Critique of the angular momentum sum rules and a new angular momentum sum rule

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We present a study of the tensorial structure of the hadronic matrix elements of the angular momentum operators $\boldsymbol{J}$. Well known results in the literature are shown to be incorrect, and we have taken pains to derive the correct expressions in three different ways, two involving explicit physical wave packets and the third, totally independent, based upon the rotational properties of the state vectors. Surprisingly it turns out that the results are very sensitive to the type of relativistic spin state used to describe the motion of the particle, i.e., whether a canonical (i.e., boost) state or a helicity state is utilized. We present results for the matrix elements of the angular momentum operators, valid in an arbitrary Lorentz frame, for both helicity states and canonical states. These results are relevant for the construction of angular momentum sum rules, relating the angular momentum of a nucleon to the spin and orbital angular momentum of its constituents. It turns out that it is necessary to distinguish carefully whether the motion of the partons is characterized via canonical or helicity spin states. Fortunately, for the simple parton model interpretation, when the proton moves along OZ, our results for the sum rule based upon the matrix elements of J_z agree with the often used sum rule found in the literature. But for the components J_x, J_y the results are different and lead to a new and very intuitive sum rule for transverse polarization.

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I. INTRODUCTION AND SUMMARY OF RESULTS

Sum rules, relating the total angular momentum of a nucleon to the spin and orbital angular momentum carried by its constituents, are interesting and important in understanding the internal structure of the nucleon. Indeed it is arguable that the main stimulus for the tremendous present day experimental activity in the field of spin-dependent structure functions was the surprising result of the European Muon Collaboration's polarized deep inelastic scattering experiment in 1988 [1], which, via such sum rules, led to what was called a "spin crisis in the parton model" [2], namely, the discovery that the spins of its quarks provide a very small contribution to the angular momentum of the proton. A key element in deriving such sum rules is a precise knowledge of the tensorial structure of the expectation values of the angular momentum operators J_i in a state $|p, \sigma\rangle$ of the nucleon, labeled by its momentum p, and with some kind of specification of its spin state, denoted here noncommitally by σ.

In a much cited paper [3], Jaffe and Manohar stressed the subtleties involved in deriving general angular momentum sum rules. As they point out, too naive an approach leads immediately to highly ambiguous divergent integrals, and a careful limiting procedure has to be introduced in order to obtain physically meaningful results. In this it is essential to work with nondiagonal matrix elements $\langle p', \sigma|\boldsymbol{J}|p, \sigma\rangle$ and, as discussed below, this can have some unexpected consequences. Jaffe and Manohar comment that to justify rigorously the steps in such a procedure requires the use of normalizable wave packets, though they do *not* do this explicitly in their paper.

In a later paper [4], Shore and White utilized the approach of Ref. [3], including an explicit treatment with wave packets, to derive some far reaching conclusions about the role of the axial anomaly in these sum rules.

We shall argue that despite all the care and attention to subtleties, there are flaws in the analysis in Ref. [3] and the results presented there are not entirely general. Indeed there are cases where the results of Ref. [3] are incorrect. This, in turn, throws doubt upon some of the conclusions reached in Ref. [4], which we will examine.

The bulk of our analysis is based on a straightforward wave-packet approach. However, as we explain, this is rather subtle for particles with nonzero spin. The key points underlying our results are

(1) Our wave packets are strictly physical, i.e., a superposition of physical plane-wave states. This requirement turns out to be incompatible with some of the steps in Ref. [3].
(2) We give a careful treatment of the Lorentz covariance properties of the matrix elements involved in the subsidiary steps of the analysis. This leads to tensorial structures which differ in some cases from those in Ref. [3].
(3) Because our results differ from Ref. [3] we have looked for and found a totally independent way to check our results. This does not use wave packets and is based upon the transformation properties of

momentum states under rotations. This very direct approach holds for arbitrary spin, whereas in the wave-packet treatment we are only able to deal with spin-$\frac{1}{2}$ particles. It also brings to light some peculiar and unintuitive properties of helicity states, and this must be taken into account when deriving spin sum rules. This is important since we have to deal with gluons in our sum rules.

Our results for the matrix elements of $\boldsymbol{J}$ are as follows:

For a massive particle of spin-$\frac{1}{2}$ with four-momentum p in a canonical spin state $|p, \boldsymbol{s}\rangle$ (i.e., in a "boost" state of the kind generally used in textbooks on field theory, e.g., in Bjorken and Drell [5], or in Peskin and Schroeder [6]), where $\boldsymbol{s}/2$ is the spin eigenvector in the rest frame ($\boldsymbol{s}^2 = 1$), we show that, for the forward matrix elements,

$$\langle p', \boldsymbol{s}|J_i|p, \boldsymbol{s}\rangle = 2p^0(2\pi)^3[\tfrac{1}{2}s_i + i(\boldsymbol{p}\times\boldsymbol{\nabla}_{\boldsymbol{p}})_i]\delta^3(\boldsymbol{p}' - \boldsymbol{p}). \tag{1.1}$$

The states $|p, \boldsymbol{s}\rangle$ are normalized conventionally to

$$\langle p', \boldsymbol{s}|p, \boldsymbol{s}\rangle = 2p^0(2\pi)^3\delta^3(\boldsymbol{p}' - \boldsymbol{p}) \tag{1.2}$$

and we note that in the rest frame

$$|0, \boldsymbol{s}\rangle = \mathcal{D}^{1/2}_{m1/2}[R(\boldsymbol{s})]|0, m\rangle, \tag{1.3}$$

where $|0, m\rangle$ has spin projection m along the z direction in the rest frame and $R(\boldsymbol{s})$ rotates a unit vector in the z direction into $\boldsymbol{s}$ by first a rotation about y and then a rotation about z.

For the purpose of deriving sum rules our result for the matrix elements *nondiagonal* in the spin label is actually more useful, namely, for a spin-$\frac{1}{2}$ particle

$$\langle p', m'|J_i|p, m\rangle = 2p_0(2\pi)^3\left[\frac{1}{2}\sigma_i + i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\right]_{m'm}\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}). \tag{1.4}$$

The generalization of these results for arbitrary spin is given in Eq. (6.13).

Helicity states are more suitable for massless particles such as gluons. Using the Jacob-Wick conventions for helicity states [7] we find a surprisingly different result, namely,

$$\langle p', \lambda'|J_i|p, \lambda\rangle = (2\pi)^3 2p_0[\lambda\eta_i(\boldsymbol{p}) + i(\boldsymbol{p}\times\boldsymbol{\nabla}_{\boldsymbol{p}})_i] \times \delta^3(\boldsymbol{p}' - \boldsymbol{p})\delta_{\lambda\lambda'}, \tag{1.5}$$

where

$$\eta_x = \cos(\phi)\tan(\theta/2), \quad \eta_y = \sin(\phi)\tan(\theta/2), \quad \eta_z = 1, \tag{1.6}$$

and (θ, ϕ) are the polar angles of $\boldsymbol{p}$.

The first term in Eq. (1.1) differs from the results of Jaffe and Manohar [3]. If we rewrite their expression [see Eq. (4.23)] in terms of the independent vectors $\boldsymbol{p}$ and $\boldsymbol{s}$, we find, for the *expectation* value

$$\langle J_i\rangle_{\rm JM} = \frac{1}{4Mp^0}\left\{(3p_0^2 - M^2)s_i - \frac{3p_0 + M}{p_0 + M}(\boldsymbol{p}\cdot\boldsymbol{s})p_i\right\} \tag{1.7}$$

to be compared to

$$\langle J_i\rangle = \tfrac{1}{2}s_i \tag{1.8}$$

arising from the first term in Eq. (1.1). In general these are different. However, one may easily check that if $\boldsymbol{s} = \hat{\boldsymbol{p}}$ the Jaffe-Manohar value agrees with Eq. (1.8), while if $\boldsymbol{s} \perp \hat{\boldsymbol{p}}$ they are not the same.

The agreement for $\boldsymbol{s} = \hat{\boldsymbol{p}}$ is consistent with the much used and intuitive sum rule

$$\tfrac{1}{2} = \tfrac{1}{2}\Delta\Sigma + \Delta G + \langle L^q\rangle + \langle L^G\rangle. \tag{1.9}$$

In the case that $\boldsymbol{s} \perp \hat{\boldsymbol{p}}$ we find a new sum rule. For a proton with transverse spin vector $\boldsymbol{s}_T$ we find

$$\tfrac{1}{2} = \tfrac{1}{2}\sum_{q,\bar{q}}\int dx\Delta_T q^a(x) + \sum_{q,\bar{q},G}\langle L_{\mathbf{s}_T}\rangle^a, \tag{1.10}$$

where L_{s_T} is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$. The structure functions $\Delta_T q^a(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon. Note that no such parton model sum rule is possible with the Jaffe-Manohar formula because, as $p \to \infty$, Eq. (1.7) for $i = x, y$ diverges.

The result Eq. (1.10) has a very intuitive appearance, very similar to Eq. (1.9).

The organization of our paper is as follows: In Sec. II we explain why the calculation of the angular momentum matrix elements is so problematical. Because of the unexpected sensitivity of the matrix elements of $\boldsymbol{J}$ to the type of spin state used, and because we are forced to use helicity states for gluons, Sec. III presents a resumé of the difference between Jacob-Wick helicity states and canonical (i.e., as in Bjorken-Drell) spin states. Further, given that our results disagree with one of the classic papers in the literature, we have felt it incumbent to summarize, in Sec. IV, the treatments of Jaffe-Manohar and Shore-White, pointing out the incorrect steps in these derivations.

In Sec. V we present a detailed wave-packet derivation of the structure of the matrix elements of $\boldsymbol{J}$ for spin-$\frac{1}{2}$, first for a relativistic Dirac particle, then in a field theoretic treatment. We comment here on the claims made in Shore-White on the role of the axial anomaly in the structure of the matrix elements. Sections IV and V are heavy going, and the reader only interested in a quick and direct derivation of the key results should skip these and read Secs. VI, VI A, VI C, and VII.

In Sec. VI we confirm the results of Sec. V in a completely independent approach, which is valid for *arbitrary* spin, based on the rotational properties of canonical

and helicity spin states. We also prove that our results are in conformity with the demands of Lorentz invariance.

In Sec. VII we derive the most general form of an angular momentum sum rule for a nucleon and show that it reduces to the standard, intuitive, sum rule for J_z when the nucleon is moving along OZ. We also derive a new sum rule for a transversely polarized nucleon.

II. THE ORIGIN OF THE PROBLEM

In the standard approach one relates the matrix elements of the angular momentum operators to those of the energy-momentum tensor.

Let $T^{\mu\nu}(x)$ be the total energy-momentum density which is conserved

$$\partial_\mu T^{\mu\nu}(x) = 0. \tag{2.1}$$

Later we shall distinguish between the conserved *canonical* energy-momentum tensor $T_{\rm C}^{\mu\nu}$, which emerges from Noether's theorem, and which is, generally, not symmetric under $\mu \leftrightarrow \nu$, and the *symmetrized* Belinfante tensor $T^{\mu\nu}$, which for QCD is given by

$$T^{\mu\nu}(x) = \tfrac{1}{2}[T_{\rm C}^{\mu\nu}(x) + T_{\rm C}^{\nu\mu}(x)] \tag{2.2}$$

and which is also conserved. For the moment, however, this distinction is irrelevant.

Being a *local* operator, $T^{\mu\nu}(x)$ transforms under translations as follows:

$$T^{\mu\nu}(x) = e^{ix\cdot P} T^{\mu\nu}(0) e^{-ix\cdot P}, \tag{2.3}$$

where the P^μ are the total momentum operators of the theory.

By contrast the various angular momentum density operators which are of interest, the orbital angular momentum densities

$$M_{\rm orb}^{\mu\nu\lambda}(x) \equiv x^\nu T_{\rm C}^{\mu\lambda}(x) - x^\lambda T_{\rm C}^{\mu\nu}(x) \tag{2.4}$$

or the version constructed using the symmetrized stress-energy tensor

$$M^{\mu\nu\lambda}(x) \equiv x^\nu T^{\mu\lambda}(x) - x^\lambda T^{\mu\nu}(x) \tag{2.5}$$

are not local operators (we shall call them *compound*) and do not transform according to Eq. (2.3).

Note that, strictly speaking, the operators relevant to the angular momentum are the components M^{0ij} where i, j are spatial indices. However, for reasons of simplicity in utilizing the Lorentz invariance of the theory, the authors of Ref. [3] prefer to deal covariantly with the entire tensor $M^{\mu\nu\lambda}$. We shall loosely refer to them also as angular momentum densities.

The total angular momentum density is

$$J^{\mu\nu\lambda}(x) = M_{\rm orb}^{\mu\nu\lambda}(x) + M_{\rm spin}^{\mu\nu\lambda}(x), \tag{2.6}$$

where the structure of $M_{\rm spin}^{\mu\nu\lambda}$ depends on the type of fields involved. From Noether's theorem $J^{\mu\nu\lambda}(x)$ is a set of conserved densities, i.e.,

$$\partial_\mu J^{\mu\nu\lambda}(x) = 0. \tag{2.7}$$

As a consequence of the densities being conserved, it follows that the total momentum operators

$$P^\nu \equiv \int d^3x T^{0\nu}(x) \tag{2.8}$$

and the total angular momentum operators $\boldsymbol{J}$,

$$J_z = J^3 = J^{12}, \quad \text{(cyclical)} \tag{2.9}$$

with

$$J^{ij} \equiv \int d^3x J^{0ij}(x) \tag{2.10}$$

are conserved quantities, independent of time.

The relationship between the $M^{\mu\nu\lambda}(x)$ constructed using the symmetrical energy-momentum density and the $J^{\mu\nu\lambda}(x)$ constructed from the canonical energy-momentum tensor is extremely interesting and will be commented on later in Sec. IV B. One can show (see, e.g., Ref. [4]) that

$$M^{0ij}(x) = J^{0ij}(x) + [\text{E of M terms}] + [\text{divergence terms}]. \tag{2.11}$$

The [E of M terms] vanish if it is permissible to use the equations of motion of the theory. The [divergence terms] are of the form $\partial_\alpha F^{\alpha 0ij}(x)$.

As mentioned we shall be primarily interested in the expectation values of the physical operators, i.e., in their forward matrix elements. If $F^{\alpha 0ij}(x)$ were a *local* operator, it would follow directly that the forward, momentum-space, matrix elements of the divergence terms in Eq. (2.11) vanish. But it is not a local operator. Nonetheless, a careful treatment using wave packets [4] demonstrates that the forward matrix elements do indeed vanish. See Sec. IV below.

Dropping, as is customary, the [E of M terms], we shall thus assume the validity of

$$\langle p,\sigma| \int d^3x M^{0ij}(\boldsymbol{x},0)|p,\sigma\rangle = \langle p,\sigma| \int d^3x J^{0ij}(\boldsymbol{x},0)|p,\sigma\rangle. \tag{2.12}$$

We shall return to this question in Secs. V A and V B.

Of primary interest are the matrix elements of the angular momentum operators J^k or, equivalently, the J^{ij}. Consider the forward matrix element, at $t = 0$,

$$\mathcal{M}^{0ij}(p,s) \equiv \langle p,s| \int d^3x M^{0ij}(\boldsymbol{x},0)|p,s\rangle \tag{2.13}$$

$$= \int d^3x \langle p,s|x^i T^{0j}(x) - x^j T^{0i}(x)|p,s\rangle$$

$$= \int d^3x x^i \langle p,s|e^{iP\cdot x} T^{0j}(0) e^{-iP\cdot x}|p,s\rangle - (i \leftrightarrow j)$$

$$= \int d^3x x^i \langle p,s|T^{0j}|p,s\rangle - (i \leftrightarrow j). \tag{2.14}$$

The integral in Eq. (2.14) is totally ambiguous, being either infinite or, by symmetry, zero.

The essential problem is to obtain a sensible physical expression, in terms of p and $\boldsymbol{s}$, for the above matrix element. The fundamental idea is to work with a nonforward matrix element and then to try to approach the forward limit. This is similar to what is usually done when dealing with non-normalizable plane-wave states and it requires the use of wave packets for a rigorous justification.

