then consider the orthogonal polynomials $p_0, p_1, p_2, \cdots$ with respect to this scalar product. The theory of orthogonal polynomials is classical matter [11,12]. They are usually considered with respect to a scalar product defined by a weighted integral, but their properties extends easily to the more general case needed here, where we do not know *a priori* if the scalar product (C13) can be given by an integral.

The polynomial p_n has degree n, and we have:

$$\langle p_n | p_{n'} \rangle = d_n^2 \delta_{n,n'}. \tag{C14}$$

It will be convenient for us to fix p_n by taking the coefficient of x^n to be 1. These polynomials can be computed recursively by the orthogonalization Schmidt process:

$$p_n = x^n - \sum_{k=0}^{n-1} \frac{\langle p_k | x^n \rangle}{\langle p_k | p_k \rangle} p_k, \tag{C15}$$

where the automatic fact that $\langle p_k | p_k \rangle \neq 0$ is essential. Also, since any polynomial of degree $\leq n$ is a linear combination of $p_0, \cdots, p_n$, we have the property:

$$\langle P | p_n \rangle = 0 \qquad \text{for any } P \text{ of degree } < \text{n}, \tag{C16}$$

which is of constant use in the following. Taking $P = 1, x, \cdots, x^{n-1}$, (C16) gives a system of n linear equations for the $n + 1$ coefficients of p_n, which, according to (C4), can be solved uniquely up to a constant, and then Cramer's formulae give an explicit expression for p_n.

We are actually interested by the zeros of p_n.

(iv) All the roots of p_n are simple and strictly positive.

In fact, let $x_1, \cdots, x_m$ be the strictly positive roots of p_n of *odd* multiplicity. We have only to show that $m = n$. If $m < n$, according to (C16), we have

$$\langle (x - x_1) \cdots (x - x_m) p_n \rangle = \langle (x - x_1) \cdots (x - x_m) | p_n \rangle = 0. \tag{C17}$$

However, the polynomial $(x - x_1) \cdots (x - x_m) p_n$ has a constant sign for $x \geq 0$, and does not vanish identically. Therefore, according to (C9) and (C17) and hence $m < n$ cannot be.

(v) We may now write explicit formulae for μ_{2k} with $0 \leq k \leq 2n-1$:

$$\mu_{2k} = \sum_{i=1}^{n} \lambda_i x_i^k \qquad (0 \leq k \leq 2n-1), \tag{C18}$$

where $x_1, \cdots, x_n$ are the roots of p_n, and the coefficients λ_i are given by:

$$\lambda_i = \frac{1}{p_n'(x_i)} \left\langle \frac{p_n(x)}{x - x_i} \right\rangle. \tag{C19}$$

To prove (C18), notice that it amounts to:

$$\langle P \rangle = \sum_{i=1}^{n} \lambda_i P(x_i) \qquad (\text{any } P \text{ of degree } \leq 2n-1). \tag{C20}$$

Performing the Euclidean division of P by p_n, we have

$$P = Qp_n + R, \tag{C21}$$

with degree $Q < n$ and degree $R < n$. We may verify (C20) separately for Qp_n and for R.

For $P = Qp_n$, the left-hand side of (C20) vanishes by (C16), and the right-hand side vanishes because the x_i are the roots of p_n.

For $P = R$, we use the identity

$$R(x) = \sum_{i=0}^{n} \frac{1}{p_n'(x_i)} \frac{p_n(x)}{x - x_i} R(x_i), \tag{C22}$$

which stems from the fact that both sides are polynomials of degree $<n$, that are equal at n points $x = x_i$. Then (C20) is satisfied due to the choice (C19) of the coefficients λ_i.

(vi) Define

$$\rho_n(x) = \sum_{i=1}^{n} \lambda_i \delta(x - x_i). \tag{C23}$$

Then (C18) writes

$$\mu_{2k} = \int_0^\infty \rho_n(x) x^k dx \qquad (0 \leq k \leq 2n-1), \tag{C24}$$

to be compared with formula (C6) to be proved, which writes

$$\mu_{2k} = \int_0^\infty \rho_N(x) x^k dx \qquad (0 \leq k \leq N), \tag{C25}$$

with

$$\rho_N(x) = 2\pi\sqrt{x}|\psi_0^{(N)}(\sqrt{x})|^2. \tag{C26}$$

We have still the problem that $\rho_n(x)$ is not a function.

