M theory model of a big crunch/big bang transition

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We consider a picture in which the transition from a big crunch to a big bang corresponds to the collision of two empty orbifold planes approaching each other at a constant nonrelativistic speed in a locally flat background space-time, a situation relevant to recently proposed cosmological models. We show that p-brane states which wind around the extra dimension propagate smoothly and unambiguously across the orbifold plane collision. In particular we calculate the quantum mechanical production of winding M2-branes extending from one orbifold to the other. We find that the resulting density is finite and that the resulting gravitational backreaction is small. These winding states, which include the string theory graviton, can be propagated smoothly across the transition using a perturbative expansion in the membrane tension, an expansion which from the point of view of string theory is an expansion in *inverse* powers of α'. The conventional description of a crunch based on Einstein general relativity, involving Kasner or mixmaster behavior is misleading, we argue, because general relativity is only the leading order approximation to string theory in an expansion in positive powers of α'. In contrast, in the M theory setup we argue that interactions should be well behaved because of the smooth evolution of the fields combined with the fact that the string coupling tends to zero at the crunch. The production of massive Kaluza-Klein states should also be exponentially suppressed for small collision speeds. We contrast this good behavior with that found in previous studies of strings in Lorentzian orbifolds.

I. INTRODUCTION

One of the greatest challenges faced by string and M theory is that of describing time-dependent singularities, such as occur in cosmology and in black holes. These singularities signal the catastrophic failure of general relativity at short distances, precisely the pathology that string theory is supposed to cure. Indeed string theory does succeed in removing the divergences present in perturbative quantum gravity about flat space-time. String theory is also known to tolerate singularities in certain static backgrounds such as orbifolds and conifolds. However, studies within string theory thus far have been unable to shed much light on the far more interesting question of the physical resolution of time-dependent singularities.

In this paper we discuss M theory in one of the simplest possible time-dependent backgrounds [1,2], a direct product of $d - 1$-dimensional flat Euclidean space R^{d-1} with two-dimensional compactified Milne space-time, $\mathcal{M}_C$, with line element

$$-dt^2 + t^2 d\theta^2. \tag{1}$$

The compactified coordinate θ runs from 0 to θ_0. As t runs from $-\infty$ to $+\infty$, the compact dimension shrinks away and reappears once more, with rapidity θ_0. Analyticity in t suggests that this continuation is unique [3].

Away from $t = 0$, $\mathcal{M}_C$ is locally flat, as can be seen by changing to coordinates $T = t \cosh\theta$, $Y = t \sinh\theta$ in which (1) is just $-dT^2 + dY^2$. Hence $\mathcal{M}_C \times R^{d-1}$ is naturally a solution of any geometrical theory whose field equations are built from the curvature tensor. However, $\mathcal{M}_C \times R^{d-1}$ is nonetheless mathematically singular at $t = 0$ because the metric degenerates when the compact dimension disappears. General relativity cannot make sense of this situation since there ceases to be enough Cauchy data to determine the future evolution of fields. In fact, the situation is worse than this: within general relativity, generic perturbations diverge as $\log|t|$ as one approaches the singularity [4], signaling the breakdown of perturbation theory and the approach to Kasner or mixmaster behavior, according to which the space-time curvature diverges as t^{-2}. Of course, this breakdown of general relativity presents a challenge: can M theory make sense of the singularity at $t = 0$?

We are interested in what happens in the immediate vicinity of $t = 0$, when the compact dimension approaches, and becomes smaller than, the fundamental membrane tension scale. The key difference between M theory (or string theory) and local field theories such as general relativity is the existence of extended objects including those stretching across compactified dimensions. Such states become very light as the compact dimensions shrink below the fundamental scale. They are known to play a central role in resolving singularities, for example, in orbifolds and in topology-changing tran-

sitions [5]. Therefore, it is very natural to ask what role such states play in big crunch/big bang space-times.

In this paper we shall show that p-branes winding uniformly around the compact dimension obey equations, obtained by canonical methods, which are completely regular at $t = 0$. These methods are naturally invariant under choices of world-volume coordinates. Therefore we claim that it is possible to unambiguously describe evolution of such states from $t < 0$ to $t > 0$, through a cosmological singularity from the point of view of the low-energy effective theory. Indeed the space-time we consider corresponds locally to one where two empty, flat, parallel orbifold planes collide, precisely the situation envisaged in recently proposed cosmological models.

Hence, the calculations we report are directly relevant to the ekpyrotic [6] and cyclic universe [7] scenarios, in which passage through a singularity of this type is taken to represent the standard hot big bang. In particular, the equation of state during the dark energy and contracting phases causes the orbifold planes to be empty, flat, and parallel as they approach to within a string length [8,9]. This setup makes it natural to split the study of the collision of orbifold planes into a separate analysis of the winding modes, which become light near $t = 0$, and other modes that become heavy there. This strategy feeds directly into the considerations in this paper.

The physical reason why winding states are well behaved is easy to understand. The obvious problem with a space-time such as compactified Milne is the blue shifting effect felt by particles which can run around the compact dimension as it shrinks away. As we shall discuss in detail, winding states wrapping around the compact dimension do not feel any blue shifting effect because there is no physical motion along their length. Instead, as their length disappears, from the point of view of the non-compact dimensions, their effective mass or tension tends to zero but their energy and momentum remain finite. When such states are quantized the corresponding fields are well behaved and the field equations are analytic at $t = 0$. In contrast, for bulk, nonwinding states, the motion in the θ direction is physical and it becomes singular as t tends to zero. In the quantum field theory of such states, this behavior results in logarithmic divergences of the fields near $t = 0$, even for the lowest modes of the field which are uniform in θ, i.e., the lowest Kaluza-Klein modes (see Sec. X and Appendix D).

We are specially interested in the case of M theory, considered as the theory of branes. As the compact dimension becomes small, the winding M2-branes we focus on are the lowest energy states of the theory, and describe a string theory in a certain time-dependent background. The most remarkable feature of this setup is that the string theory includes a graviton and, hence, describes perturbative gravity near $t = 0$. In this paper, we show these strings, when considered as winding M2-branes,

follow smooth evolution (see Sec. VII) across the singularity, even though the string-frame metric degenerates there. Furthermore we show that this good behavior is only seen in a perturbation expansion in the membrane tension, corresponding from the string theory point of view to an expansion in *inverse powers of* α'. We argue that the two-dimensional nonlinear sigma model describing this situation is renormalizable in such an expansion. The good behavior of the relevant string theory contrasts sharply with the bad behavior of general relativity. There is no contradiction, however, because general relativity is only the first approximation to string theory in an expansion in positive powers of α'. Such an expansion is valid when t is much larger than the fundamental membrane scale, but it fails near the singularity where, as mentioned, the theory is regular in the opposite $(\alpha')^{-1}$ expansion. The logarithmic divergences of perturbations found using the Einstein equations are, thereby, seen to be due to the failure of the α' expansion, and not of M or string theory *per se.*

When the M theory dimension is small, the modes of the theory are neatly partitioned into light θ-independent modes and heavy θ-dependent modes. The former set consists of winding membranes, which describe a string theory including perturbative gravity. This is the sector within which cosmological perturbations lie, and which will be our prime focus in this paper.

The θ-dependent modes are likely to be harder to describe. The naive argument that these modes are problematic because they are blue shifted and, hence, infinitely amplified as $t \rightarrow 0$ is suspect because it relies on conventional Einstein gravity, Here we argue that, close to the brane collision, Einstein gravity is a poor approximation and, instead, perturbative gravity is described by the nonsingular winding sector. The latter does not exhibit blue shifting behavior near $t = 0$, so the naive argument does not apply.

Witten has argued [10] that the massive Kaluza-Klein modes of the 11-dimensional theory map onto nonperturbative black hole states in the effective string theory. Even though these black hole states are likely to be hard to describe in detail, we will explain in Section II why their overall effect is likely to be small. First, in the cosmological scenarios of interest, the universe enters the regime where perturbative gravity is described by the winding modes (i.e., the branes are close) with a negligible density of Kaluza-Klein massive modes. This suppression is a result of the special equation of state in the contracting phase that precedes this regime [8]. Second, the density of black holes quantum produced due to the time-dependent background in the vicinity of the collision is likely to be negligible because they are so massive and so large.

For these reasons, we focus at present on the propagation of the perturbative gravity sector near $t = 0$ corre-

sponding to the winding M2-brane states. In Sec. II, we introduce the compactified Milne background metric that describes the collision between orbifold planes in the big crunch/big bang transition. We also discuss the motivation for the initial conditions that will be assumed in this paper. The canonical Hamiltonian description of p-branes in curved space is given in Sec. III and applied to winding modes in the compactified Milne background in Sec. IV. Section V discusses the key difference in the Hamiltonian description between winding and bulk states that accounts for their different behavior near the big crunch/big bang transition.

Section VI is the consideration of a toy model in which winding strings are produced as the branes collide. The winding modes are described semiclassically, and their quantum production at the bounce is computed. Section VII presents the analogous semiclassical description of winding M2-branes. Although we cannot solve the theory exactly, we show the 11-dimensional theory is well behaved near $t = 0$ and explain how the apparent singularity in the dimensionally-reduced string theory is resolved in the membrane picture. Then, Sec. VIII makes clear the difference between our calculation, an expansion in inverse powers of α', versus Einstein gravity, the leading term in an expansion in positive powers of α'. This argument is key to explaining why we think the transition is calculable even though it appears to be poorly behaved when described by Einstein gravity. Section IX, then, uses Euclidean instanton methods to study the quantum production of winding M2-branes (in analogy to the case of winding strings in Sec. VI) induced by passage through the singularity, obtaining finite and physically sensible results. In particular, the resultant density tends to zero as the speed of contraction of the compact dimension is reduced. We estimate the gravitational backreaction and show it is small provided θ_0, the rapidity of contraction of the compact dimension, is small.

In Sec. X, we comment on why our M theory setup is better behaved than the Lorentzian orbifold case [11] considered in some previous investigations of the big crunch/big bang transition. The fundamental problem with the latter case, we argue, is that perturbative gravity lies within the bulk sector and not the winding sector as far as the compactified Milne singularity is concerned. Therefore, it is susceptible to the blueshifting problem mentioned above, rendering the string equations singular.

II. THE BACKGROUND BIG CRUNCH/BIG BANG SPACE-TIME

The $d + 1$-dimensional space-time we consider is a direct product of $d - 1$-dimensional Euclidean space, R^{d-1}, and a two-dimensional time-dependent space-time known as compactified Milne space-time, or $\mathcal{M}_C$. The line element for $\mathcal{M}_C \times R^{d-1}$ is thus

$$ds^2 = -dt^2 + t^2 d\theta^2 + d\vec{x}^2, \qquad 0 \le \theta \le \theta_0, \tag{2}$$
$$-\infty < t < \infty,$$

where $\vec{x}$ are Euclidean coordinates on R^{d-1}, θ parameterizes the compact dimension and t is the time. The compact dimension may either be a circle, in which case we identify θ with $\theta + \theta_0$, or a Z_2 orbifold in which case we identify θ with $\theta + 2\theta_0$ and further identify θ with $2\theta_0 - \theta$. The fixed points $\theta = 0$ and $\theta = \theta_0$ are then interpreted as tensionless Z_2-branes approaching at rapidity θ_0, colliding at $t = 0$ to reemerge with the same relative rapidity.

The orbifold reduction is the case of prime interest in the ekpyrotic/cyclic models, originally motivated by the construction of heterotic M theory from 11-dimensional supergravity [12,13]. In these models, the boundary branes possess nonzero tension. However, the tension is a subdominant effect near $t = 0$ and the brane collision is locally well modeled by $\mathcal{M}_C \times R^{d-1}$ (See Ref. [14]).