It will turn out that the results are sensitive to the type of relativistic spin state employed, so in Sec. III we present a brief resumé of the properties of relativistic spin states. We then proceed to discuss the approaches of Refs. [3,4] in Sec. IV, where we shall comment on the dubious steps in these treatments. The most crucial error in these treatments is the mishandling of the matrix elements of a covariant tensor operator. If $T^{\mu\lambda}$ transforms as a second-rank tensor its *nonforward* matrix elements do *not* transform covariantly. This was the motivation, decades ago, for Stapp to introduce M functions [8]. Namely, the covariance is spoiled, for canonical spin states by the Wigner rotation, and, for helicity states by the analogous Wick helicity rotation [9]. Only by first factoring out the wave functions (in our case Dirac spinors), i.e., by writing

$$\langle p', S'|T^{\mu\lambda}|p, S\rangle = \bar{u}(p', S')\mathcal{T}^{\mu\nu}(p', p)u(p, S), \tag{2.15}$$

does the remaining M function, in this case $\mathcal{T}^{\mu\nu}(p', p)$, transform covariantly. For local operators the transformations of the spinors u and $\bar{u}$ cancel between themselves for forward matrix elements and so the result does have the naively expected tensor expansion. This is not true in general for compound operators, in particular, the angular momentum and boost operators.

III. RELATIVISTIC SPIN STATES

The definition of a spin state for a particle in motion, in a relativistic theory, is nontrivial and is convention dependent. Namely, starting with the states of a particle at rest, which we shall denote by $|0, m\rangle$, where m is the spin projection in the z direction, one defines states $|p, \sigma\rangle$ for a particle with four-momentum p by acting on the rest-frame states with various boosts and rotations, and the choice of these is convention dependent. The states are on-shell so $p^2 = M^2$.

There are three conventions in general use [10]:

(a) Canonical or boost states as used, e.g., in Bjorken and Drell [5] or Peskin and Schroeder [6]

$$|p, m\rangle = B(\boldsymbol{v})|0, m\rangle, \tag{3.1}$$

where $B(\boldsymbol{v})$ is a pure boost along $\boldsymbol{v} = \boldsymbol{p}/p_0$, and $\boldsymbol{p} = (p, \theta, \phi)$ denotes the three-vector part of p^μ;

(b) Jacob-Wick helicity states [7]

$$|p, \lambda\rangle_{\mathrm{JW}} = R_z(\phi)R_y(\theta)R_z(-\phi)B_z(v)|0, m = \lambda\rangle, \tag{3.2}$$

where B_z is a boost along OZ, and the later introduced, somewhat simpler;

(c) Wick helicity states [9]

$$|p, \lambda\rangle = R_z(\phi)R_y(\theta)B_z(v)|0, m = \lambda\rangle. \tag{3.3}$$

From the canonical states of spin-$\frac{1}{2}$ one can construct the states

$$|p, \boldsymbol{s}\rangle = B(\boldsymbol{v})|0, \boldsymbol{s}\rangle = B(\boldsymbol{v})\mathcal{D}^{1/2}_{m1/2}[R(\boldsymbol{s})]|0, m\rangle \tag{3.4}$$

which, in the rest frame, are eigenstates of spin with spin eigenvector along the unit vector $\boldsymbol{s}$. [The rotation $R(\boldsymbol{s})$ was explained after Eq. (1.3).]

The canonical states, with their reference to a rest frame, are clearly not suitable for massless particles like gluons. Helicity states, on the other hand, can be used for both massive and massless particles. However, it turns out that the results for the canonical states are much more intuitive, so we will generally use them for $M \neq 0$.

The reason we are emphasizing this distinction between canonical and helicity states is that the matrix elements of the angular momentum operators between helicity states are quite bizarre. Since, for arbitrary p, helicity states are just linear superpositions of canonical states, one may wonder why this is so. It results from the facts (i) that the coefficients in the linear superposition are p dependent, i.e., depend upon the polar angles of $\boldsymbol{p}$ and (ii) that the matrix elements of the angular momentum operators contain derivatives of δ functions, and these, as usual, must be interpreted in the sense of partial integration, i.e.,

$$f(p, p')\frac{\partial}{\partial p_i}\delta^3(p - p') = -\delta^3(p - p')\frac{\partial}{\partial p_i}f(p, p'). \tag{3.5}$$

The technical details are explained in Sec. VI.

In almost all studies of hard processes, where a mixture of perturbative and nonperturbative QCD occurs, nucleons are taken to be in helicity states moving with high energy along the z axis, and typically one is utilizing

matrix elements of local products of quark or gluon field operators between these states. For these operators there is no problem in dealing with diagonal matrix elements. But when it comes to an angular momentum sum rule for the nucleon, care must be taken to decide whether one is dealing with helicity states $|p_z, \lambda\rangle$, where $p_z = (E, 0, 0, p)$, or with canonical states $|p_z, s_z\rangle$, where $s_z = (0, 0, 2\lambda)$. The point is that even though the initial states are the same,

$$|p_z, \lambda\rangle = |p_z, s_z\rangle \tag{3.6}$$

the singular nature of J_i forces one to deal with non-diagonal matrix elements, i.e., to utilize $\langle p', \sigma|$ where $\boldsymbol{p}'$ is not along the z axis, and for these

$$\langle p', \lambda| \neq \langle p', s_z|. \tag{3.7}$$

In this paper we show that it is possible to give a rigorous derivation of the structure of the expectation values for canonical states

$$\langle p, \boldsymbol{s}|J_i|p, \boldsymbol{s}\rangle \equiv \mathcal{L}_{p'\to p}\langle p', \boldsymbol{s}|J_i|p, \boldsymbol{s}\rangle, \tag{3.8}$$

where $\boldsymbol{s}$ is a unit vector along the rest-frame spin eigenvector, and for helicity states

$$\langle p, \lambda|J_i|p, \lambda\rangle \equiv \mathcal{L}_{p'\to p}\langle p', \lambda|J_i|p, \lambda\rangle. \tag{3.9}$$

In general, for the arbitrary ith component of $\boldsymbol{J}$, for spin-$\frac{1}{2}$

$$\langle p, \lambda|J_i|p, \lambda\rangle \neq \langle p, \boldsymbol{s} = 2\lambda\hat{\boldsymbol{p}}|J_i|p, \boldsymbol{s} = 2\lambda\hat{\boldsymbol{p}}\rangle, \tag{3.10}$$

even though $\boldsymbol{s}$ lies along the direction of $\hat{p}$ in both cases, and even if $\boldsymbol{p}$ is along OZ where Eq. (3.6) holds. Only for the specific component of $\boldsymbol{J}$ along $\hat{\boldsymbol{p}}$ do the matrix elements agree, i.e., for arbitrary $\boldsymbol{p}$,

$$\langle p, \lambda|\boldsymbol{J}\cdot\boldsymbol{p}|p, \lambda\rangle = \langle p, \boldsymbol{s} = 2\lambda\hat{\boldsymbol{p}}|\boldsymbol{J}\cdot\boldsymbol{p}|p, \boldsymbol{s} = 2\lambda\hat{\boldsymbol{p}}\rangle. \tag{3.11}$$

In using the sum rules based on Eq. (3.8) or Eq. (3.9) for arbitrary i to test any model of the nucleon in terms of its constituents, it is essential to construct wave functions appropriate to the type of spin state being used for the nucleon. The equations Eqs. (3.8) and (3.9) contain delta functions and *derivatives* of delta functions and this is the reason for the special care required. Throughout this paper, with the exception of the discussion in Sec. VI and in Sec. VII, we will use canonical spin states. In these latter sections we shall utilize Jacob-Wick helicity states for the massless gluons.

In Sec. IV we summarize the treatments of Jaffe-Manohar and Shore-White. The approaches in Refs. [3,4] are different, so we shall present both in some detail and will comment upon the dubious steps.

IV. THE JAFFE-MANOHAR AND SHORE-WHITE APPROACHES

These authors employ the standard approach of trying to relate the matrix elements of the angular momentum operators to those of the energy-momentum density operator, and utilize the version Eq. (2.5) based on the symmetrized energy-momentum tensor.

A. The Jaffe-Manohar treatment

In order to make efficient use of the Lorentz invariance the authors of Ref. [3] prefer to label their states using the covariant spin four-vector S and to consider the entire tensor $J^{\mu\nu\lambda}$ and to integrate the tensor densities over four-dimensional space-time, i.e., they consider

$$\mathcal{M}^{\mu\nu\lambda}(p, k, p', S) \equiv \int d^4x e^{ik\cdot x}\langle p', S' = S|M^{\mu\nu\lambda}(x)|p, S\rangle \tag{4.1}$$

and eventually take the limit $k^\mu \to 0$. In Ref. [3] the left-hand side is written in the abbreviated form $\mathcal{M}^{\mu\nu\lambda}(p, k, s)$, but we shall use the above notation for clarity. Note that Jaffe and Manohar use the notation s to mean the covariant spin which we denote by S. It is $2/M$ times the expectation value of the Pauli-Lubanski operator; see, e.g., [10]:

$$\mathcal{W}_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma} \tag{4.2}$$

and

$$(2\pi)^3 p_0 M\delta^3(\boldsymbol{p}' - \boldsymbol{p})S_\mu = \langle p', \boldsymbol{s}|\mathcal{W}_\mu|p, \boldsymbol{s}\rangle. \tag{4.3}$$

In terms of components

$$S_0 = \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{M}, \qquad S^i = s^i + p^i\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{M(p_0 + M)}. \tag{4.4}$$

Also we take $S^2 = -1$ while Ref. [3] takes the covariant normalization to be $-M^2$.

The Lorentz invariant normalization of the states is conventional

$$\langle p', S|p, S\rangle = (2\pi)^3 2p_0\delta^3(\boldsymbol{p}' - \boldsymbol{p}). \tag{4.5}$$

The extra integration in Eq. (4.1) $\int dt$, is argued to be harmless, leading to an infinite $\delta(0)$ which cancels out when calculating a genuine expectation value.

Comment 1.—It will be seen in Sec. IV B that the choice $S' = S$ as done in Ref. [3] is not consistent in a proper wave-packet treatment.

Analogously to the steps leading to Eq. (2.14) we have

$$\mathcal{M}^{\mu\nu\lambda}(p,k,p',S) = \int d^4x e^{ix\cdot(k-p+p')}x^\nu\langle p',S|T^{\mu\lambda}(0)|p,S\rangle - (\nu\leftrightarrow\lambda)$$
$$= \int d^4x\left[-i\frac{\partial}{\partial k_\nu}e^{ix\cdot(k-p+p')}\right]\langle p',S|T^{\mu\lambda}(0)|p,S\rangle - (\nu\leftrightarrow\lambda)$$
$$= -i(2\pi)^4\frac{\partial}{\partial k_\nu}[\delta^4(k-p+p')\langle p',S|T^{\mu\lambda}(0)|p,S\rangle] - (\nu\leftrightarrow\lambda). \quad (4.6)$$

Comment 2.—The last step in Eq. (4.6) can only be justified if p' in Eq. (4.1) is considered an independent variable. We may not take $p' = p - k$. Once this is recognized it is no longer so evident that Eq. (4.1) followed by the limit $k^\mu \to 0$ provides a natural definition of the ambiguous forward matrix element.

Continuing from Eq. (4.6) one has

$$\mathcal{M}^{\mu\nu\lambda}(p,k,p',S) = -i(2\pi)^4\Big[\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle\frac{\partial}{\partial k_\nu}\delta^4(k-p+p')$$
$$+\delta^4(k-p+p')\frac{\partial}{\partial k_\nu}\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle\Big] - (\nu\leftrightarrow\lambda). \quad (4.7)$$

In Ref. [3] the limit $k^\mu \to 0$ is given as

$$\mathcal{M}^{\mu\nu\lambda}(p,0,p',S) = -i(2\pi)^4\Big[\langle p,S|T^{\mu\lambda}(0)|p,S\rangle\partial^\nu\delta^4(0) + \delta^4(0)\frac{\partial}{\partial k_\nu}\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle\Big] - (\nu\leftrightarrow\lambda). \quad (4.8)$$

This form is a little puzzling, given that $p' \neq p - k$ in Eq. (4.1), as discussed in Comment 2. We thus prefer to write Eq. (4.8) in the form

$$\mathcal{M}^{\mu\nu\lambda}(p,0,p',S) = -i(2\pi)^4\Big[\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle\lim_{k^\mu\to0}\frac{\partial}{\partial k_\nu}\delta^4(k-p+p')$$
$$+\delta^4(p'-p)\frac{\partial}{\partial k_\nu}\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle|_{k^\mu\to0}\Big] - (\nu\leftrightarrow\lambda). \quad (4.9)$$

The highly singular first term in Eq. (4.9) can only be understood in a wave-packet analysis. It corresponds to the angular momentum about the origin arising from the motion of the center of mass of the wave packet, and has nothing to do with the internal structure of the nucleon. Thus we will take as the definition of the ambiguous forward matrix element in the Jaffe-Manohar (JM) approach

$$\mathcal{M}^{\mu\nu\lambda}(p,0,p',S) = -i(2\pi)^4\delta(0)\delta^3(\boldsymbol{p}'-\boldsymbol{p})\frac{\partial}{\partial k_\nu}$$
$$\times\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle|_{k^\mu\to0}$$
$$-(\nu\leftrightarrow\lambda), \quad (4.10)$$

where we have used the fact that for the on mass-shell momenta $\boldsymbol{p}' = \boldsymbol{p}$ forces $p_0' = p_0$.

The last part of the analysis concerns the structure of the matrix element of $T^{\mu\lambda}(0)$. For Eq. (4.10) we require an expansion in k^μ of $\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle$ up to terms linear in k^μ. It is at this point that the choice in Eq. (4.1) of $S' = S$ becomes significant: it greatly simplifies the tensorial structure of the expansion.

Following Ref. [3] one writes, with $P^\mu = p^\mu + \frac{1}{2}k^\mu$,

$$\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle = A_0(k^2)P^\mu P^\lambda + iA_1(k^2)$$
$$\times[\epsilon^{\mu\alpha\beta\sigma}P^\lambda$$
$$+\epsilon^{\lambda\alpha\beta\sigma}P^\mu]k_\alpha P_\beta S_\sigma + O(k^2), \quad (4.11)$$

where A_0 and A_1 are scalar functions of k^2. The terms in Eq. (4.11) are chosen to respect the relation

$$k_\mu\langle p-k,S|T^{\mu\lambda}(0)|p,S\rangle = 0. \quad (4.12)$$

[To see that $k\cdot P = 0$ one should recall that the nucleon states are physical states with $(p-k)^2 = p^2 = M^2$.]

For the forward matrix element

$$\langle p,S|T^{\mu\nu}(0)|p,S\rangle = A_0(0)p^\mu p^\lambda. \quad (4.13)$$

Comment 3.—While Eq. (4.12) *is* correct the crucial expansion (4.11) is not. The reason is the following. The right-hand side of Eq. (4.11) has been constructed to transform under Lorentz transformations as a genuine second-rank tensor on the grounds that $T^{\mu\lambda}(0)$ transforms as a second-rank tensor. But the *nonforward* matrix elements of a tensor operator do *not* transform covariantly, as was explained in the discussion of Eq. (2.15). We shall see the consequences of this in Sec. V A.

Continuing with the derivation in Ref. [3], we have from Eqs. (2.8) and (4.5)

$$\langle p',S|\int d^3xT^{00}(x)|p,S\rangle = \langle p',S|H|p,S\rangle$$
$$= 2p_0^2(2\pi)^3\delta^3(\boldsymbol{p}'-\boldsymbol{p}), \quad (4.14)$$

where we have used the fact that P^0 is the total energy or Hamiltonian operator. But the left-hand side, taking $t = 0$, equals

$$\int d^3x e^{i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')}\langle p', S|T^{00}(0)|p, S\rangle = (2\pi)^3\delta^3(\boldsymbol{p}'-\boldsymbol{p})A_0(0)p_0^2 \tag{4.15}$$

from Eq. (4.13).

