Quantum effects can render $w < -1$ on cosmological scales

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We report on a revision of our previous computation of the renormalized expectation value of the stress-energy tensor of a massless, minimally coupled scalar with a quartic self-interaction on a locally de Sitter background. This model is important because it demonstrates that quantum effects can lead to violations of the weak energy condition on cosmological scales—on average, not just in fluctuations—although the effect in this particular model is far too small to be observed. The revision consists of modifying the propagator so that dimensional regularization can be used when the dimension of the renormalized theory is not four. Although the finite part of the stress-energy tensor does not change (in $D = 4$) from our previous result, the counterterms do. We also speculate that a certain, finite and separately conserved part of the stress tensor can be subsumed into a natural correction of the initial state from free Bunch-Davies vacuum.

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Caldwell [1] was the first to point out that the original supernova acceleration data [2,3] are consistent with a dark energy equation of state $w \equiv p/\rho$ less than minus one, which would violate the weak energy condition. Subsequent analyses of better and more abundant data have confirmed this possibility in the context of an evolving dark energy equation of state whose current value is less than minus one [4–10]. However, it should be noted that realizing this possibility generally implies accepting a somewhat low value for the current Hubble parameter and a somewhat high value for the fraction of the critical density currently comprised by dark matter [11]. When combined data sets are used, which restrict these two parameters, the data are well fit by a simple cosmological constant with $w = -1$ [11–13].

If the current phase of acceleration is actually driven by dark energy which violates the weak energy condition it would pose an excruciating problem for fundamental theory because the universe has existed over 13 Gyr [14–36]. One can get $w < -1$ by using scalars with a negative kinetic term, however, such models are unstable against the production of positive-negative energy particles. This instability obviously grows worse as the negative energy particle is endowed with interactions with more species of positive energy particles. The minimal case is for it to interact only with gravity. For a specific model of this type Carroll, Hoffman and Trodden [14] estimated that such a scalar would decay into two gravitons and three scalars over the lifetime of the universe unless the interaction is cut off, by fiat, at about 100 MeV. A more stringent and model-independent bound was obtained by Cline, Jeon and Moore [15] by considering the process whereby a graviton loop produces two scalars and two photons in empty space. They conclude that the diffuse gamma ray background will be too high unless the interaction is cut off at about 3 MeV. More recently Hsu, Jenkins and Wise have shown [16] that instabilities occur in any scalar theory which exhibits $w < -1$, irrespective of how this is achieved. Clearly, the observed persistence of the universe can only be consistent with a relatively brief phase of $w < -1$.

One way to achieve such a self-limiting phase—without violating classical stability—is through quantum effects. Four years before the first supernova data appeared Starobinsky and Yokoyama studied a model which does this [37]. It consists of a massless, minimally coupled scalar with a quartic self-interaction which is released in free Bunch-Davies vacuum on a locally de Sitter background. By applying Starobinsky's technique of stochastic inflation [38], they were able to show that the scalar initially moves up its potential, which would violate the weak energy condition by increasing the Hubble parameter. Eventually the upward push from inflationary particle production is compensated by the downward classical force and the Hubble parameter asymptotes to a constant value. The time for the process goes like the inverse square root of the coupling constant.

The solution of Starobinsky and Yokoyama [37] is nonperturbative, but it includes only the leading logarithms of the scale factor at each order. (We thank A. A. Starobinsky for pointing this out.) One can see that the vacuum energy increases this way, but it is not possible to either verify stress-energy conservation or to directly check that $\rho + p$ is negative. We recently computed the fully renormalized expectation value of this model's stress-energy tensor at one and two loops [39]. Although our analysis was explicitly perturbative it produced the complete result at one and two loop orders, thereby allowing verification of conservation and a direct check that $\rho + p$ is in fact negative.