The line element (2) is of particular interest because it is locally flat and, hence, an exact solution not only of $d + 1$-dimensional Einstein gravity but of any higher dimensional gravity theory whose field equations are constructed from curvature invariants with no cosmological constant. And even if a small cosmological constant were present, it would not have a large effect locally so that solutions with a similar local structure in the vicinity of the singularity would be expected to exist.

Consider the description of (2) within $d + 1$-dimensional general relativity. When the compact dimension is small, θ-dependent states become massive and it makes sense to describe the system using a low-energy effective field theory. This may be obtained by the well-known procedure of dimensional reduction. The $d + 1$-dimensional line element (2) may be rewritten in terms of a d-dimensional Einstein-frame metric, $g^{(d)}_{\mu\nu}$, and a scalar field ϕ:

$$ds^2 = e^{2\phi\sqrt{(d-2)/(d-1)}}d\theta^2 + e^{-2\phi/\sqrt{(d-2)(d-1)}}g^{(d)}_{\mu\nu}dx^\mu dx^\nu. \tag{3}$$

The numerical coefficients are chosen so that if one substitutes this metric into the $d + 1$-dimensional Einstein action and assumes that ϕ and $g^{(d)}_{\mu\nu}$ are both θ-independent, one obtains d-dimensional Einstein gravity with a canonically normalized massless, minimally coupled scalar field ϕ. (Here we choose units in which the coefficient of the Ricci scalar in the d-dimensional Einstein action is $\frac{1}{2}$. We have also ignored Kaluza-Klein vectors, which play no role in this argument and are in any case projected out in the orbifold reduction.)

From the viewpoint of the low-energy effective theory, the $d + 1$-dimensional space-time $\mathcal{M}_C \times R^{d-1}$ is reinterpreted as a d-dimensional cosmological solution where t plays the role of the conformal time. Comparing (2) and

(3), the d-dimensional Einstein-frame metric $g^{(d)}_{\mu\nu} = a^2 \eta_{\mu\nu}$ with $a \propto |t|^{1/(d-2)}$ and the scalar field $\phi = \sqrt{(d-1)/(d-2)}\ln|t|$. From this point of view $t = 0$ is a spacelike curvature singularity of the standard big bang type where the scalar field diverges, and passing through $t = 0$ would seem to be impossible. However, by lifting to the higher dimensional viewpoint one sees that the situation is not really so bad. The line element (2) is in fact static at all times in the noncompact directions $\vec{x}$. So, for example, matter localized on the branes would see no blue shifting effect as the singularity approaches [7]. As we discuss in detail in Sec. IV, winding states do not see a blue shifting effect either.

In this paper, we consider an M theory picture with two empty, flat, parallel colliding orbifold planes and we are interested in the dynamics of the collision region from the point where the planes are roughly a string length apart. The assumed initial conditions are important for two reasons. First, they correspond to the simple compactified Milne background discussed above. Second, as mentioned in the introduction, this initial condition means that the excitations neatly divide into light winding modes that are becoming massless and heavy Kaluza-Klein modes that are becoming massive and decoupling from the low-energy effective theory.

What we want to show now is that initial conditions with negligible heavy Kaluza-Klein modes present are naturally produced in cosmological scenarios such as the cyclic model [7,8]. This justifies our focus on the winding modes throughout the remainder of the paper. However, the argument is inessential to the rest of the paper and readers willing to accept the initial conditions without justification may wish to proceed straight away to the next section.

The cyclic model assumes a nonperturbative potential that produces an attractive force between the orbifold planes. When the branes are far apart, perhaps 10^4 Planck lengths, the potential energy is positive and small, acting as the dark energy that causes the currently observed accelerated expansion. In the dark energy dominated phase, the branes stretch by a factor of 2 in linear dimensions every 14×10^9 years or so, causing the branes to become flat, parallel and empty. In the low-energy effective theory, the total energy is dominated by the scalar field ϕ whose value determines the distance between branes. As the planes draw together, the potential energy $V(\phi)$ of this field decreases and becomes increasingly negative until the expansion stops and a contracting phase begins.

A key point is that this contracting phase is described by an attractor solution, which has an equation of state parameter $w \equiv P/\rho \gg 1$. The Einstein-frame energy density of the scalar field ϕ scales as

$$\rho_\phi \propto a^{-(d-1)(1+w)} \tag{4}$$

in this phase. This is a very rapid increase, causing the density in ϕ to come to dominate over curvature, anisotropy, matter, or radiation [9]. We now show that ϕ comes to dominate over-the massive Kaluza-Klein modes. The latter scale as

$$\rho_{KK} \propto a^{-(d-1)} L^{-(d-1)/(d-2)}, \tag{5}$$

where L is the size of the extra dimension. The first factor is the familiar inverse volume scaling which all particles suffer. The second factor indicates the effective mass of the Kaluza-Klein modes. The $d + 1$-dimensional mass is L^{-1}, but this must be converted to a d-dimensional mass using the ratios of square roots of the 00 components of the $d + 1$-dimensional metric and the d-dimensional metric. This correction produces the second factor in (5).

From (3), we have $L \propto e^{\phi\sqrt{(d-2)/(d-1)}}$. Now the key point is that this scales much more slowly with a than the potential $V(\phi)$ which scales as ρ_ϕ in (4). Neglecting the scaling with L, the density of massive Kaluza-Klein modes scales as $\rho_\phi^{1/(1+w)}$. The final suppression of the density of massive modes relative to the density in ϕ is therefore $\sim (V_i/V_f)^{w/(1+w)}$, where V_i and V_f are the magnitudes of the scalar potential when the $w \gg 1$ phase begins and ends. For large w, which we need in order to obtain scale-invariant perturbations, this is an exponentially large factor [7,8].

The massive Kaluza-Klein modes are, hence, exponentially diluted when the $w \gg 1$ phase ends and the Milne phase begins. During the Milne phase, the scalar field is massless and has an equation of state $w = 1$, so ρ_ϕ scales as $a^{-2(d-1)}$ as the distance between the orbifold planes shrinks to zero. In this regime, the Kaluza-Klein massive mode density scales in precisely the same way. Therefore, their density remains an exponentially small fraction of the total density right up to collision: meaning from the string theory point of view that the black hole states remain exponentially rare.

So we need only worry about black holes produced in the vicinity of the brane collision itself. From the point of view of the higher dimensional theory, the oscillation frequency of the massive Kaluza-Klein modes $\omega \sim |\theta_0 t|^{-1}$ changes adiabatically, $\dot\omega/\omega^2 \sim \theta_0 \ll 1$ for small θ_0, all the way to $t = 0$. Therefore, one expects little particle production before or after $t = 0$. From the dimensionally-reduced point of view, the mass of the string theory black holes is larger than the Hubble constant $\sim t^{-1}$, by the same factor θ_0^{-1}. From either analysis, production of such states should be suppressed by a factor e^{-1/θ_0}, making it negligible for small θ_0.

In sum, for the cosmological models of interest, the Kaluza-Klein modes are exponentially rare when the Milne phase begins, and, since their mass increases as the collision approaches, they should not be generated by the orbifold plane motion. (They effectively decouple

from the low-energy effective theory.) Hence, all the properties we want at the outset of our calculation here are naturally achieved by the contracting phase with $w \gg 1$, as occurs in the cyclic model [8].

III. GENERAL HAMILTONIAN FOR p-BRANES IN CURVED SPACE

The classical and quantum dynamics of p-branes may be treated using canonical methods, indeed p-branes provide an application *par excellence* of Dirac's general method. As Dirac himself emphasized [15], one of the advantages of the canonical approach is that it allows a completely general choice of gauge. In contrast, gauge-fixed methods tie one to a choice of gauge before it is apparent whether that gauge is or is not a good choice. In the situation of interest here, where the background space-time is singular, the question of gauge choice is especially delicate. Hence, the canonical approach is preferable.

In this section we provide an overview of the main results. The technical details are relegated to Appendix A. Our starting point is the Howe-Tucker action for a p-brane described by embedding coordinates x^μ in a background space-time with metric $g_{\mu\nu}$:

$$S_p = -\frac{1}{2}\mu_p \int d^{p+1}\sigma\sqrt{-\gamma}[\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - (p-1)],$$ (6)

where μ_p is a mass per unit p-volume. The p-brane world volume has coordinates σ^α, where $\sigma^0 = \tau$ is the time and σ^i, $i = 1..p$, are the spatial coordinates.

Variation of the action with respect to $\gamma_{\alpha\beta}$ yields the constraint that for $p \neq 1$, $\gamma_{\alpha\beta}$ equals the induced metric $\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}$ whereas for $p = 1$, $\gamma_{\alpha\beta}$ is conformal to the induced metric. Substituting these results back into the action one obtains the Nambu action for the embedding coordinates $x^\mu(\sigma^\alpha)$, i.e., $-\mu_p$ times the induced p-brane world volume. We shall go back and forth between the Howe-Tucker and Nambu forms in this paper. The former is preferable for quantization but the latter is still useful for discussing classical solutions.

The simplest case of (6) is $p = 0$, a 0-brane or massive particle. Writing $\gamma_{00} = -e^2$ with e the "einbein," one obtains

$$S_0 = \frac{1}{2}m\int d\tau(e^{-1}\dot{x}^\mu\dot{x}^\nu g_{\mu\nu} - e),$$ (7)

where we have set $\mu_0 = m$ and the dot above a variable indicates a derivative with respect to τ. Variation with respect to e yields the constraint $e^2 = -\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}$. The canonical momentum is $p_\mu = mg_{\mu\nu}\dot{x}^\nu e^{-1}$ and the constraint implies the familiar mass shell condition $g^{\mu\nu}p_\mu p_\nu = -m^2$.

The canonical treatment for general p is explained in Appendix A. The main result is that a p-brane obeys $p +$ 1 constraints, reading

$$C \equiv \pi_\mu \pi_\nu g^{\mu\nu} + \mu_p^2 \mathrm{Det}(x_{,i}^\mu x_{,j}^\nu g_{\mu\nu}) \approx 0,$$
$$C_i \equiv x_{,i}^\mu \pi_\mu \approx 0,$$ (8)

where "≈ 0" means "weakly zero" in the sense of the Dirac canonical procedure (see Appendix A). Here the brane embedding coordinates are x^μ and their conjugate momentum densities are π_μ. The spatial world-volume coordinates are σ^i, $i = 1..p$, and the corresponding partial derivatives are denoted $x_{,i}^\mu$. The quantity $x_{,i}^\mu x_{,j}^\nu g_{\mu\nu}$ is the induced spatial metric on the p-brane. In Appendix B we calculate the Poisson bracket algebra of the constraints (8), showing that the algebra closes and hence the constraints are all first class. The constraints (8) are invariant under world-volume coordinate transformations.

The Hamiltonian giving the most general evolution in world-volume time τ is then given by

$$H = \int d^p\sigma\left(\frac{1}{2}AC + A^iC_i\right),$$ (9)

with C and C_i given in (8). The functions A and A^i are completely arbitrary, reflecting the arbitrariness in the choice of world-volume time and space coordinates. All coordinate choices related by nonsingular coordinate transformations give equivalent physical results.

For $p = 0$, anything with a spatial index i can be ignored, except the determinant in (8) which is replaced by unity. The first constraint is then the usual mass shell condition, and the Hamiltonian is an arbitrary function of τ times the constraint. The case of $p = 1$, i.e., a string, in Minkowski space-time, $g_{\mu\nu} = \eta_{\mu\nu}$ is also simple and familiar. In this case, the constraints and the Hamiltonian (9) are quadratic. The resulting equations of motion are linear and hence exactly solvable. The constraints (8) amount to the usual Virasoro conditions. In general, the $p + 1$ constraints (8) together with the $p + 1$ free choices of gauge functions A and A_i reduce the number of physical coordinates and momenta to $2(d + 1) - 2(p + 1) = 2(d - p)$, the correct number of transverse degrees of freedom for a p-brane in d spatial dimensions.