Comparing Eqs. (4.14) and (4.15) yields

$$A_0(0) = 2. \tag{4.16}$$

Now one uses Eq. (4.11) to calculate the derivative needed in Eq. (4.10). The result in Ref. [3] is

$$\mathcal{M}^{\mu\nu\lambda}(p, 0, p', S) = (2\pi)^4\delta(0)\delta^3(\boldsymbol{p}'-\boldsymbol{p})A_1(0) \times [2p^\mu\epsilon^{\lambda\nu\beta\sigma} - p^\nu\epsilon^{\mu\lambda\beta\sigma} + p^\lambda\epsilon^{\mu\nu\beta\sigma}]p_\beta S_\sigma. \tag{4.17}$$

Last, the value of $A_1(0)$ is found by choosing a nucleon state at rest and spin along OZ. This is an eigenstate of J_z

$$J_z|0, \hat{z}\rangle = \tfrac{1}{2}|0, \hat{z}\rangle. \tag{4.18}$$

Then with $\mathring{S} = (0, 0, 0, 1)$

$$\langle p', \hat{z}|\int d^4x M^{012}(x)|0, \hat{z}\rangle = \int dt\langle p', \mathring{S}|J_z|0, \mathring{S}\rangle = \tfrac{1}{2}(2\pi)^4 2M\delta^3(\boldsymbol{p}')\delta(0). \tag{4.19}$$

But from Eq. (4.17) the left-hand side is just equal to[1]

$$\mathcal{M}^{012}(0, 0, p', \mathring{S}) = (2\pi)^4\delta(0)\delta^3(\boldsymbol{p}')A_1(0)2M^2 \tag{4.20}$$

so

$$A_1(0) = \frac{1}{2M}. \tag{4.21}$$

Finally, then, in the Jaffe-Manohar treatment the interpretation given to the forward matrix element of the angular momentum operator is

$$\langle p', S|\int d^4x M^{\mu\nu\lambda}(x)|p, S\rangle = \mathcal{M}^{\mu\nu\lambda}(p, 0, p', S) = \frac{(2\pi)^4\delta(0)\delta^3(\boldsymbol{p}'-\boldsymbol{p})}{2M} \times [2p^\mu\epsilon^{\lambda\nu\beta\sigma} - p^\nu\epsilon^{\mu\lambda\beta\sigma} + p^\lambda\epsilon^{\mu\nu\beta\sigma}]p_\beta S_\sigma. \tag{4.22}$$

Equation (4.22) is meant to provide a general basis for angular momentum sum rules. So, for example, if we have a theory of the nucleon in terms of quark and gluon fields and we construct the operator $M^{\mu\nu\lambda}$ from these fields, then the requirement that our $M^{\mu\nu\lambda}$ satisfy Eq. (4.22) for an arbitrary state of the nucleon yields a set of conditions on some of the elements of the theory.

[1]There is a factor of M^2 missing in the value of $A_1(0)$ given in Ref. [3].

As indicated in the comments, there are flaws in the derivation and Eq. (4.22) is incorrect. We shall present the correct result, in a wave-packet treatment in Secs. V A and V B. There our states or wave functions will be normalized to 1 so that we calculate actual expectation values. Moreover, as will be explained in Sec. IV B, the wave-packet approach seems only able to deal with the physically relevant operators $\int d^3x M^{0ij}(x)$. For these conserved densities the $\int dt$ in Eq. (4.1) is simply equivalent to the factor $2\pi\delta(0)$ in Eq. (4.17). Thus, dividing the expression (4.22) by this factor and by the normalization given in Eq. (4.5) we have, for the expectation values in the JM treatment

$$\left.\frac{\langle p, S|\int d^3x M^{0ij}(x)|p, S\rangle}{\langle p, S|p, S\rangle}\right|_{\text{JM}} = \frac{1}{4Mp_0}[2p^0\epsilon^{ji\beta\sigma} - p^i\epsilon^{0j\beta\sigma} + p^j\epsilon^{0i\beta\sigma}]p_\beta S_\sigma \tag{4.23}$$

which can be compared directly with the result we shall obtain in Secs. V A and V B. In Sec. VI we shall give a completely independent corroboration of these results.

We turn now to the treatment of Shore and White, which basically follows the pattern of Ref. [3], but attempts to put the argument on a rigorous footing via the use of wave packets.

B. The Shore-White treatment

As already mentioned, the authors of Ref. [3] remark that a wave-packet approach is needed to justify some of the manipulations involved, more precisely, to get rid of the unwelcome derivatives of δ functions. This is done in Ref. [4], but, as we shall see, the treatment also suffers from some of the incorrect elements commented on in Sec. IV A. The notation in Ref. [4] differs somewhat from that of Ref. [3], so we will rephrase the notation in Ref. [4] to match as closely as possible the development in Ref. [3].

The authors of Ref. [4] try to give a sensible definition to the forward matrix elements defined in Eq. (2.14), i.e.,

$$\mathcal{M}^{0ij}(p, S) = \langle p, S|\int d^3x[x^iT^{0j}(x) - x^jT^{0i}(x)]|p, S\rangle \tag{4.24}$$

by utilizing a wave packet. Thus they define

$$|\phi(p, S)\rangle = \int \frac{d^3q}{(2\pi)^3\sqrt{2q_0}}\phi[(\boldsymbol{q}-\boldsymbol{p})^2]|q, S\rangle \tag{4.25}$$

where ϕ drops rapidly to zero as $|\boldsymbol{q}-\boldsymbol{p}| \to \infty$, and they interpret Eq. (4.24), modulo normalization, as

$$\mathcal{M}^{0ij}_{\text{SW}}(p, S) = \langle\phi(p, S)|\int d^3x[x^iT^{0j}(x) - x^jT^{0i}(x)]|p, S\rangle. \tag{4.26}$$

It turns out to be sufficient to use just one wave packet, either for the initial or the final state.

Note that this differs from $\mathcal{M}^{\mu\nu\lambda}(p, 0, p', S)$ in two respects. First, consideration is given here only to the spatial elements i, j of the tensor and second, the integral is over three-dimensional space. The latter difference is not significant given that the operators in Eq. (4.24) are supposed to be time independent.

The derivation then runs as follows:

$$\mathcal{M}^{0ij}_{\rm SW}(p, S) = \int \frac{d^3q}{(2\pi)^3}\,\phi[(\boldsymbol{q}-\boldsymbol{p})^2]\int d^3x[x^i e^{i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{q})}\langle q, S|T^{0j}(0)|p, S\rangle - (i \leftrightarrow j)]$$
$$= i\int \frac{d^3q}{(2\pi)^3}\,\phi[(\boldsymbol{q}-\boldsymbol{p})^2]\int d^3x\left(\frac{\partial}{\partial q_i}e^{i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{q})}\right)\langle q, S|T^{0j}(0)|p, S\rangle - (i \leftrightarrow j). \tag{4.27}$$

Exchanging the order of integration, then integrating by parts with respect to $\boldsymbol{q}$, and discarding the surface terms at $q_i = \pm\infty$,

$$\mathcal{M}^{0ij}_{\rm SW}(p, S) = -i\int d^3x\int \frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{x}\cdot(\boldsymbol{q}-\boldsymbol{p})}\frac{\partial}{\partial q_i}\{\phi[(\boldsymbol{q}-\boldsymbol{p})^2]\langle q, S|T^{0j}(0)|p, S\rangle\}$$
$$= -i\int d^3q\,\delta^3(\boldsymbol{q}-\boldsymbol{p})\left[\left(\frac{\partial\phi}{\partial q_i}\right)\langle q, S|T^{0j}(0)|p, S\rangle + \phi\frac{\partial}{\partial q_i}\langle q, S|T^{0j}(0)|p, S\rangle\right]$$
$$= -i\frac{\partial\phi[(\boldsymbol{q}-\boldsymbol{p})^2]}{\partial q_i}\bigg|_{\boldsymbol{q}=\boldsymbol{p}}\langle p, S|T^{0j}(0)|p, S\rangle + \phi(0)\frac{\partial}{\partial q_i}\langle q, S|T^{0j}(0)|p, S\rangle|_{\boldsymbol{q}=\boldsymbol{p}}, \tag{4.28}$$

all antisymmetrized under $i \leftrightarrow j$.

Now

$$\frac{\partial\phi[(\boldsymbol{q}-\boldsymbol{p})^2]}{\partial q_i}\bigg|_{\boldsymbol{q}=\boldsymbol{p}} = 0, \tag{4.29}$$

so in the Shore-White (SW) approach

$$\mathcal{M}^{0ij}_{\rm SW}(p, S) = -i\phi(0)\left[\frac{\partial}{\partial q_i}\langle q, S|T^{0j}(0)|p, S\rangle|_{\boldsymbol{q}=\boldsymbol{p}} - (i \leftrightarrow j)\right], \tag{4.30}$$

which is identical to $\mathcal{M}^{0ij}_{\rm JM}(p, 0, p', S)$ in Eq. (4.10), aside from normalization, which is irrelevant, since, at the end, it cancels out when computing actual expectation values.

Comment 4.—It is actually *not* possible in Eq. (4.30) to take the same S in both initial and final states, as in Eq. (4.1). The reason is the following. A wave packet is, by definition, a superposition of *physical* states. But for a spin-1/2 particle in a physical state

$$q^2 = M^2 \quad \text{and} \quad q\cdot S = 0 \tag{4.31}$$

and we cannot integrate independently over each component of $\boldsymbol{q}$. That is one reason why we have chosen to define the states using the rest-frame spin vector $\boldsymbol{s}$ instead. The correct procedure is then to take the same rest frame $\boldsymbol{s}$ for the initial and final states, not the same S. Thus the wave packet Eq. (4.25) should be modified to

$$|\phi(p, \boldsymbol{s})\rangle = \int \frac{d^3q}{(2\pi)^3}\,\phi[(\boldsymbol{q}-\boldsymbol{p})^2]|q, \boldsymbol{s}\rangle \tag{4.32}$$

and all three components of $\boldsymbol{q}$ can then be integrated over independently.

The use of the Lorentz tensor form, as in Eq. (4.22), is central to some of the most interesting arguments given by Shore and White—for example, their conclusion that the axial charge does not contribute to the angular momentum sum rules. But, as pointed out in Comment 3, this form is based on the incorrect expansion Eq. (4.11), and so one must be skeptical about their results. We shall see, however, in Sec. V B, that the conclusion regarding the role of the axial charge in the sum rules *is* correct in spite of the erroneous argument.

To summarize: the Shore-White wave-packet analysis provides a justification for the manipulations in the Jaffe-Manohar treatment for the spatial components of the angular momentum tensor. But the analysis is not correct in general because it utilizes the impermissible simplification $S = S'$ and also makes an incorrect use of Lorentz covariance.

Given these criticisms, it is interesting to consider one important practical case of a sum rule that can be derived from Eq. (4.22). Namely, for a longitudinally polarized proton moving along OZ, i.e., in a state of definite helicity $\frac{1}{2}$ with $\boldsymbol{p}$ along OZ one has the sum rule

$$\tfrac{1}{2} = \tfrac{1}{2}\Delta\Sigma + \Delta G + \langle L^q\rangle + \langle L^G\rangle \tag{4.33}$$

where $\Delta\Sigma, \Delta G$ are the first moments of the polarized quark flavor-singlet and the polarized gluon densities, respectively, and $\langle L^q\rangle, \langle L^G\rangle$ are contributions from the quark and gluon internal orbital momenta. This sum rule is so manifestly intuitively right that it is almost impossible to contemplate it being incorrect [11]. And indeed, it is correct. This is the one unique case where the flaws in the deduction are irrelevant, as will be seen later.

In Sec. V we will present what we believe to be a correct evaluation of the wave-packet definition of the recalcitrant forward matrix element.

V. A DETAILED WAVE-PACKET TREATMENT

Given our critical stance *vis-à-vis* the treatments of Refs. [3,4] it is incumbent upon us to proceed with caution. We shall therefore present a detailed wave-packet treatment for two separate situations, for a relativistic quantum-mechanical Dirac particle and for the field-theoretical case. It will emerge that there is complete agreement between the results for the two cases and that they differ, in general, from Eq. (4.23).

We shall also address the question as to the validity of Eq. (2.12), which was assumed in the derivations in Refs. [3,4]. In order to do this we shall need to distinguish between $M^{\mu\nu\lambda}_{\rm orb}(x)$ constructed out of the canonical energy-momentum density $T^{\mu\lambda}_{\rm C}(x)$ as in Eq. (2.4) and $M^{\mu\nu\lambda}(x)$ used in Secs. IV A and IV B based upon the symmetrized $T^{\mu\lambda}(x)$, as in Eq. (2.5).

A. Relativistic quantum-mechanical Dirac particle

We construct the wave function corresponding to a superposition of physical states centered around momentum $\boldsymbol{p}$, all of which have rest-frame spin vector $\boldsymbol{s}$:

$$\psi_{p,s}(\boldsymbol{x},t)=\frac{\bar{N}}{(2\pi)^3}\int\frac{d^3q}{q^0}\,e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2}e^{i(\boldsymbol{q}\cdot\boldsymbol{x}-q^0t)}u(q,\boldsymbol{s}), \tag{5.1}$$

where $q^0=\sqrt{\boldsymbol{q}^2+M^2}$, and

$$u(q,\boldsymbol{s})=\sqrt{\frac{q^0+M}{2M}}\begin{pmatrix}1\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}}{q^0+M}\end{pmatrix}\chi(\boldsymbol{s}) \tag{5.2}$$

with

$$\chi^\dagger(\boldsymbol{s})\boldsymbol{\sigma}\chi(\boldsymbol{s})=\boldsymbol{s}. \tag{5.3}$$

The constant $\bar{N}$, whose value is irrelevant for the moment, is adjusted so that $\psi_{\boldsymbol{p},\boldsymbol{s}}$ is normalized to 1, i.e.,

$$\int d^3x\psi^\dagger_{p,s}(x)\psi_{p,s}(x)=1. \tag{5.4}$$

Since our aim is to provide a sensible prescription for a plane-wave state of definite momentum $\boldsymbol{p}$, we shall, at the end, take the limit $\lambda\to\infty$. In this limit the values of $\boldsymbol{q}$ that contribute in the integral Eq. (5.1) are forced towards $\boldsymbol{p}$, so we make a Taylor expansion of $(1/q^0)u(q,\boldsymbol{s})$ about the point $\boldsymbol{p}$ and keep only those terms that survive ultimately in the limit $\lambda\to\infty$.

Note that we are unable to carry out the analysis if t can be arbitrarily large. The point of using the wave packet is to produce a cutoff in the divergent spatial integrals in Eq. (2.14), but this does not produce a cutoff in t. Thus the wave-packet approach cannot be used if the operator densities are integrated over four-dimensional space-time. Since we are only interested in conserved densities, we will take advantage of the time independence to choose $t=0$ in Eq. (5.1).

It turns out to be sufficient to keep just the first two terms of the Taylor expansion, which yield:

$$\frac{1}{q^0}u(q,\boldsymbol{s})=\frac{1}{p^0}\Big\{u(p,\boldsymbol{s})-\frac{1}{p^0+M}\Big[\frac{p^0+2M}{2p^{02}}u(p,\boldsymbol{s})\boldsymbol{p}+\sqrt{\frac{p^0+M}{2M}}\begin{pmatrix}0\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\boldsymbol{p}}{p^0(p^0+M)}-\boldsymbol{\sigma}\end{pmatrix}\chi(\boldsymbol{s})\Big]\cdot(\boldsymbol{q}-\boldsymbol{p})\Big\} \tag{5.5}$$

with $p^0=\sqrt{\boldsymbol{p}^2+M^2}$.