What made our calculation possible was a relatively simple form for the D-dimensional scalar propagator, which allowed us to employ dimensional regularization. The scalar propagator is constrained to obey the equation,

$$\frac{\partial}{\partial x^\mu}[\sqrt{-g(x)}g^{\mu\nu}(x)\frac{\partial}{\partial x^\nu}i\Delta(x;x')] = i\delta^D(x-x'). \quad (1)$$

Were de Sitter invariance maintained one could express $i\Delta(x;x')$ entirely in terms of the geodesic length $\ell(x;x')$. However, Allen and Follaci long ago showed that the massless, minimally coupled scalar possesses no normalizable, de Sitter invariant states [40]. We chose to introduce the inevitable breaking of de Sitter invariance in a manner consistent with the homogeneity and isotropy of cosmology. In our conformal coordinate system the invariant element is,

$$g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + d\vec{x}\cdot d\vec{x}), \quad a(\eta) \equiv -\frac{1}{H\eta}, \quad (2)$$

where $\Lambda = (D-1)H^2$ relates the Hubble constant H to the cosmological constant Λ in D dimensions. Our solution to (1) depends upon $a \equiv a(\eta)$ and $a' \equiv a(\eta')$ in addition to the de Sitter invariant length function,

$$\begin{aligned} y(x;x') &\equiv 4\sin^2\left[\frac{1}{2}H\ell(x;x')\right] \\ &= aa'H^2[\|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2], \end{aligned} \quad (3)$$

where δ is a positive real number. We normalize the scale factor to $a = 1$ when the state is released, so that $a > 1$ throughout the computation.

Our previous computation [39] was based upon the solution,

$$\begin{aligned} i\Delta_{\rm old} = \frac{H^{D-2}}{(4\pi)^{D/2}}\Bigg\{&-\sum_{n=0}^{\infty}\frac{1}{n-\frac{D}{2}+1}\frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+1)}\left(\frac{y}{4}\right)^{n-(D/2)+1} \\ &+\frac{2^{4-D}}{2-\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right) + \frac{\Gamma(\frac{D}{2})}{2^{D-4}}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \\ &\times\left(\frac{y}{4}\right)^n + \frac{\Gamma(D-1)}{2^{D-4}}\ln(aa')\Bigg\}. \end{aligned} \quad (4)$$

It consists of four terms: (i) An infinite series of D-dependent powers of $\frac{y}{4}$; (ii) A D-dependent constant; (iii) An infinite series of integer powers of $\frac{y}{4}$ and (iv) The $\ln(aa')$ term. The normalization of (i) is set by getting the delta function. The constant term (ii) is a homogeneous solution and can be added for free. It was chosen to cancel the singularity at $D = 4$ in the $n = 1$ term of the series (i). Although the sum of (i) and (ii) is finite for small y, it diverges at $y = 4$ and beyond. The point of the second infinite series (iii) is to cancel this divergence at $D = 4$. However, the series (iii) does not solve the homogeneous equation. The de Sitter breaking term (iv) must be added for this purpose.

In addition to the ϕ^4 stress-energy tensor [39], the old propagator (4) was used in [41,42] to compute the one loop vacuum polarization from scalar QED. While computing the one loop self-energy of a Yukawa-coupled fermion [43] the propagator was recently modified to make it valid for regulating a theory whose dimension will not ultimately be taken to $D = 4$,

$$\begin{aligned} i\Delta_{\rm new} = \frac{H^{D-2}}{(4\pi)^{D/2}}\Bigg\{&-\sum_{n=0}^{\infty}\frac{1}{n-\frac{D}{2}+1}\frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+1)}\left(\frac{y}{4}\right)^{n-(D/2)+1} \\ &-\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\left(\pi\frac{D}{2}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \\ &\times\left(\frac{y}{4}\right)^n + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\ln(aa')\Bigg\}. \end{aligned} \quad (5)$$

This change makes a few insignificant alterations in the finite part of the one loop vacuum polarization [44]. The purpose of this brief report is to consider possible changes from using the modified propagator to recompute the expectation value of the ϕ^4 stress-energy tensor. We also take this opportunity to correct a minor error—in the normalization of the de Sitter breaking term (iv)—in the previously published expressions for the new propagator [43,44]. This error has no effect on the one loop computations for which the new propagator was previously employed but it becomes quite significant at higher loops.