IV. WINDING p-BRANES IN $\mathcal{M}_C \times R^{d-1}$

In this paper, we shall study the dynamics of branes which wind around the compact dimension in $\mathcal{M}_C \times R^{d-1}$, the line element for which is given in (2). This space-time possesses an isometry $\theta \to \theta +$ constant, so one can consistently truncate the theory to consider p-branes which wind uniformly around the θ direction. Such configurations may be described by identifying one of the p-brane spatial coordinates (the p'th spatial coordinate, σ^p say) with θ and to simultaneously insist that $\partial_p x^\mu = \partial_p \pi_\mu = 0$, for $\mu = 0..d - 1$.

Through Hamilton's equations, the constraint $\theta = \sigma^p$ implies that $\pi_\theta = 0$. This suggests that we can set $\theta = \sigma^p$ and $\pi_\theta = 0$ and, hence, dimensionally reduce the p-brane to a $(p - 1)$-brane. Detailed confirmation that this is indeed consistent proceeds as follows. We compute the Poisson brackets between all the constraints C, C_i, $\theta - \sigma^p$, and π_θ. Following the Dirac procedure, we then attempt to build a maximal set of first class constraints. The constraints C and C_i commute with each other, for all σ^i, but not with $\theta - \sigma^p$ and π_θ. The solution is to remove all the π_θ and $\theta_{,p}$ terms from C and C_θ by adding terms involving π_θ and $\theta_{,p} - 1 = (\theta - \sigma^p)_{,p}$. The new C and C_i are now first class since they have weakly vanishing Poisson brackets with all the constraints, and the remaining second class constraints are $\theta - \sigma^p$ and π_θ. For these, construction of the Dirac bracket is trivial and it amounts simply to canceling the θ and π_θ derivatives from the Poisson bracket. The conclusion is that we can indeed consistently set $\theta = \sigma^p$ and $\pi_\theta = 0$. We shall see in the following section that eliminating θ and π_θ in this way results directly in the good behavior of the winding modes as $t \rightarrow 0$, in contrast with the bad behavior of bulk modes.

The surviving first class constraints for winding p-branes are those obtained by substituting $\theta = \sigma^p$ and $\pi_\theta = 0$ into the p-brane constraints (8), namely

$$C \equiv \pi_\mu \pi_\nu \eta^{\mu\nu} + \mu_p^2 \theta_0^2 t^2 \mathrm{Det}(x^\mu_{,i} x^\nu_{,j} \eta_{\mu\nu}) \approx 0;$$
$$C_i \equiv x^\mu_{,i} \pi_\mu \approx 0, \tag{10}$$

where i and j now run from 1 to $p - 1$ and μ and ν from 0 to $d - 1$. The t^2 term comes from the $\theta\theta$ component of the $\mathcal{M}_C \times R^{d-1}$ background metric (2). We have also redefined the momentum density π_μ for the $p - 1$-brane to be θ_0 times the momentum density for the p-brane so that the new Poisson brackets are correctly normalized to give a $p - 1$-dimensional delta function. The Hamiltonian is again given by the form (9) with the integral taken over the remaining $p - 1$ spatial coordinates.

For $p = 1$, the reduced string is a particle moving in d dimensions. π_μ is now the momentum p_μ and the determinant appearing in (10) should be interpreted as unity. The second constraint is trivial since there are no remaining spatial directions. The general Hamiltonian reads:

$$H_0 = A(\tau)(p_\mu p_\nu \eta^{\mu\nu} + \mu_1^2 \theta_0^2 t^2), \tag{11}$$

where μ, ν run from 0 to $d - 1$ and $A(\tau)$ is an arbitrary function of τ. We shall study the quantum field theory for this Hamiltonian in Sec. VI.

Comparing (10) with (8), we see that a p-brane which winds around the compact dimension in $\mathcal{M}_C \times R^{d-1}$ behaves like a $p - 1$-brane in Minkowski space-time with a time-dependent effective tension $\mu_p \theta_0 |t|$, i.e., the

p-brane tension times the size of the compact dimension, $\theta_0 |t|$.

V. WINDING STATES VERSUS BULK STATES

We have discussed in detail how in the canonical treatment the coordinate θ and conjugate momentum density π_θ may be eliminated for p-branes winding uniformly around the compact dimension. This is physically reasonable, since motion of a winding p-brane along its own length (i.e., along θ) is meaningless. This is a crucial difference from bulk states. Whereas the metric on the space of coordinates for bulk states includes the $t^2 d\theta^2$ term, the metric on the space of coordinates for winding states does not. As we discuss in detail in Appendix D, when we quantize the system the square root of the determinant of the metric on the space of coordinates appears in the quantum field Hamiltonian. For bulk modes the metric on the space of coordinates inherits the singular behavior of the background metric (2), degenerating at $t = 0$ and causing the field equations to become singular at $t = 0$ even for θ-independent field modes (see Sec. X). Conversely, for winding modes the Hamiltonian operator is regular at $t = 0$.

The metric on the space of coordinates is defined by the kinetic energy term in the action: if the action reads $S = \frac{1}{2} \int d\tau g_{IJ} \dot{x}^I \dot{x}^J + \ldots$, where x^I are the coordinates, then g_{IJ} is the metric on the space of coordinates. (The sum over I includes integration over worldsheet spatial coordinates.) This superspace metric is needed for quantizing the theory. For example, in the coordinate representation one needs an inner product on Hilbert space and this involves integration over coordinates. The determinant of g_{IJ} is needed in order to define this integral (see Appendix D).

The simplest way to identify the physical degrees of freedom is to choose a gauge, for example $A = $ constant, $A_i = 0$. We then determine the corresponding gauge-fixed action and read off g_{IJ} from the kinetic terms. For bulk particles, the gauge-fixed action is (7) with $e = 1$ and the metric is just the background metric.

We have already derived the Hamiltonian for winding states, and showed how through the use of Dirac brackets the θ coordinate may be discarded. If we choose the gauge $A = 1$, $A_i = 0$ in the Hamiltonian (9) with constraints given in (10), we can construct the corresponding gauge-fixed action:

$$S_{gf} = \int d\tau d^{p-1}\sigma \frac{1}{2}[\dot{x}^\mu \dot{x}^\mu \eta_{\mu\nu}$$
$$- \mu_{p-1}^2 \theta_0^2 t^2 \mathrm{Det}(x^\mu_{,i} x^\nu_{,j} \eta_{\mu\nu})], \tag{12}$$

where μ, ν run over 0 to $d - 1$ and i, j run from 1 to $p - 1$. One may check that the classical equations following from the action (12) are the correct Lagrangian equations for the p-brane in a certain world-volume co-

ordinate system and that these equations preserve the constraints (10) (see Appendix C).

The metric on the space of coordinates may be inferred from the kinetic term in (12), and it is just the Minkowski metric. In contrast, as discussed, the metric on the space of coordinates for bulk states involves the full background metric (2) which degenerates at $t = 0$. The difference means that whereas the quantum fields describing winding states are regular in the neighborhood of $t = 0$, those describing bulk states exhibit logarithmic divergences. In the penultimate section of this paper we argue that these divergences are plausibly the origin of the bad perturbative behavior displayed by strings and particles propagating on Lorentzian orbifolds, behavior we do not expect to be exhibited in M2-brane winding states in M theory.

VI. TOY MODEL: WINDING STRINGS IN $\mathcal{M}_C \times R^{d-1}$

Before approaching the problem of quantizing winding membranes, we start with a toy model consisting of winding string states propagating in $\mathcal{M}_C \times R^{d-1}$. This problem has also been considered by others [16,17] and in more detail than we shall do here. They point out and exploit interesting analogies with open strings in an electric field. Our focus will be somewhat different and will serve mainly as a warmup for case of winding M2-branes in which we are more interested.[1]

Strings winding uniformly around the compact θ dimension in (2) appear as particles from the d-dimensional point of view. To study the classical behavior of these particles, it is convenient to start from the Nambu action for the string

$$S = -\mu \int d^2\sigma \sqrt{-\mathrm{Det}(\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu})}, \qquad (13)$$

where μ is the string tension (to avoid clutter we set $\mu_1 = \mu$ for the remainder of this section). The string worldsheet coordinates are $\sigma^\alpha = (\tau, \sigma)$.

For the winding states we consider, we can set $\theta = \sigma$, so $0 \leq \sigma \leq \theta_0$. We insist that the other space-time coordinates of the string $x^\mu = (t, \vec{x})$ do not depend on σ. It is

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[1]We emphasize that this toy model of winding string states is just that: we are not discussing the case of a full string theory on this background, as is done in Ref. [17]. In string theory, generically there are massless, point-like states, including the dilaton and graviton, which propagate in the bulk. The metric on the configuration space of such states degenerates at $t = 0$, leading to logarithmic divergences in the massless fields [3,12]. In contrast, for the M theory setup of interest to us, as t tends to zero the light states in the theory (including the graviton and the dilaton) are entirely accounted for by winding number one membrane states: there are no additional massless bulk states to be considered. As we have shown, the metric on configuration space for these winding states is non-singular at $t = 0$, so there should be no correspondig divergences. (We thank Ben Craps and Boris Pioline for helpful remarks on this point.)

convenient also to choose the gauge $t = \tau$, in which the action (13) reduces to

$$S = -\mu\theta_0 \int dt |t| \sqrt{1 - \dot{\vec{x}}^2}, \qquad (14)$$

in which t is now the time, not a coordinate. This is the usual square root action for a relativistic particle, but with a time-dependent mass $\mu\theta_0|t|$. The canonical momentum is $\vec{p} = \mu\theta_0|t|\dot{\vec{x}}/\sqrt{1 - \dot{\vec{x}}^2}$ and the classical Hamiltonian generating evolution in the time t is $H = \sqrt{\vec{p}^2 + (\mu\theta_0 t)^2}$. This is regular at $t = 0$, indicating that the classical equations should be regular there.

Because of translation invariance, the canonical momentum $\vec{p}$ is a constant of the motion. Using this, one obtains the general solution

$$\vec{x} = \vec{x}_0 + \frac{\vec{p}}{\mu\theta_0} \sinh^{-1}(\mu\theta_0 t/|\vec{p}|), \qquad -\infty < t < \infty, \qquad (15)$$

according to which the particle moves smoothly through the singularity. At early and late times the large mass slows the motion to a crawl. However, at $t = 0$ the particle's mass disappears and it instantaneously reaches the speed of light. The key point for us is that these winding states have completely unambiguous evolution across $t = 0$, even though the background metric (2) is singular there.

Now we turn to quantizing the theory, as a warmup for the membranes we shall consider in the next section. The relevant classical Hamiltonian was given in (11): it describes a point particle with a mass $\mu\theta_0|t|$. In a general background space-time, ordering ambiguities appear, which are reviewed in Appendix D. However, in the case at hand, there are no such ambiguities. The metric on the space of coordinates is the Minkowski metric $\eta_{\mu\nu}$. The standard expression for the momentum operator $p_\nu = -i\hbar(\partial/\partial x^\nu)$, and the Hamiltonian H given in (11) are clearly Hermitian under integration over coordinate space, $\int d^d x$. Finally, the background curvature R vanishes for our background so there is no curvature term ambiguity either.