The term $(\boldsymbol{q}-\boldsymbol{p})$ is removed from under the integral in Eq. (5.1) via

$$(\boldsymbol{q}-\boldsymbol{p})e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2}=\frac{\boldsymbol{\nabla}_{\boldsymbol{p}}}{2\lambda^2}e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2}. \tag{5.6}$$

The remaining integral is just the Fourier transform of a Gaussian, yielding a factor $\exp[-\boldsymbol{x}^2/(4\lambda^2)]\exp(i\boldsymbol{p}\cdot\boldsymbol{x})$, and the $\boldsymbol{\nabla}_{\boldsymbol{p}}$ on the right-hand side of Eq. (5.6) then produces a term $i\boldsymbol{x}$. The result is

$$\psi_{p,s}(\boldsymbol{x},0)=Ne^{-\boldsymbol{x}^2/4\lambda^2}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}u_{p,s}(x) \tag{5.7}$$

where

$$N^2=\frac{M}{p^0}\Big(\frac{1}{\sqrt{2\pi}\lambda}\Big)^3 \tag{5.8}$$

and

$$u_{p,s}(x)=\Big[1-\frac{i\boldsymbol{x}\cdot\boldsymbol{p}}{2\lambda^2}\,\frac{p^0+2M}{2p^{02}(p^0+M)}\Big]u(p,\boldsymbol{s})+\frac{i}{2\lambda^2(p^0+M)}\times\sqrt{\frac{p^0+M}{2M}}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{x}-\frac{\boldsymbol{x}\cdot\boldsymbol{p}\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p^0(p^0+M)}\\ 0\end{pmatrix}\chi(\boldsymbol{s}). \tag{5.9}$$

The structure of Eq. (5.9) is extremely instructive in understanding both the difference between matrix elements of local operators like $T^{\mu\lambda}(x)$ and compound ones like $x^\rho T^{\mu\lambda}(x)$, and the question as to how many terms of the Taylor expansion are necessary.

On the one hand the Gaussian implies that the effective values of $\boldsymbol{x}$ satisfy $|\boldsymbol{x}|\le 2\lambda$. On the other, the nth term in the Taylor expansion provides a term of order $(|\boldsymbol{x}|/\lambda^2)^n$. For a local operator there are no other factors of x present, so even the first-order term of a Taylor expansion can be ignored in the limit $\lambda\to\infty$. For the angular momentum, on the contrary, there is one explicit factor of x. The first-order term in the Taylor expansion is then essential, but higher order terms can be disregarded as $\lambda\to\infty$. Of course, this is totally analogous to what happened in Sec. IV A, where the results involved a first derivative.

The difference is that here the derivative has been calculated accurately and without recourse to an incorrect assumption of Lorentz covariance.

Now that a sufficiently accurate wave function has been obtained, we turn to the calculation of the expectation values of the operators. In Dirac theory the canonical energy-momentum density is [12]

$$T_{\rm C}^{\mu\lambda}(x) = \frac{i}{2}\bar{\psi}(x)\gamma^{\mu}\partial^{\lambda}\psi(x) + \text{H.c.}, \tag{5.10}$$

and the orbital angular momentum density is

$$M_{\rm orb}^{\mu\nu\lambda}(x) = x^{\nu}T_{\rm C}^{\mu\lambda}(x) - x^{\mu}T_{\rm C}^{\nu\lambda}(x) \tag{5.11}$$

and the spin density is

$$M_{\rm spin}^{\mu\nu\lambda}(x) = \tfrac{1}{2}\bar{\psi}(x)\gamma^{\mu}\sigma^{\nu\lambda}\psi(x). \tag{5.12}$$

The orbital angular momentum operators $M_{\rm orb}^{ij}$ and the spin angular momentum $M_{\rm spin}^{ij}$ are the space integrals of $M_{\rm orb}^{0ij}$ and $M_{\rm spin}^{0ij}$, respectively, calculated, in our case, at $t = 0$ and are not separately time independent.

The calculation of the spatial derivatives of the wave function (5.7) needed in Eq. (5.10) and the subsequent Dirac algebra is straightforward, but laborious and will not be spelled out here. Helpful is the fact that odd functions of $\boldsymbol{x}$ will vanish under integration $\int d^3x$ since the nonsymmetric term $\exp(i\boldsymbol{p}\cdot\boldsymbol{x})$ cancels out in constructing Eq. (5.10). The terms in the spatial derivative that survive give

$$\partial^{j}\psi_{p,s}(\boldsymbol{x},0) = iNe^{-\boldsymbol{x}^2/4\lambda^2}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}u_{p,s}^{j}(\boldsymbol{x}) \tag{5.13}$$

where

$$u_{p,s}^{j}(\boldsymbol{x}) = \left\{p^{j}\left[1 - \frac{i(p^0+2M)}{2\lambda^2(p^0+M)}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{2p^{02}}\right] + \frac{ix^{j}}{2\lambda^2}\right\}u(p,s) + \frac{ip^{j}}{2\lambda^2(p^0+M)}\sqrt{\frac{p^0+M}{2M}}\begin{pmatrix} 0 \\ \boldsymbol{\sigma}\cdot\boldsymbol{x} - \frac{\boldsymbol{p}\cdot\boldsymbol{x}\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p^0(p^0+M)}\end{pmatrix}\chi(\boldsymbol{s}). \tag{5.14}$$

Then, keeping only the terms that survive in the limit $\lambda \to \infty$, we find ultimately:

$$M_{\rm orb}^{ij} = \frac{1}{2p^0(p^0+M)}[p^{j}(\boldsymbol{p}\times\boldsymbol{s})^{i} - p^{i}(\boldsymbol{p}\times\boldsymbol{s})^{j}] \tag{5.15}$$

or via the analog of Eq. (2.9), we express the orbital angular momentum vector $\boldsymbol{L}$ in terms of the independent vectors $\boldsymbol{p}$ and $\boldsymbol{s}$:

$$\boldsymbol{L} = -\frac{1}{2p^0(p^0+M)}[\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{s})] = \frac{1}{2p^0(p^0+M)}[\boldsymbol{p}^2\boldsymbol{s} - (\boldsymbol{p}\cdot\boldsymbol{s})\boldsymbol{p}]. \tag{5.16}$$

For the spin angular momentum we find

$$M_{\rm spin}^{ij} = \frac{\epsilon^{ijk}}{2p^0}\left[Ms^{k} + \frac{(\boldsymbol{p}\cdot\boldsymbol{s})p^{k}}{p^0+M}\right] = \frac{\epsilon^{ijk}}{2p^0}S^{k}M \tag{5.17}$$

where we have used Eq. (4.4).

For the spin vector $\boldsymbol{S}$ of the system Eq. (5.17) yields

$$\boldsymbol{S} = \frac{1}{2p^0}\left[M\boldsymbol{s} + \frac{(\boldsymbol{p}\cdot\boldsymbol{s})}{p^0+M}\boldsymbol{p}\right]. \tag{5.18}$$

Adding Eqs. (5.16) and (5.18) we have the remarkable result that the term proportional to $\boldsymbol{p}$ cancels out, and

$$\boldsymbol{J} = \tfrac{1}{2}\boldsymbol{s}. \tag{5.19}$$

For later use we write this as

$$J^{ij} = M_{\rm orb}^{ij} + M_{\rm spin}^{ij} = \tfrac{1}{2}\epsilon^{ijk}s^{k}. \tag{5.20}$$

In the above we have used the canonical form of the total angular momentum, where it is split naturally into an orbital and a spin part. In Sec. IV, however, following Refs. [3,4] we utilized the angular momentum tensor built from the symmetrized Belinfante energy-momentum density. According to Eq. (2.12) this should yield the same forward matrix element as the total angular momentum in the canonical approach. For our quantum-mechanical example we can check whether indeed $M^{ij} = J^{ij}$.

For the symmetrical energy-momentum density we have $T^{\mu\lambda} = \frac{1}{2}(T_{\rm C}^{\mu\lambda} + T_{\rm C}^{\lambda\mu})$ so that from Eqs. (2.5) and (5.10)

$$M^{\mu\nu\lambda}(x) = \tfrac{1}{2}M_{\rm orb}^{\mu\nu\lambda}(x) + \tfrac{1}{2}\left\{\frac{i}{2}[x^{\nu}\bar{\psi}(x)\gamma^{\lambda}\partial^{\mu}\psi(x) - x^{\lambda}\bar{\psi}(x)\gamma^{\nu}\partial^{\mu}\psi(x)] + \text{H.c.}\right\}. \tag{5.21}$$

For $M^{0ij}(x)$ we now require the time derivative $\partial\psi_{p,s}(\boldsymbol{x},t)/\partial t|_{t=0}$ of the wave function (5.1). The time derivative gives a factor $-iq^0$ which is expanded as

$$-iq_0 = -i\left[p_0 + \frac{\boldsymbol{p}\cdot(\boldsymbol{q}-\boldsymbol{p})}{p_0}\right] \tag{5.22}$$

and via the steps explained after Eq. (5.5) this becomes a multiplicative factor:

$$\frac{\partial}{\partial t}\psi_{p,s}(\boldsymbol{x},t)|_{t=0} = -i\left[p_0 + \frac{i\boldsymbol{p}\cdot\boldsymbol{x}}{2\lambda^2 p_0}\right]\psi_{p,s}(\boldsymbol{x},0). \tag{5.23}$$

After further laborious Dirac algebra we find

$$M^{ij} = \tfrac{1}{2}\epsilon^{ijk}s^{k} \tag{5.24}$$

in complete agreement with Eq. (5.20).

Thus despite the fact that $M^{0ij}(x)$ and $J^{0ij}(x)$ differ by the divergence of a *compound* operator, and despite the fact that the definition of the forward matrix element

involves nonforward ones, it seems that Eq. (2.12) is indeed valid, in a wave-packet approach, as was claimed in Ref. [4].

In the next section we shall see that we obtain the same, to us surprising, result Eq. (5.20) in field theory, and we shall then compare it to the results in Secs. IV A and IV B.

B. Field theoretic treatment

Analogously to the quantum-mechanical case we construct a wave-packet state $|\Psi_{p,s}\rangle$ as a linear superposition of physical (canonical spin) plane-wave states of momentum q all of which have the same rest-frame spin vector $\boldsymbol{s}$:

$$|\Psi_{p,s}\rangle = \frac{N}{\sqrt{(2\pi^3)}} \int d^3q e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} |q, \boldsymbol{s}\rangle. \tag{5.25}$$

Note that we label the momentum eigenstates by $\boldsymbol{s}$, not by the covariant spin vector S. As before we shall consider the limit $\lambda \to \infty$ and we normalize the state to 1.

$$\langle \Psi_{p,s} | \Psi_{p,s} \rangle = 1. \tag{5.26}$$

It follows that

$$N^2 = \left(\frac{2\lambda}{\sqrt{2\pi}}\right)^3 \frac{1}{2p_0} \tag{5.27}$$

with $p_0 = \sqrt{p^2 + M^2}$.

Let us first consider the general structure of the matrix elements of some local operator density $O(x)$ between the states $|\Psi_{p,s}\rangle$. We have

$$\langle \Psi_{p',s} | O(\boldsymbol{x}, 0) | \Psi_{p,s} \rangle = \frac{N^2}{(2\pi)^3} \int d^3q d^3q' e^{-\lambda^2(\boldsymbol{p}-\boldsymbol{q})^2} e^{-\lambda^2(\boldsymbol{p}'-\boldsymbol{q}')^2} e^{i\boldsymbol{x}\cdot(\boldsymbol{q}-\boldsymbol{q}')} \langle q', \boldsymbol{s} | O(0) | q, \boldsymbol{s} \rangle \tag{5.28}$$

where we have temporarily kept p' distinct from p as an aid to calculation.

The matrix element on the right-hand side is expanded in a Taylor series in q and q' about the points p and p', respectively, and to the required order is of the form

$$\langle q', \boldsymbol{s} | O(0) | q, \boldsymbol{s} \rangle = f_1(p, \boldsymbol{s}) + (\boldsymbol{q} - \boldsymbol{p}) \cdot \boldsymbol{f}_2(p, \boldsymbol{s}) + (\boldsymbol{q}' - \boldsymbol{p}') \cdot \boldsymbol{f}_3(p', \boldsymbol{s}). \tag{5.29}$$

Factors like $\boldsymbol{q} - \boldsymbol{p}$ are transformed into $\boldsymbol{\nabla}_p / 2\lambda^2$, etc., under the integral in Eq. (5.28), so that

$$\langle \Psi_{p',s} | O(\boldsymbol{x}, 0) | \Psi_{p,s} \rangle = \left[f_1 + \boldsymbol{f}_2 \cdot \frac{\boldsymbol{\nabla}_p}{2\lambda^2} + \boldsymbol{f}_3 \cdot \frac{\boldsymbol{\nabla}'_p}{2\lambda^2} \right] N^2 \int d^3q d^3q' e^{-\lambda^2(\boldsymbol{q}-\boldsymbol{p})^2} e^{-\lambda^2(\boldsymbol{q}'-\boldsymbol{p}')^2} e^{i\boldsymbol{x}\cdot(\boldsymbol{q}-\boldsymbol{q}')}. \tag{5.30}$$

Putting $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{p}$, $\boldsymbol{r}' = \boldsymbol{q}' - \boldsymbol{p}'$ the integral in Eq. (5.30) becomes

$$e^{i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')} \int d^3r e^{-\lambda^2 r^2} e^{i\boldsymbol{r}\cdot\boldsymbol{x}} \int d^3r' e^{-\lambda^2 r'^2} e^{-i\boldsymbol{r}'\cdot\boldsymbol{x}} \tag{5.31}$$

so that $\boldsymbol{\nabla}_p$ can be replaced by $i\boldsymbol{x}$ and $\boldsymbol{\nabla}'_p$ by $-i\boldsymbol{x}$ in Eq. (5.30).

Carrying out the integrals in Eq. (5.30) and putting $\boldsymbol{p}' = \boldsymbol{p}$ once again, we end up with

$$\langle \psi_{p,s} | O(\boldsymbol{x}, 0) | \psi_{p,s} \rangle = \left\{ f_1(p, s) + \frac{i\boldsymbol{x}}{2\lambda^2} \cdot [\boldsymbol{f}_2(p, \boldsymbol{s}) - \boldsymbol{f}_3(p, \boldsymbol{s})] \right\} \frac{\bar{C}}{\lambda^3} e^{-x^2/2\lambda^2} \tag{5.32}$$

where, via Eq. (5.27)

$$\frac{\bar{C}}{\lambda^3} = \frac{N^2}{(\sqrt{2}\lambda)^6} = \frac{1}{(2\pi)^{3/2}} \frac{1}{2p_0} \frac{1}{\lambda^3}, \tag{5.33}$$

so that $\bar{C} = [1/(2\pi)^{3/2}](1/2p_0)$.

We now apply this to the case where $O(x)$ is the canonical energy-momentum density $T_C^{\mu\nu}(x)$. The most general structure of the matrix elements of the conserved operator $T_C^{\mu\nu}(0)$ is

$$\begin{aligned} \langle q', \boldsymbol{s} | T_C^{\mu\nu}(0) | q, \boldsymbol{s} \rangle = \bar{u}(q', \boldsymbol{s}) \{ & \mathbb{G}(q^\mu q^\nu + q'^\mu q'^\nu) \\ & + \mathbb{H}(q^\mu q'^\nu + q^\nu q'^\mu) + M\mathbb{S}[(q+q')^\mu \gamma^\nu \\ & + (q+q')^\nu \gamma^\mu] + (q \cdot q' - M^2) \\ & \times (\mathbb{G} - \mathbb{H}) g^{\mu\nu} + M\mathbb{A}[(q+q')^\mu \gamma^\nu \\ & - (q+q')^\nu \gamma^\nu] \} u(q, \boldsymbol{s}) \end{aligned} \tag{5.34}$$

where the $u(q, \boldsymbol{s})$, $u(q', \boldsymbol{s})$ are the usual canonical Dirac spinors normalized to $\bar{u}u = 1$ and $\mathbb{G}$, $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{A}$ are Lorentz scalars. Note that all terms, except the $\mathbb{A}$ term, are symmetric in $\mu \leftrightarrow \nu$. In order to identify the function f_1 and the vector function $\boldsymbol{f}_2 - \boldsymbol{f}_3$ in Eq. (5.29) we have to expand Eq. (5.34) about $\boldsymbol{q} = \boldsymbol{p}$ and $\boldsymbol{q}' = \boldsymbol{p}$. The Gordon decomposition

$$\begin{aligned} \bar{u}(q')\gamma^\mu u(q) = & \frac{(q+q')^\mu}{2M} \bar{u}(q')u(q) \\ & + \frac{i(q'-q)_\rho}{2M} \bar{u}(q')\sigma^{\rho\mu}u(q) \end{aligned} \tag{5.35}$$

is helpful, because to the accuracy required we can replace q and q' by p in the factors multiplying $(q'-q)_\rho$ and then use

$$\bar{u}(p, \boldsymbol{s})\sigma^{\rho\mu}u(p, \boldsymbol{s}) = \frac{1}{M} \epsilon^{\rho\mu\alpha\beta} S_\alpha p_\beta \tag{5.36}$$

where S is given by Eq. (4.4) and the convention is $\epsilon^{0123} = +1$. Then

$$\begin{aligned}\langle q', s|T_C^{\mu\nu}(0)|q, s\rangle = [&\mathbb{B}(q^\mu q^\nu + q'^\mu q'^\nu) + (q\cdot q' - M^2)\\ &\times(\mathbb{B}-\mathbb{C})g^{\mu\nu} + \mathbb{C}(q^\mu q'^\nu + q^\nu q'^\mu)]\\ &\times \bar{u}(q', s)u(q, s) + \frac{i(q'-q)_\rho}{M}\\ &\times(\mathbb{S}[p^\mu \epsilon^{\rho\nu\alpha\beta} + p^\nu \epsilon^{\rho\mu\alpha\beta}]\\ &+ \mathbb{A}[p^\mu \epsilon^{\rho\nu\alpha\beta} - p^\nu \epsilon^{\rho\mu\alpha\beta}])S_\alpha p_\beta\end{aligned} \tag{5.37}$$

where $\mathbb{B} = \mathbb{G} + \mathbb{S}$ and $\mathbb{C} = \mathbb{H} + \mathbb{S}$.