When the new propagator (5) is employed (with $D = 4 - \epsilon$) to recompute the expectation value of the ϕ^4 stress-energy tensor we find the same fully renormalized result but totally different results for the mass-squared, the conformal and the cosmological constant counterterms. We report the various changes below, giving the previously reported results [39] with subscript "old", followed by the new results with subscript "new",

$$\begin{aligned} \delta m^2_{\rm old} &= -\frac{\lambda H^{2-\epsilon}}{2^4\pi^{2-(\epsilon/2)}}\frac{1}{\epsilon}\Gamma\left(3-\frac{\epsilon}{2}\right) + O(\lambda^2) \\ &\longrightarrow -\frac{\lambda H^{2-\epsilon}}{2^{5-\epsilon}\pi^{2-(\epsilon/2)}}\frac{\Gamma(3-\epsilon)}{\Gamma(2-\frac{\epsilon}{2})}\pi\cot\left(\frac{\pi}{2}\epsilon\right) + O(\lambda^2) \\ &\equiv \delta m^2_{\rm new}, \end{aligned} \quad (6)$$

$$\begin{aligned} \delta\xi_{\rm old} &= \frac{\lambda}{2^4\pi^2}\left(\frac{H^2}{\pi}\right)^{\epsilon/2}\left\{\frac{\zeta_{\rm old}}{2\epsilon\Gamma(4-\epsilon)} + \delta\xi^{\rm old}_{\rm fnt}\right\} \\ &\longrightarrow \frac{\lambda}{2^4\pi^2}\left(\frac{H^2}{4\pi}\right)^{\epsilon/2}\left\{\frac{\Gamma(2-\frac{\epsilon}{2})}{\Gamma(4-\epsilon)}\frac{\zeta_{\rm new}}{2\epsilon} + \delta\xi^{\rm new}_{\rm fnt}\right\} \\ &\equiv \delta\xi_{\rm new}, \end{aligned} \quad (7)$$

where

$$\begin{aligned} \zeta_{\rm old} &= \left(\frac{\pi}{\mu H}\right)^\epsilon\left(1-\frac{\epsilon}{2}\right)^2\Gamma(1-\epsilon)\Gamma\left(1-\frac{\epsilon}{2}\right) \\ &\longrightarrow \left(\frac{2\pi}{\mu H}\right)^\epsilon\left(1-\frac{\epsilon}{2}\right)\Gamma(1-\epsilon) \equiv \zeta_{\rm new}, \end{aligned} \quad (8)$$

and

$$\frac{\delta\Lambda_{\rm old}}{8\pi G} = \frac{\lambda H^4}{2^6\pi^4}\left\{\left(\frac{\pi}{H^2}\right)^{\epsilon}\frac{\Gamma^2(3-\frac{\epsilon}{2})}{8\epsilon^2} - \frac{\zeta_{\rm old}}{2\epsilon^2}\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(3-\epsilon)} - \frac{\delta\xi^{\rm old}_{\rm fnt}}{\epsilon}\times(3-\epsilon)\Gamma\left(3-\frac{\epsilon}{2}\right) + \delta\Lambda^{\rm old}_{\rm fnt}\right\}$$
$$\longrightarrow \frac{\lambda H^4}{2^6\pi^4}\left\{\left(\frac{4\pi}{H^2}\right)^{\epsilon}\frac{\Gamma^2(3-\epsilon)}{\Gamma^2(2-\frac{\epsilon}{2})}\frac{\pi^2}{32}\cot^2\left(\frac{\pi}{2}\epsilon\right) - \frac{\zeta_{\rm new}}{4\epsilon}\times\pi\cot\left(\frac{\pi}{2}\epsilon\right) - \delta\xi^{\rm new}_{\rm fnt}\frac{\Gamma(4-\epsilon)}{\Gamma(2-\frac{\epsilon}{2})}\frac{\pi}{2}\cot\left(\frac{\pi}{2}\epsilon\right) + \delta\Lambda^{\rm new}_{\rm fnt}\right\} \equiv \frac{\delta\Lambda_{\rm new}}{8\pi G}. \tag{9}$$

We make the same choices as before for the arbitrary finite parts of the cosmological and conformal counterterms,

$$\delta\xi^{\rm new}_{\rm fnt} \equiv -\frac{7}{36} + \frac{1}{12}\ln\left(\frac{2\mu}{H}\right) = \delta\xi^{\rm old}_{\rm fnt}$$
$$\delta\Lambda^{\rm new}_{\rm fnt} \equiv \frac{1}{18} - \frac{\pi^2}{12} = \delta\Lambda^{\rm old}_{\rm fnt}. \tag{10}$$