Quantization now proceeds by setting $p_\mu = -i\partial_\mu$ (we use units in which $\hbar$ is unity) in the Hamiltonian constraint (11) which is now an operator acting on the quantum field ϕ. Fourier transforming with respect to $\vec{x}$, we obtain

$$\ddot{\phi} = -[\vec{p}^2 + (\mu\theta_0 t)^2]\phi, \qquad (16)$$

i.e., the Klein Gordon equation for a particle with a mass $\mu\theta_0|t|$.

Equation (16) is the parabolic cylinder equation. Its detailed properties are discussed in Ref. [18], whose notation we follow. We write the time-dependent frequency as $\omega \equiv \sqrt{\vec{p}^2 + (\mu\theta_0 t)^2}$. At large times $\mu\theta_0|t| \gg |\vec{p}|$, ω is slowly varying: $\dot{\omega}/\omega^2 \ll 1$ so all modes follow

WKB evolution. The general solution behaves as a linear combination of $\omega^{-1/2} \exp(\pm i \int \omega dt) \approx t^{-1/2} \exp(\pm i \frac{1}{2} \times [\mu\theta_0 t^2 + (\vec{p}^2 \ln t)/(\mu\theta_0)])$.

For large momentum, $\vec{p}^2 \gg \mu\theta_0$, the WKB approximation remains valid for all time since $\dot{\omega}/\omega^2$ is never large. In the WKB approximation there is no mode mixing and no particle production. Therefore for large momentum one expects little particle production. Departures from WKB are nonperturbative in $\dot{\omega}/\omega^2$, as explicit calculation verifies, the result scaling as $\sim \exp(-|\omega^2/\dot{\omega}|_{\max}) \sim \exp(-\vec{p}^2/\mu\theta_0)$, at large $\vec{p}^2$.

The parabolic cylinder functions which behave as positive and negative frequency modes at large times are denoted $E(a, x)$ and $E^*(a, x)$, where $x = \sqrt{2\mu\theta_0}t$ and $a = -\vec{p}^2/(2\mu\theta_0)$. For positive x they behave, respectively, as $x^{-1/2} \exp[+i(\frac{1}{4}x^2 - a \ln x)]$ and $x^{-1/2} \exp[-i(\frac{1}{4}x^2 - a \ln x)]$. Both $E(a, x)$ and $E^*(a, x)$ are analytic at $x = 0$. They are uniquely continued to negative values through the relation

$$E(a, -x) = -ie^{\pi a}E(a, x) + i\sqrt{1 + e^{2\pi a}}E^*(a, x). \quad (17)$$

For $t < 0$, $E(a, -\sqrt{2\mu\theta_0}t)$ is the positive frequency incoming mode. As we extend t to positive values, (17) yields the outgoing solution consisting of a linear combination of the positive frequency solution E^* and the negative frequency solution E. The Bogoliubov coefficient [19] β for modes of momentum $\vec{p}$ is read off from (17): $= -ie^{-\pi\vec{p}^2/(2\mu\theta_0)}$.

$$\beta = -ie^{\pi a} = -ie^{-\pi\vec{p}^2/(2\mu\theta_0)}. \quad (18)$$

The result is exponentially suppressed at large $\vec{p}$, hence, the total number of particles per unit volume created by passage through the singularity is

$$\int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} |\beta|^2 = \left(\frac{\mu\theta_0}{2\pi}\right)^{(d-1)/2}, \quad (19)$$

which is finite and tends to zero as the rapidity of the brane collision is diminished.

It is interesting to ask what happens if we attempt to attach the $t < 0$ half of $\mathcal{M}_C \times R^{d-1}$, (2) with rapidity parameter θ_0^{in}, to the upper half of $\mathcal{M}_C \times R^{d-1}$ with a different rapidity parameter θ_0^{out}. After all, the field equations for general relativity break down at $t = 0$ and there is insufficient Cauchy data to uniquely determine the solution to the future. It might seem that we have the freedom to attach a future compactified Milne with any parameter θ_0^{out}, since this would still be locally flat away from $t = 0$ and, hence, a legitimate string theory background. However it is quickly seen that this is not allowed. By matching the field ϕ and its first time derivative $\partial_t \phi$ across $t = 0$, we can determine the particle production in this case. We find that due to the jump in θ_0, the Bogoliubov coefficient β behaves like $[(\theta_0^{\text{in}})^{1/2} - (\theta_0^{\text{out}})^{1/2}]/\sqrt{\theta_0^{\text{in}}\theta_0^{\text{out}}}$, at large momentum, independent of $\vec{p}$. This implies divergent particle production, and indicates that to lowest order one only obtains sensible results in the analytically—continued background, which has $\theta_0^{\text{out}} = \theta_0^{\text{in}}$. We conclude that to retain a physically sensible theory we must have $\theta_0^{\text{out}} = \theta_0^{\text{in}}$, at least at lowest order in the interactions.

VII. DYNAMICS OF WINDING M2-BRANES

We now turn to the more complicated but far more interesting case of winding membranes in $\mathcal{M}_C \times R^{d-1}$. We have in mind 11-dimensional M theory, where the 11th, M theory dimension shrinks away to a point. When this dimension is small but static, well-known arguments [10] indicate that M theory should tend to a string theory: Type IIA for circle compactification, heterotic string theory for orbifold compactification. It is precisely the winding membrane states we are considering which map onto the string theory states as the M theory dimension becomes small. What makes this case specially interesting is that the string theory states include the graviton and the dilaton. Hence, by describing string propagation across $t = 0$ we are describing the propagation of perturbative gravity across a singularity, which as explained in Sec. II is a Friedmann-Robertson-Walker cosmological singularity from the d-dimensional point of view.

A winding 2-brane is a string from the d-dimensional point of view. As explained in Sec. III, the Hamiltonian for such strings may be expressed in a general gauge as

$$H = \int d\sigma \left[\frac{A}{2} (\pi_\mu \pi_\nu \eta^{\mu\nu} + \mu_2^2 \theta_0^2 t^2 \eta_{\mu\nu} x^{\mu\prime} x^{\nu\prime}) + A^1 x^{\mu\prime} \pi_\mu \right], \quad (20)$$

where A and A^1 are arbitrary functions of the world-sheet coordinates σ and τ, and prime represents the derivative with respect to σ. Here μ, ν run over $0, 1, .., d - 1$, and primes denote derivatives with respect to σ. The Hamiltonian is supplemented by the following first class constraints

$$\pi_\mu \pi_\nu \eta^{\mu\nu} + \mu_2^2 \theta_0^2 t^2 \eta_{\mu\nu} x^{\mu\prime} x^{\nu\prime} \approx 0; \qquad x^{\mu\prime} \pi_\mu \approx 0, \quad (21)$$

which ensure the Hamiltonian is weakly zero. The latter constraint is the familiar requirement that the momentum density is normal to the string.

The key point for us is that the Hamiltonian and the constraints are regular in the neighborhood of $t = 0$, implying that for a generic class of world-sheet coordinates the solutions of the equations of motion are regular there.

As we did with particles, it is instructive to examine the classical theory from the point of view of the Nambu

action. We may directly infer the classical action for winding membranes on $\mathcal{M}_C \times R^{d-1}$ by setting $\theta = \sigma^2$ and $\partial_2 t = \partial_2 \vec{x} = 0$. The Nambu action for the 2-brane then becomes

$$S = -\mu_2 \theta_0 \int d^2\sigma |t| \sqrt{-\mathrm{Det}(\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu})}, \quad (22)$$

where $\sigma^\alpha = (\tau, \sigma)$ and μ runs over $0, ..d - 1$. This is precisely the action for a string (13) in a time-dependent background $g_{\mu\nu} = \theta_0 |t| \eta_{\mu\nu}$, the appropriate string-frame background corresponding to $\mathcal{M}_C \times R^{d-1}$ in M theory [1].

As a prelude to quantization, let us discuss the classical evolution of winding membranes across $t = 0$. We can pick world-sheet coordinates so that $x^0 \equiv t = \tau$, and $\dot{\vec{x}} \cdot \vec{x}' = 0$. In this gauge the Nambu action is:

$$S = -\mu_2 \theta_0 \int dt d\sigma |t| |\vec{x}'| \sqrt{1 - \dot{\vec{x}}^2}, \quad (23)$$

and the classical equations are

$$\partial_t(\epsilon \dot{\vec{x}}) = \partial_\sigma\left(\frac{t^2}{\epsilon} \partial_\sigma \vec{x}\right), \qquad \partial_t \epsilon = t \frac{(\vec{x}')^2}{\epsilon}, \quad (24)$$

where

$$\epsilon = |t| \sqrt{\vec{x}'^2/(1 - \dot{\vec{x}}^2)} \quad (25)$$

the energy density along the string is $\mu_2 \epsilon$. It may be checked that the solutions of these equations are regular at $t = 0$, with the energy and momentum density being finite there. The local speed of the string hits unity at $t = 0$, but there is no ambiguity in the resulting solutions. The data at $t = 0$ consists of the string coordinates $\vec{x}(\sigma)$ and the momentum density $\vec{\pi}(\sigma) = \mu_2 \theta_0 \epsilon \dot{\vec{x}}$, which must be normal to $\vec{x}'(\sigma)$ but is otherwise arbitrary. This is the same amount of initial data as that pertaining at any other time.

Equations (25) describe strings in a d-dimensional flat FRW cosmological background with $ds^2 = a(t)^2 \eta_{\mu\nu}$ and scale factor $a(t) \propto t^{1/2}$. The usual cosmological intuition is helpful in understanding the string evolution. The comparison one must make is between the curvature scale on the string and the comoving Hubble radius $|t|$. When $|t|$ is larger than the comoving curvature scale on the string, the string oscillates as in flat space-time, with fixed proper amplitude and frequency. However, when $|t|$ falls below the string curvature scale, the string is "frozen" in comoving coordinates. This point of view is useful in understanding the qualitative behavior we shall discuss in the next section.

There is one final point we wish to emphasize. In Sec. II we discussed the general Hamiltonian for a p-brane in curved space. As we have just seen, a winding M2-brane on $\mathcal{M}_C \times R^{d-1}$ has the same action as a string in the background $g_{\mu\nu} = |t| \eta_{\mu\nu}$. We could have considered this

case directly using the methods of Sec. II. The constraints (8) for this case read:

$$(\theta_0 |t|)^{-1} \eta^{\mu\nu} \pi_\mu \pi_\nu + \mu_2^2 \theta_0 |t| \eta_{\mu\nu} x^{\mu\prime} x^{\nu\prime} \approx 0;$$
$$x^{\mu\prime} \pi_\mu \approx 0, \quad (26)$$

and the Hamiltonian would involve an arbitrary linear combination of the two. These constraints may be compared with those coming directly from our analysis of winding 2-branes, given by (10) with $p = 2$. Only the first constraint differs, and *only by multiplication by $\theta_0 |t|$*. For all nonzero $\theta_0 |t|$, the difference is insignificant: the constraints are equivalent. Multiplication of the Hamiltonian by any function of the canonical variables merely amounts to a redefinition of world-sheet time. This is just as it should be: dimensionally-reduced membranes are strings. However, the membrane viewpoint is superior in one respect, namely, that the background metric (2) is nonsingular in the physical coordinates (which do not include θ). That is why the membrane Hamiltonian constraint is nonsingular for these states. The membrane viewpoint tells us we should multiply the string Hamiltonian by $\theta_0 |t|$ in order to obtain a string theory which is regular at $t = 0$. Without knowing about membranes, the naive reaction might have been to discard the string theory on the basis that the string-frame metric is singular there.