Putting

$$\begin{aligned}u(q, s) &= u(p, s) + (q-p)^k u^k(p, s),\\ \bar{u}(q', s) &= \bar{u}(p, s) + (q'-p)^k \bar{u}^k(p, s)\end{aligned} \tag{5.38}$$

where $u^k(p, s) = \partial/\partial p^k u(p, s)$ and similarly for $\bar{u}^k$, we find, after much algebra, that for $T_C^{\mu\nu}(0)$ the function f_1 and the kth component of $\boldsymbol{f}_2 - \boldsymbol{f}_3$, needed in Eq. (5.32) are

$$f_1 = 2\mathbb{D}p^\mu p^\nu \tag{5.39}$$

where

$$\mathbb{D} = \mathbb{B} + \mathbb{C} \tag{5.40}$$

and

$$\begin{aligned}(f_2 - f_3)^k = &2\mathbb{D}p^\mu p^\nu[\bar{u}(p, s)u^k(p, s) - \bar{u}^k(p, s)u(p, s)]\\ &- 2i\frac{(\mathbb{S}+\mathbb{A})}{M}p^\mu\left[\frac{p^k}{p_0}\epsilon^{0\nu\alpha\beta} - \epsilon^{k\nu\alpha\beta}\right]S_\alpha p_\beta\\ &- 2i\frac{(\mathbb{S}-\mathbb{A})}{M}p^\nu\left[\frac{p^k}{p_0}\epsilon^{0\mu\alpha\beta} - \epsilon^{k\mu\alpha\beta}\right]S_\alpha p_\beta\end{aligned} \tag{5.41}$$

so that from Eq. (5.32)

$$\begin{aligned}\langle\psi_{p,s}|T_C^{0j}(\boldsymbol{x}, 0)|\psi_{p,s}\rangle = \frac{\bar{C}}{\lambda^3}e^{-\boldsymbol{x}^2/2\lambda^2}\Big(&2\mathbb{D}p^0p^j + i\frac{x^k}{2\lambda^2}\Big\{2p^0p^j[\bar{u}(p, s)u^k(p, s) - \bar{u}^k(p, s)u(p, s)]\\ &- 2i\frac{(\mathbb{S}+\mathbb{A})}{M}p^0\Big(\frac{p^k}{p^0}\epsilon^{0j\alpha\beta} - \epsilon^{kj\alpha\beta}\Big)S_\alpha p_\beta + 2i\frac{(\mathbb{S}-\mathbb{A})}{M}p^j\epsilon^{k0\alpha\beta}S_\alpha p_\beta\Big\}\Big).\end{aligned} \tag{5.42}$$

Consider now the matrix element[2]

$$\begin{aligned}\langle\psi_{p,s}|M_{\rm orb}^{ij}|\psi_{p,s}\rangle &= \langle\psi_{p,s}|\int d^3x M_{\rm orb}^{0ij}(\boldsymbol{x}, 0)|\psi_{p,s}\rangle\\ &= \int d^3x x^i\langle\psi_{p,s}|T_C^{0j}(\boldsymbol{x}, 0)|\psi_{p,s}\rangle - (i\leftrightarrow j).\end{aligned} \tag{5.43}$$

We see that integrating over space kills the first term in Eq. (5.42), and the second only contributes if $k = i$. The spatial integral is then simply

$$\frac{\bar{C}}{\lambda^3}\int d^3x(x^i)^2e^{-x^2/2\lambda^2} = \frac{\lambda^2}{2p_0} \tag{5.44}$$

where we have used Eq. (5.33) for $\bar{C}$. The λ^2 in Eq. (5.44) cancels the remaining $1/\lambda^2$ in Eq. (5.42), so that we can take the limit $\lambda\to\infty$ and have

$$\begin{aligned}\langle\psi_{p,s}|M_{\rm orb}^{ij}|\psi_{p,s}\rangle = \frac{i}{2p_0}\Big\{&\mathbb{D}p^0p^j[\bar{u}(p,s)u^i(p,s)\\ &- \bar{u}^i(p,s)u(p,s)] - \frac{i\mathbb{S}}{M}[p^i\epsilon^{0j\alpha\beta}\\ &- p_0\epsilon^{ij\alpha\beta} + p^j\epsilon^{0i\alpha\beta}]S_\alpha p_\beta - \frac{i\mathbb{A}}{M}\\ &\times[p^i\epsilon^{0j\alpha\beta} - p_0\epsilon^{ij\alpha\beta} - p^j\epsilon^{0i\alpha\beta}]S_\alpha p_\beta\Big\}\\ &- (i\to j).\end{aligned} \tag{5.45}$$

Part of the term multiplying $\mathbb{S}$ is symmetric under $(i\leftrightarrow j)$, so cancels out. The other terms multiplying $\mathbb{S}$ and $\mathbb{A}$ are antisymmetric so just get doubled under $(i\leftrightarrow j)$. Also

$$\begin{aligned}&\bar{u}(p, s)u^i(p, s) - \bar{u}^i(p, s)u(p, s)\\ &= \bar{u}(p, s)u^i(p, s) - [\bar{u}(p, s)u^i(p, s)]^*\\ &= 2i\mathrm{Im}[\bar{u}(p, s)u^i(p, s)].\end{aligned} \tag{5.46}$$

Thus Eq. (5.45) becomes

$$\begin{aligned}\langle\psi_{p,s}|M_{\rm orb}^{ij}|\psi_{p,s}\rangle = &-\mathbb{D}\{p^j\mathrm{Im}[\bar{u}(p, s)u^i(p, s)] - (i\leftrightarrow j)\}\\ &- \frac{\mathbb{S}}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta + \frac{\mathbb{A}}{Mp_0}\\ &\times(p^i\epsilon^{0j\alpha\beta} - p^j\epsilon^{0i\alpha\beta} - p_0\epsilon^{ij\lambda\beta})S_\alpha p_\beta.\end{aligned} \tag{5.47}$$

Finally we expand the spinors as in Eq. (5.38) and find that

$$\mathrm{Im}[\bar{u}(p, s)u^i(p, s)] = \frac{1}{2M(p_0+M)}\epsilon^{ilm}p_l s_m \tag{5.48}$$

and comparing Eq. (4.13) with $A_0(0) = 2$ [(see Eq. (4.16)] with Eq. (5.37), we see that $\mathbb{B} + \mathbb{C} = 1$ and so, via Eq. (5.40), $\mathbb{D} = 1$. Thus

$$\begin{aligned}\langle\psi_{p,s}|M_{\rm orb}^{ij}|\psi_{p,s}\rangle = &\frac{1}{2M(p_0+M)}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i]\\ &- \frac{\mathbb{S}}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta + \frac{\mathbb{A}}{Mp_0}(p^i\epsilon^{0j\alpha\beta}\\ &- p^j\epsilon^{0i\alpha\beta} - p_0\epsilon^{ij\alpha\beta})S_\alpha p_\beta.\end{aligned} \tag{5.49}$$

[2]Note that here, in the general field theoretic case, $M_{\rm orb}$ is actually the sum of the quark orbital angular momentum plus the full angular momentum of the gluons.

We shall return to consider the interpretation of this result presently. First, though, we use it to deduce the structure of the matrix elements of J^{ij}, which, by Eq. (2.12), are the same as those of M^{ij}. The latter are built from the symmetric energy-momentum density $T^{\mu\nu} = \frac{1}{2}(T_C^{\mu\nu} + T_C^{\nu\mu})$. Since in Eq. (5.34) all terms are symmetric in μ, ν except the $\mathbb{A}$ term which is antisymmetric, we obtain the matrix elements of $T^{\mu\nu}$ by simply putting $\mathbb{A} = 0$. Then from Eq. (5.49)

$$\langle\psi_{p,s}|M^{ij}|\psi_{p,s}\rangle = \frac{1}{2M(p_0+M)}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i] - \frac{\mathbb{S}}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta. \tag{5.50}$$

We obtain the value of $\mathbb{S}$ by choosing a wave packet with $\boldsymbol{p} = (0, 0, p)$ and $\boldsymbol{s} = (0, 0, 1)$. This is then a helicity state $|\psi_{1/2}\rangle$ and should be an eigenstate of J_z with eigenvalue $1/2$. Hence for this state

$$\langle\psi_{1/2}|J_z|\psi_{1/2}\rangle = \langle\psi_{1/2}|M^{12}|\psi_{1/2}\rangle = 1/2 \tag{5.51}$$

which implies

$$\frac{1}{2} = -\frac{\mathbb{S}}{M}\epsilon^{1203}(S_0p_3 - S_3p_0) = \frac{\mathbb{S}}{M}(S_0p^3 - S^3p_0). \tag{5.52}$$

Now from Eq. (4.4), for the present case,

$$S_0 = p/M, \quad S^3 = 1 + \frac{p^2}{M(p_0+M)} = p_0/M. \tag{5.53}$$

Hence Eq. (5.52) becomes

$$\frac{1}{2} = \frac{\mathbb{S}}{M^2}(p^2 - p_0^2) = -\mathbb{S}. \tag{5.54}$$

Putting this into Eq. (5.50) gives

$$\langle\psi_{p,s}|M^{ij}|\psi_{p,s}\rangle = \frac{1}{2M}\left\{\frac{1}{p_0+M}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i] + \epsilon^{ij\alpha\beta}S_\alpha p_\beta\right\}. \tag{5.55}$$

Let us first compare this result to what we obtained in Eqs. (5.20) and (5.24) for the relativistic quantum-mechanical Dirac particle. The last term in Eq. (5.55) can be written

$$\begin{aligned}\epsilon^{ij\alpha\beta}S_\alpha p_\beta &= \epsilon^{ij0\beta}[S_0p_\beta - S_\beta p_0] \\ &= \epsilon^{ijk}\left[p_0S^k - \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{M}p^k\right] \\ &= \epsilon^{ijk}\left[p_0\left(s^k + \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{M(p_0+M)}p^k\right) - \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{M}p^k\right] \\ &= \epsilon^{ijk}\left[p_0s^k - \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p_0+M}p^k\right].\end{aligned} \tag{5.56}$$

Putting this into Eq. (5.55) we have, finally,

$$\begin{aligned}\langle\psi_{p,s}|M^{ij}|\psi_{p,s}\rangle &= \frac{\epsilon^{ijk}}{2M}\left\{\frac{1}{p_0+M}[\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{s})]^k + p_0s^k - \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p_0+M}p^k\right\} \\ &= \frac{\epsilon^{ijk}}{2M}\left(p_0 - \frac{\boldsymbol{p}^2}{p_0+M}\right)s^k = \frac{\epsilon^{ijk}}{2}s^k.\end{aligned} \tag{5.57}$$

This is exactly the result found in Eqs. (5.20) and (5.24). Taken in conjunction with Eqs. (2.9), (2.10), and (2.12), the result Eq. (5.57) corresponds to the first term in Eq. (1.1). The derivative term in the latter equation is missing here because of our use of a wave packet. We shall compare Eq. (5.57) to the results of Refs. [3,4] in Sec. VI.

Let us return now to the canonical form of the angular momentum. It is shown in Ref. [4] for QCD that aside from the terms which vanish by virtue of the equations of motion or which give no contribution to forward matrix elements

$$M^{\mu\nu\lambda}(x) = M^{\mu\nu\lambda}_{\rm orb}(x) + M^{\mu\nu\lambda}_{\rm axial}(x) \tag{5.58}$$

where in the convention $\epsilon^{0123} = +1$, the axial density is

$$M^{\mu\nu\lambda}_{\rm axial}(x) = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\alpha}\bar\psi(x)\gamma_5\gamma_\alpha\psi(x) \tag{5.59}$$

and a sum over the color labels of the quark fields is implied.

Note that Eq. (5.59) is not the same as the spin density in Eq. (5.12), for arbitrary values of μ, ν, λ. But Eqs. (5.12) and (5.59) are equal if $\mu \neq \nu, \lambda$. Thus

$$M^{0ij}_{\rm axial}(x) = M^{0ij}_{{\rm spin}f}(x) \tag{5.60}$$

where "spin f" means the fermionic spin terms in the angular momentum operator. This is the basis for the result in Eq. (2.12). The axial density is a local operator, so that

$$\langle p', \boldsymbol{s}|\int d^3x M^{\mu\nu\lambda}_{\rm axial}(x)|p, \boldsymbol{s}\rangle = (2\pi)^3\delta^3(\boldsymbol{p}' - \boldsymbol{p}) \times \langle p', \boldsymbol{s}|M^{\mu\nu\lambda}_{\rm axial}(0)|p, \boldsymbol{s}\rangle. \tag{5.61}$$

The only possible structure for the matrix element on the right-hand side is

$$\langle p, \boldsymbol{s}|M^{\mu\nu\lambda}_{\rm axial}(0)|p, \boldsymbol{s}\rangle = -a_0\epsilon^{\mu\nu\lambda\rho}S_\rho \tag{5.62}$$

where a_0 is known as the axial charge. Thus for the expectation value

$$\frac{\langle p', \boldsymbol{s}|\int d^3x M^{\mu\nu\lambda}_{\rm axial}(x, 0)|p, \boldsymbol{s}\rangle}{\langle p', \boldsymbol{s}|p, \boldsymbol{s}\rangle} = -\frac{a_0}{2p_0}\epsilon^{\mu\nu\lambda\rho}S_\rho. \tag{5.63}$$

From Eq. (5.60) we thus have for the spin density

$$\frac{\langle p', s| \int d^3x M^{0ij}_{\text{spin}f}(x,0)|p, s\rangle}{\langle p', s|p, s\rangle} = -\frac{a_0}{2p_0}\epsilon^{ijk}S_k. \quad (5.64)$$

After a little algebra, using Eq. (4.4), we write the fermionic part of the spin vector $\mathbf{S}^f$ of the nucleon in terms of the independent vectors and $\boldsymbol{p}$ and $\boldsymbol{s}$[3]:

$$S^f_k = \frac{a_0}{2p_0}\left[Ms_k + \frac{(\boldsymbol{p}\cdot\boldsymbol{s})}{p_0+M}p_k\right]. \quad (5.65)$$

Similarly, for the vector representing the fermionic orbital angular momentum vector plus the full gluon angular momentum, Eq. (5.49) yields

$$L^f_k + J^G_k = \left(\frac{1}{2} - \mathbb{A}\frac{M}{p_0}\right)s_k - \mathbb{A}\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p_0(p_0+M)}p_k. \quad (5.66)$$

Adding Eqs. (5.65) and (5.66) gives

$$\mathbf{J} = \left[\frac{1}{2} - \frac{M}{p_0}\left(\mathbb{A} - \frac{a_0}{2}\right)\right]\boldsymbol{s} - \left(\mathbb{A} - \frac{a_0}{2}\right)\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{p_0(p_0+M)}\boldsymbol{p}. \quad (5.67)$$

Then Eq. (5.57) implies that

$$\mathbb{A} = \frac{a_0}{2} \quad (5.68)$$

and the contribution from the axial charge cancels against the antisymmetric term in the fermionic orbital plus full gluonic angular momentum.