The idea to solve this problem is to vary μ_{4n-2}, keeping fixed μ_{2k} for $0 \leq k \leq 2n-2$. The polynomial p_n then depends on μ_{4n-2} as a parameter, and as well its zeros $x_i(\mu_{4n-2})$ and the coefficients $\lambda_i(\mu_{4n-2})$ defined by (C19). Then formula (C24) is lost for μ_{4n-2}, but remains valid for $0 \leq k \leq 2n-2$, and in fact gives a whole family of formulae

$$\mu_{2k} = \int_0^\infty \rho_n(\mu_{4n-2}, x) x^k dx \qquad (0 \leq k \leq 2n-2), \tag{C27}$$

with a weight distribution

$$\rho_n(\mu_{4n-2}, x) = \sum_{i=1}^{n} \lambda_i(\mu_{4n-2}) \delta[x - x_i(\mu_{4n-2})], \tag{C28}$$

depending on μ_{4n-2}. We may then take the mean value of (C27) over any interval $[\mu_{4n-2}^{(1)}, \mu_{4n-2}^{(2)}]$ in which the constraints are satisfied, obtaining

$$\mu_{2k} = \int_0^\infty \rho_n(x) x^k dx \qquad (0 \leq k \leq 2n-2), \tag{C29}$$

with

$$\rho_n(x) = \frac{1}{\mu_{4n-2}^{(2)} - \mu_{4n-2}^{(1)}} \int_{\mu_{4n-2}^{(1)}}^{\mu_{4n-2}^{(2)}} \rho_n(\mu_{4n-2}, x) d\mu_{4n-2}. \tag{C30}$$

Now, $\rho_n(x)$ defined by (C30) has a good chance to be a genuine function, because integrating a δ distribution over a parameter usually gives a function.

However, there is an obvious case in which this does not hold, namely, when the point where the δ distribution is concentrated does not depend on the parameter. So we still have to show that *each* zero of p_n does vary with μ_{4n-2}. Let us consider the orthogonal polynomials $\tilde{p}_0, \tilde{p}_1, \cdots$ with respect to the new scalar product $\widetilde{\langle P|P' \rangle}$ associated to new values $\tilde{\mu}_0, \tilde{\mu}_2, \cdots$ of the moments, with $\tilde{\mu}_{2k} = \mu_{2k}$ for $0 \leq k \leq 2n-2$, and $\tilde{\mu}_{4n-2} \neq \mu_{4n-2}$. Note that the new scalar product of two polynomials is the same as the original one when the sum of the degrees is $\leq 2n-2$. It follows that $\tilde{p}_k = p_k$ for $0 \leq k \leq n-1$, and also that

$$\langle \tilde{p}_n | p_k \rangle = \langle \tilde{p}_n | \tilde{p}_k \rangle = 0 \qquad \text{for} \quad 0 \leq \mathrm{k} \leq \mathrm{n} - 2. \tag{C31}$$

Therefore, the expansion of $\tilde{p}_n$ over the p_k writes:

$$\tilde{p}_n = p_n + c p_{n-1}. \tag{C32}$$

And one has $c \neq 0$. Indeed, since $\langle \tilde{p}_n | \tilde{\tilde{p}}_{n-1} \rangle = 0$, one has

$$\begin{aligned} c &= \frac{1}{\langle p_{n-1} | p_{n-1} \rangle} (\langle \tilde{p}_n | p_{n-1} \rangle - \widetilde{\langle \tilde{p}_n | \tilde{p}_n - 1 \rangle}) \\ &= \frac{1}{\langle p_{n-1} | p_{n-1} \rangle} (\langle x^n | x^{n-1} \rangle - \widetilde{\langle x^n | x^{n-1} \rangle}), \end{aligned} \tag{C33}$$

so that

$$c = \frac{1}{\langle p_{n-1} | p_{n-1} \rangle} (\mu_{4n-2} - \tilde{\mu}_{4n-2}). \tag{C34}$$

The fact that a zero of p_n cannot be a zero of $\tilde{p}_n$ now follows from (C32) and the fact that a zero of p_n cannot be a zero of p_{n-1}.

This last point is a well known property of orthogonal polynomials, which can be proved directly as follows. Assume that a zero x_i of p_n is also a zero of p_{n-1}. Then we have

$$\left\langle p_{n-1} \left| \frac{p_n}{x - x_i} \right.\right\rangle = \left\langle \frac{p_{n-1}}{x - x_i} \right| \left. p_n \right\rangle = 0 \qquad \text{(C35)}$$

by (C16). On the other hand, writing $\frac{p_n}{x-x_i} = ax^{n-1} + \cdots$, where $a \neq 0$, we have again by (C16):

$$\left\langle p_{n-1} \left| \frac{p_n}{x - x_i} \right.\right\rangle = a\langle p_{n-1}|x^{n-1}\rangle = a\langle p_{n-1}|p_{n-1}\rangle, \qquad \text{(C36)}$$

which cannot vanish, contradicting (C35).

We are now in a position to complete the proof of (C6). Indeed, using the implicit functions theorem, one can infer from (C32) and (C34) that, for a small enough interval $[\mu^{(1)}_{4n-2}, \mu^{(2)}_{4n-2}]$, each function $x_i(\mu_{4n-2})$ is a diffeomorphism of this interval to an interval $[x_i^{(1)}, x_i^{(2)}]$ in x. Then introducing the reciprocal function $x_i \rightarrow \mu_i(x_i)$ of $\mu_{4n-2} \rightarrow x_i(\mu_{4n-2})$, the integral of a δ function is computed by changing the variable of integration μ_{4n-2} to $x_i = x_i(\mu_{4n-2})$:

$$\begin{aligned}\int_{\mu^{(1)}_{4n-2}}^{\mu^{(2)}_{4n-2}} &\lambda_i(\mu_{4n-2})\delta[x - x_i(\mu_{4n-2})]d\mu_{4n-2} \\ &= \int_{x_i^{(1)}}^{x_i^{(2)}} \lambda_i[\mu_i(x_i)]\delta(x - x_i)|\mu_i'(x_i)|dx_i \\ &= \chi_{[x_i^{(1)},x_i^{(2)}]}(x)\lambda_i[\mu_i(x)]|\mu_i'(x)|, \end{aligned} \qquad \text{(C37)}$$

where χ_I is the characteristic function of an interval I, namely $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ for $x \notin I$. Then (C30) gives

$$\rho_n(x) = \frac{1}{\mu^{(2)}_{4n-2} - \mu^{(1)}_{4n-2}} \sum_{i=1}^{n} \chi_{[x_i^{(1)},x_i^{(2)}]}(x)\lambda_i(\mu_i(x)|\mu_i'(x)|, \qquad \text{(C38)}$$

and this is a genuine positive function, which can therefore be written as the square of a wave function.

APPENDIX D: EXPLICIT LOWER LIMITS FOR THE EVEN DERIVATIVES

In this appendix we generalize to all even derivatives $\xi^{(2n)}(1)$ the proof of the bound (50) that we have given in Ref. [3].

From expression (21):

$$\xi^{(m)}_{\rm NR}(1) = (-1)^m 2^m \frac{m!}{(2m)!}(m_q)^{2m}\langle 0|z^{2m}|0\rangle, \qquad \text{(D1)}$$

using rotational invariance we obtain

$$\xi^{(m)}_{\rm NR}(1) = (-1)^m 2m \frac{m!}{(2m)!}(m_q)^{2m}\frac{1}{2m+1}\langle 0|r^{2m}|0\rangle, \qquad \text{(D2)}$$

and from

$$\langle 0|r^{2m}|0\rangle = |\langle 0|r^m|0\rangle|^2 + \sum_{n\neq 0}|\langle n|r^m|0\rangle|^2. \qquad \text{(D3)}$$

Using again spherical symmetry

$$\begin{aligned}\langle 0|r^{2m}|0\rangle = (m+1)^2&|\langle 0|z^m|0\rangle|^2 \\ &+(m+1)^2\sum_{n=0,\rm rad}|\langle n|z^m|0\rangle|^2,\end{aligned} \qquad \text{(D4)}$$

one obtains

$$\begin{aligned}\xi^{(m)}_{\rm NR}(1) = (-1)^m 2m &\frac{m!}{(2m)!}(m_q)^{2m}\frac{(m+1)^2}{2m+1} \\ &\times\left[|\langle 0|z^{2m}|0\rangle|^2 + \sum_{n\neq 0,\rm rad}|\langle n|z^m|0\rangle|^2\right],\end{aligned} \qquad \text{(D5)}$$

and therefore

$$(-1)^m\xi^{(m)}_{\rm NR}(1) > \frac{m!}{(2m)!}2m(m_q)^{2m}\frac{(m+1)^2}{2m+1}|\langle 0|z^m|0\rangle|^2. \qquad \text{(D6)}$$

This expression demonstrates that $\xi_{\rm NR}(w)$ is an alternate series in powers of $(w-1)$.

Assuming m to be even, $m = 2n$, one gets

$$\xi^{(2n)}_{\rm NR}(1) > \frac{(2n)!}{(4n)!}2^{2n}(m_q)^{4n}\frac{(2n+1)^2}{4n+1}|\langle 0|z^{2n}|0\rangle|^2. \qquad \text{(D7)}$$

The moment $\langle 0|z^{2n}|0\rangle$ can be expressed in terms of $\xi^{(n)}_{\rm NR}(1)$ though (D1), giving finally

$$\xi^{(2n)}_{\rm NR}(1) > \frac{[(2n)!]^3}{[n!]^2(4n)!}\frac{(2n+1)^2}{4n+1}[\xi^{(n)}_{\rm NR}(1)]^2 \qquad (n \geq 0). \qquad \text{(D8)}$$

We obtain, for the lower values of n,

$$\begin{aligned} n = 1 \qquad &\xi^{(2)}_{\rm NR}(1) \geq \tfrac{3}{5}\,[\xi^{(1)}_{\rm NR}(1)]^2, \\ n = 2 \qquad &\xi^{(4)}_{NR}(1) \geq \tfrac{5}{21}\,[\xi^{(2)}_{NR}(1)]^2. \end{aligned} \qquad \text{(D9)}$$

The formula (D8) generalizes the result (50) to all even derivatives.

For the odd derivatives one gets, with the present method, from (D6), the weaker result

$$-\xi^{(2n+1)}_{\rm NR}(1) > 0 \qquad (n \geq 0). \qquad \text{(D10)}$$

We see that we had obtained in Secs. III and IV a much stronger result for $\xi^{(4)}_{\rm NR}(1)$ than (D9), and non trivial results for $-\xi^{(3)}_{\rm NR}(1)$ and $-\xi^{(5)}_{\rm NR}(1)$. However, we have obtained here an explicit lower bound for $\xi^{(2n)}_{\rm NR}(1)$ (D8).

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