VIII. EINSTEIN GRAVITY VERSUS AN EXPANSION IN $1/\alpha'$

We are interested in the behavior of M theory, considered as a theory of M2-branes, in the vicinity of $t = 0$. The first question is, for what range of $|t|$ is the string description valid? The effective string tension μ_1 is given in terms of the M2-brane tension μ_2 by

$$\mu_1 = \mu_2 \theta_0 |t|. \quad (27)$$

The mass scale of stringy excitations is the string-scale $\mu_1^{1/2}$. For the stringy description to be valid, this scale must be smaller than the mass of Kaluza-Klein excitations, $(\theta_0 |t|)^{-1}$. This condition reads

$$|t| < \mu_2^{-1/3} \theta_0^{-1}. \quad (28)$$

The second question is: for what range of $|t|$ may the string theory be approximated by d-dimensional Einstein-dilaton gravity? Recall that the string-frame metric is $g_{\mu\nu} = a^2 \eta_{\mu\nu} = |t| \eta_{\mu\nu}$, so the curvature scale is $\dot{a}/a \sim 1/|t|$. The approximation holds when the curvature scale is smaller than the string-scale $\mu_1^{1/2}$, which implies

$$|t| > \mu_2^{-1/3} \theta_0^{-1/3}. \quad (29)$$

We conclude that for small θ_0 there are three relevant regimes. In units where the membrane tension μ_2 is unity,

they are as follows. For $|t| > \theta_0^{-1}$ the description of M theory in terms of 11-dimensional Einstein gravity should hold. As $|t|$ falls below θ_0^{-1}, the size of the extra dimension falls below the membrane scale and we go over to the ten-dimensional string theory description. In the cosmological scenarios of interest, the incoming state is very smooth [8,9], hence this state should be well described by ten-dimensional Einstein-dilaton gravity, which of course agrees with the 11-dimensional Einstein theory reduced in the Kaluza-Klein fashion. However, when $|t|$ falls below $\theta_0^{-1/3}$, the Einstein-dilaton description fails and we must employ the fundamental description of strings in order to obtain regular behavior at $t = 0$.

Let us consider, then, the relevant string action. As we explained in Sec. IV, for winding M2-branes a good gauge choice is $A = 1$, $A^1 = 0$. The corresponding gauge-fixed world-sheet action was given in (12): for $p = 2$ this reads

$$S_{gf} = \int d\tau d\sigma \frac{1}{2} \eta_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu - \mu_2^2 \theta_0^2 t^2 x^{\mu\prime} x^{\nu\prime}), \quad (30)$$

where the fields $x^\mu = (t, \vec{x})$ depend on σ and τ. This action describes a two-dimensional field theory with a quartic interaction.

We confine ourselves to some preliminary remarks about the perturbative behavior of (30) before proceeding to nonperturbative calculations in the next section. In units where $\hbar$ is unity, the action must be dimensionless. From the d-dimensional point of view, the coordinates x^μ have dimensions of inverse mass and μ has dimensions of mass cubed, so σ/τ must have dimensions of mass squared. However, the dimensional analysis relevant to the quantization of (30) considered as a two-dimensional field theory is quite different: the fields x^μ are dimensionless. From the quartic term we see that $\mu_2 \theta_0$ is a dimensionless coupling. suggesting that perturbation theory in $\mu_2 \theta_0$ should be renormalizable. The string tension at a time t_1 is $\mu_1 = \mu_2 \theta_0 |t_1|$; the usual α' expansion is then an expansion in the Regge slope parameter $\alpha' = 1/(2\pi\mu_1)$, i.e., in negative powers of $\mu_2 \theta_0$. Conversely, the perturbation expansion we are discussing is an *expansion in inverse powers of α'*.

These considerations point the way to the resolution of an apparent conflict between two facts. Winding M2-brane evolution is as we have seen smooth through $t = 0$. We expect M2-branes to be described by strings near $t = 0$. The low-energy approximation to string theory is Einstein-dilaton gravity. Yet, as noted above, in Einstein-dilaton gravity, generic perturbations diverge logarithmically with $|t|$ as t tends to zero. The resolution of the paradox is that general relativity is the leading term in an expansion in α' for the string theory. As we have shown, however, the good behavior of string theory is only apparent in the opposite expansion, in *inverse* powers of α'.

In order to evolve the incoming state defined at large $|t|$ where general relativity is a good description, through small $|t|$ where string theory is still valid but general relativity fails, we must match the standard α' expansion onto the new $1/\alpha'$ expansion we have been discussing. We defer a detailed discussion of this fascinating issue to a future publication.

The action (30) describes a string with a time-dependent tension, which goes to zero at $t = 0$. There is an extensive literature on the zero-tension limit of string theory (see, for example, [20]), in Minkowski space-time. At zero tension the action is (30) but without the second term. This is the action for an infinite number of massless particles with no interactions. Quantum mechanically, there is no central charge and no critical dimension. However, as the tension is introduced, the usual central charge and critical dimension appear [21].

IX. WORLD-SHEET INSTANTON CALCULATION OF LOOP PRODUCTION

The string theory we are discussing, with action (30), is nonlinear and therefore difficult to solve. We can still make substantial progress on questions of physical interest by employing nonperturbative instanton methods. One of the most interesting questions is whether one can calculate the quantum production of M2-branes as the universe passes through the big crunch/big bang transition. As we shall now show, this is indeed possible through Euclidean instanton techniques.

First, let us reproduce the result obtained in our toy model of winding string production, Eq. (18). Equation (16) may be reinterpreted as a time-independent Schrodinger equation, with t being the coordinate, describing an over-the-barrier wave in a upside-down harmonic potential. The Bogoliubov coefficient is then just the ratio of the reflection coefficient R to the transmission coefficient T. For large momentum $|\vec{p}|$, R is exponentially small and T is close to unity since the WKB approximation holds. To compute R we employ the following approach, described in the book by Heading [22]. (For related approximation schemes, applied to string production, see Refs. [23,24].)

The method is to analytically continue the WKB approximate solutions in the complex t-plane. Defining the WKB frequency, $\omega = \sqrt{p^2 + (\mu\theta_0 t)^2}$, one observes there are zeros at $t = \pm ip/(\mu\theta_0)$, where the WKB approximation must fail and where the WKB approximate solutions possess branch cuts. Heading shows that one can compute the reflection coefficient by running the branch cut from $t = -ip/(\mu\theta_0)$ up the imaginary t axis and out along the negative real t axis. Then if one continues an incoming WKB solution, defined below the cut, in from $t = -\infty$, below the branch point at $t = -ip/(\mu\theta_0)$ and back out towards $t = -\infty$ just above the cut, it becomes the outgoing reflected wave. In the leading WKB approximation,
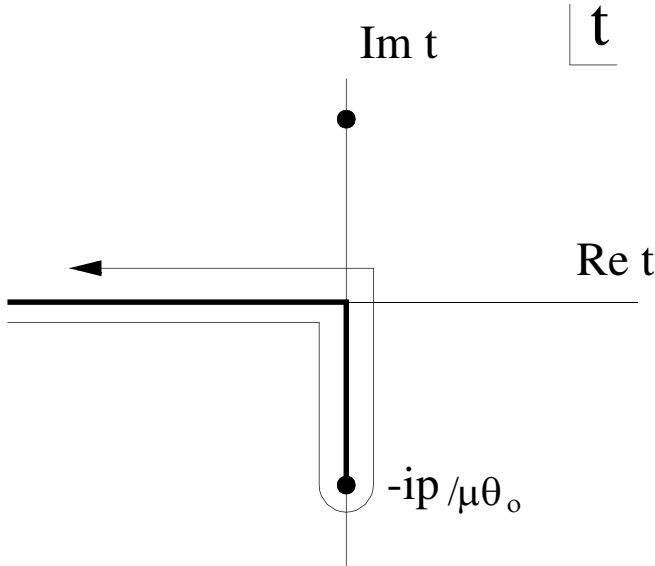
FIG. 1. The contour for computation of Bogoliubov coefficients in string/membrane production.

the wave is given by $w^{-\frac{1}{2}}e^{-i\int wdt}$. Continuing this expression along the stated contour, shown in Fig. 1, the exponent acquires a real contribution on the parts running parallel to the imaginary axis, where $t = -i\tau$. Hence, the magnitude of the reflection amplitude R is given in the first approximation by $e^{(-2\int_0^p wd\tau)}$ where the factor of 2 arises from the two contributions on either side of the axis. With $w = \sqrt{p^2 - (\mu\theta_0\tau)^2}$, one easily sees that the exponent agrees precisely with that in (18).

A more direct method for getting the exponent in the Bogoliubov coefficient is to start not from the field Eq. (16) but from the original action for the fundamental string, (13). We look for an imaginary-time solution (i.e., an instanton) corresponding to the WKB continuation described in the previous section. The gauge-fixed particle Hamiltonian was given in (11), as $H = \frac{1}{2}[\vec{p}^2 - p_0^2 + (\mu\theta_0 t)^2]$. The corresponding gauge-fixed action is, in first order form,

$$S = \int d\tau(-p_0\dot{t} + \vec{p}\cdot\dot{\vec{x}} - H),\qquad(31)$$

where as usual dots over a variable denote τ derivatives. Before continuing to imaginary-time, it is important to realize that the spatial momentum $\vec{p} = \dot{\vec{x}}$ is conserved (by translation invariance). Hence, all states, and, in particular, the asymptotic states we want, are labeled by $\vec{p}$. We are interested in the transition amplitude for fixed initial and final $\vec{p}$, not $\vec{x}$, and we must use the appropriate action which is not (31), but rather

$$S = \int d\tau(-p_0\dot{t} - \dot{\vec{p}}\cdot\vec{x} - H),\qquad(32)$$

related by an integration by parts. The $\dot{\vec{p}}\cdot\vec{x}$ term contributes only a phase in the Euclidean path integral (because $\vec{p}$ remains real) and the $\vec{x}$ integration produces a delta function for overall momentum conservation. Notice that if we instead had used the naive action $\int d\tau\frac{1}{2}\dot{\vec{x}}^2$, we would have obtained a $\vec{p}^2$ term in the Euclidean action of the opposite sign. Similar considerations have been noted elsewhere [25].

Now we continue the action (32) to imaginary time, setting $t = -it_E$ and $\tau = -i\tau_E$. Eliminating p_0, the Euclidean action $S_E \equiv -iS$) is found to be

$$S_E = \int d\tau_E\Big(\frac{1}{2}[\dot{t}_E^2 - (\mu\theta_0 t_E)^2 + \vec{p}^2] + i\dot{\vec{p}}\cdot\vec{x}\Big),\quad(33)$$

where dots now denote derivatives with respect to τ_E. The amplitude we want involves t_E running from 0 to $|\vec{p}|/(\mu\theta_0)$ and back again: (33) is just the action for a simple harmonic oscillator and the required instanton is $t_E = p\cos(\mu\theta_0\tau_E)$, $-\pi/2 < \mu\theta_0\tau_E < \pi/2$. The corresponding Euclidean action is

$$S_E = \frac{\pi}{2}\frac{\vec{p}^2}{\mu\theta_0},\qquad(34)$$

giving precisely the exponent in (18).