This supports the claim in Ref. [4] that the axial contribution totally cancels out when taking forward matrix elements of Eq. (5.58), at least for the $0ij$ elements of $M^{\mu\nu\lambda}$. So although the proof given in Ref. [4] is invalid because it suffers from the problems mentioned in Comments 1 and 3, it is correct in the contention that the axial anomaly cancels out of the angular momentum sum rules.[4] As the authors of Ref. [4] realize [cf. their Eq. (21)], this must occur: because of the nonrenormalization of the energy-momentum tensor, the total angular momentum must not be anomalous. See also Ref. [13].

VI. AN INDEPENDENT APPROACH

We have argued that the forms for the angular momentum tensor given in Refs. [3,4] are incorrect, in spite of their appearance of explicit Lorentz invariance. On the other hand our procedure has led to the surprising result Eq. (5.57) that for a spin-$\frac{1}{2}$ particle described by a canonical spin state

$$\langle p', s|M^{ij}|p, s\rangle = 2p_0(\epsilon^{ijk}s_k/2 + ip^i\partial^j - ip^j\partial^i) \times (2\pi)^3\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}), \quad (6.1)$$

though the derivatives of the delta functions shown here disappeared in our wave-function treatment, via partial integration, for the kind of wave packet chosen in Sec. V. The key part of Eq. (6.1), namely, the first term on the right-hand side, was derived in several different ways, where always careful attention was paid to the definition of matrix elements required by the compound nature of the angular momentum density operator. However, it does not share the explicit Lorentz invariance of the form Eq. (4.23) and so one may question its correctness.

In this section we will confront this issue by deriving Eq. (6.1) by a totally different method, *valid for arbitrary spin*, which is based on the rotational properties of states, and which circumvents completely the use of the energy-momentum tensor, and demonstrate that Eq. (6.1) is exactly what is expected on very general grounds. We shall then show that our result does have the correct Lorentz transformation properties. Finally we shall derive the analog of Eq. (6.1) for particles described by Jacob-Wick helicity states.

A. Canonical spin state matrix elements

In order to utilize the rotational properties of the canonical or boost spin states we need to display explicitly the Wigner boost operators used in defining the states of a moving particle in terms of the rest-frame spin states quantized in the z direction $|0, m\rangle$. Then the transformation properties of the states become explicit.

We will use the definitions of the Lorentz group generators given in Weinberg [14]. In this section we will write the three-vector operators in terms of the integrated tensor M^{ij}

$$J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk}, \quad (6.2)$$

$$K_i = M^{i0} \quad (6.3)$$

where on the left-hand side $i = x, y, z$. For a particle of mass M in motion the boost (or canonical) state is defined by

$$|p, m\rangle = B(\boldsymbol{v})|0, m\rangle = \exp(i\zeta\hat{\boldsymbol{p}}\cdot\boldsymbol{K})|0, m\rangle, \quad (6.4)$$

where $\boldsymbol{v} = \boldsymbol{p}/p_0$, $\cosh\zeta = p_0/M$, and $\hat{\boldsymbol{p}}$ is the unit vector along $\boldsymbol{p}$.

Now consider a rotation about axis i through an angle β. The unitary operator which effects this is given in terms of the angular momentum operator J_i:

$$R_i(\beta) = \exp(-i\beta J_i) \quad (6.5)$$

and for a particle of arbitrary spin s

$$R_i(\beta)|p, m\rangle = |R_i(\beta)p, n\rangle\mathcal{D}^s_{nm}[R_W(p, \beta)], \quad (6.6)$$

where $R_W(p, \beta)$ is the Wigner rotation. In a slight abuse of

[3]Note that we are deliberately not calling $\mathbf{S}^f$ the quark spin contribution to the nucleon spin. In an interacting theory with an anomaly the connection between $\mathbf{S}^f$ and the spin carried by the quarks is subtle since a_0 contains an anomalous gluon contribution, as explained in Ref. [10].

[4]This does not mean that the spin carried by the quarks does not contribute to the spin as will become clear in Sec. VII.

notation, we will use the symbol R to denote both the unitary operator for rotations in Hilbert space and the corresponding rotation matrix in Minkowski space. The same will be done for boosts, B. For a pure rotation the Wigner rotation R_W is very simple

$$R_W(p, \beta) = R_i(\beta);$$

independent of p; cf., for example, Ref. [10]. Therefore,

$$\begin{aligned}\langle p', m'|R_i(\beta)|p, m\rangle &= \langle p', m'|R_i(\beta)p, n\rangle \mathcal{D}^s_{nm}[R_i(\beta)] \\ &= 2p_0(2\pi)^3\delta^{(3)}[\boldsymbol{p}' - R_i(\beta)\boldsymbol{p}] \\ &\quad \times \mathcal{D}^s_{m'm}[R_i(\beta)],\end{aligned} \tag{6.7}$$

using the conventional normalization

$$\langle p', m'|p, m\rangle = 2p_0(2\pi)^3\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\delta_{m'm}. \tag{6.8}$$

Thus

$$\langle p', m'|J_i|p, m\rangle = i\frac{\partial}{\partial\beta}\langle p', m'|R_i(\beta)|p, m\rangle|_{\beta=0} \tag{6.9}$$

$$\begin{aligned}&= 2p_0(2\pi)^3\Big\{i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\delta_{m'm} \\ &\quad + i\frac{\partial}{\partial\beta}\mathcal{D}^s_{m'm}[R_i(\beta)]|_{\beta=0}\Big\}\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}).\end{aligned} \tag{6.10}$$

Now [15]

$$i\frac{\partial}{\partial\beta}\mathcal{D}^s_{m'm}[R_i(\beta)]|_{\beta=0} = (\mathrm{S}_i)_{m'm} \tag{6.11}$$

where the three $(2s + 1)$ dimensional matrices S_i are the spin matrices for spin s which satisfy

$$[\mathrm{S}_j, \mathrm{S}_k] = i\epsilon_{jkl}\mathrm{S}_l. \tag{6.12}$$

Thus, our final result for the matrix elements of the angular momentum operators for arbitrary spin, from Eq. (6.10), becomes

$$\begin{aligned}\langle p', m'|J_i|p, m\rangle &= 2p_0(2\pi)^3\Big[\mathrm{S}_i + i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\Big]_{m'm} \\ &\quad \times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}).\end{aligned} \tag{6.13}$$

For spin-$\frac{1}{2}$, of course, the S_i are just $\frac{1}{2}$ times the Pauli matrices σ_i. For arbitrary spin they are still very simple:

$$\begin{aligned}(\mathrm{S}_z)_{m'm} &= m\delta_{m'm}, \\ (\mathrm{S}_x)_{m'm} &= \frac{1}{2}[C(s, m)\delta_{m',m+1} + C(s, -m)\delta_{m',m-1}], \\ (\mathrm{S}_y)_{m'm} &= \frac{-i}{2}[C(s, m)\delta_{m',m+1} - C(s, -m)\delta_{m',m-1}],\end{aligned} \tag{6.14}$$

where

$$C(s, m) = \sqrt{(s - m)(s + m + 1)}. \tag{6.15}$$

For the case of spin-$\frac{1}{2}$ Eq. (6.13) is exactly equivalent to the result quoted in Eq. (1.4) and which we obtained in Sec. V after much labor using the wave-packet approach. It is completely general. The second term will vanish if integrated over symmetric wave packets. However it must be kept for analyzing the transformation properties, as we will see, and must, as usual, always be interpreted in the sense of partial integration. It is very easy to verify that the form Eq. (6.13) satisfies the usual commutation relations and so is consistent with rotational invariance.

Combining the result Eq. (6.13) for the case of spin-$\frac{1}{2}$ with Eq. (1.3) leads directly to the result quoted in Eq. (1.1).

B. Lorentz invariance

We shall now demonstrate that, despite appearances to the contrary, Eq. (6.13) is consistent with Lorentz invariance. Because of the complicated algebra involved, we shall present the proof just for a particle of spin-$\frac{1}{2}$ and mass M. Under Lorentz transformations, $M^{i0} = -M^{0i} = K_i$ are brought in so we need the matrix elements of K_i. These can be obtained just as those for the angular momentum. For a boost of magnitude and direction $\boldsymbol{\omega}$

$$B(\boldsymbol{\omega}) = \exp(i\boldsymbol{\omega}\cdot\boldsymbol{K}). \tag{6.16}$$

Here we are using a natural shorthand for a boost Eq. (6.4) with velocity $\boldsymbol{v} = \hat{\boldsymbol{\omega}}\tanh(\omega) \approx \boldsymbol{\omega}$ for small $|\boldsymbol{\omega}|$. Proceeding as before we have

$$\langle p', m'|B(\boldsymbol{\omega})|p, m\rangle = \langle p', m'|B(\boldsymbol{\omega})p, n\rangle\mathcal{D}^{1/2}_{nm}[R_W(\boldsymbol{p}, \boldsymbol{\omega})]. \tag{6.17}$$

The Wigner rotation for this case is defined by

$$B[B(\boldsymbol{\omega})p]R_W(\boldsymbol{p}, \boldsymbol{\omega}) = B(\boldsymbol{\omega})B(\boldsymbol{p}) \tag{6.18}$$

and is much more complicated than for rotations. However, we will only need the Wigner angle for small Lorentz transformations for our discussion. By doing the matrix multiplication explicitly for small $\boldsymbol{\omega}$ we find the Wigner rotation is given, in magnitude and direction, by

$$\boldsymbol{\alpha} = \frac{\boldsymbol{p}\times\boldsymbol{\omega}}{p_0 + M}. \tag{6.19}$$

In this case the Wigner rotation angle depends on $\boldsymbol{p}$ and this leads to the necessarily more complicated commutation properties involving the boost generators. Differentiating Eq. (6.17) with respect to ω and then putting $\omega = 0$ produces

$$\begin{aligned}\langle p', m'|K_i|p, m\rangle &= 2p_0(2\pi)^3\Big[-ip_0\partial_i + \frac{1}{2}\frac{(\boldsymbol{p}\times\boldsymbol{\sigma})_i}{p_0 + M}\Big]_{m'm} \\ &\quad \times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}).\end{aligned} \tag{6.20}$$

We will now verify that the matrix elements of J_i and K_i transform among each other correctly. The Lorentz transformation of J_i under a boost $\boldsymbol{\omega}$ requires that

$$\langle p', m'|J_i|p, m\rangle = \langle B(\boldsymbol{\omega})p', n'|B(\boldsymbol{\omega})J_iB(\boldsymbol{\omega})^{-1}|B(\boldsymbol{\omega})p, n\rangle \times \mathcal{D}^{1/2*}_{n',m'}[R_W(\boldsymbol{p}', \boldsymbol{\omega})]\mathcal{D}^{1/2}_{n,m}[R_W(\boldsymbol{p}, \boldsymbol{\omega})]. \tag{6.21}$$

Notice that the two Wigner rotations appearing in Eq. (6.21) are not the same as long as $p \neq p'$. It is essential to keep this is mind because of the derivatives of δ functions that enter. If we write this out to terms linear in $\boldsymbol{\omega}$ and use

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \tag{6.22}$$

Eq. (6.21) becomes

$$\langle p', m'|J_i|p, m\rangle = \langle B(\boldsymbol{\omega})p', n'|J_i + \epsilon_{ijk}\omega_jK_k|B(\boldsymbol{\omega})p, n\rangle + \left(1 - i\frac{1}{2}\epsilon_{abc}\sigma_a\frac{\omega_b p'_c}{p_0 + M}\right)_{m'n'} \times \left(1 + i\frac{1}{2}\epsilon_{abc}\sigma_a\frac{\omega_b p_c}{p_0 + M}\right)_{nm}. \tag{6.23}$$

We must also expand the matrix element to first order in ω. Using Eqs. (6.20) and (6.13) with S_i replaced by $\sigma_i/2$, and recalling that $p_0\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}) = p'_0\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})$ is invariant under Lorentz transformations we find that

$$\langle B(\boldsymbol{\omega})p', n'|J_i + \epsilon_{ijk}\omega_jK_k|B(\boldsymbol{\omega})p, n\rangle = \left[i\epsilon_{ijk}p_j\partial_k + \frac{1}{2}\sigma_i + \frac{1}{2}\frac{[\boldsymbol{\omega} \times (\boldsymbol{p} \times \boldsymbol{\sigma})]_i}{p_0 + M}\right]_{n'n} 2p_0(2\pi)^3\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}). \tag{6.24}$$

If one inserts Eq. (6.24) into Eq. (6.23) and evaluates the terms proportional to ω one easily finds three components: one coming from the second term in Eq. (6.24) combined with the rotation matrices in Eq. (6.23), one coming from the third term in Eq. (6.24), and one from the first term in Eq. (6.24) acting on the rotation matrices in Eq. (6.23). It is essential to carry out the derivatives in this last component *before* setting $p' = p$:

$$\frac{[\boldsymbol{\omega} \times (\boldsymbol{p} \times \boldsymbol{\sigma})]_i}{2(p_0 + M)} - i\frac{[\boldsymbol{\sigma} \cdot (\boldsymbol{\omega} \times \boldsymbol{p}), \sigma_i]}{4(p_0 + M)} + i\epsilon_{ijk}p_j\partial_k \times \left[1 - i\frac{\boldsymbol{\sigma} \cdot (\boldsymbol{\omega} \times \boldsymbol{p}')}{2(p_0 + M)}\right]\left[1 + i\frac{\boldsymbol{\sigma} \cdot (\boldsymbol{\omega} \times \boldsymbol{p})}{2(p_0 + M)}\right]. \tag{6.25}$$

Use of the commutation relations of the Pauli matrices and the vector double cross product identity shows that the sum of these three pieces vanishes leaving just the matrix element of J_i as was to be shown in order to satisfy Eqs. (6.21) or (6.23).

Using the same techniques, one can show that the matrix elements of K_i as given in Eq. (6.20) also transform correctly under boosts and so Eqs. (6.13) and (6.20) form a representation of the Lorentz group.

C. Helicity state matrix elements

We now turn to the case of helicity states which have some rather surprising properties. One can proceed just as here; the main difference is that the Wigner rotation becomes a Wick helicity rotation, always about the z axis. This simplifies things somewhat; all the complication is in calculating the angle that results, the analog of Eq. (6.19). The result is also convention dependent, depending on whether one uses the original Jacob and Wick definition [7] or the later one due to Wick [9] [see Eqs. (3.2) and (3.3)]. We give here the result for the first case. The result of this messy calculation is that, for $\boldsymbol{p} = (p, \theta, \phi)$

$$\langle p', \lambda'|J_i|p, \lambda\rangle_{\mathrm{JW}} = (2\pi)^3 2p_0[\lambda\eta_i + i(\boldsymbol{p} \times \boldsymbol{\nabla})_i] \times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\delta_{\lambda'\lambda} \tag{6.26}$$

where

$$\eta_x = \cos(\phi)\tan(\theta/2), \quad \eta_y = \sin(\phi)\tan(\theta/2), \quad \eta_z = 1. \tag{6.27}$$

Although these components look a little odd—the singularity at $\theta = \pi$ results from the ambiguity of Jacob and Wick helicity states at that point—it is easy to verify some important properties: they are manifestly diagonal in λ, which is required since rotations preserve the helicity, and they satisfy

$$\langle p', \lambda'|\hat{\boldsymbol{p}} \cdot \boldsymbol{J}|p, \lambda\rangle_{\mathrm{JW}} = \lambda 2p_0(2\pi)^3\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\delta_{\lambda'\lambda} \tag{6.28}$$

and no orbital angular momentum piece survives as expected.

