Severe constraints on the loop-quantum-gravity energy-momentum dispersion relation from the black-hole area-entropy law

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(Received 8 June 2004; published 9 November 2004)

We explore a possible connection between two aspects of loop quantum gravity which have been extensively studied in the recent literature: the black-hole area-entropy law and the energy-momentum dispersion relation. We observe that the original Bekenstein argument for the area-entropy law implicitly requires information on the energy-momentum dispersion relation and on the position-momentum uncertainty relation. Recent results show that in first approximation black-hole entropy in loop quantum gravity depends linearly on the area, with small correction terms which have logarithmic or inverse-power dependence on the area. And it has been argued that in loop quantum gravity the dispersion relation should include terms that depend linearly on the Planck length, while no evidence of modification of the position-momentum uncertainty relation has been found. We observe that this scenario with Planck-length-linear modification of the dispersion relation and unmodified position-momentum uncertainty relation is incompatible with the black-hole-entropy results, since it would give rise to a term in the entropy formula going like the square root of the area.

DOI: 10.1103/PhysRevD.70.107501 PACS numbers: 04.60.Pp, 04.70.Dy

I. INTRODUCTION

The intuition that the entropy of a black hole should be proportional to its (horizon-surface) area, up to corrections that can be neglected when the area A is much larger than the square of the Planck length L_p, has provided an important element of guidance for quantum gravity research. It is noteworthy that, as shown by Bekenstein [1], this contribution to black-hole entropy can be obtained from very simple ingredients. One starts from the general-relativity result [2] that the minimum increase of area when the black hole absorbs a classical particle of energy E and size s is $\Delta A \simeq 8\pi L_p^2 E s$ (in "natural units" with $\hbar = c = 1$). Taking into account the quantum properties of particles one can estimate s as roughly given by the position uncertainty δx, and, since a particle with position uncertainty δx should at least [3] have energy $E \sim 1/\delta x$, this leads to the conclusion[1,4] that the minimum change in the black-hole area must be of order L_p^2, independent of the size of the area. Then using the fact that, also independent of the size of the area, this minimum increase of area should correspond to the minimum ("1 bit") change of entropy one easily obtains [1] the proportionality between black-hole entropy and area.

It is remarkable that, in spite of the humble ingredients of this Bekenstein analysis, the entropy-area relation introduced such a valuable constraint for quantum gravity research. And a rather challenging constraint, since attempts to reproduce the entropy-area-linearity result using directly some quantum properties of black holes were unsuccessful for nearly three decades. But over the last few years both in string theory and in loop quantum gravity the needed techniques for the analysis of entropy on the basis of quantum properties of black holes were developed. These results [5–8] now go even beyond the entropy-area-proportionality contribution: they establish that the leading correction should be of log-area type, so that one expects (for $A \gg L_p^2$) an entropy-area relation for black holes of the type

$$S = \frac{A}{4L_p^2} + \rho \ln\frac{A}{L_p^2} + O\left(\frac{L_p^2}{A}\right). \tag{1}$$

For the case of loop quantum gravity, which is here of interest, there is still no consensus on the coefficient of the logarithmic correction, ρ, but it is established [6–8] that there are no correction terms with stronger-than-logarithimic dependence on the area.

We observe that the availability of results on the log-area correction might provide motivation for reversing the Bekenstein argument: the knowledge of black-hole entropy up to the leading log correction can be used to establish the Planck-scale modifications of the ingredients of the Bekenstein analysis.

In particular, the mentioned role of the relation $E \geq 1/\delta x$ in the Bekenstein analysis appears to provide an opportunity to put under scrutiny some scenarios proposed for loop quantum gravity. Several recent studies have tentatively argued that the loop-quantum-gravity dispersion relation might involve a term with a linear dependence on the Planck length, while no evidence of modification of the position-momentum uncertainty relation has been found. As we observe in Sec. II, this scenario leads to a Planck-length modification of the relation $E \geq 1/\delta x$ between the energy and position uncertainty of a particle, and in turn, as we show in Sec. III following the Bekenstein argument, the modification of the $E \geq 1/\delta x$ relation leads to a contribution to black-hole entropy that goes like the square root of the area. Since such a square-root contribution is, as mentioned, excluded by direct analysis of black-hole entropy in loop quantum gravity, we conclude that there is an inconsistency in the loop-quantum-gravity scenario in which the position-momentum uncertainty relation is not modified while the energy-momentum dispersion relation includes a term with linear dependence on the Planck length.