Now we wish to apply this method to calculating the production of winding membrane states, described by the action (30). As in the case of analogous calculations of vacuum bubble nucleation within field theory [26], it is plausible that objects with the greatest symmetry are produced since nonsymmetrical deformations will generally yield a larger Euclidean action. Therefore one might guess that the dominant production mechanism is the production of circular loops. Let us start by considering this case. The constraint $\pi_\mu x^{\mu\prime} = 0$ implies that the plane of such loops must be perpendicular to their center of mass momentum p_μ. As in the particle production process previously considered, loops must be produced in pairs carrying equal and opposite momentum. The Hamiltonian for such circular loops is straightforwardly found to be $H = \frac{1}{2}[\vec{p}^2 + p_R^2 - p_0^2 + (2\pi R\mu\theta_0 t)^2]$, Following the same steps that led to (33), we infer that the appropriate Euclidean action is

$$S_E = \int d\tau_E\Big(\frac{1}{2}[\dot{t}_E^2 + \dot{R}^2 - (2\pi\mu_2\theta_0 Rt_E)^2 + \vec{p}^2]$$
$$+ i\dot{\vec{p}}\cdot\vec{x}\Big).\qquad(35)$$

This action describes 2 degrees of freedom t_E and R interacting via a positive potential $t_E^2 R^2$. Up to the trivial symmetries $t_E \rightarrow -t_E$, $R \rightarrow -R$, there is only one classical solution which satisfying the boundary conditions we want, namely, starting and ending at $t_E = 0$, and running up to the zero of the WKB frequency function at $2\pi\mu\theta_0 Rt_E = |\vec{p}|$. This solution has $t_E = R$ and the

Euclidean action is found to be

$$S_E = \frac{(2|\vec{p}|)^{3/2}}{(2\pi\mu_2\theta_0)^{1/2}} \int_0^1 dx\sqrt{1 - x^4}, \qquad (36)$$

where the last integral is $\Gamma[\frac{1}{4}]^2/(6\sqrt{2\pi})$, a constant of order unity which we shall denote I.

The Euclidean action grows like $|\vec{p}|^{3/2}$ at large momentum: this means that the total production of loops is finite. Neglecting a possible numerical prefactor in the Bogoliubov coefficient, we can estimate the number density of loops produced per unit volume,

$$n \sim \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} e^{-2S_E} = (\mu_2\theta_0)^{(d-1)/3}$$

$$\times \frac{2^{(13-7d)/3}\Gamma[2(d-1)/3]}{3\pi^{(d-1)/6}I^{2(d-1)/3}\Gamma[(d-1)/2]}, \qquad (37)$$

where I is given above.

From the instanton solution, the characteristic size of the loops and the time when they are produced are both of the same order, $R \sim |t| \sim (|\vec{p}|/\mu_2\theta_0)^{1/2} \sim (\mu_2\theta_0)^{-1/3}$. The effective string tension when they are produced is $\sim \mu_2\theta_0|t| \sim (\mu_2\theta_0)^{2/3}$.

We have restricted attention so far to the production of circular loops. It is also important to ask whether long, irregular strings are also copiously produced. Even though such strings would be disfavoured energetically, there is an exponentially large density of available states which could in principle compensate. An estimate may be made along the lines of Ref. [24], by simply replacing $\vec{p}^2$ with $\vec{p}^2 + \mu_1 N$, where μ_1 is the effective string tension and N is the level number of the string excitations. This picture only makes sense for times greater than the string time, so we use the tension at the string time, $\mu_1 \sim (\mu_2\theta_0)^{2/3}$. The density of string states scales as $e^{\sqrt{N}}$ hence one should replace (37) with a sum over N:

$$\sim \sum_N e^{\sqrt{N}} \int d^{d-1}\vec{p}\, e^{-[(\vec{p}^2+\mu_1 N)^{3/4}/\mu_1^{3/4}]}. \qquad (38)$$

The $N^{3/4}$ beats the $\sqrt{N}$ so the sum is dominated by modest N, indicating that the production of long strings is suppressed. According to this result, the universe emerges at the string time with of order one string-scale loop per string-scale volume, i.e., at a density comparable to but below the Hagedorn density.

Another key question is whether gravitational back-reaction effects are likely to be significant at the transition. As the universe fills with string loops, what is their effect on the background geometry? We estimate this as follows. Consider a string loop of radius R in M theory frame. Its mass M is $2\pi R$ times the effective string tension $\mu_2 L$, where L is the size of the extra dimension. The effective Einstein-frame gravitational coupling (the inverse of the coefficient of $R/2$ in the Lagrangian density) is given by $\kappa_d^2 = \kappa_{d+1}^2/L$. The gravitational potential produced by such a loop in d space-time dimensions is [27]

$$\Phi = -\kappa_d^2 \frac{M}{(d-2)A_{d-2}R^{d-3}}, \qquad (39)$$

where A_D is the area of the unit D-sphere, $A_D = 2\pi^{\frac{D+1}{2}}/\Gamma[(D+1)/2]$. Specializing to the case of interest, namely, 2-branes in 11-dimensional M theory, the tension μ_2 is related to the 11-dimensional gravitational coupling by a quantization condition relating to the four-form flux, reading [28]

$$\mu_2^3 = 2\pi^2/(n\kappa_{11}^2), \qquad (40)$$

with n an integer. Eqs. (39) and (40) then imply that the typical gravitational potential around a string loop is

$$\Phi = -\frac{105}{64\pi\mu_2^2 R^6 n} \sim -(\mu_2^2 R^6 n)^{-1} \sim -\theta_0^2/n \qquad (41)$$

up to numerical factors.

We conclude that the gravitational potential on the scale of the loops is of order θ_0^2 and therefore is consistently small for small collision rapidity. Since the mean separation of the loops when they are produced is of order their size R, this potential Φ is the typical gravitational potential throughout space. Multiplying the tt component of the background metric (2) by $1 + 2\Phi$ and redefining t, we conclude that the outgoing metric has an expansion rapidity of order $\sim \theta_0(1 + C\theta_0^2)$ with C a constant of order unity. We conclude that for small θ_0 the gravitational backreaction due to string loop production is small. Note that loop production is a quantum mechanical effect taking place smoothly over a time scale of order $(\mu_2\theta_0)^{-1/3}$. Therefore if the rapidity of the outgoing branes alters as we have estimated, it happens smoothly and not like the jump in θ_0 discussed at the end of Sec. VI. Therefore the picture of loop production is consistent with the comments made there.

X. STRINGS ON LORENTZIAN ORBIFOLDS ARE NOT REGULAR

We have shown that strings constructed as winding M2-branes on $\mathcal{M}_C \times R^9$ are analytic in the neighborhood of $t = 0$. This was the setup originally envisaged in the ekpyrotic model, where collapse of the M theory dimension was considered. Subsequently, a number of authors investigated the simpler case of string theory on $\mathcal{M}_C \times R^8$, considered as a Lorentzian orbifold solution of ten-dimensional string theory. This is a simpler, but different setting, hence we expressed misgivings [14,29] about drawing conclusions from these reduced models. We shall now explain why the behavior in the Lorentzian orbifold models is significantly worse than in M theory and,

hence, why no negative conclusion should be drawn on the basis of the failed perturbative calculations.

Consider string theory on the background (2). Let us choose the gauge $x^0 = t = \tau$, and $g_{\mu\nu}\dot{x}^\mu \dot{x}'^\mu = 0$. In this gauge we express the string solution as $\theta(t, \sigma)$ and $\vec{x}(t, \sigma)$. If a classical solution in this gauge possesses a singularity at $t = 0$ in the complex t-plane, for all σ, then there can be no choice of world-sheet coordinates τ and σ which can render the solution analytic in the neighborhood of $t = 0$. For if such a choice existed, one could reexpress τ in terms of t and, hence, $\vec{x}(t, \sigma)$ would be analytic. We shall show that generically, for strings on Lorentzian orbifolds, the solutions possess logarithmic singularities, i.e., branch points, rendering them ambiguous as one circumvents the singularity in the complex t-plane.

The demonstration is straightforward, and our argument is similar to that in earlier papers[2]. We are only interested in the classical equations of motion and we may compute the relevant Hamiltonian from the Nambu action,

$$H \equiv \int d\sigma \mathcal{H} = \int d\sigma \sqrt{\frac{\pi_\theta^2}{\mu t^2} + \frac{\vec{\pi}^2}{\mu} + \mu[(t\theta')^2 + \vec{x}'^2]}. \tag{42}$$

The Hamiltonian equations allow generic solutions in which $\pi_\theta(\sigma)$ tends to any function of σ as t tends to zero. From its definition, the Hamiltonian density $\mathcal{H}$ then diverges as t^{-1}. The equation of motion for θ is $\dot{\theta} = \pi_\theta/(t^2\mathcal{H})$, implying that $\dot{\theta} \rightarrow \pm t^{-1}$ independent of σ. This implies a leading term $\theta \sim \pm \log t$, independent of σ. Recalling that $\mathcal{M}_C$ may be rewritten in flat coordinates by setting $T = t\cosh\theta$ and $Y = t\sinh\theta$, one readily understands this behavior. A geodesic in the (T, Y) coordinates is just $Y = VT$, with V a constant. At small t this requires e^θ or $e^{-\theta}$ to diverge as t^{-1}, which is just the result we found.

We conclude that generic solutions to the Hamiltonian equations possess branch points at $t = 0$ meaning that the solutions to the classical equations are ambiguous as one continues around $t = 0$ in the complex t-plane. This is a much worse situation that encountered for winding M2-branes in M theory.

The second problem occurs when we quantize and construct the associated field theory. Then the bad behavior at $t = 0$ corresponds to a diverging energy density which renders perturbation theory invalid. Since the previous problem can be seen even in pointlike states, let us focus attention on those. As we discuss in detail in Appendix D, one needs to use the metric on the space of coordinates in order to construct the quantum Hamiltonian. In this case, the metric on the space of coordinates is the background metric, (2). The field equation for point particles is then given from (D5): Fourier transforming with respect to $\vec{x}$ and θ, it reads

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} = -\vec{k}^2\phi - \frac{k_\theta^2}{t^2}\phi, \tag{43}$$

where k_θ is quantized in the usual way. This is the equation studied in earlier work[3] on quantum field theory on $\mathcal{M}_C \times R^{d-1}$. The generic solutions of (43) behave for small t as $\log t$ for $k_\theta = 0$, or t^{ik_θ} for $k_\theta \neq 0$. In both cases, the kinetic energy density $\dot{\phi}^2$ diverges as t^{-2}. Similar behavior is found for linearized vector and tensor fields on $\mathcal{M}_C \times R^{d-1}$. These divergences lead to the breakdown of perturbation theory in classical perturbative gravity, an effect which is plausibly the root cause of the bad behavior of the associated string theory scattering amplitudes [11]. As we have stressed, in the sector of M theory considered as a theory of membranes, describing perturbative gravity, this effect does not occur. The field equations are regular in the neighborhood of $t = 0$ and there is no associated divergence in the energy density.

With hindsight, one can now see that directly constructing string theory on Lorentzian orbifolds sheds little light on the M theory case of interest in the ekpyrotic and cyclic models. Whilst the orbifolding construction provides a global map between incoming and outgoing free fields, it does not avoid the blueshifting effect which such fields generically suffer as they approach $t = 0$, which seems to lead to singular behavior in the interactions (although Ref. [30] argues that a resummation may cure this problem).

An alternate approach involving analytic continuation around $t = 0$ has been simultaneously developed, but so far only implemented successfully in linearized cosmological perturbation theory [3,14]. This method may in fact turn out to work even in the M theory context. The point is that by circumnavigating $t = 0$ in the complex t-plane, maintaining a sufficient distance from the singularity, one may still retain the validity of linear perturbation theory and the use of the Einstein equations all the way along the complex time contour. The principles behind this would be similar to those familiar in the context of WKB matching via analytic continuation.

In any case, the main point we wish to make is that now that we have what seems like a consistent microscopic theory for perturbative gravity, valid all the way through $t = 0$, we have a reliable foundation for such investigations.

XI. CONCLUSIONS AND COMMENTS

We have herein proposed an M theoretic model for the passage through a cosmological singularity in terms of a collision of orbifold planes in a compactified Milne $\mathcal{M}_C \times R^9$ background. The model begins with two empty, flat, parallel branes a string length apart approaching one another at constant rapidity θ_0. With this initial condition, we have argued that the excitations naturally bifurcate into light winding M2-brane modes, and a set of massive

modes including the Kaluza-Klein massive states. It is plausible that the massive modes decouple as their mass diverges. The light modes incorporate perturbative gravity and, hence, describe the space-time throughout the transition. Our finding that they are produced with finite density following dynamical equations that propagate smoothly through the transition supports the idea that this M theory picture is well behaved and predictive. Our model also suggests a string theory explanation of what goes wrong with Einstein gravity near the singularity: Einstein gravity is the leading approximation in an expansion in α', but the winding mode picture is a perturbative expansion in $1/\alpha'$.