It is enlightening to consider these amplitudes from a different direction: comparing the definitions of canonical (boost) states to helicity states we have for the case of spin-$\frac{1}{2}$

$$\begin{aligned}|p, s\rangle &= |p, m\rangle\mathcal{D}^{1/2}_{m1/2}[R(s)] \\ &= |p, \lambda\rangle_{\mathrm{JW}}\mathcal{D}^{1/2}_{\lambda m}[R^{-1}(\boldsymbol{p})]\mathcal{D}^{1/2}_{m1/2}[R(s)] \\ &= |p, \lambda\rangle_{\mathrm{JW}}\mathcal{D}^{1/2}_{\lambda 1/2}[R^{-1}(\boldsymbol{p})R(s)].\end{aligned} \tag{6.29}$$

This has the appearance of an ordinary unitary change of basis, but because of the compound nature of J_i when we apply this to the canonical form, using the spin-$\frac{1}{2}$ version of Eq. (6.13), we get

$$\langle p', \lambda'|J_i|p, \lambda\rangle_{\mathrm{JW}} = (2\pi)^3 2p_0\mathcal{D}^{1/2}_{m\lambda}[R(\boldsymbol{p})]\mathcal{D}^{1/2}_{m'\lambda'}[R(\boldsymbol{p}')]^* \times \left[i\epsilon_{ijk}p_j\partial_k + \frac{1}{2}\sigma_i\right]_{m'm}\delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}). \tag{6.30}$$

We cannot use the unitarity of the $\mathcal{D}$'s because $\boldsymbol{p} \neq \boldsymbol{p}'$,

and we must first pass the first $\mathcal{D}^{1/2}[R(\boldsymbol{p})]$ through the derivative before setting them equal. This produces an extra term

$$-(2\pi)^3 2p_0 i\epsilon_{ijk}\mathcal{D}^{1/2}_{m'\lambda'}[R(\boldsymbol{p}')]^* p_j \partial_k \mathcal{D}^{1/2}_{m\lambda}[R(\boldsymbol{p})]\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{p}) \tag{6.31}$$

which is tedious to evaluate in the general case. The result of this labor is identical to Eqs. (6.26) and (6.27).

VII. APPLICATIONS: SUM RULES

Equipped with the expressions for the matrix elements of J_i derived in Secs. V and VI and summarized explicitly in Eqs. (1.4), (1.5), and (1.6), we will derive the general form for angular momentum sum rules for the nucleon and, in particular, will derive a new sum rule for transverse polarization. This differs from the sum rule that would be derived from Eq. (4.23).

A. The matrix elements of J: the J_z sum rule

The first term in our result Eq. (1.1) differs from the results of Jaffe and Manohar [3]. If we rewrite their expression Eq. (4.23) in terms of the independent vectors $\boldsymbol{p}$ and $\boldsymbol{s}$, we find, for the expectation value

$$\langle J_i\rangle_{\rm JM} = \frac{1}{4Mp^0}\left\{(3p_0^2 - M^2)s_i - \frac{3p_0+M}{p_0+M}(\boldsymbol{p}\cdot\boldsymbol{s})p_i\right\} \tag{7.1}$$

to be compared to

$$\langle J_i\rangle = \tfrac{1}{2}s_i \tag{7.2}$$

arising from the first term in Eq. (1.1). In general these are different. However, one may easily check that if $\boldsymbol{s} = \hat{\boldsymbol{p}}$ the Jaffe-Manohar value agrees with Eq. (7.2), while if $\boldsymbol{s} \perp \hat{\boldsymbol{p}}$ they are not the same.

The agreement for $\boldsymbol{s} = \hat{\boldsymbol{p}}$ is consistent with the much used and intuitive sum rule

$$\tfrac{1}{2} = \tfrac{1}{2}\Delta\Sigma + \Delta G + \langle L^q\rangle + \langle L^G\rangle \tag{7.3}$$

based on the matrix elements of J_z, for a proton moving along the z axis with helicity $\lambda = \frac{1}{2}$ (this, as explained in Sec. III, coincides with a canonical spin state), relating the component along $\boldsymbol{p}$ of the spin and orbital angular momentum of the quarks and gluons to the helicity of the proton.

B. General structure of sum rules

Consider a nucleon with momentum along OZ, $\boldsymbol{p} = (0, 0, p)$, in a canonical spin state with rest-frame spin eigenvector along $\boldsymbol{s}$, where $\boldsymbol{s}$ could be longitudinal $\boldsymbol{s}_L$ or transverse $\boldsymbol{s}_T$. Sum rules can be constructed by equating the expression Eq. (1.4) for the nucleon matrix elements $\langle \boldsymbol{p}', m'|J_i|\boldsymbol{p}, m\rangle$ with the expression obtained when the nucleon state is expressed in terms of the wave functions of its constituents (partons; quarks and gluons).

There is great interest in sum rules in which the partonic quantities can be related to other physically measurable quantities. The classic example of this is Eq. (7.3). We will now investigate other similar possibilities, using Eq. (1.4) as the relevant starting point.

We have stressed the importance of a wave-packet approach in order to deal with the derivative of the delta function in the equations above. As it happens, however, when constructing sum rules, the expression in terms of constituents automatically produces a term which cancels the delta function, irrespective of the actual model wave functions used.

The nucleon state is expanded as a superposition of n-parton Fock states, wherein, for the purpose of showing the structure of the sum rules, we will not display flavor and color labels. We use the "instant" form rather than the commonly used "light-cone" form [16] since it is more suitable for discussing rotational properties. We will use the original $p \to \infty$ limit in order to obtain the parton model sum rules.

$$\begin{aligned}|\boldsymbol{p}, m\rangle = [(2\pi)^3 2p_0]^{1/2}\sum_n\sum_{\{\sigma\}}\int \frac{d^3\boldsymbol{k}_1}{\sqrt{(2\pi)^3 2k_1^0}}\cdots\frac{d^3\boldsymbol{k}_n}{\sqrt{(2\pi)^3 2k_n^0}} \\ \times\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1,\sigma_1,\ldots\boldsymbol{k}_n,\sigma_n) \\ \times\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}_1-\ldots-\boldsymbol{k}_n)|\boldsymbol{k}_1,\sigma_1,\ldots\boldsymbol{k}_n,\sigma_n\rangle\end{aligned} \tag{7.4}$$

where σ_i denotes either the spin projection on the z axis or the helicity, as appropriate. $\psi_{\boldsymbol{p},m}$ is the partonic wave function of the nucleon normalized so that

$$\sum_{\{\sigma\}}\int d^3\boldsymbol{k}_1\ldots d^3\boldsymbol{k}_n|\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1,\sigma_1,\ldots\boldsymbol{k}_n,\sigma_n)|^2\times \delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}_1-\ldots-\boldsymbol{k}_n) = \mathcal{P}_n \tag{7.5}$$

with $\mathcal{P}_n$ denoting the probability of the n-parton state.

The n-parton contribution is then

$$\begin{aligned}\langle \boldsymbol{p}', m'|J_i|\boldsymbol{p}, m\rangle^{\text{n-parton}} = (2\pi)^3 2p_0\sum_{\sigma,\sigma'}\int[d^3k_1']\ldots[d^3k_n'][d^3k_1]\ldots[d^3k_n]\psi^*_{\boldsymbol{p}',m}(\boldsymbol{k}_1',\sigma_1',\ldots\boldsymbol{k}_n',\sigma_n') \\ \times\langle\boldsymbol{k}_1',\sigma_1',\ldots,\boldsymbol{k}_n',\sigma_n'|J_i|\boldsymbol{k}_1,\sigma_1,\ldots,\boldsymbol{k}_n,\sigma_n\rangle\psi_{\boldsymbol{p}m}(\boldsymbol{k}_1,\sigma_1,\ldots\boldsymbol{k}_n,\sigma_n)\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{k}_1'-\ldots-\boldsymbol{k}_n') \\ \times\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}_1-\ldots-\boldsymbol{k}_n)\end{aligned} \tag{7.6}$$

where we use the notation

$$[d^3k] = \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3 2k^0}}. \tag{7.7}$$

We take for the Fock-state matrix elements

$$\langle \boldsymbol{k}'_1, \sigma'_1, \ldots, \boldsymbol{k}'_n, \sigma'_n | J_i | \boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}_n, \sigma_n \rangle = \sum_r \langle \boldsymbol{k}'_r, \sigma'_r | J_i | \boldsymbol{k}_r, \sigma_r \rangle \prod_{l \neq r} (2\pi)^3 2k^0_l \delta^{(3)}(\boldsymbol{k}'_l - \boldsymbol{k}_l) \delta_{\sigma'_l \sigma_l}. \tag{7.8}$$

so

$$\langle \boldsymbol{p}', m' | J_i | \boldsymbol{p}, m \rangle = (2\pi)^3 2p_0 \sum_{n,r} \sum_{\sigma_i} \sum_{\sigma'_r} \int [d^3k'_r] \int d^3k_1 \ldots [d^3k_r] \ldots d^3k_n \langle \boldsymbol{k}'_r, \sigma'_r | J_i | \boldsymbol{k}_r, \sigma_r \rangle \psi^*_{\boldsymbol{p}', m'}(\boldsymbol{k}_1, \sigma_1, \ldots \boldsymbol{k}'_r, \sigma'_r, \ldots \boldsymbol{k}_n, \sigma_n)$$
$$\times \psi_{\boldsymbol{p}, m}(\boldsymbol{k}_1, \sigma_1, \ldots \boldsymbol{k}_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n) \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{k}_1 - \boldsymbol{k}_2 \ldots - \boldsymbol{k}'_r \ldots - \boldsymbol{k}_n) \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \boldsymbol{k}_2 \ldots - \boldsymbol{k}_r \ldots - \boldsymbol{k}_n). \tag{7.9}$$

After some manipulation this can be written as:

$$\langle \boldsymbol{p}', m' | J_i | \boldsymbol{p}, m \rangle = (2\pi)^3 2p_0 \sum_n \sum_{\sigma, \sigma'} \int d^3k d^3k' \times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p} + \boldsymbol{k} - \boldsymbol{k}') \rho^{m'm}_{\sigma'\sigma}(\boldsymbol{k}', \boldsymbol{k})^a \times \frac{1}{\sqrt{(2\pi)^3 2k'_0}} \langle \boldsymbol{k}', \sigma' | J_i | \boldsymbol{k}, \sigma \rangle \frac{1}{\sqrt{(2\pi)^3 2k_0}} \tag{7.10}$$

where we have introduced a density matrix for the internal motion of type "a" partons in a proton of momentum $\boldsymbol{p}$:

$$\rho^{m'm}_{\sigma'\sigma}(\boldsymbol{k}', \boldsymbol{k})^a \equiv \sum_{n, r(a)} \sum_{\sigma_i} \sum_{\sigma'_r} \delta_{\sigma\sigma_r} \delta_{\sigma'\sigma'_r} \times \int d^3k'_r d^3k_1 \ldots d^3k_r \ldots d^3k_n \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}_r) \times \delta^{(3)}(\boldsymbol{k}' - \boldsymbol{k}'_r) \psi^*_{p'm'}(\boldsymbol{k}_1, \sigma_1, \ldots \boldsymbol{k}'_r, \sigma'_r, \ldots \boldsymbol{k}_n, \sigma_n) \times \psi_{pm}(\boldsymbol{k}_1, \sigma_1, \ldots \boldsymbol{k}_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n) \times \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \ldots - \boldsymbol{k}_r \ldots - \boldsymbol{k}_n). \tag{7.11}$$

Here a, which we will frequently suppress, denotes the type of parton: quark, antiquark or gluon. The sum goes over all Fock states and, within these states, over the spin and momentum labels r corresponding to the parton type a. Equations (7.10) and (7.11) are the basis for the angular momentum sum rules.

The two terms in Eq. (1.4) applied to the parton matrix elements in Eq. (7.10) suggest a spin part and an orbital part for quarks and gluons. This decomposition is a little misleading at this level, as we will see, but we will use it here to organize the various pieces. First consider the spin part of the matrix element when k is the momentum carried by a quark.

$$\langle \boldsymbol{p}', m' | J_i | \boldsymbol{p}, m \rangle^{\text{quarkspin}} = (2\pi)^3 2p_0 \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}) \times \int d^3k d^3k' \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \times \sum_{\sigma, \sigma'} \frac{1}{2} (\boldsymbol{\sigma}_i)_{\sigma'\sigma} \rho^{m'm}_{\sigma'\sigma}(\boldsymbol{k}', \boldsymbol{k})^q, \tag{7.12}$$

where here $\boldsymbol{\sigma}_i$ denotes the Pauli spin matrix of Eq. (1.4).

The spin part for the gluons is completely analogous, but now σ and σ' in Eq. (7.10) refer to the gluon helicity λ. From Eq. (1.5), which is diagonal in helicity, we obtain

$$\langle \boldsymbol{p}', m' | J_i | \boldsymbol{p}, m \rangle^{\text{gluonspin}} = (2\pi)^3 2p_0 \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p}) \times \int d^3k d^3k' \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \times \eta_i \lambda \rho^{m'm}_{\lambda\lambda}(\boldsymbol{k}', \boldsymbol{k})^G. \tag{7.13}$$

The orbital part is somewhat different because of the derivative of the δ function that enters, and we have stressed the need for a proper wave-packet treatment, as carried out in Sec. V. However in Eq. (7.9) or (7.10) the partons are not in plane-wave states and the partonic wave function ψ plays the role of a wave packet. Thus we may proceed directly by inserting the orbital piece of Eq. (1.4) into Eq. (7.9).

We have

$$\langle \boldsymbol{p}', m' | J_i | \boldsymbol{p}, m \rangle^{\text{orbital}} = (2\pi)^3 2p_0 \sum_n \sum_{\{\sigma\}} \sum_r \int d^3k'_r d^3k_1 \ldots d^3k_r \ldots d^3k_n \sqrt{\frac{k^0_r}{k'^0_r}} \psi^*_{\boldsymbol{p}', m'}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}'_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n)$$
$$\times \psi_{\boldsymbol{p}, m}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n) \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{k}_1 - \ldots - \boldsymbol{k}'_r - \ldots \boldsymbol{k}_n) \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \ldots - \boldsymbol{k}_r - \ldots \boldsymbol{k}_n)$$
$$\times i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i \delta^{(3)}(\boldsymbol{k}'_r - \boldsymbol{k}_r). \tag{7.14}$$

Integrating over $\boldsymbol{k}_r$ by parts yields

$$\langle \boldsymbol{p}', m'|J_i|\boldsymbol{p}, m\rangle^{\text{orbital}} = -(2\pi)^3 2p_0 \sum_n \sum_{\{\sigma\}} \sum_r \int d^3k'_r d^3k_1 \ldots d^3k_r \ldots d^3k_n \sqrt{\frac{k_r^0}{k_r'^0}} \psi^*_{\boldsymbol{p}',m'}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}'_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n)$$
$$\times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{k}_1 \ldots - \boldsymbol{k}'_r \ldots - \boldsymbol{k}_n)\delta^{(3)}(\boldsymbol{k}'_r - \boldsymbol{k}_r) i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i [\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_r \ldots, \boldsymbol{k}_r, \sigma_r \ldots \boldsymbol{k}_n, \sigma_n)$$
$$\times \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \ldots - \boldsymbol{k}_r \ldots - \boldsymbol{k}_n)]. \tag{7.15}$$

The derivative produces two terms. The one arising from the derivative of the delta function is

$$= -(2\pi)^3 2p_0 \sum_n \sum_{\{\sigma\}} \int d^3k_1 \ldots\ldots d^3k_n \psi^*_{\boldsymbol{p},m'}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}_n, \sigma_n)\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}_n, \sigma_n)$$
$$\times \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{k}_1 \ldots - \boldsymbol{k}_n) \sum_r i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \ldots - \boldsymbol{k}_n). \tag{7.16}$$

Now it is easy to check that

$$\sum_r i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \ldots - \boldsymbol{k}_r - \ldots - \boldsymbol{k}_n)$$
$$= -(\boldsymbol{p} \times \boldsymbol{\nabla}_p)_i \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \ldots - \boldsymbol{k}_n) \tag{7.17}$$

and putting this into Eq. (7.16) and using the normalization and orthogonality of the wave functions, this term produces

$$2p_0(2\pi)^3 i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\delta_{mm'} \tag{7.18}$$

which just cancels the derivative of the delta function in Eq. (1.4).