II. LOOP-QUANTUM-GRAVITY DISPERSION RELATION AND ITS IMPLICATIONS FOR THE $E \geq 1/\delta x$ RELATION

The possibility of Planck-scale modifications of the dispersion relation has been considered extensively in the recent quantum-gravity literature [9–11] and, in particular, in loop quantum gravity [12–15].

Some calculations in loop quantum gravity [12,13] provide support for the idea of an energy-momentum dispersion relation that for a particle of high energy would take the approximate form

$$E \simeq p + \frac{m^2}{2p} + \alpha L_p E^2, \tag{2}$$

where α is a coefficient of order 1. The same studies found no evidence of modification of the $\delta x \delta p \geq 1$ Heisenberg uncertainty relation.

These results must be viewed as preliminary [14,15] since at best they describe perturbations of some examples of states of the theory ("weave states" in the case of Ref. [12] and some heuristic semiclassical states in the case of Ref. [13]) rather than perturbations of the ground state of the theory. It is not surprising (and therefore not necessarily insightful) that there would be some states of the theory whose excitations have a modified spectrum. If instead a relation of the type (2) was applicable to excitations of the ground state of the theory this would provide a striking characteristic of the loop-quantum-gravity approach.

Work attempting to address this issue by reaching a deeper level of analysis of the formalism is in progress [14,15], but it is confronted with severe technical challenges. Therefore there has been a sizeable research effort attempting to constrain in other ways this loop-quantum-gravity scenario with modified dispersion relation and unmodified position-momentum uncertainty relation. Several papers have been devoted to the derivation of tighter and tighter experimental limits on coefficients of the α type for loop quantum gravity (see, e.g., Ref. [16] and references therein). As announced, we intend to show here that the scenario with linear-in-L_p modification of the dispersion relation and unmodified position-momentum uncertainty relation can be excluded already on theoretical grounds, because of an inconsistency with the black-hole-entropy results.

In this section we start by observing that a modified dispersion relation implies a modification of the relation $E \geq 1/\delta x$ between the energy of a particle and its position uncertainty. We can see this by simply following the familiar derivation [3] of the relation $E \geq 1/\delta x$, substituting, where applicable, the standard special-relativistic dispersion relation with the Planck-scale-modified dispersion relation. We instead maintain the standard $\delta x \delta p \geq 1$ uncertainty relation, consistently with the assumptions of the mentioned loop-quantum-gravity scenario.

It is convenient to focus first [3] on the case of a particle of mass M at rest, whose position is being measured by a procedure involving a collision with a photon of energy E_γ and momentum p_γ. In order to measure the particle position with precision δx one should use a photon with momentum uncertainty $\delta p_\gamma \geq 1/\delta x$. Following the standard argument [3], one takes this $\delta p_\gamma \geq 1/\delta x$ relation and converts it into the relation $\delta E_\gamma \geq 1/\delta x$, using the special-relativistic dispersion relation, and then the relation $\delta E_\gamma \geq 1/\delta x$ is converted into the relation $M \geq 1/\delta x$ because the measurement procedure requires[1] $M \geq \delta E_\gamma$. If indeed loop quantum gravity hosts a Planck-scale-modified dispersion relation of the form (2), it is easy to see that, following the same reasoning, one would obtain from $\delta p_\gamma \geq 1/\delta x$ the requirement $M \geq (1/\delta x)[1 + 2\alpha(L_p/\delta x)]$.

These results strictly apply only to the measurement of the position of a particle at rest, but they can be straightforwardly generalized [3] (simply using a boost) to the case of measurement of the position of a particle of energy E. In the case of the standard dispersion relation (without Planck-scale modification) one obtains the familiar $E \geq 1/\delta x$. In the case of (2) one instead easily finds that

$$E \geq \frac{1}{\delta x}\left(1 + 2\alpha \frac{L_p}{\delta x}\right). \tag{3}$$

III. A REQUIREMENT OF CONSISTENCY WITH THE BLACK-HOLE ENTROPY ANALYSIS

We now intend to show that the linear-in-L_p modification of the relation between the energy of a particle and its position uncertainty, which we derived in the previous section for the loop-quantum-gravity scenario here of interest, should be disallowed in loop quantum gravity since it leads to a contribution to the black-hole entropy-

[1]One must take into account the fact [3] that the measurement procedure should ensure that the relevant energy uncertainties are not large enough to possibly produce extra copies of the particle whose position one intends to measure.

area relation which has already been excluded in direct black-hole-entropy analyses.