Our considerations have been almost entirely classical or semiclassical, although we believe the canonical approach we have adopted is a good starting point for a full quantum theory. Much remains to be done to fully establish the consistency of the picture we are advocating. In particular, first we need to understand the sigma model in (A3), and make sure that it is consistent quantum mechanically in the critical dimension. Second, we need to learn how to match the standard α' expansion to the $1/\alpha'$ expansion which we have argued should be smooth around $t = 0$. Third, we need to incorporate string interactions. Although the vanishing of the string theory coupling around $t = 0$ suggests that scattering plays a minor role, the task of fully constructing string perturbation theory in this background remains.

Assuming these nontrivial tasks can be completed, can we say something about the significance of this example? The most obvious application is to the cyclic and ekpyrotic universe models which motivated these investigations, and which produce precisely the initial conditions required. As has been argued elsewhere [1,8], the vicinity of the collision is well modeled by compactified Milne times flat space. Within this model, the calculations reported here yield estimates of the reheat temperature immediately after the brane collision, i.e., at the beginning of the hot big bang. Furthermore, if our arguments of Sec. VIII are correct, they should in principle provide complete matching rules for evolving cosmological perturbations through singularities of the type occurring in cyclic/ekpyrotic models. As we discussed briefly in Sec. X, it is plausible that this matching rule will recover the results obtained earlier for long-wavelength modes by the analytic continuation method [3,14], although that remains to be demonstrated in detail.

Many other questions are raised by our work. Can we extend the treatment to other time-dependent singularities? The background considered here, $\mathcal{M}_C \times R^9$ is certainly very special being locally flat. Although the string-frame metric is singular, it is conformally flat. One would like to study more generic string backgrounds corresponding to black holes, or Kasner/mixmaster space-times in general relativity. As we have argued, there is

no reason to take the latter solutions seriously within M theory since they are only solutions of the low-energy effective theory, which fails in the relevant regime. Nevertheless, they presumably have counterparts in M theory, and it remains a challenge to find them. We would like to believe that by constructing one consistent example, namely, M theory on $\mathcal{M}_C \times R^9$ we would be opening the door to an attack on the generic case.

Such a program is admittedly ambitious. However, should it succeed, we believe it would (and should) completely change our view of cosmology. If the best theories of gravity allow for a smooth passage through time-dependent singularities, this must profoundly alter our interpretation of the big bang, and of the major conceptual problems of the standard hot big bang cosmology.

ACKNOWLEDGMENTS

APPENDIX A: CANONICAL TREATMENT OF p-BRANES IN CURVED SPACE

In this appendix we review the canonical treatment of p-Brane dynamics in curved space. A similar approach is taken in the recent work of Capovilla *et al.* [31]. Earlier treatments include Refs. [32,33] .

We start from the action in Polyakov form. No square roots appear and the ensuing general Hamiltonian involves only polynomial interactions. The action for a p-brane embedded in a background space-time with coordinates x^μ and metric $g_{\mu\nu}$ is

$$S = -\frac{1}{2} \mu_p \int d^{p+1}\sigma \sqrt{-\gamma} [\gamma^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - (p-1)], \tag{A1}$$

where μ_p is a mass per unit p-volume. The p-brane world volume has coordinates σ^α, $\alpha = 0, 1, \ldots, p$ in which the metric is $\gamma_{\alpha\beta}$.

One way to proceed is to vary (A1) with respect to $\gamma_{\alpha\beta}$, hence, obtaining the constraint expressing the world-volume metric $\gamma_{\alpha\beta}$ in terms of the induced metric $\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}$. Substituting back into (A1) one obtains the Nambu action for the embedding coordinates $x^\mu(\sigma)$, i.e., $-\mu_p$ times the induced p-brane world volume.

One can do better, however, by not eliminating the world-volume metric so soon. Instead it is better to retain the $\gamma_{\alpha\beta}$ as independent variables and derive the corresponding Hamiltonian and constraints corresponding to evolution in world-volume time τ. It is convenient to express $\gamma_{\alpha\beta}$ in the form frequently used in canonical general relativity: the world-sheet line element is written

$$\gamma_{\alpha\beta}d\sigma^\alpha d\sigma^\beta = (-\alpha^2 + \beta_k\beta^k)d\tau^2 + 2\beta_i d\tau d\sigma^i + \overline{\gamma}_{ij}d\sigma^i d\sigma^j, \tag{A2}$$

where σ^i, $i = 1, \ldots, p$ are the spatial world-volume coordinates, β^k is the shift vector, and α is the lapse function. The good property of this representation is that the metric determinant simplifies: $\gamma = -\alpha^2\overline{\gamma}$.

In the following discussion we shall for the most part assume $p > 1$ and then comment on the amendments needed for $p = 0, 1$. Using (A2) the action (A1) becomes

$$S = -\frac{1}{2}\mu_p \int d\tau d^p\sigma\, \alpha\overline{\gamma}^{1/2}\Bigg[-\frac{1}{\alpha^2}\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}$$
$$+ 2\frac{\beta^i}{\alpha^2}\dot{x}^\mu x^\nu_{,i}g_{\mu\nu} + \left(\overline{\gamma}^{ij} - \frac{\beta^i\beta^j}{\alpha^2}\right)x^\mu_{,i}x^\nu_{,j}g_{\mu\nu}$$
$$- (p - 1)\Bigg]. \tag{A3}$$

The canonical momenta conjugate to x^μ are found to be

$$\pi_\mu = \mu_p \frac{\overline{\gamma}^{1/2}}{\alpha}(\dot{x}^\nu - \beta^i x^\nu_{,i})g_{\mu\nu}. \tag{A4}$$

No time derivatives of α, β^i and $\overline{\gamma}^{ij}$ appear in the Lagrangian, hence, the corresponding conjugate momenta vanish. In Dirac's language [15], these are the primary constraints.

$$\pi_\alpha \approx 0, \qquad \pi_i \approx 0, \qquad \pi_{ij} \approx 0. \tag{A5}$$

Since Poisson brackets between momenta vanish, these constraints are first class. The total Hamiltonian H then consists of the usual expression $H \equiv \int d^p\sigma[\pi_A\dot{x}^A - L]$,

$$H = \int d^p\sigma\Bigg(\frac{\alpha}{2\mu_p\overline{\gamma}^{1/2}}\pi_\mu\pi_\nu g^{\mu\nu} + \beta^i x^\mu_{,i}\pi_\mu$$
$$+ \frac{\mu_p\alpha\overline{\gamma}^{1/2}}{2}[\overline{\gamma}^{ij}x^\mu_{,i}x^\nu_{,j}g_{\mu\nu} - (p - 1)]\Bigg), \tag{A6}$$

plus an arbitrary linear combination of the primary first class constraints (A5).

Additional secondary constraints are obtained from the Hamiltonian equations for $\dot{\pi}_\alpha$, $\dot{\pi}_i$ and $\dot{\pi}_{ij}$. Insisting these vanish as they must for consistency with (A5), one finds

$$C \equiv \pi_\mu\pi_\nu g^{\mu\nu} + \mu_p^2\overline{\gamma}[\overline{\gamma}^{ij}x^\mu_{,i}x^\nu_{,j}g_{\mu\nu} - (p - 1)] \approx 0,$$
$$C_i \equiv x^\mu_{,i}\pi_\mu \approx 0,$$
$$C_{ij} \equiv \gamma_{ij} - x^\mu_{,i}x^\nu_{,j}g_{\mu\nu} \approx 0. \tag{A7}$$

Following Dirac's procedure, we can now try to eliminate variables using second class constraints (constraints whose Poisson brackets with the other constraints does not vanish). In particular, it makes sense to eliminate γ_{ij} since the corresponding momenta π_{ij} vanish weakly. It is easy to see that C_{ij} are second class since their Poisson brackets with π_{ij} are nonzero. Hence, we eliminate γ_{ij} using the C_{ij} constraint, obtaining

$$C \equiv \pi_\mu\pi_\nu g^{\mu\nu} + \mu_p^2\text{Det}(x^\mu_{,i}x^\nu_{,j}g_{\mu\nu}) \approx 0,$$
$$C_i \equiv x^\mu_{,i}\pi_\mu \approx 0, \tag{A8}$$

as our new constraints, on the remaining variables x^μ and π_μ. The matrix $x^\mu_{,i}x^\nu_{,j}g_{\mu\nu}$ is the induced spatial metric on the brane. When written this way, the C and C_i constraints have zero Poisson brackets with the remaining primary constraints π_α and π_i. A lengthy but straightforward calculation establishes that all Poisson brackets between C and C_i are weakly vanishing (see Appendix B) and, hence, that we have a complete set of first class constraints consisting of π_α, π_i, C_i and C.

The canonical Hamiltonian (A6) is now seen to be a linear combination of C and C_i, with coefficients depending on α and β^i respectively. The general Hamiltonian consists of a sum of this term plus an arbitrary linear combination of the first class constraints, $\int d\sigma(v_\alpha\pi_\alpha + v^i\pi_i)$ where v_α and v_i are arbitrary functions of the world-volume coordinates. From Hamilton's equations, one infers that $\dot{\alpha} = v_\alpha$, and $\dot{\beta}^i = v^i$. Therefore, α and β are completely arbitrary functions of time. As Dirac emphasizes, one can then forget about α, π_α, β^i and π_i and just write the total Hamiltonian for the surviving coordinates as x^μ and π_μ as

$$H = \int d^p\sigma\Bigg(\frac{A}{2}[\pi_\mu\pi_\nu g^{\mu\nu} + \mu_p^2\text{Det}(x^\mu_{,i}x^\nu_{,j}g_{\mu\nu})]$$
$$+ A^i x^\mu_{,i}\pi_\mu\Bigg), \tag{A9}$$

i.e., a linear combination of the constraints (A8) with arbitrary coefficients A and A^i. Different choices of A and A^i then correspond to different choices of world-volume coordinates.

We count the surviving physical degrees of freedom as follows. We start with the $2(d + 1)$ coordinates x^μ and momenta π^μ, each functions of the $p + 1$ world-volume coordinates. Then we impose the $p + 1$ constraints $C = C_i = 0$. Finally, in order to specify time evolution we must pick $p + 1$ arbitrary functions A and A_i. The remaining physical degrees of freedom are $2(d + 1) - 2(p + 1) = 2(d - p)$ in number, the right number of transverse coordinates and momenta for the p-brane.

We have tacitly assumed $p > 1$ in the above analysis. The following minor amendments are needed for $p = 0$ and 1. For $p = 0$, the world-volume metric involves α only and one can ignore anything with an i index except the determinant, which is replaced by unity.

In particular there is no integration over σ in the Hamiltonian, and the canonical momentum density π_μ is replaced by the momentum p_μ. The only constraint is

$$C \equiv p_\mu p_\nu g^{\mu\nu} + \mu_0^2 \approx 0, \qquad (A10)$$

which is just the usual mass shell condition. The general Hamiltonian consists of an arbitrary function of τ times C.