The other term in the differentiation in Eq. (7.16) yields

$$2p_0(2\pi)^3 \delta^{(3)}(\boldsymbol{p}' - \boldsymbol{p})\langle L_i\rangle^a_{m'm} \tag{7.19}$$

where $\langle L_i\rangle^a_{m'm}$ is the contribution from the internal orbital angular momentum arising from partons of type a, given by

$$\langle L_i\rangle^a_{m'm} = \sum_n \sum_{\{\sigma\}} \int d^3k_1 \ldots\ldots d^3k_n \psi^*_{\boldsymbol{p},m'}(\boldsymbol{k}_1, \sigma_1, \ldots, \boldsymbol{k}_n, \sigma_n) \sum_{r(a)} \{[-i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i]$$
$$\times \psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, \ldots \boldsymbol{k}_r, \sigma_r, \ldots \boldsymbol{k}_n, \sigma_n)\}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - \ldots - \boldsymbol{k}_n) \tag{7.20}$$

where the sum over $r(a)$ means a sum over those r values corresponding to partons of type a. Note that a can refer to both quarks and gluons; the structure of Eq. (7.20) is the same for both. Note also that the orbital angular momentum defined in this way is not in general the same as that given by the matrix element M^{ij}_{orb}, Eqs. (5.49) and (5.66), which contain gluon spin-dependent parts which here are included in the spin part of the matrix elements of J_i. This difference is important for transverse polarization but not for longitudinal polarization. Furthermore, it is important to realize that the orbital angular momentum defined in this way depends on the basis states used for the partons. In particular, because of the momentum dependence of the transformation from canonical to helicity basis, Eq. (6.29),

$$\rho^{m'm}_{\lambda'\lambda}(\boldsymbol{k}', \boldsymbol{k}) = \mathcal{D}^{1/2}_{m'_r\lambda'}[R(\boldsymbol{k}')]^* \rho^{m'm}_{m'_r m_r}(\boldsymbol{k}', \boldsymbol{k}) \mathcal{D}^{1/2}_{m_r\lambda}[R(\boldsymbol{k})] \tag{7.21}$$

the orbital angular momentum will not be the same in the two bases; cf. Eqs. (6.30) and (6.31). Of course, for $k_z \to \infty$ with finite k_x, k_y the z component will be the same but the other components will not be.

In this discussion we have used a fixed-axis quantization for the quarks since it has a more transparent connection to the polarization states of the proton, which in most cases are more naturally described in that way. At the same time, because the gluons are massless it is most natural to use helicity states to describe them, so we are led to a mixed notation. There is no real problem with this, but there may be occasions where one wishes to treat the quarks in helicity states as well.

Putting Eqs. (7.19), (7.12), and (7.13) into Eq. (7.10), utilizing Eq. (1.4) for its left-hand side, and cancelling the factors $2p_0(2\pi)^3\delta(\boldsymbol{p}' - \boldsymbol{p})$, we end up with the general sum rule for a spin-$\frac{1}{2}$ nucleon:

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i)_{m'm} = \int d^3\boldsymbol{k} [\tfrac{1}{2}(\boldsymbol{\sigma}_i)_{\sigma'\sigma} \rho^{m'm}_{\sigma'\sigma}(\boldsymbol{k}, \boldsymbol{k})^{q+\bar{q}}$$
$$+ \lambda \eta_i(\boldsymbol{k}) \rho^{m'm}_{\lambda\lambda}(\boldsymbol{k}, \boldsymbol{k})^G] + \langle L_i\rangle^{q+\bar{q}}_{m'm} + \langle L_i\rangle^G_{m'm} \tag{7.22}$$

where η_i is given in Eq. (6.27).

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C. A new sum rule

For proton matrix elements of J_z, Eq. (7.22) is nonvanishing only for $m' = m$. It then becomes the classic sum rule Eq. (7.3). There is one other independent sum rule that can be obtained from this general one; one way to obtain it is to consider the matrix elements of J_x which are nonvanishing only for $m' = -m$. The left-hand side of Eq. (7.22) is then equal to $\frac{1}{2}$. The quark spin contribution to the right-hand side (an identical expression holds for the antiquarks) is

$$\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}[\rho^{+-}_{+-} + \rho^{+-}_{-+} + \rho^{-+}_{-+} + \rho^{-+}_{+-}]^q \tag{7.23}$$

where $+/-$ refers to $\pm\frac{1}{2}$. By rotating the system through π about the z axis, it is easy to see that elements of $\rho^{m',m}_{\sigma',\sigma}$ with $(-1)^{m-m'-\sigma+\sigma'} = -1$ are odd under this rotation and so will integrate to zero when integrated over $\boldsymbol{k}_T$. This enables us to rewrite the expression (7.23), the quark contribution, in a way that has a nice interpretation, viz.

$$\tfrac{1}{2}\int d^3\boldsymbol{k}\tfrac{1}{2}[\rho^{++}_{+-} + \rho^{++}_{-+} + \rho^{--}_{+-} + \rho^{--}_{-+} + \rho^{+-}_{+-} + \rho^{+-}_{-+} + \rho^{-+}_{-+} + \rho^{-+}_{+-}]^q. \tag{7.24}$$

Consider the proton state with spin oriented along OX, perpendicular to the direction of motion

$$|\boldsymbol{p}, \boldsymbol{s}_x\rangle = \frac{1}{\sqrt{2}}\{|\boldsymbol{p}, m = 1/2\rangle + |\boldsymbol{p}, m = -1/2\rangle\}. \tag{7.25}$$

To understand the content of expression (7.24) write schematically

$$\rho^{m'm}_{\sigma'\sigma} = \sum_{X=\text{all}} \psi^*_{m'}(\sigma', X)\psi_m(\sigma, X). \tag{7.26}$$

Now the number density of quarks with spin along or opposite to OX, denoted by $\pm\hat{\boldsymbol{s}}_x$ in a proton spinning along OX is

$$q_{\pm\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(\boldsymbol{k}) = \sum_{X=\text{all}} |\psi_{\boldsymbol{s}_x}(\pm\hat{\boldsymbol{s}}_x, X)|^2 \tag{7.27}$$

where

$$\psi_{\boldsymbol{s}_x}(\pm\hat{\boldsymbol{s}}_x) = \tfrac{1}{2}[\psi_+(+) \pm \psi_+(-) + \psi_-(+) \pm \psi_-(-)] \tag{7.28}$$

so that (suppressing the $\sum_{X=\text{all}}$)

$$q_{\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(\boldsymbol{k}) - q_{-\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(\boldsymbol{k}) = \text{Re}\{[\psi_+(+) + \psi_-(+)]^* \times [\psi_-(-) + \psi_+(-)]\} \tag{7.29}$$

which, via Eq. (7.26), is exactly the integrand in Eq. (7.24). Thus the expression (7.23) is equal to

$$\tfrac{1}{2}\int d^3\boldsymbol{k}[q_{\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(\boldsymbol{k}) - q_{-\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(\boldsymbol{k})] = \tfrac{1}{2}\int dx d^2\boldsymbol{k}_T[q_{\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(x, \boldsymbol{k}_T) - q_{-\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(x, \boldsymbol{k}_T)] \tag{7.30}$$

and there is an analogous term for the antiquarks.

Now it is known [17] that

$$q^a_{\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(x, \boldsymbol{k}_T) - q^a_{-\hat{\boldsymbol{s}}_x/\boldsymbol{s}_x}(x, \boldsymbol{k}_T) = \Delta'_T q^a(x, k_T^2) + \cos 2\phi \frac{k_T^2}{2M^2} h_{1T}^{\perp a}(x, k_T^2) + \sin\phi \frac{k_T}{M} h_1^{\perp a}(x, k_T^2), \tag{7.31}$$

where ϕ is the azimuthal angle of $\boldsymbol{k}_T$, $\Delta'_T q^a(x, k_T^2)$ is the same as $h_1^a(x, k_T^2)$ in the notation of Ref. [18], and

$$\Delta_T q^a(x) \equiv h_1^a(x) = \int d^2\boldsymbol{k}_T \Delta'_T q^a(x, k_T^2). \tag{7.32}$$

Substituting Eq. (7.31) into Eq. (7.30) and integrating over the direction of $\boldsymbol{k}_T$, we end up with the quark contribution to the right-hand side of Eq. (7.22):

$$\tfrac{1}{2}\int dx h_1(x) = \tfrac{1}{2}\int dx \sum_{a,\bar{a}} h_1^a(x) \equiv \tfrac{1}{2}\int dx \sum_{a,\bar{a}} \Delta_T q^a(x). \tag{7.33}$$

We turn now to the gluon contribution to the right-hand side of Eq. (7.22), which is

$$\int d^3\boldsymbol{k}\,\eta_x(\boldsymbol{k})(\tfrac{1}{2}[\rho^{++}_{11} - \rho^{++}_{-1-1} + \rho^{--}_{11} - \rho^{--}_{-1-1} + \rho^{+-}_{11} - \rho^{+-}_{-1-1} + \rho^{-+}_{11} - \rho^{-+}_{-1-1}]), \tag{7.34}$$

where ± 1 refers to the gluon helicity. Once again we have added in terms which integrate to zero in order to get a nice interpretation in terms of densities. [Recall that η_x contains the factor $\cos\phi$, and the factors $\rho^{m',m}_{\lambda'\lambda}$ with $(-1)^{m'-m-\lambda'+\lambda} = +1$ are even under $\phi \to \pi \pm \phi$.]

Now consider

$$\Delta G_{h/\boldsymbol{s}_x} \equiv G_{1/\boldsymbol{s}_x} - G_{-1/\boldsymbol{s}_x} = \sum_{X=\text{all}} \{|\psi_{\boldsymbol{s}_x}(1, X)|^2 - |\psi_{\boldsymbol{s}_x}(-1, X)|^2\}. \tag{7.35}$$

Carrying out the analog of Eq. (7.28) we find that the right-hand side of Eq. (7.35) is exactly equal to the terms in parenthesis in Eq. (7.34). Thus the gluon contribution to the right-hand side of Eq. (7.22) is

$$\int d^3\boldsymbol{k}\,\eta_x(\boldsymbol{k})\Delta G_{h/\boldsymbol{s}_x}(\boldsymbol{k}) = \int dx d^2\boldsymbol{k}_T \eta_x(\boldsymbol{k})\Delta G_{h/\boldsymbol{s}_x}(x, \boldsymbol{k}_T). \tag{7.36}$$

It is easy to see, geometrically, that $\Delta G_{h/\boldsymbol{s}_x}(x, \boldsymbol{k}_T)$ contains a factor k_x and we make this explicit by writing [18]

$$\Delta G_{h/s_x}(x, \boldsymbol{k}_T) = \frac{k_x}{M} g_{1T}^G(x, k_T^2). \tag{7.37}$$

Then the contribution of the gluon spin to the right-hand side of Eq. (7.22), i.e., to the proton whose spin is in the x direction, is

$$\begin{aligned}\Delta G_{h/s_x} &= \int dx d^2\boldsymbol{k}_T \eta_x \frac{k_x}{M} g_{1T}^G(x, k_\perp^2) \\ &= \pi \int dx k_T dk_T \frac{\sqrt{x^2p^2 + k_T^2} - xp}{M} g_{1T}^G(x, k_T^2),\end{aligned} \tag{7.38}$$

where we have used Eq. (6.27). As $p \to \infty$ this piece vanishes and so the gluon spin does not contribute to the transverse spin sum rule.

Finally, the internal orbital angular momentum terms $\langle L_x \rangle_{s_x}^q$ and $\langle L_x \rangle_{s_x}^G$ are obtained from Eq. (7.20) by the replacement

$$\psi_{\boldsymbol{p},m} \to \psi_{\boldsymbol{p},s_x} = \frac{1}{\sqrt{2}}[\psi_{\boldsymbol{p},1/2} + \psi_{\boldsymbol{p},-1/2}]. \tag{7.39}$$

Putting together the various pieces of the right-hand side of Eq. (7.22) we obtain a new, transverse spin sum rule. Since the same result holds when considering J_y with the proton polarized along OY, we prefer to state the result in the more general form: for a proton in an eigenstate of transverse spin with eigenvector along $\boldsymbol{s}_T$

$$\tfrac{1}{2} = \tfrac{1}{2}\sum_{q,\bar{q}} \int dx \Delta_T q^a(x) + \sum_{q,\bar{q},G} \langle L_{s_T} \rangle^a \tag{7.40}$$

where L_{s_T} is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$. No such sum rule is possible with the Jaffe-Manohar formula because, as $p \to \infty$, Eq. (7.1) for $i = x, y$ diverges.

This has a very intuitive appearance, very similar to Eq. (7.3), but there are a couple of points to be made. First of all, the right-hand side of this sum rule is distinct from the expression for the tensor charge δ of Jaffe and Ji [19] because here the quarks and antiquarks *add* whereas in δ the difference enters. It is possible that this difference may be of some use in disentangling the quark and antiquark transverse spin structure functions. Second, the integral does not correspond to the matrix element of the axial vector current, which via Eq. (5.65), vanishes for $p \to \infty$ when $\boldsymbol{p} \cdot \boldsymbol{s} = 0$.

The structure functions $\Delta_T q^a(x) \equiv h_1^q(x)$ are most directly measured in doubly polarized Drell-Yan reactions where the asymmetry is proportional to

$$\sum_a e_a^2[\Delta_T q^a(x_1) \Delta_T \bar{q}^a(x_2) + (1 \leftrightarrow 2)]. \tag{7.41}$$

For a detailed discussion see Ref. [17].

They can also be determined from the asymmetry in semi-inclusive hadronic interactions like

$$p + p(s_T) \to H + X$$

where H is a detected hadron, typically a pion [20], and in semi-inclusive deep inelastic scattering reactions with a transversely polarized target [21,22]

$$\ell + p(\boldsymbol{s}_T) \to \ell + H + X.$$

The problem is that in these semi-inclusive reactions $\Delta_T q^a(x)$ always occurs multiplied by a function which depends on the largely unknown Collins fragmentation function. But progress is being made and we expect to have a reasonable estimate of $\Delta_T q^a(x)$ soon [23].

VIII. SUMMARY

The standard derivation of the tensorial structure of the expectation value of the angular momentum $\boldsymbol{J}$, for a relativistic spin-s particle, in which the matrix elements of the angular momentum operators are related to the matrix elements of the energy-momentum tensor, is rendered difficult by the singular nature of the operators involved. The fact that the operators are space integrals of densities that contain explicit factors of x^μ means that the densities do not transform like local operators (we have called these *compound* operators) and has the consequence that the matrix elements of $\boldsymbol{J}$ are highly singular, containing derivatives of delta functions. Consequently the evaluation of the expectation values of $\boldsymbol{J}$ requires a careful limiting procedure, beginning with the off-diagonal elements $\langle p', \sigma|\boldsymbol{J}|p, \sigma\rangle$, where σ labels the kind of spin state under consideration. We have shown that the results in the literature are incorrect, and have derived the correct expressions in three different ways, two of them based on a careful wave-packet treatment of the standard approach, and the third, quite independent, based on the known rotational properties of the spin states, which circumvents the use of the energy-momentum tensor. All three methods yield the same results, given in Eqs. (1.1), (1.4), and (1.5). We have emphasized that these matrix elements are not Lorentz tensors, but nevertheless the resulting forms have the correct Lorentz transformation properties when taken in conjunction with the corresponding forms for the matrix elements of the boost operators. We have given some attention to the relation between the matrix elements for canonical spin states and helicity states, which is not obvious, and requires some care because of the derivatives of the delta functions which enter. Using a Fock-space picture of the proton, we have used our results to obtain a new sum rule for a transversely polarized nucleon, Eq. (1.10), which involves the transverse spin or transversity distribution $\Delta_T q(x) \equiv h_1(x)$, and which is similar in form to the classic longitudinal spin sum rule.

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