We do this by following the original Bekenstein argument [1]. As done in Ref. [1] we take as starting point the general-relativistic result which establishes that the area of a black hole changes according to $\Delta A \geq 8\pi E s L_p^2$ when a classical particle of energy E and size s is absorbed. In order to describe the absorption of a quantum particle one must describe the size of the particle in terms of the uncertainty in its position [1,4], $s \sim \delta x$, and take into account a "calibration[2] factor" [17–19] $(\ln 2)/2\pi$ that connects the $\Delta A \geq 8\pi E s L_p^2$ classical-particle result with the quantum-particle estimate $\Delta A \geq 4(\ln 2) L_p^2 E \delta x$. Following the original Bekenstein argument [1] one then enforces the relation $E \geq 1/\delta x$ (and this leads to $\Delta A \geq 4(\ln 2) L_p^2$), but we must take into account the Planck-length modification in (3), obtaining

$$\Delta A \geq 4(\ln 2)\left[L_p^2 + 2\frac{\alpha L_p^3}{\delta x}\right] \simeq 4(\ln 2)\left[L_p^2 + 2\frac{\alpha L_p^3}{R_S}\right] \simeq 4(\ln 2)\left[L_p^2 + \frac{\alpha 4\sqrt{\pi} L_p^3}{\sqrt{A}}\right],$$

where we also used the fact that in falling in the black hole the particle acquires [18,21,22] position uncertainty $\delta x \sim R_S$, where R_S is the Schwarzschild radius (and of course $A = 4\pi R_S^2$).

Next, following again Bekenstein [1], one assumes that the entropy depends only on the area of the black hole, and one uses the fact that according to information theory the minimum increase of entropy should be ln2, independent of the value of the area:

$$\frac{dS}{dA} \simeq \frac{\min(\Delta S)}{\min(\Delta A)} \simeq \frac{\ln 2}{4(\ln 2) L_p^2 [1 + \alpha 4\sqrt{\pi}\frac{L_p}{\sqrt{A}}]} \simeq \left(\frac{1}{4L_p^2} - \frac{\alpha\sqrt{\pi}}{L_p\sqrt{A}}\right). \tag{4}$$

From this one easily obtains (up to an irrelevant constant contribution to entropy):

$$S \simeq \frac{A}{4L_p^2} - 2\alpha\sqrt{\pi}\frac{\sqrt{A}}{L_p}. \tag{5}$$

We therefore conclude that if a quantum-gravity theory predicts that the position-momentum uncertainty relation is unmodified while energy-momentum dispersion relation includes a term linear in L_p then it should correspondingly predict the presence of $\sqrt{A}$ contribution to black-hole entropy. Since in loop quantum gravity such a $\sqrt{A}$ contribution to black-hole entropy has already been excluded [6–8] in direct black-hole entropy studies, we conclude that in loop quantum gravity it is not possible to introduce a linear-in-L_p contribution to the energy-momentum dispersion relation while preserving an unmodified position-momentum uncertainty relation.

It is instead possible for loop quantum gravity to host a linear-in-L_p contribution to the energy-momentum dispersion relation if there is a corresponding linear-in-L_p modification of the position-momentum uncertainty relation. In fact, as the careful reader can easily verify following our line of analysis, the two linear-in-L_p modifications can combine (if the coefficients are appropriately adjusted) to give no net $\sqrt{A}$ contribution to black-hole entropy.

And of course, it is also possible that loop quantum gravity would host (even without assuming any modification of the position-momentum uncertainty relation) a dispersion relation of the type

$$E \simeq p + \frac{m^2}{2p} + \tilde{\alpha} L_p^2 E^3, \tag{6}$$

with a quadratic-in-L_p contribution. In fact, adapting our analysis to the case of the dispersion relation (6), one finds that the quadratic-in-L_p contribution to the dispersion relation ultimately leads to a leading correction to the black-hole-entropy formula which is of log-area type, consistent with the indications obtained in direct black-hole entropy studies [6–8].

ACKNOWLEDGMENTS

G. A.-C. gratefully acknowledges conversations with O. Dryer, D. Oriti, C. Rovelli, and L. Smolin. The work of M. A. was supported by The Graduate School of The University of North Carolina. M. A. also thanks the Department of Physics of the University of Rome for hospitality.

[2]Clearly some calibration is needed in order to adapt the classical-gravity result for absorption of a classical particle to the case of a quantum black hole absorbing a quantum particle. In particular, a calibration should arise in the description of a quantum particle with position uncertainty δx in terms of a classical particle of size s. A direct evaluation of the calibration coefficient within quantum gravity is presently beyond reach; however, several authors (see, e.g., Refs. [17–19]) have used the independent analysis of black-hole entropy by Hawking [20] to infer indirectly this calibration needed in the Bekenstein argument. We adopt this calibration for consistency with previous literature, but the careful reader will notice that this calibration does not affect our line of analysis (the calibration could be reabsorbed in the free parameter α).

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