For $p = 1$ the action (A1) is invariant under conformal transformations of the world-volume metric, and hence only two independent combinations of the three world-volume metric variables appear in the decomposition (A2). The corresponding two momenta vanish and these are the primary constraints. Through Hamilton's equations one finds the following secondary constraints:

$$C \equiv \pi_\mu \pi_\nu g^{\mu\nu} + \mu_1^2 x^{\mu\prime} x^{\nu\prime} g_{\mu\nu} \approx 0,$$
$$C_1 \equiv \pi_\mu x^{\mu\prime} \approx 0, \qquad (A11)$$

where primes denote derivatives with respect to $\sigma^1 \equiv \sigma$. The general Hamiltonian again takes the form (A9), with $p = 1$.

APPENDIX B: POISSON BRACKET ALGEBRA OF THE CONSTRAINTS

In the canonical theory [15], one considers arbitrary functions of the canonical variables, and of the time τ. In our case, the canonical variables are fields $x^\mu(\sigma)$ and $\pi_\mu(\sigma)$ depending upon σ, which is regarded as a continuous index labeling an infinite number of canonical variables. In particular, the constraints in (A8) are infinite in number. In this appendix we shall show that the Poisson bracket algebra of the constraints closes, and, hence, that, in Dirac's terminology, they are first class.

The Poisson bracket between any two quantities M and N, which may be arbitrary functions of the canonical variables (local or nonlocal in σ) and of the time τ, is given by

$$\{M, N\} \equiv \int d^p\sigma \left(\frac{\partial M}{\partial x^\mu(\sigma)} \frac{\partial N}{\partial \pi_\mu(\sigma)} - \frac{\partial N}{\partial x^\mu(\sigma)} \frac{\partial M}{\partial \pi_\mu(\sigma)} \right), \qquad (B1)$$

where $[\partial x^\mu(\sigma')/\partial x^\nu(\sigma)] = [\partial \pi_\nu(\sigma')/\partial \pi_\mu(\sigma)] = \delta^\mu_\nu \delta^p(\sigma - \sigma')$, with other partial derivatives being zero.

One way to calculate the Poisson brackets between a set of constraints C and C_i, is to start from a putative Hamiltonian

$$H = \int d^p\sigma \left(\frac{A}{2} C + A^i C_i \right), \qquad (B2)$$

where A and A^i are arbitrary functions of σ, and then compute Hamilton's equations for the τ derivatives of x^μ and π_μ. We then use these to determine the corresponding τ derivatives of C and C_i. Setting these equal to $\{C, H\}$ and $\{C_i, H\}$ with H given by (B2), we are able to infer the

Poisson brackets between the constraints. For the Hamiltonian (B2) with C and C_i given in (A8), Hamilton's equations read

$$\dot{x}^\mu = A g^{\mu\nu} \pi_\nu + A^i x^\mu_{,i},$$
$$\dot{\pi}_\mu = (A^i \pi_\mu)_{,i} - \tfrac{1}{2} A g^{\lambda\nu}_{,\mu} \pi_\lambda \pi_\nu$$
$$+ \mu_p^2 (A \overline{\gamma} \overline{\gamma}^{ij} x^\nu_{,j})_{,i} g_{\mu\nu} + \mu_p^2 A g_{\mu\nu} \Gamma^\nu_{\lambda\epsilon} \overline{\gamma} \overline{\gamma}^{ij} x^\lambda_{,i} x^\epsilon_{,j}, \qquad (B3)$$

where dots denote τ derivatives and $\overline{\gamma}_{ij} = x^\mu_{,i} x^\nu_{,j} g_{\mu\nu}$ is the induced spatial metric on the brane and $\overline{\gamma}$ its determinant. We have made use of the formula $d\overline{\gamma} = \overline{\gamma}\overline{\gamma}^{ij} d\overline{\gamma}_{ij}$.

Using (B3), it is a matter of straightforward algebra to compute $\dot{C}$ and $\dot{C}_i$ and hence infer all of the Poisson brackets. We find

$$\{C(\sigma), C(\sigma')\} = \left[(8\mu_p^2 \overline{\gamma}\overline{\gamma}^{ij} C_j)(\sigma) \frac{\partial}{\partial \sigma^i} \right.$$
$$\left. + 4\mu_p^2 (\overline{\gamma}\overline{\gamma}^{ij} C_j)_{,i}(\sigma) \right] \delta^p(\sigma - \sigma')$$

$$\{C(\sigma), C_i(\sigma')\} = \left[2C(\sigma) \frac{\partial}{\partial \sigma^i} + C_{,i}(\sigma) \right] \delta^p(\sigma - \sigma') \qquad (B4)$$

$$\{C_i(\sigma), C_j(\sigma')\} = \left[C_i(\sigma) \frac{\partial}{\partial \sigma^j} + C_j(\sigma) \frac{\partial}{\partial \sigma^i} \right.$$
$$\left. + \frac{\partial C_i}{\partial \sigma^j}(\sigma) \right] \delta^p(\sigma - \sigma').$$

The right-hand side consists of linear combinations of the constraints and, hence, it vanishes weakly. We conclude that the constraint algebra closes and, hence that the constraints are first class. Notice that the case of strings, $p = 1$, is specially simple since $\overline{\gamma}\gamma^{11} = 1$ and the Poisson bracket algebra is linear, with field-independent structure constants.

The calculation also provides a consistency check on our Hamiltonian (9), which is precisely of the form (B2), since it implies the constraints are preserved under time evolution in τ.

APPENDIX C: EQUIVALENCE OF GAUGE-FIXED HAMILTONIAN AND LAGRANGIAN EQUATIONS

In this appendix we establish that the Lagrangian equations following from the gauge-fixed action (30) for winding $M2$-branes are equivalent to the Lagrangian equations for a string in the time-dependent background $g_{\mu\nu} = |t|\eta_{\mu\nu}$, in a certain string world-sheet coordinate system. This is in accord with our general arguments.

The equations of motion following from the gauge-fixed action (30) are

$$\ddot{\vec{x}} = t^2 \vec{x}'' + 2tt' \vec{x}', \qquad \ddot{t} = t\vec{x}'^2 + tt'^2 + t''t^2. \qquad (C1)$$

and the constraints take the form

$$t\dot{t}' = \dot{\vec{x}} \cdot \vec{x}'; \qquad \dot{t}^2 = \dot{\vec{x}}^2 + t^2(\vec{x}'^2 - t'^2). \qquad (C2)$$

We want to compare these equations with the Lagrangian equations of motion following from the Polyakov action (6), with $p = 1$. These are

$$\partial_\tau[(-\gamma)^{1/2}\gamma^{\tau\tau}\partial_\tau x^\mu] + \partial_\sigma[(-\gamma)^{1/2}\gamma^{\sigma\sigma}\partial_\sigma x^\mu] +$$
$$(-\gamma)^{1/2}\Gamma^\mu_{\nu\lambda}(\gamma^{\tau\tau}\partial_\tau x^\nu \partial_\tau x^\lambda + \gamma^{\sigma\sigma}\partial_\sigma x^\nu \partial_\sigma x^\lambda) = 0,$$
$$(C3)$$

where $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol for the background metric.

We also have the constraints that the world-sheet metric $\gamma_{\alpha\beta}$ is conformal to the induced metric on the string. We have the freedom to choose world-sheet coordinates on the string, but since the equations are conformally invariant, only the conformal class matters. The choice $\gamma_{\alpha\beta} = \Omega^2 \text{diag}(-t^2, 1)$ is found to yield the two constraints (C2).

For our background, $g_{\mu\nu} = |t|\eta_{\mu\nu}$, we have nonzero Christoffel symbols $\Gamma^0_{00} = 1/(2t)$, $\Gamma^i_{j0} = \Gamma^i_{0j} = \delta_{ij}/(2t)$, $\Gamma^0_{ij} = \delta_{ij}/(2t)$, where i runs over-the background spatial indices 1 to $d - 1$. The string equations of motion (C3) are then found to be equivalent to (C1), for all nonzero t.

From the string point of view, this choice of gauge would seem arbitrary, and indeed it would appear to be degenerate at $t = 0$. Yet, as we have seen, this gauge choice is just $A = 1$ and $A^i = 0$, which is entirely natural from the canonical point of view. It has the desirable property that the equations of motion and the constraints are regular at $t = 0$, and from the general properties of the canonical formalism we are guaranteed the existence of an infinite class of coordinate systems, related by nonsingular coordinate transformations, in which the equations of motion will remain regular.

APPENDIX D: ORDERING AMBIGUITIES AND THEIR RESOLUTION FOR RELATIVISTIC PARTICLES

In the main text we have discussed the canonical Hamiltonian treatment of relativistic particles and p-branes. When one comes to quantize these theories in a general background, certain ordering ambiguities appear which must be resolved. Here we provide a brief overview, following the more comprehensive discussion in Ref. [34].

The field equation for a relativistic particle is simply the expression of the quantum Hamiltonian constraint $H = 0$, in a coordinate space representation. The first task is to determine the representation of the momentum operator p_μ in this representation, and then that of the Hamiltonian operator H. As we shall now discuss, this requires knowledge of the metric on the space of coordinates. We shall only deal with the point particle case.

The classical Hamiltonian constraint for a massive particle in a background metric $g_{\mu\nu}$ reads

$$g^{\mu\nu}p_\mu p_\nu + m^2 \approx 0. \qquad (D1)$$

First we attempt to determine the coordinate space representation of p_μ, consistent with the quantum bracket:

$$[x^\mu, p_\nu] = i\hbar\delta^\mu_\nu. \qquad (D2)$$

One choice is $p_\mu = -i\hbar\partial_\mu$ but this is not unique: the representation $p_\mu = -i\hbar(\partial_\mu + f_\mu)$, with f_μ any function of the coordinates x^μ and τ, is equally good as far as (D2) is concerned.

We now show how f_μ may be determined from the additional requirement that p_μ be Hermitian, i.e., that the momentum be real. In the coordinate space representation, this requirement reads

$$\langle\chi|p_\mu|\phi\rangle \equiv \int d^d x[-g(x)]^{1/2}(\chi^* p_\mu \phi) = \langle\phi|p_\mu|\chi\rangle^*$$
$$\equiv \int d^d x[-g(x)](\phi^* p_\mu \chi)^*, \qquad (D3)$$

where the integration runs over-the space of coordinates and $g_{\mu\nu}$ is the metric on that space. It is straightforward to check that the naive operator $-i\hbar\partial_\mu$ is in fact *not* Hermitian for general $g_{\mu\nu}$, but that

$$p_\mu = -i\hbar(\partial_\mu + \tfrac{1}{4}[\partial_\mu \ln(-g)])$$
$$= -i\hbar(-g)^{-1/4}\partial_\mu(-g)^{1/4} \qquad (D4)$$

is. This discussion uniquely determines the real part of f_μ: an imaginary part may be absorbed in an unobservable phase of the wave function [34].

Similarly, when we consider the Hamiltonian constraint (D1), the questions arise of where to place the $g^{\mu\nu}$ relative to the p_μ's, and whether to include any factors of the metric determinant g. The resolution is familiar: if we write the Hamiltonian in covariant derivatives on the space of coordinates, it will be Hermitian since we can integrate by parts ignoring the $\sqrt{-g}$ factor in the measure. This suggests setting the first term in (D1) equal to the scalar Laplacian:

$$g^{\mu\nu}p_\mu p_\nu \rightarrow -\hbar^2(-g)^{-1/2}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu$$
$$= (-g)^{-1/2}p_\mu(-g)^{1/2}g^{\mu\nu}p_\nu(-g)^{-1/4}. \qquad (D5)$$

It is straightforward to check that this is the only choice of ordering and powers of $(-g)$ which is Hermitian and has the correct classical limit. Nevertheless, this ordering is not immediately apparent! More generally, one can also include terms involving commutators of p_μ which are zero in the classical limit, but which produce the Ricci scalar R in the quantum Hamiltonian [34]. In the spacetime we consider R is zero. Hence, such terms do not arise.

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