The renormalized energy density and pressure are unchanged from their previous values,

$$\rho_{\rm ren} = \frac{\Lambda}{8\pi G} + \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a) + \frac{2}{9}a^{-3} - \frac{1}{2}\times\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^2}a^{-n-1}\right\} + O(\lambda^2), \tag{11}$$

$$p_{\rm ren} = -\frac{\Lambda}{8\pi G} - \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a) + \frac{1}{3}\ln(a) + \frac{1}{6}\times\sum_{n=1}^{\infty}\frac{n^2-4}{(n+1)^2}a^{-n-1}\right\} + O(\lambda^2). \tag{12}$$

Hence

$$\rho_{\rm ren} + p_{\rm ren} = \frac{\lambda H^4}{2^6\pi^4}\left\{-\frac{1}{3}\ln(a) + \frac{2}{9}a^{-3} - \frac{1}{6}\times\sum_{n=1}^{\infty}\frac{n+2}{n+1}a^{-n-1}\right\} + O(\lambda^2), \tag{13}$$

violates the weak energy condition on cosmological scales.

Before concluding we wish to make three comments. First, $w+1$ is unobservably small in this model. From (11) and (12) we compute,

$$w \equiv \frac{p_{\rm ren}}{\rho_{\rm ren}} = -\left\{1 + \frac{\lambda H^2 G}{(2\pi)^3}\left[\frac{1}{9}\ln(a) + O(a^{-2})\right] + O(\lambda^2)\right\}. \tag{14}$$

Using $H_0 \simeq 71$ km/(s $\cdot$ Mpc) one finds the dimensionless number $GH_0^2 \equiv GH_0^2(\hbar/c^5) \simeq 1.5\times 10^{-122}$. One might hope this minuscule prefactor could be enhanced by the coupling constant λ or by the secular factor of $\ln(a)$. However, our analysis has been perturbative—which rules out $\lambda > 1$—and the data shows that acceleration only began at about $z \simeq 1$—which means that $a_0 \simeq 2$ if we assume the process began when the deceleration became negative. In any case, the nonperturbative solution of Starobinsky and Yokoyama [37] shows that w approaches -1 after $\ln(a) \simeq 1/\sqrt{\lambda}$, so the weak energy condition is never violated by very much in this model. What the model does establish, in a simple setting and beyond the point of dispute, is that quantum effects can induce a self-limiting phase in which a classically stable theory violates the weak energy condition on cosmological scales. Once this is accepted one can search for other models in which the effect may be observable. Such a model has been proposed by Parker and Raval [45,46], and slightly modified by Parker and Vanzella [47].

Our second comment concerns the exponentially falling portions of the stress-energy tensor,

$$\rho_{\rm falling} \equiv \frac{\lambda H^4}{2^7\pi^4}\left\{\frac{4}{9}a^{-3} - \sum_{n=1}^{\infty}\frac{n+2}{(n+1)^2}a^{-n-1}\right\}, \tag{15}$$

$$p_{\rm falling} \equiv -\frac{\lambda H^4}{2^7 3\pi^4}\sum_{n=1}^{\infty}\frac{n^2-4}{(n+1)^2}a^{-n-1}. \tag{16}$$

Note that these terms are separately conserved,

$$\dot{\rho}_{\rm falling} = -3H(\rho_{\rm falling} + p_{\rm falling}). \tag{17}$$

We conjecture that these terms can be subsumed into a modification of the initial free Bunch-Davies vacuum at $a=1$. Even in flat space one can see that the free state wave functional,

$$\Omega[\phi] = N\exp\left[-\frac{1}{2}\int d^3x\,\phi(\vec{x})\sqrt{-\nabla^2}\,\phi(\vec{x})\right], \tag{18}$$

must suffer nonlocal corrections of order $\lambda\phi^4$. We propose that using this perturbatively corrected initial state would cancel the falling portions of the stress-energy leaving only the infrared logarithms,

$$\rho_{\rm conj} = \frac{\Lambda}{8\pi G} + \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a)\right\} + O(\lambda^2), \tag{19}$$

$$p_{\rm conj} = -\frac{\Lambda}{8\pi G} - \frac{\lambda H^4}{2^6\pi^4}\left\{\frac{1}{2}\ln^2(a) + \frac{1}{3}\ln(a)\right\} + O(\lambda^2). \tag{20}$$

Our final comment is that quantum fluctuations of the stress-energy operator will of course violate the weak energy condition for a classical background such as de Sitter which is right on the boundary $\rho + p = 0$ [48,49]. The model we have considered gives a more serious violation, in the *average value* of the stress-energy tensor, rather than in fluctuations about an average which obeys the condition.

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